Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 4, 1085–1100

Persistent URL: `http://dml.cz/dmlcz/143046`

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DECOMPOSITION OF ℓ-GROUP-VALUED MEASURES

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(Received August 28, 2011)

Abstract. We deal with decomposition theorems for modular measures $\mu\colon L \to G$ defined on a D-lattice with values in a Dedekind complete ℓ-group. Using the celebrated band decomposition theorem of Riesz in Dedekind complete ℓ-groups, several decomposition theorems including the Lebesgue decomposition theorem, the Hewitt-Yosida decomposition theorem and the Alexandroff decomposition theorem are derived. Our main result—also based on the band decomposition theorem of Riesz—is the Hammer-Sobczyk decomposition for ℓ-group-valued modular measures on D-lattices. Recall that D-lattices (or equivalently lattice ordered effect algebras) are a common generalization of orthomodular lattices and of MV-algebras, and therefore of Boolean algebras. If L is an MV-algebra, in particular if L is a Boolean algebra, then the modular measures on L are exactly the finitely additive measures in the usual sense, and thus our results contain results for finitely additive G-valued measures defined on Boolean algebras.

Keywords: D-lattice, measure, lattice ordered group, decomposition, Hammer-Sobczyk decomposition

MSC 2010: 28B15, 06C15

1. Introduction

In this article we deal with decomposition theorems for modular measures $\mu\colon L \to G$ (see Definition 2.9) defined on a D-lattice with values in a Dedekind complete ℓ-group.

D-lattices (or equivalently lattice ordered effect algebras) are a common generalization of orthomodular lattices and of MV-algebras, and therefore of Boolean algebras. As for the significance of this structure we refer to [8]. If L is an MV-algebra, then every measure is modular. In particular, if L is a Boolean algebra, then the modular measures on L according to Definition 2.9 are exactly the finitely additive measures in the usual sense.

To obtain decomposition theorems, we first observe that the space $b(L,G)$ of all bounded G-valued modular measures on L is a Dedekind complete ℓ-group. Therefore, by the band decomposition Theorem 2.3, for every band A in $b(L,G)$, any $\mu \in b(L,G)$ has a unique decomposition $\mu = \lambda + \nu$ with $\lambda \in A$ and $\nu \in A^{\perp}$. Different bands lead to different decomposition theorems. This method to obtain decomposition theorems is used e.g. in [5] for real-valued measures and in [7], as far as we know, the first time for ℓ-group-valued measures on Boolean algebras; see also [13], [14]. The idea of this method goes back to the paper by Riesz, see [12]. Our main result is the Hammer-Sobczyk decomposition for $\mu \in b(L,G)$. For ℓ-group-valued measures on Boolean algebras this theorem is contained in the unpublished Ph.D. thesis of the second author written under the supervision of the third author. A decomposition theorem of this type was first proved by Hammer and Sobczyk [11, Theorem 4.2] for bounded real-valued measures defined on an algebra of sets.

This article is organized as follows. In Section 2.1 we first present some basic facts on ℓ-groups. We then deduce from the band decomposition theorem 2.3 that, if A is a solid subset of G, any $x \in G$ can be written as $x = y + \sum\limits_{\gamma\in\Gamma} z_\gamma$ where $y \in A^{\perp}$ and $(z_\gamma)_{\gamma\in\Gamma}$ is an orthogonal family in A (see Theorem 2.7). This becomes a basic tool in Section 4. In Section 2.2 we recall the definition of D-lattices and some of their basic properties. In Section 3 we study lattice properties of $b(L,G)$, in particular we obtain the Jordan decomposition and several decomposition *à* la Hewitt-Yosida, Alexandroff and Lebesgue. Section 4 contains our main result, namely the Hammer-Sobczyk decomposition theorem for measures of $b(L,G)$ (see Theorems 4.1 and 4.8).

All in all, the paper is written in a way that makes it well readable for readers interested mainly in measures defined on Boolean algebras.

2. Preliminaries

In this section we shall give some basic definitions and preliminary results. The books by Birkhoff [4], by Glass and Holland [10] and by Dvurečenskij and Pulmannová [8] can be used for more information on ℓ-groups and effect algebras.

2.1. ℓ-groups. The triple $G = (G, +, \leqslant)$ will always denote an additively written ℓ-group; i.e. $(G,+)$ is a group and $(G,\leqslant)$ is a lattice such that the group translations are isotone.

For $x \in G$, let $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| = x \vee (-x)$ denote the positive part, the negative part and the absolute value of x, respectively. Then $|x| = x^+ + x^-$ and $x = x^+ - x^-$. Moreover, x^+, x^- are the unique elements x_1, $x_2 \in G^+ := \{y \in G\colon\ y \geqslant 0\}$ such that $x = x_1 - x_2$ and $x_1 \wedge x_2 = 0$; this corresponds to the Jordan decomposition.

A subset A of G is called *solid* if $x \in A$, $y \in G$ and $|y| \leqslant |x|$ imply $y \in A$. A solid subgroup of G is called an *ideal* of G. An ideal A of G is called a *band* if for any $M \subseteq A$ for which $\sup M$ exists in G we have $\sup M \in A$.

Elements $x, y \in G$ are called *orthogonal* (we write $x \perp y$) if $|x| \wedge |y| = 0$. For $A \subseteq G$, the disjoint complement $A^{\perp} := \{x \in G \colon x \perp a \text{ for any } a \in A\}$ is a band of G and $A^{\perp\perp} := (A^{\perp})^{\perp}$ is a band containing A. We write $y \ll x$ if $y \in \{x\}^{\perp\perp}$. The disjoint complement of a solid subset can be characterized as follows:

Proposition 2.1. *For a solid subset A of G and $x \in A$, the following conditions are equivalent:*

(1) $x \in A^{\perp}$;

(2) $y \in A$ *and* $y \ll x$ *imply* $y = 0$;

(3) $y \in A$ *and* $|y| \leqslant |x|$ *imply* $y = 0$;

(4) $y \in A$ *and* $0 \leqslant y \leqslant |x|$ *imply* $y = 0$.

Consequently, $x \notin A^{\perp}$ iff there exists an element $y \in A$ with $0 < y \leqslant |x|$.

Proof. (1) $\Rightarrow$ (2): If $x \in A^{\perp}$, then $A \subseteq \{x\}^{\perp}$, hence $A^{\perp} \supseteq \{x\}^{\perp\perp}$. Thus $A \cap \{x\}^{\perp\perp} \subseteq A \cap A^{\perp} = \{0\}$, and $A \cap \{x\}^{\perp\perp} = \{0\}$ is equivalent to (2).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1): Let $a \in A$. Then $y := |a| \wedge |x| \in A$ and $0 \leqslant y \leqslant |x|$. Thus $y = 0$ by (4), i.e. $a \perp x$. Therefore $x \in A^{\perp}$. $\square$

A net $(x_\gamma)_{\gamma\in\Gamma} \in G$ is said to *order converge* to $x \in G$ (in symbols $x_\gamma \xrightarrow{o} x$) if there are nets $(y_\gamma)_{\gamma\in\Gamma}$, $(z_\gamma)_{\gamma\in\Gamma}$ in G such that $y_\gamma \leqslant x_\gamma \leqslant z_\gamma$ and $y_\gamma \uparrow x$, $z_\gamma \downarrow x$; that is (y_γ) is increasing and $\sup y_\gamma = x$, and (z_γ) is decreasing and $\inf z_\gamma = x$. One has $x_\gamma \xrightarrow{o} x$ iff there exists a net (p_γ) in G such that $|x - x_\gamma| \leqslant p_\gamma$ and $p_\gamma \downarrow 0$. Moreover, if $x_\gamma \xrightarrow{o} x$ and $y_\gamma \xrightarrow{o} y$, then $x_\gamma + y_\gamma$, $x_\gamma \vee y_\gamma$, $x_\gamma \wedge y_\gamma$, $-x_\gamma$, $|x_\gamma|$ order converge to $x + y$, $x \vee y$, $x \wedge y$, $-x$, $|x|$, respectively.

A family $(x_\gamma)_{\gamma\in\Gamma}$ is called *summable* if the net of finite partial sums $\sum\limits_{\gamma\in F} x_\gamma$, F is finite $\subseteq \Gamma$, order converges; the order limit is then denoted by $\sum\limits_{\gamma\in\Gamma} x_\gamma$. Obviously, any summable net is order bounded.

G is *Dedekind complete* if every subset of G which is bounded from above has a supremum in G.

G is *super Dedekind complete* if G is Dedekind complete and every set $A \subseteq G$ having a least upper bound contains a countable subset B with $\sup B = \sup A$.

If G is Dedekind complete, then G is commutative and Archimedean, i.e. for every $x \in G \setminus \{0\}$ the set $\{nx \colon n \in \mathbb{N}\}$ is not bounded.

Proposition 2.2. *Let G be Dedekind complete and $(x_\gamma)_{\gamma\in\Gamma}$ an order bounded orthogonal family in G (i.e. $x_\alpha \perp x_\beta$ if $\alpha,\beta\in\Gamma$ and $\alpha\neq\beta$).*

Then (x_γ), (x_γ^+), (x_γ^-), $(|x_\gamma|)$ are summable and

$$\Big(\sum_{\gamma\in\Gamma} x_\gamma\Big)^+ = \sum_{\gamma\in\Gamma} x_\gamma^+ = \sup_{\gamma\in\Gamma} x_\gamma^+, \quad \Big(\sum_{\gamma\in\Gamma} x_\gamma\Big)^- = \sum_{\gamma\in\Gamma} x_\gamma^- = \sup_{\gamma\in\Gamma} x_\gamma^-,$$
$$\Big|\Big(\sum_{\gamma\in\Gamma} x_\gamma\Big)\Big| = \sum_{\gamma\in\Gamma} |x_\gamma| = \sup_{\gamma\in\Gamma} |x_\gamma|.$$

Proof. Since the family (x_γ) is orthogonal, for any finite $F\subseteq\Gamma$ we have $\sum\limits_{\gamma\in F} x_\gamma^+ = \sup\limits_{\gamma\in F} x_\gamma^+$. Hence $(x_\gamma^+)_{\gamma\in\Gamma}$ is summable and $\sum\limits_{\gamma\in\Gamma} x_\gamma^+ = \sup\limits_{\gamma\in F} x_\gamma^+$.

Analogously, (x_γ^-) and $(|x_\gamma|)$ are summable and $\sum\limits_{\gamma\in\Gamma} x_\gamma^- = \sup\limits_{\gamma\in F} x_\gamma^-$ and $\sum\limits_{\gamma\in\Gamma} |x_\gamma| = \sup\limits_{\gamma\in F} |x_\gamma|$. It follows that $(x_\gamma) = (x_\gamma^+) - (x_\gamma^-)$ is summable and $\sum\limits_{\gamma\in\Gamma} x_\gamma = \sum\limits_{\gamma\in\Gamma} x_\gamma^+ - \sum\limits_{\gamma\in\Gamma} x_\gamma^-$. Since $\sum\limits_{\gamma\in\Gamma} x_\gamma^+$ and $\sum\limits_{\gamma\in\Gamma} x_\gamma^-$ are positive and orthogonal, it follows that $\Big(\sum\limits_{\gamma\in\Gamma} x_\gamma\Big)^+ = \sum\limits_{\gamma\in\Gamma} x_\gamma^+$ and $\Big(\sum\limits_{\gamma\in\Gamma} x_\gamma\Big)^- = \sum\limits_{\gamma\in\Gamma} x_\gamma^-$. □

The most important tool in our approach to measure decomposition theorems is the following band decomposition theorem.

Theorem 2.3. *Let G be Dedekind complete and A a band in G. Then*

$$A\oplus A^\perp = G.$$

It follows that for G Dedekind complete $A^{\perp\perp}$ is the band generated by A for any $A\subseteq G$; in particular, for $u\in G$ the set $\{u\}^{\perp\perp}$ is the smallest band containing u. Therefore Theorem 2.3 yields the following version of Lebesgue's decomposition theorem.

Corollary 2.4. *Let G be Dedekind complete and $u\in G$. Then any $x\in G$ can be written in a unique way as $x = y+z$ where $y,z\in G$, $y\perp u$ and $z\ll u$.*

In Corollary 2.4 an element $x\in G$ is decomposed into two orthogonal elements y and z. Our next aim is to decompose x into an infinite sum of orthogonal elements, see Corollary 2.6 and Theorem 2.7. Before presenting these results we determine the band generated by a family $(A_\gamma)_{\gamma\in\Gamma}$ of orthogonal bands, i.e. $x\perp y$ whenever $x\in A_\beta$, $y\in A_\gamma$, $\beta\neq\gamma$ and $\beta,\gamma\in\Gamma$.

Proposition 2.5. *Let G be Dedekind complete and $(A_\gamma)_{\gamma\in\Gamma}$ be a family of orthogonal bands of G.*

Then $A := \left\{ \sum\limits_{\gamma\in\Gamma} x_\gamma \colon\ x_\gamma \in A_\gamma$ for $\gamma \in \Gamma$ and $(x_\gamma)_{\gamma\in\Gamma}$ is order bounded$\right\}$ is the band generated by $\bigcup\limits_{\gamma\in\Gamma} A_\gamma$. Moreover, for $x \in A$ the representation $x = \sum\limits_{\gamma\in\Gamma} x_\gamma$ with $x_\gamma \in A_\gamma$ is unique.

Proof. The proof is based on Proposition 2.2. We first prove the uniqueness statement. Let (x_γ), (y_γ) be order bounded nets with $x_\gamma \in A_\gamma$, $y_\gamma \in A_\gamma$ and

$$\sum_{\gamma\in\Gamma} x_\gamma = \sum_{\gamma\in\Gamma} y_\gamma.$$

Then $0 = \left|\sum\limits_{\gamma\in\Gamma}(x_\gamma - y_\gamma)\right| = \sum\limits_{\gamma\in\Gamma} |x_\gamma - y_\gamma|$, hence $|x_\gamma - y_\gamma| = 0$, i.e. $x_\gamma = y_\gamma$ for all $\gamma \in \Gamma$.

Obviously, A is a subgroup of G. To show that A is a band we claim:

(i) $x \in A$ implies $x^+ \in A$;
(ii) $0 \leqslant y \leqslant x \in A$ implies $y \in A$;
(iii) $0 \leqslant x^{(\alpha)} \in A$ and $x^{(\alpha)} \uparrow x$ imply $x \in A$.

(i) If $x \in A$ and $x = \sum x_\gamma$ with $x_\gamma \in A_\gamma$, then $x^+ = \sum x_\gamma^+ \in A$.
(ii) If $0 \leqslant y \leqslant x = \sum x_\gamma$ with $x_\gamma \in A_\gamma$, then $y = y \wedge x = y \wedge \sup x_\gamma = \sup(y \wedge x_\gamma) = \sum(y \wedge x_\gamma)$ and $y \wedge x_\gamma \in A_\gamma$, hence $y \in A$.
(iii) Let $0 \leqslant x^{(\alpha)} \uparrow x$ and $x^{(\alpha)} = \sum\limits_{\gamma\in\Gamma} x_\gamma^{(\alpha)}$ with $x_\gamma^{(\alpha)} \in A_\gamma$. Then $(x_\gamma^{(\alpha)})_\alpha$ is an increasing net in $[0, x] \cap A_\gamma$. Hence $x_\gamma := \sup\limits_\alpha x_\gamma^{(\alpha)} \in A_\gamma$ and $x = \sup\limits_\alpha \sup\limits_\gamma x_\gamma^{(\alpha)} = \sup\limits_\gamma \sup\limits_\alpha x_\gamma^{(\alpha)} = \sup\limits_\gamma x_\gamma = \sum x_\gamma \in A$.

We have proved that A is a band. It now clear that A is the band generated by $\bigcup\limits_{\gamma\in\Gamma} A_\gamma$. $\square$

Corollary 2.6. *Let G be Dedekind complete and $(u_\gamma)_{\gamma\in\Gamma}$ an orthogonal net in G. Then any $x \in G$ can be written in a unique way as $x = y + \sum\limits_{\gamma\in\Gamma} z_\gamma$ where y, $z_\gamma \in G$, $z_\gamma \ll u_\gamma$ and $y \perp u_\gamma$ for all $\gamma \in \Gamma$.*

Proof. Let A be the band generated by $\{u_\gamma\colon\ \gamma \in \Gamma\}$. Then x has a unique decomposition as $x = y + z$, $y \in A^\perp$, $z \in A$. Applying Theorem 2.3 with $A_\gamma = \{u_\gamma\}^{\perp\perp}$ one sees that z has a unique representation as $z = \sum\limits_{\gamma\in\Gamma} z_\gamma$ with $z_\gamma \in \{u_\gamma\}^{\perp\perp}$. $\square$

The following theorem is an important tool for the Hammer-Sobczyk decomposition of Section 4.

Theorem 2.7. *Let G be Dedekind complete, let A be a nonempty solid subset of G and $x \in G$. Then there exist an element $y \in A^{\perp}$ and an order bounded orthogonal family $(z_\gamma)_{\gamma\in\Gamma}$ in A such that $x = y + \sum\limits_{\gamma\in\Gamma} z_\gamma$.*

Proof. We may assume that $A \neq (0)$. Let $(u_\gamma)_{\gamma\in\Gamma}$ be a maximal orthogonal family in $A\setminus\{0\}$. In view of Corollary 2.6 we have only to verify that $\{u_\gamma\colon\ \gamma\in\Gamma\}^{\perp} = A^{\perp}$.

Suppose the equality were false. Then we could find $x \in \{u_\gamma\colon\ \gamma\in\Gamma\}^{\perp} \setminus A^{\perp}$ and by Proposition 2.1 an element $y \in A$ with $0 < y \leqslant |x|$. Then $y \perp u_\gamma$ for all $\gamma\in\Gamma$, a contradiction to the maximality of $(u_\gamma)_{\gamma\in\Gamma}$. $\square$

2.2. D-lattices. In this section let $(L, \leqslant, 0, 1, \ominus)$ be a *D-lattice*, i.e. $(L, \leqslant)$ is a lattice with a smallest element 0 and a greatest element 1, and $\ominus$ is a partial operation on L such that $b \ominus a$ is defined iff $a \leqslant b$, and for all $a, b, c \in L$:

If $a \leqslant b$ then $b \ominus a \leqslant b$ and $b \ominus (b \ominus a) = a$.

If $a \leqslant b \leqslant c$ then $c \ominus b \leqslant c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

One defines in L a partial operation $\oplus$ as follows:

$$a \oplus b \text{ is defined and } a \oplus b = c \ \text{ iff } \ c \ominus b \text{ is defined and } c \ominus b = a.$$

The operation $\oplus$ is well-defined by the cancellation law [8, on p. 13] ($a \leqslant b$, $a \leqslant c$ and $b \ominus a = c \ominus a$ imply $b = c$), and $(L, \oplus, 0, 1)$ is an effect algebra (see [8, Theorem 1.3.4]), i.e. $\oplus$ is a commutative and associative partial operation, $a \oplus 1$ is only defined for $a = 0$, and $a^{\perp} := 1 \ominus a$ is the unique element such that $a \oplus a^{\perp}$ is defined and $a \oplus a^{\perp} = 1$.

Elements $a, b \in L$ are called *orthogonal* (in symbols $a \perp b$) if $a \leqslant b^{\perp}$. Thus $a \oplus b$ is defined iff $a \perp b$. A finite family $a_1, \ldots, a_n$ of (not necessarily different) elements is *orthogonal* if $a_1 \oplus \ldots \oplus a_n$ exists, where the sum is inductively defined by $a_1 \oplus \ldots \oplus a_k = (a_1 \oplus \ldots \oplus a_{k-1}) \oplus a_k$.

A *D-ideal* in L is a nonempty subset N of L such that $a, b \in N$ and $a \perp b$ imply $a \oplus b \in N$, and $(a \vee c) \ominus c \in N$ for every $a \in N$ and $c \in L$.

If N is a D-ideal, then $b \leqslant a \in N$ implies $b \in N$, and $a \vee b \in N$ whenever $a, b \in N$.

There is a natural bijection between D-ideals and congruence relations in L (see [1]). If $\equiv$ is a congruence with respect to $\wedge, \vee$ and $\ominus$, then $N := \{x \in L\colon\ x \equiv 0\}$ is the corresponding D-ideal and $a \equiv b$ iff $(a \vee b) \ominus (a \wedge b) \in N$.

The set of equivalence classes $L/N := L/\equiv$ is then in a natural way a D-lattice.

Remark 2.8. As observed in the Introduction, any Boolean algebra A is a D-lattice. The difference $\ominus$ is the usual difference in A, i.e. $b \ominus a = b \setminus a$ if $a \leqslant b$. The sum $a \oplus b$ is defined iff a and b are disjoint, and in this case $a \oplus b = a \vee b$. A D-ideal N in A in an ideal in the usual sense, and A/N is a Boolean quotient algebra.

Definition 2.9. A group-valued function μ on L is called a *measure* if $\mu(a \oplus b) = \mu(a) + \mu(b)$ for every $a, b \in L$ with $a \perp b$, or equivalently, if $\mu(b \ominus a) = \mu(b) - \mu(a)$ whenever $a, b \in L$ with $a \leqslant b \in L$.

Function μ is called *modular* if $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$ for every $a, b \in L$.

Proposition 2.10. *If μ is a modular measure, then*

$$N(\mu) := \{a \in L\colon\ \mu(x) = 0 \quad \text{for all} \quad x \leqslant a\}$$

is a D-ideal and the quotient $L/N(\mu)$ is a modular D-lattice.

For the modularity in Proposition 2.10, see [9].

For $a \in L$, the *height* $h(a)$ of a is the supremum of the lengths of the chains $0 = a_0 < a_1 < \ldots < a_n = a$. The lattice L is *of finite length* if $h(1)$ is finite.

If L is a modular D-lattice of finite length, then h is a modular measure on L (see [4, on p. 41] and [3, Proposition 2.15]).

Important in Section 4 is the next result, which follows from [3, Proposition 5.4 and 2.14].

Proposition 2.11. *Let L be a modular irreducible D-lattice of finite length and $\mu\colon L \to G$ a group-valued modular measure. Then L is atomic and there is an element $t \in G$ such that $\mu(x) = h(x)t$ for any $x \in L$.*

A D-lattice of finite length is irreducible iff $\{0\}$ and L are the only D-ideals in L. Later on we use only the obvious part ($\Leftarrow$) of this fact.

If L is a Boolean algebra, then—in contrast to the general case of D-lattices—Proposition 2.11 becomes obvious since any irreducible Boolean algebra has at most two elements.

3. The space $b(L, G)$ of bounded modular measures

In this section, let L be a D-lattice and let G be a Dedekind complete ℓ-group.

A G-valued function is called *bounded* if its range is (order) bounded.

Our aim is to obtain various decomposition theorems for bounded modular measures based on the band decomposition theorem 2.3.

Theorem 3.1. *The space $b(L,G)$ of all G-valued bounded modular measures on L is a Dedekind complete ℓ-group. The positive part of $\mu \in b(L,G)$ is given by*

$$\mu^+(a) = \sup\{\mu(b)\colon\ b \leqslant a\}.$$

If G is super Dedekind complete, then $b(L,G)$ is super Dedekind complete, too.

Proof. By [2, Corollary 2.4] $b(L,G)$ is an ℓ-group. Moreover, the formula for μ^+ given above is proved there.

To prove the Dedekind completeness of $b(L,G)$, let $(\mu_\gamma)_{\gamma\in\Gamma}$ be an increasing net in $b(L,G)$ and $\lambda \in b(L,G)$ such that $0 \leqslant \mu_\gamma \leqslant \lambda$. Put $\mu(a) := \sup\limits_{\gamma\in\Gamma} \mu_\gamma(a)$, i.e. $\mu_\gamma(a) \xrightarrow{o} \mu(a)$ for $a \in L$. It immediately follows from the continuity of the addition in G with respect to the order convergence that μ is a modular measure. Since the pointwise supremum of (μ_γ) is a modular measure, it is clear that it is the supremum in $b(L,G)$.

If G is super Dedekind complete, then there is an increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $\mu_{\gamma_n}(1) \xrightarrow{o} \mu(1)$. Thus, for any $a \in L$ we have $(\mu - \mu_{\gamma_n})(a) \leqslant (\mu - \mu_{\gamma_n})(1) \xrightarrow{o} 0$, therefore $\sup\limits_{n\in\mathbb{N}} \mu_{\gamma_n}(a) = \mu(a)$. Hence $\mu = \sup\limits_{n\in\mathbb{N}} \mu_{\gamma_n}$. □

In contrast to Theorem 3.1 the partially ordered group of all G-valued bounded (not necessarily modular) measures on L is in general not an ℓ-group, even if L is an orthomodular lattice.

Example 3.2. Let MO2 (sometimes called the Chinese lantern) be the orthomodular lattice $\{0,1,x,x^\perp,y,y^\perp\}$ of six elements and height 2. Let μ,ν be the real-valued measures on MO2 defined by $\mu(1) = \nu(1) = 1$, $\mu(x) = \mu(y) = \nu(x) = 1/2$, $\nu(y) = 1/4$. Then μ and ν do not have a least upper bound.

Various important bands of $b(L,G)$ can be described in the following way.

Proposition 3.3. *Let $\mathcal{R}$ be a system of nets in L such that for any $(a_\gamma) \in \mathcal{R}$ and $a \in L$ the net $(a_\gamma \wedge a)_{\gamma\in\Gamma}$ belongs to $\mathcal{R}$. Then*

$$b_{\mathcal{R}}(L,G) := \{\mu \in b(L,G)\colon\ \mu(a_\gamma) \xrightarrow{o} 0 \quad \textit{for every} \quad (a_\gamma) \in \mathcal{R}\}$$

is a band in $b(L,G)$.

Proof. Obviously, $b_{\mathcal{R}}(L,G)$ is a subgroup of $b(L,G)$ and $0 \leqslant \nu \leqslant \mu \in b_{\mathcal{R}}(L,G)$ implies $\nu \in b_{\mathcal{R}}(L,G)$.

To prove that $b_{\mathcal{R}}(L,G)$ is an ideal in $b(L,G)$ it suffices to show that $\mu \in b_{\mathcal{R}}(L,G)$ implies $\mu^+ \in b_{\mathcal{R}}(L,G)$. Let $(a_\gamma) \in \mathcal{R}$. Then for any $a \in L$ we have

$$\begin{aligned}\mu^+(a_\gamma) &= \mu^+(a \vee a_\gamma) - \mu^+((a \vee a_\gamma) \ominus a_\gamma)\\ &\leqslant \mu^+(1) - \mu((a \vee a_\gamma) \ominus a_\gamma) = \mu^+(1) - \mu(a) + \mu(a \wedge a_\gamma).\end{aligned}$$

Since $(a \wedge a_\gamma) \in \mathcal{R}$ and $\mu \in b_{\mathcal{R}}(L,G)$, it follows that

$$\limsup \mu^+(a_\gamma) \leqslant \mu^+(1) - \mu(a) + \limsup \mu(a \wedge a_\gamma) = \mu^+(1) - \mu(a)$$

and finally $\limsup \mu^+(a_\gamma) \leqslant \inf\limits_{a \in L}(\mu^+(1) - \mu(a)) = 0$. Thus $\mu^+(a_\gamma) \xrightarrow{o} 0$, i.e. $\mu^+ \in b_{\mathcal{R}}(L,G)$.

Let now (μ_α) be a bounded increasing net in $b_{\mathcal{R}}(L,G)$ and μ its supremum in $b(L,G)$. It remains to show that $\mu \in b_{\mathcal{R}}(L,G)$. For that, let $(a_\gamma) \in \mathcal{R}$. Then $(\mu-\mu_\alpha)(a_\gamma) \leqslant (\mu-\mu_\alpha)(1)$, hence $\mu(a_\gamma) \leqslant \mu(1)-\mu_\alpha(1)+\mu_\alpha(a_\gamma)$, thus $\limsup\limits_\gamma \mu(a_\gamma) \leqslant \mu(1) - \mu_\alpha(1) + \limsup\limits_\gamma \mu_\alpha(a_\gamma) = \mu(1) - \mu_\alpha(1)$ and finally $\limsup \mu(a_\gamma) \leqslant \inf\limits_\alpha(\mu(1) - \mu_\alpha(1)) = 0$, thus $\mu(a_\gamma) \xrightarrow{o} 0$, i.e. $\mu \in b_{\mathcal{R}}(L,G)$. □

Corollary 3.4. *Let $\mathcal{R}$ be a system of nets in L. Then*

$$\{\mu \in b(L,G)\colon\ |\mu|(a_\gamma) \xrightarrow{o} 0 \quad \text{for every} \quad (a_\gamma) \in \mathcal{R}\}$$

is a band in $b(L,G)$.

Proof. Let $\mathcal{R}_0 = \{(a_\gamma \wedge a)\colon\ (a_\gamma) \in \mathcal{R},\ a \in L\}$. By Proposition 3.3, it is enough to observe that

$$\{\mu \in b(L,G)\colon\ |\mu|(a_\gamma) \xrightarrow{o} 0 \quad \text{for every} \quad (a_\gamma) \in \mathcal{R}\} = b_{\mathcal{R}_0}(L,G).$$

In fact, if $\mu \in b_{\mathcal{R}_0}(L,G)$, then $|\mu| \in b_{\mathcal{R}_0}(L,G)$ by Proposition 3.3, hence $|\mu|(a_\gamma) \xrightarrow{o} 0$ for $(a_\gamma) \in \mathcal{R}$. Vice versa, if $|\mu|(a_\gamma) \xrightarrow{o} 0$ and $a \in L$, then $|\mu|(a_\gamma \wedge a) \xrightarrow{o} 0$, hence $\mu(a_\gamma \wedge a) \xrightarrow{o} 0$. □

Specifying $\mathcal{R}$ in Proposition 3.3 and Corollary 3.4, one obtains various bands in $b(L,G)$ of particular interest; and any of these bands gives rise to a decomposition theorem. First we explain this with the Hewitt-Yosida decomposition.

A function $\mu\colon\ L \to G$ is called *σ-order continuous* if $\mu(a_n) \xrightarrow{o} 0$ for any decreasing sequence (a_n) in L with $\inf a_n = 0$. Obviously, a measure $\mu\colon\ L \to G$ is σ-order continuous iff it is σ-additive, i.e. $\mu(a) = \sum\limits_{n \in \mathbb{N}} \mu(a_n)$ whenever (a_n) is an orthogonal sequence in L and $a = \sup\limits_n \bigoplus\limits_{i=1}^{n} a_i$. Taking in Proposition 3.3 for $\mathcal{R}$ the system of all decreasing sequences in L with infimum 0 one sees that the space of all σ-order continuous functions of $b(L,G)$ is a band.

We say that $\mu \in b(L,G)$ is *purely finitely additive* if the zero measure is the only σ-additive measure $\nu \in b(L,G)$ such that $0 \leqslant \nu \leqslant |\mu|$ (cf. Proposition 2.1 for equivalent conditions).

From the band decomposition theorem 2.3 it now follows:

Theorem 3.5 (Hewitt-Yosida decomposition). *Let $\mu \in b(L, G)$. Then there are unique measures $\mu_1, \mu_2 \in b(L, G)$ such that $\mu = \mu_1 + \mu_2$, μ_1 is σ-additive and μ_2 is purely finitely additive.*

We now give further examples for bands in $b(L, G)$.

A function $\mu\colon L \to G$ is called *order continuous* if $\mu(a_\gamma) \xrightarrow{o} 0$ for any decreasing net (a_γ) in L with $\inf a_\gamma = 0$. Taking for $\mathcal{R}$ the system of all decreasing nets in L with infimum 0 one sees that the space of all order continuous functions of $b(L, G)$ is a band.

Let K and H be an upwards directed and, respectively, a downwards directed subset of L with $0 \in K$ and $1 \in H$. For $a \in L$ put $K_a := \{k \in K\colon k \leqslant a\}$ and $H_a := \{h \in H\colon a \leqslant h\}$. Then $K_a \times H_a$ becomes a directed system defining $(k_1, h_1) \leqslant (k_2, h_2)$ if $k_1 \leqslant k_2$ and $h_1 \geqslant h_2$. We call a function $\mu\colon L \to G$ *regular* (with respect to (K, H)) if the net

$$(|\mu|(h \ominus k))_{h \in H_a, k \in K_a}$$

order converges to 0. By Corollary 3.4 the space of all regular measures of $b(L, G)$ is a band. Thus the band decomposition theorem 2.3 yields the following decomposition:

Any $\mu \in b(L, G)$ has a unique decomposition $\mu = \mu_1 + \mu_2$ where $\mu_1, \mu_2 \in b(L, G)$, μ_1 is regular and μ_2 is “antiregular”.

Let K be a subset of L with $0 \in K$. A function $\mu\colon L \to G$ is called *K-smooth* if $|\mu|(a_\gamma) \xrightarrow{o} 0$ whenever (a_γ) is a decreasing net in K with $\inf a_\gamma = 0$. The space of all K-smooth functions of $b(L, G)$ is a band: Take in Corollary 3.4 for $\mathcal{R}$ the system of all decreasing nets (a_γ) in K with $\inf a_\gamma = 0$. Thus the band decomposition theorem 2.3 yields a generalization of the *Alexandroff decomposition.*

At the end of the section we deal with Lebesgue decompositions, i.e. decomposition of $\mu \in b(L, G)$ as $\mu = \mu_1 + \mu_2$ where μ_1 is “λ-continuous” and μ_2 is “λ-singular”.

Let $\lambda, \mu \in b(L, G)$. We say that μ is *order continuous with respect to λ* if $|\lambda|(a_\gamma) \xrightarrow{o} 0$ implies $\mu(a_\gamma) \xrightarrow{o} 0$ for any decreasing net (a_γ) in L. We call μ *order singular with respect to λ* if the zero measure is the only measure $\nu \in b(L, G)$ such that $0 \leqslant \nu \leqslant |\mu|$ and ν is order continuous with respect to λ.

By Proposition 3.3, $\{\mu \in b(L, G)\colon \mu$ is order continuous with respect to $\lambda\}$ is a band in $b(L, G)$. Thus the band decomposition theorem 2.3 yields the following version of the Lebesgue decomposition theorem.

Theorem 3.6 (Lebesgue decomposition theorem). *Let $\lambda, \mu \in b(L, G)$. Then there are unique measures $\mu_1, \mu_2 \in b(L, G)$ such that $\mu = \mu_1 + \mu_2$, μ_1 is order continuous with respect to λ and μ_2 is order singular with respect to λ.*

Another version of Lebesgue's decomposition theorem was already given in Corollary 2.4: For $\lambda, \mu \in b(L,G)$, there are unique measures $\mu_1, \mu_2 \in b(L,G)$ such that $\mu = \mu_1 + \mu_2$, $\mu_1 \ll \lambda$ and $\mu_2 \perp \lambda$.

We now compare several continuity conditions with respect to λ, each of which yields another version of Lebesgue's decomposition theorem.

Proposition 3.7. *For $\lambda, \mu \in b(L,G)$ consider the following conditions*

(1) $\mu \ll \lambda$;
(2) $\mu(a_\gamma) \xrightarrow{o} 0$ *whenever (a_γ) is a net in L with $|\lambda|(a_\gamma) \xrightarrow{o} 0$;*
(3) *μ is order continuous with respect to λ;*
(4) $N(\mu) \supseteq N(\lambda)$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

P r o o f. By Corollary 3.4 the set $\{\mu \in b(L,G)\colon \mu$ satisfies (2)$\}$ is a band containing λ and therefore contains $\{\lambda\}^{\perp\perp}$, the band generated by λ. This proves (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious. $\square$

In Proposition 3.8 we give additional assumptions under which conditions (3) and (4) of Proposition 3.7 are equivalent.

Proposition 3.8. *Let G be super Dedekind complete, L σ-complete and let $\lambda, \mu \in b(L,G)$ be σ-additive. Then μ is order continuous with respect to λ iff $N(\mu) \supseteq N(\lambda)$.*

P r o o f. We prove the nontrivial implication $\Leftarrow$: Let $a_\gamma \downarrow$ with $|\lambda|(a_\gamma) \xrightarrow{o} 0$, i.e. $\inf |\lambda|(a_\gamma) = 0$. Since G is super Dedekind complete there exists an increasing sequence (γ_n) in Γ such that $\inf |\lambda|(a_{\gamma_n}) = 0$. Since L is σ-complete there exists $a \in L$ such that $a = \inf a_{\gamma_n}$. By the σ-additivity of $|\lambda|$ we get $\inf |\lambda|(a_{\gamma_n}) = |\lambda|(a)$. It follows that $|\lambda|(a) = 0$. Hence $a \in N(\lambda)$, therefore $a \in N(\mu)$, in other words $|\mu|(a) = 0$. Since $|\mu|$ is σ-additive, we get $\inf |\mu|(a_{\gamma_n}) = |\mu|(a) = 0$, hence $\inf |\mu|(a_\gamma) = 0$ and so $\mu(a_\gamma) \xrightarrow{o} 0$. $\square$

Easy examples show that also under the additional assumptions of Proposition 3.8, even when L is a Boolean algebra, (3) does not imply (2) and (2) does not imply (1) in Proposition 3.7:

Example 3.9. Let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and m the Lebesgue measure on $\mathcal{A}$.

(a) Then $\lambda(A) := m(A)\chi_{[0,1]}$ and $\mu(A) := \chi_A$ define bounded σ-additive measures $\lambda, \mu\colon \mathcal{A} \to L_1(m)$. μ is order continuous with respect to λ. On the other hand, if

$A_n := \{x \in [0,1] \colon \exists y \in [s_n, s_{n+1}]$ with $x \equiv y \bmod \mathbb{Z}\}$ where $s_n = \sum_{i=1}^{n} 1/i$, then $|\lambda|(A_n) \xrightarrow{o} 0$, but the sequence $\mu(A_n)$ does not order converge to 0. Thus the condition (2) of Proposition 3.7 is not satisfied.

(b) Let $\lambda, \mu \colon \mathbb{P}(\mathbb{N}) \to \mathbb{R}^2$ be defined by $\lambda := (m, 0)$ and $\mu := (0, m)$. Then μ, λ satisfy the condition (2) of Proposition 3.7, but $\mu \ll \lambda$ is not true.

If follows from Proposition 3.3 that, for $\lambda \in b(L,G)$, also the sets $\{\mu \in b(L,G) \colon \mu$ satisfies (2) of Proposition 3.7$\}$ and $\{\mu \in b(L,G) \colon N(\mu) \supseteq N(\lambda)\}$ are bands of $b(L,G)$. We omit the explicit formulation of the corresponding versions of Lebesgue's decomposition theorem.

Corollary 3.4 and Theorem 2.3 will be applied further on in Proposition 4.11 and Theorem 4.12.

4. The Hammer-Sobczyk decomposition

As in Section 3, we assume that L is a D-lattice and G a Dedekind complete ℓ-group.

Recall that a finitely additive probability measure μ on a Boolean algebra A is *strongly continuous* if for every $\varepsilon > 0$ the maximal element of A has a finite decomposition $a_1, \ldots, a_n \in A$ such that $\mu(a_i) < \varepsilon$ $(i = 1, \ldots, n)$, see [5]. There are different natural generalizations of this concept for ℓ-group-valued measures, see Remark 4.10.

We call a measure $\mu \colon L \to G$ *strongly continuous* if for every $t \in G$ with $t > 0$ there is a finite orthogonal family $a_1, \ldots a_n$ in L such that $\bigoplus_{i=1}^{n} a_i = 1$ and $|\mu|(a_i) \not\geq t$ $(i = 1, \ldots, n)$.

This definition is justified by the fact that, for a Boolean algebra L, a finitely additive measure $\mu \colon L \to G$ is strongly continuous (according to our definition) iff it is orthogonal with respect to any two-valued measure from L to G, see also Proposition 4.7.

We first formulate the Hammer-Sobczyk decomposition theorem for ℓ-group-valued measures on Boolean algebras.

Theorem 4.1. *Let A be a Boolean algebra and $\mu \colon A \to G$ a finitely additive bounded measure. Then there are two-valued finitely additive measures $\nu_\gamma \colon A \to G$ $(\gamma \in \Gamma)$ and a strongly continuous bounded finitely additive measure $\lambda \colon A \to G$ such that $(\nu_\gamma(a))_{\gamma \in \Gamma}$ is summable for every $a \in A$ and $\mu = \lambda + \sum_{\gamma \in \Gamma} \nu_\gamma$.*

To prove the Hammer-Sobczyk decomposition for $\mu \in b(L,G)$, we introduce the following notation.

Notation 4.2. For $\mu \in b(L,G)$ and $t \in G^+$, let $N(\mu,t)$ be the set of all elements $\bigoplus_{i=1}^{n} a_i$ where $n \in \mathbb{N}$ and $a_1,\dots,a_n$ is an orthogonal family in L such that $|\mu|(a_i) \not\geqslant t$ $(i = 1,\dots,n)$.

Obviously, μ is strongly continuous iff $1 \in N(\mu,t)$ for every $t > 0$. Since $N(\mu,t)$ is a D-ideal as shown in Lemma 4.3, μ is strongly continuous iff $N(\mu,t) = L$ for every $t > 0$.

Lemma 4.3. *Let $\mu \in b(L,G)$ and $t \in G$, $t > 0$.*

(a) *Then $N(\mu,t)$ is a D-ideal containing $N(\mu)$.*

(b) *The quotient $L/N(\mu,t)$ is a modular D-lattice of finite length.*

Proof. (a) Obviously, $N(\mu) \subseteq N(\mu,t)$ and $a \oplus b \in N(\mu,t)$ whenever $a, b \in N(\mu,t)$ and $a \perp b$.

Let now $a \in N(\mu,t)$ and $c \in L$. It remains to show that $(a \vee c) \ominus c \in N(\mu,t)$. Since $a \in N(\mu,t)$, there exists a finite chain $0 = a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n = a$ with $|\mu|(a_i \ominus a_{i-1}) \not\geqslant t$ $(i = 1,\dots,n)$. Let $b_i := (a_i \vee c) \ominus c$. Then $0 = b_0 \leqslant b_1 \ldots \leqslant b_n = (a \vee c) \ominus c$ and $|\mu|(b_i \ominus b_{i-1}) = |\mu|(a_i \vee c) - |\mu|(a_{i-1} \vee c) = (|\mu|(a_i) + |\mu|(c) - |\mu|(a_i \wedge c)) - (|\mu|(a_{i-1}) + |\mu|(c) - |\mu|(a_{i-1} \wedge c)) = |\mu|(a_i) - |\mu|(a_{i-1}) - (|\mu|(a_i \wedge c) - |\mu|(a_{i-1} \wedge c)) \leqslant |\mu|(a_i) - |\mu|(a_{i-1}) \not\geqslant t$, hence $|\mu|(b_i \ominus b_{i-1}) \not\geqslant t$. This proves $(a \vee c) \ominus c \in N(\mu,t)$.

(b) By Proposition 2.10 $L/N(\mu)$ is modular. Since $N(\mu) \subseteq N(\mu,t)$, the quotient $L/N(\mu,t)$ is an epimorphic image of $L/N(\mu)$, hence modular, too.

Now suppose that $L/N(\mu,t)$ has infinite length. Then, for any $n \in \mathbb{N}$, there is a chain $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n$ in $L/N(\mu,t)$. Choose $a_i \in \alpha_i$ such that $0 = a_0 < a_1 < \ldots < a_n$. Then $a_i \ominus a_{i-1} \notin N(\mu,t)$, hence $|\mu|(a_i \ominus a_{i-1}) \geqslant t$ and $|\mu|(1) \geqslant |\mu|\Big(\bigoplus_{i=1}^{n}(a_i \ominus a_{i-1})\Big) = \sum_{i=1}^{n} |\mu|(a_i \ominus a_{i-1}) \geqslant nt$. We have seen that $nt \leqslant |\mu|(1)$ for any $n \in \mathbb{N}$. This contradicts the fact that G is Archimedean. □

Notation 4.4. Let $m(L,G)$ be the set of all $\delta \in b(L,G)$ such that $N(\delta)$ is a maximal D-ideal and $L/N(\delta)$ is of finite length.

From Proposition 2.10 and Proposition 2.11 immediately follows:

Proposition 4.5. *For $\delta \in m(L,G)$ the quotient $L/N(\delta)$ is a modular irreducible atomic D-lattice and for some $t \in G \setminus \{0\}$ one has $\delta(x) = h(\hat{x})t$ for $x \in \hat{x} \in L/N(\delta)$ where h denotes the height function on $L/N(\delta)$.*

If L is a Boolean algebra, the measures belonging to $m(L,G)$ are exactly the two-valued measures or equivalently the ultrafilter measures.

Proposition 4.6. *Let $\mu \in b(L,G)$. Then μ is not strongly continuous iff there is a measure $\delta \in m(L,G)$ such that $0 < \delta \leqslant |\mu|$.*

Proof. ($\Rightarrow$): If μ is not strongly continuous, then $N(\mu,t) \neq L$ for some $t > 0$. Let M be a maximal D-ideal containing $N(\mu,t)$. Since L/M is an epimorphic image of $L/N(\mu,t)$ and $L/N(\mu,t)$ has finite length by Lemma 4.3(b), L/M has finite length, too.

Define $\delta\colon L \to G$ by $\delta(x) = h(\hat{x})t$ where $x \in \hat{x} \in L/M$ and h is the height function on L/M. Then $\delta \in b(L,G)$ and $N(\delta) = M$, hence $\delta \in m(L,G)$.

We claim that $\delta \leqslant |\mu|$. Let $a \in L\setminus\{0\}$ and $a_i \in L$ with $0 = a_0 < a_1 < \ldots < a_n = a$ such that $0 = \hat{a}_0 < \hat{a}_1 < \ldots < \hat{a}_n = \hat{a}$ is a maximal chain in $[0,\hat{a}]$. Then $h(\hat{a}) = n$, thus $\delta(a) = nt$. Since $b_i := a_i \ominus a_{i-1} \notin M$, hence $b_i \notin N(\mu,t)$, we have $|\mu|(b_i) \geqslant t$. Therefore $|\mu|(a) = |\mu|\Big(\bigoplus\limits_{i=1}^{n} b_i\Big) = \sum\limits_{i=1}^{n} |\mu|(b_i) \geqslant nt = \delta(a)$.

($\Leftarrow$): Let $\delta \in m(L,G)$ with $0 < \delta \leqslant |\mu|$. By Proposition 4.5 there is an element $t \in G$, $t > 0$ such that $\delta(x) = h(\hat{x})t$ where $x \in \hat{x} \in L/N(\delta)$ and h is the height function on $L/N(\delta)$.

Suppose that μ is strongly continuous. Then there is an orthogonal family $a_1,\ldots,a_n$ such that $1 = \bigoplus\limits_{i=1}^{n} a_i$ and $|\mu|(a_i) \ngeqslant t$ $(i = 1,\ldots,n)$. Since $N(\delta) \neq L$, at least one of the elements $a_1,\ldots,a_n$ does not belong to $N(\delta)$. If $a_j \notin N(\delta)$, then $|\mu|(a_j) \geqslant \delta(a_j) = h(\hat{a_j})t \geqslant t$, a contradiction. □

Proposition 4.7. (a) *$m(L,G) \cup \{0\}$ is a solid subset of $b(L,G)$;*
(b) *$\mu \in b(L,G)$ is strongly continuous iff $\mu \in m(L,G)^{\perp}$.*

Proof. (a) Let $\delta \in m(L,G)$ and $\mu \in b(L,G)$ with $|\mu| \leqslant |\delta|$. If $\mu \neq 0$, then $N(\mu) = N(\delta)$, thus $\mu \in m(L,G)$.

Item (b) now follows from Propositions 4.6 and 2.1. □

Proposition 4.7 and Theorem 2.7 immediately yield:

Theorem 4.8 (Hammer-Sobczyk decomposition)**.** *Let $\mu \in b(L,G)$. Then there is a strongly continuous $\lambda \in b(L,G)$ and an orthogonal summable family $(\nu_\gamma)_{\gamma\in\Gamma}$ in $m(L,G)$ such that $\mu = \lambda + \sum\limits_{\gamma\in\Gamma} \nu_\gamma$.*

If $(\nu_\gamma)_{\gamma\in\Gamma}$ is a summable family in $b(L,G)$ and $\nu = \sum\limits_{\gamma\in\Gamma} \nu_\gamma$, then—as can easily be verified—for any $a \in L$ the family $(\nu_\gamma(a))_{\gamma\in\Gamma}$ is summable and $\nu(a) = \sum\limits_{\gamma\in\Gamma} \nu_\gamma(a)$.

If L is a Boolean algebra, then, as observed before, $m(L,G)$ is the space of all two-valued measures on L with values in G. Therefore Theorem 4.1 is a special case of Theorem 4.8.

Remark 4.9. If G is super Dedekind complete, then every orthogonal summable family is at most countable. Hence, if G is super Dedekind complete, the set Γ in Theorems 2.7, 4.1 and 4.8 is at most countable.

We shall compare our version of the Hammer-Sobczyk decomposition (Theorem 4.1) with the version of Boccuto and Candeloro [6, Theorem 4.6].

From now on let A be a Boolean algebra with a maximal element 1. We denote by $\mathcal{D}$ the system of all finite disjoint partitions of 1, and by $c(A,G)$ the space of all finitely additive bounded measures $\mu\colon A \to G$ such that

$$\inf_{D\in\mathcal{D}} \sup_{d\in D} |\mu|(d) = 0.$$

The last condition was introduced by Boccuto and Candeloro [6, Section 3]. They call (positive) measures belonging to $c(A,G)$ “continuous”. Obviously, for real-valued measures “continuity” and “strong continuity” are equivalent properties, but this is not true in general for G-valued measures.

Remark 4.10. (a) It immediately follows from the definition that any measure belonging to $c(A,G)$ is strongly continuous.

(b) Let $\mathcal{A}$ be the Borel algebra of $[0,1]$ and let m be the Lebesgue measure on $\mathcal{A}$. Then $\mu\colon \mathcal{A} \to L_1(m)$ defined by $\mu(A) = \chi_A$ is strongly continuous, but it does not belong to $c(\mathcal{A}, L_1(m))$ (i.e. μ is not continuous in the sense of [6]).

Proposition 4.11. *The set $c(A,G)$ is a band in $b(A,G)$.*

Proof. Define on $\Gamma := \{(D,d)\colon D \in \mathcal{D}, d \in D\}$ a relation by $(D_1,d_1) \leqslant (D_2,d_2)$ iff D_2 is a refinement of D_1. Then $(\Gamma,\leqslant)$ is upwards directed. We set $a_\gamma := d$ if $\gamma = (D,d) \in \Gamma$. Then $(a_\gamma)_{\gamma\in\Gamma}$ is a net and

$$c(A,G) = \{\mu \in b(A,G)\colon\ |\mu|(a_\gamma) \xrightarrow{o} 0\}.$$

Therefore $c(A,G)$ is a band by Corollary 3.4. $\square$

Proposition 4.11 and the band decomposition theorem 2.3 (together with Proposition 2.1) immediately yield:

Theorem 4.12. *Let $\mu \in b(A,G)$. Then there are unique measures $\mu_1 \in c(A,G)$ and $\mu_2 \in b(A,G)$ such that $\mu = \mu_1 + \mu_2$ and μ_2 has the property that the zero measure is the unique measure $\nu \in c(A,G)$ with $0 \leqslant \nu \leqslant |\mu_2|$.*

Theorem 4.12 was proved in [6, Theorem 4.6] for positive measures under the additional assumption that G is super Dedekind complete and weakly σ-distributive.

The proof of Candeloro and Boccuto is completely different from ours; in particular, they use a Stone isomorphism technique and a measure extension theorem. This makes the additional assumptions understandable.

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