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AN IDENTITY INVOLVING DEDEKIND SUMS AND GENERALIZED KLOOSTERMAN SUMS

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Abstract. The various properties of classical Dedekind sums $S(h,q)$ have been investigated by many authors. For example, Yanni Liu and Wenpeng Zhang: A hybrid mean value related to the Dedekind sums and Kloosterman sums, Acta Mathematica Sinica, 27 (2011), 435–440 studied the hybrid mean value properties involving Dedekind sums and generalized Kloosterman sums $K(m,n,r;q)$. The main purpose of this paper, is using the analytic methods and the properties of character sums, to study the computational problem of one kind of hybrid mean value involving Dedekind sums and generalized Kloosterman sums, and give an interesting identity.

Keywords: Dedekind sum, Kloosterman sum, Dirichlet character, analytic method, Gauss sum, identity

MSC 2010: 11M20

1. Introduction

For a positive integer q and an arbitrary integer h, the classical Dedekind sum $S(h,q)$ is defined by

$$S(h,q)=\sum_{a=1}^{q}\Big(\Big(\frac{a}{q}\Big)\Big)\Big(\Big(\frac{ah}{q}\Big)\Big),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

The various properties of $S(h,q)$ were investigated by many authors, see [2], [3] and [4]. For example, L. Carlitz [3] obtained a reciprocity theorem of $S(h,q)$.

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J. B. Conrey et al. [4] studied the mean value distribution of $S(h,q)$, and proved an interesting asymptotic formula. Yanni Liu and Wenpeng Zhang [7] studied the hybrid mean value properties involving Dedekind sums and generalized Kloosterman sums $K(m,n,r;q)$, which are defined as follows (see [5] and [9]):

$$K(m,n,r;q)=\sideset{}{'}\sum_{b=1}^{q} e\Big(\frac{mb^r+n\bar{b}^r}{q}\Big),$$

where $e(y)=\mathrm{e}^{2\pi \mathrm{i} y}$, $\bar{b}$ denotes the solution of the equation $x\cdot b\equiv 1 \bmod q$. They proved the following result:

Let q be a square-full number (i.e. $p \mid q$ if and only if $p^2 \mid q$). Then we have

$$\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q} K(m,a,1;q)\overline{K(m,b,1;q)}S(a\bar{b},q)=\frac{1}{12}\cdot q\cdot\varphi^2(q)\prod_{p\mid q}\Big(1+\frac{1}{p}\Big), \tag{1.1}$$

where $\sideset{}{'}\sum_{a=1}^{q}$ denotes the summation over all $1\leqslant a\leqslant q$ such that $(a,q)=1$, $\prod_{p\mid q}$ denotes the product over all distinct prime divisors p of q, $\varphi(q)$ is the Euler function, and $\overline{f(n)}$ denotes the complex conjugation of $f(n)$.

In this paper, we use analytic methods to study another kind of hybrid mean value involving Dedekind sums and generalized Kloosterman sums, and give an identity very similar to (1.1). That is, we shall prove the following

Theorem. *Let q be an odd square-full number such that $p\equiv 3 \bmod 4$ for all prime divisors p of q. Then we have the identity*

$$\sideset{}{'}\sum_{u=1}^{q}\sideset{}{'}\sum_{v=1}^{q} K(u,u,2;q)K(v,v,2;q)S(u\bar{v},q)=\frac{-1}{12}\cdot q\cdot\varphi^2(q)\cdot 4^{\omega(q)}\cdot\prod_{p\mid q}\Big(1+\frac{1}{p}\Big),$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q.

For general integers r and $q\geqslant 3$, whether there exists an identity for

$$\sideset{}{'}\sum_{u=1}^{q}\sideset{}{'}\sum_{v=1}^{q} K(u,u,r;q)K(v,v,r;q)S(u\bar{v},q)$$

is an interesting problem.

2. Several lemmas

To complete the proof of our Theorem, we need the following lemmas.

Lemma 1. *Let $q > 2$ be an integer, then for any integer a with $(a,q) = 1$ we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d \mid q} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1,\chi)|^2,$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to the character $\chi \bmod d$.

Proof. See Lemma 2 of [8]. □

Lemma 2. *Let p be an odd prime, let k and α be two integers with $k \mid p-1$ and $\alpha \geqslant 2$. Then for any integer n with $(p,n) = 1$ and any non-primitive character $\chi \bmod p^\alpha$ we have the identity*

$$\sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{na^k}{p^\alpha}\Big) = 0.$$

Proof. It is clear that if χ is a non-primitive character $\bmod\ p^\alpha$, then it is also a character $\bmod\ p^{\alpha-1}$. From the properties of the trigonometric sums we know that for any positive integer $q \geqslant 2$ and integer n with $(n,q) = 1$ we have the identity

$$\sum_{u=0}^{q-1} e\Big(\frac{un}{q}\Big) = 0. \tag{1.2}$$

Note that if $k \mid p-1$, then $(k,p) = 1$. From (1.2) and the definition of the reduced residue system modulo p^α we have

$$\begin{aligned}
\sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{na^k}{p^\alpha}\Big) &= \sum_{u=0}^{p-1} \sideset{}{'}\sum_{v=1}^{p^{\alpha-1}} \chi(up^{\alpha-1}+v) e\Big(\frac{n(up^{\alpha-1}+v)^k}{p^\alpha}\Big) \\
&= \sum_{u=0}^{p-1} \sideset{}{'}\sum_{v=1}^{p^{\alpha-1}} \chi(v) e\Big(\frac{knuv^{k-1}p^{\alpha-1}+v^k}{p^\alpha}\Big) \\
&= \sum_{v=1}^{p^{\alpha-1}} \chi(v) e\Big(\frac{v^k}{p^\alpha}\Big) \sum_{u=0}^{p-1} e\Big(\frac{knuv^{k-1}}{p}\Big) = 0.
\end{aligned}$$

This proves Lemma 2. □

Lemma 3. *Let p be an odd prime, and $\alpha \geqslant 2$ an integer. Then for any primitive character $\chi \bmod p^\alpha$ we have the identity*

$$\Big(\frac{\tau(\chi\chi_2)}{\tau(\chi)}\Big)^2 = 1,$$

where $\chi_2(a) = \big(\frac{a}{p}\big)$ is Legendre's symbol, $\tau(\chi) = \sum\limits_{a=1}^{p^\alpha} \chi(a) e\big(\frac{a}{p^\alpha}\big)$ denotes the classical Gauss sum, and $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$.

Proof. First, for any primitive character $\chi \bmod p^\alpha$, from the properties of Gauss sums (see [1] and [6]) we know that

$$\begin{aligned}
(1.3) \qquad \sum_{r=1}^{p-1} \overline{\chi}(p^{\alpha-1}r+1)\Big(\frac{r}{p}\Big) &= \frac{1}{\tau(\chi)} \sum_{r=1}^{p-1} \Big(\frac{r}{p}\Big) \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{a(p^{\alpha-1}r+1)}{p^\alpha}\Big) \\
&= \frac{1}{\tau(\chi)} \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{a}{p^\alpha}\Big) \sum_{r=1}^{p-1} \Big(\frac{r}{p}\Big) e\Big(\frac{ar}{p}\Big) \\
&= \frac{\sum\limits_{r=1}^{p-1} \big(\frac{r}{p}\big) e\big(\frac{r}{p}\big)}{\tau(\chi)} \sum_{a=1}^{p^\alpha} \chi(a) \Big(\frac{a}{p}\Big) e\Big(\frac{a}{p^\alpha}\Big) \\
&= \frac{\tau(\chi\chi_2)}{\tau(\chi)} \sum_{r=1}^{p-1} \Big(\frac{r}{p}\Big) e\Big(\frac{r}{p}\Big) = \frac{\tau(\chi\chi_2)}{\tau(\chi)} \cdot G(p),
\end{aligned}$$

where $G(p) = \sum\limits_{r=1}^{p-1} \big(\frac{r}{p}\big) e\big(\frac{r}{p}\big)$ and $\overline{\chi}$ denotes the conjugation of χ.

Similarly, we can also deduce the identity

$$(1.4) \qquad \sum_{r=1}^{p-1} \chi(p^{\alpha-1}r-1)\Big(\frac{r}{p}\Big) = \chi(-1)\Big(\frac{-1}{p}\Big) \frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})} \cdot G(p).$$

Note that $(p^{\alpha-1}r+1)(p^{\alpha-1}r-1) \equiv -1 \bmod p^\alpha$ for any integer $1 \leqslant r \leqslant p-1$, so we have $\chi(p^{\alpha-1}r+1) \cdot \chi(p^{\alpha-1}r-1) = \chi(-1)$ or $\chi(p^{\alpha-1}r-1) = \chi(-1) \cdot \overline{\chi}(p^{\alpha-1}r+1)$. From these identities and (1.3) we have

$$(1.5) \qquad \sum_{r=1}^{p-1} \chi(p^{\alpha-1}r-1)\Big(\frac{r}{p}\Big) = \chi(-1) \cdot \frac{\tau(\chi\chi_2)}{\tau(\chi)} \cdot G(p).$$

Due to the identities $\overline{\tau(\chi)} = \overline{\chi}(-1) \cdot \tau(\overline{\chi})$, $\overline{\tau(\chi\chi_2)} = \overline{\chi}(-1)\big(\frac{-1}{p}\big) \cdot \tau(\overline{\chi}\chi_2)$, $\chi(-1) = \overline{\chi}(-1) = 1$ or -1, and $G(p) \neq 0$, from (1.4) and (1.5) we have

$$(1.6) \qquad \overline{\Big(\frac{\tau(\chi\chi_2)}{\tau(\chi)}\Big)} = \frac{\overline{\tau(\chi\chi_2)}}{\overline{\tau(\chi)}} = \Big(\frac{-1}{p}\Big) \cdot \frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})} = \frac{\tau(\chi\chi_2)}{\tau(\chi)}.$$

Then from (1.6) we know that $\tau(\chi\chi_2)/\tau(\chi)$ is a real number.

On the other hand, since χ is a primitive character mod p^α with $\alpha \geqslant 2$, so $\chi\chi_2$ is also a primitive character mod p^α, hence from the properties of Gauss sums we have $|\tau(\chi\chi_2)/\tau(\chi)| = 1$. Because $\tau(\chi\chi_2)/\tau(\chi)$ is a real number, we may immediately deduce that $\tau(\chi\chi_2)/\tau(\chi) = 1$ or -1. This proves Lemma 3. □

Lemma 4. *Let p be an odd prime, and let $\alpha \geqslant 2$ be an integer. Then for any integers m, n with $(mn, p) = 1$ and any primitive character $\chi \bmod p^\alpha$, we have the following identities:*

(A) *If χ is a primitive odd character* mod p^α *and $p \equiv 1 \bmod 4$, then*

$$\sum_{a=1}^{p^\alpha}{}' \chi(ma^2 + na^{-2}) = 0.$$

(B) *If χ is a primitive odd character* mod p^α, $p \equiv 3 \bmod 4$ *and $\overline{\chi}\chi_2 = \overline{\chi}_1^2$, then*

$$\sum_{a=1}^{p^\alpha}{}' \chi(ma^2 + na^{-2}) = \frac{\chi_1(mn)\tau(\overline{\chi}_1\chi_2)\tau(\overline{\chi}_1)}{\tau(\overline{\chi})}\Big(\Big(\frac{m}{p}\Big) + \Big(\frac{n}{p}\Big)\Big).$$

(C) *If χ is a primitive even character* mod p^α, $p \equiv 3 \bmod 4$ *and $\overline{\chi} = \overline{\chi}_1^2$, then*

$$\sum_{a=1}^{p^\alpha}{}' \chi(ma^2 + na^{-2}) = \frac{\chi_1(mn)\tau^2(\overline{\chi}_1)}{\tau(\overline{\chi})}\Big(1 + \Big(\frac{mn}{p}\Big)\Big).$$

Proof. Since χ is a primitive character mod p^α, so from the properties of Gauss sums and Legendre's symbol we have

$$\begin{aligned}
(1.7)\qquad \sum_{a=1}^{p^\alpha}{}' \chi(ma^2 + na^{-2}) &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha} \overline{\chi}(b) e\Big(\frac{b(ma^2 + na^{-2})}{p^\alpha}\Big) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{p^\alpha}{}' \overline{\chi}(a^2) \sum_{b=1}^{p^\alpha} \overline{\chi}(b) e\Big(\frac{b(ma^4 + n)}{p^\alpha}\Big) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{b=1}^{p^\alpha} \overline{\chi}(b) e\Big(\frac{nb}{p^\alpha}\Big) \sum_{a=1}^{p^\alpha} \overline{\chi}(a^2) e\Big(\frac{bma^4}{p^\alpha}\Big) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{b=1}^{p^\alpha} \overline{\chi}(b) e\Big(\frac{nb}{p^\alpha}\Big) \sum_{a=1}^{p^\alpha} (1 + \chi_2(a)) \overline{\chi}(a) e\Big(\frac{bma^2}{p^\alpha}\Big).
\end{aligned}$$

(A) If χ is a primitive odd character mod p^α and $p \equiv 1 \bmod 4$, then $\chi_2(-1) = 1$, so $\overline{\chi}$ and $\overline{\chi}\chi_2$ are also two odd characters mod p^α. Therefore, we have

$$\sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{bma^2}{p^\alpha}\Big) = \sum_{a=1}^{p^\alpha} \overline{\chi}(-a) e\Big(\frac{bm(-a)^2}{p^\alpha}\Big) = -\sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{bma^2}{p^\alpha}\Big)$$

or

$$\sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{bma^2}{p^\alpha}\Big) = 0, \tag{1.8}$$

$$\sum_{a=1}^{p^\alpha} \overline{\chi}\chi_2(a) e\Big(\frac{bma^2}{p^\alpha}\Big) = 0. \tag{1.9}$$

From (1.7), (1.8) and (1.9) we may deduce the identity

$$\sum_{a=1}^{p^\alpha}{}' \chi(ma^2 + na^{-2}) = 0. \tag{1.10}$$

(B) If χ is a primitive odd character $\bmod p^\alpha$ and $p \equiv 3 \bmod 4$, then $\chi_2(-1) = -1$, so $\overline{\chi}\chi_2$ is a primitive even character $\bmod p^\alpha$. Therefore, there exits one and only one primitive character χ_1 such that $\overline{\chi}\chi_2 = \overline{\chi}_1^2$. This time we have

$$\sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{bma^2}{p^\alpha}\Big) = 0 \tag{1.11}$$

and

$$\begin{aligned} \sum_{a=1}^{p^\alpha} \overline{\chi}\chi_2(a) e\Big(\frac{bma^2}{p^\alpha}\Big) &= \sum_{a=1}^{p^\alpha} \overline{\chi}_1(a^2) e\Big(\frac{bma^2}{p^\alpha}\Big) \\ &= \sum_{a=1}^{p^\alpha} (1 + \chi_2(a))\overline{\chi}_1(a) e\Big(\frac{bma}{p^\alpha}\Big) \\ &= \chi_1(mb)\tau(\overline{\chi}_1) + \chi_1\chi_2(mb)\tau(\overline{\chi}_1\chi_2). \end{aligned} \tag{1.12}$$

Then from (1.7), (1.11), (1.12) and Lemma 3 we have

$$\sum_{a=1}^{p^\alpha}{}' \chi(ma^2 + na^{-2}) = \frac{\chi_1(mn)\tau(\overline{\chi}_1\chi_2)\tau(\overline{\chi}_1)}{\tau(\overline{\chi})} \Big(\Big(\frac{m}{p}\Big) + \Big(\frac{n}{p}\Big)\Big). \tag{1.13}$$

(C) If χ is a primitive even character $\bmod p^\alpha$ and $p \equiv 3 \bmod 4$, then $\chi_2(-1) = -1$, so $\overline{\chi}\chi_2$ is a primitive odd character $\bmod p^\alpha$. Therefore, there exits one and only one primitive character χ_1 such that $\overline{\chi} = \overline{\chi}_1^2$. This time we have

$$\sum_{a=1}^{p^\alpha} \overline{\chi}\chi_2(a) e\Big(\frac{bma^2}{p^\alpha}\Big) = 0 \tag{1.14}$$

and

$$(1.15)\qquad \sum_{a=1}^{p^\alpha}\overline{\chi}(a)e\Big(\frac{bma^2}{p^\alpha}\Big) = \sum_{a=1}^{p^\alpha}\overline{\chi}_1(a^2)e\Big(\frac{bma^2}{p^\alpha}\Big)$$
$$= \sum_{a=1}^{p^\alpha}(1+\chi_2(a))\overline{\chi}_1(a)e\Big(\frac{bma}{p^\alpha}\Big)$$
$$= \chi_1(mb)\tau(\overline{\chi}_1) + \chi_1\chi_2(mb)\tau(\overline{\chi}_1\chi_2).$$

Then from (1.7), (1.14), (1.15) and Lemma 3 we have

$$(1.16)\qquad \sum_{a=1}^{p^\alpha}{}' \chi(ma^2+na^{-2}) = \frac{\chi_1(mn)\tau^2(\overline{\chi}_1)}{\tau(\overline{\chi})}\Big(1+\Big(\frac{mn}{p}\Big)\Big(\frac{\tau(\overline{\chi}_1\chi_2)}{\tau(\overline{\chi}_1)}\Big)^2\Big)$$
$$= \frac{\chi_1(mn)\tau^2(\overline{\chi}_1)}{\tau(\overline{\chi})}\Big(1+\Big(\frac{mn}{p}\Big)\Big).$$

Now Lemma 4 follows from (1.10), (1.13) and (1.16). □

Lemma 5. *Let k_1 and k_2 be two positive integers with $(k_1,k_2)=1$, let χ_1 be a Dirichlet character* mod k_1 *and χ_2 a Dirichlet character* mod k_2. *Then for any integers m and n with $(mn,k_1k_2)=1$ we have*

$$\sum_{a=1}^{k_1k_2}{}'\chi_1\chi_2(ma^2+na^{-2}) = \chi_1^2(k_2)\chi_2^2(k_1)\sum_{a=1}^{k_1}{}'\chi_1(ma^2+n_1a^{-2})\sum_{b=1}^{k_2}{}'\chi_2(mb^2+n_2b^{-2}),$$

where n_1 and n_2 are two integers such that $(n_1n_2,k_1k_2)=1$ and $n\equiv n_1k_2^4+n_2k_1^4 \bmod k_1k_2$.

P r o o f. Since $(n,k_1k_2)=1$, there exist two integers n_1 and n_2 such that $n\equiv n_1k_2^4+n_2k_1^4 \bmod k_1k_2$ and $(n_1n_2,k_1k_2)=1$. Then from the properties of the reduced residue system $\bmod k_1k_2$ we have

$$\sum_{a=1}^{k_1k_2}{}'\chi_1\chi_2(ma^2+na^{-2})$$
$$= \sum_{a=1}^{k_1}{}'\sum_{b=1}^{k_2}{}'\chi_1(m(ak_2+bk_1)^2+n(ak_2+bk_1)^{-2})\chi_2(m(ak_2+bk_1)^2+n(ak_2+bk_1)^{-2})$$
$$= \sum_{a=1}^{k_1}{}'\chi_1(ma^2k_2^2+na^{-2}k_2^{-2})\sum_{b=1}^{k_2}{}'\chi_2(mb^2k_1^2+nb^{-2}k_1^{-2})$$

$$= \sum_{a=1}^{k_1}{}' \chi_1(ma^2k_2^2 + n_1k_2^2a^{-2}) \sum_{b=1}^{k_2}{}' \chi_2(mk_1^2b^2 + n_2k_1^2b^{-2})$$

$$= \chi_1^2(k_2)\chi_2^2(k_1) \sum_{a=1}^{k_1}{}' \chi_1(ma^2 + n_1a^{-2}) \sum_{b=1}^{k_2}{}' \chi_2(mb^2 + n_2b^{-2}).$$

This proves Lemma 5. □

Lemma 6. *Let $q > 2$ be an odd square-full number. Then we have the identity*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\varphi^3(q)}{q^2}\prod_{p|q}\Big(1+\frac{1}{p}\Big),$$

where $\sum\limits_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*}$ *denotes the summation over all odd primitive characters* $\chi \bmod q$.

P r o o f. From the definition of Dedekind sums, Lemma 1 and the Möbius inversion formula (see Theorem 2.9 of [1]) we have

$$(1.17)\qquad \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2 = \frac{\varphi(q)}{q^2}\pi^2 \sum_{d|q} \mu(d)\frac{q}{d} S\Big(a, \frac{q}{d}\Big)$$

$$= \pi^2 \frac{\varphi(q)}{q} \sum_{d|q} \frac{\mu(d)}{d} S\Big(a, \frac{q}{d}\Big).$$

If $a = 1$, then it is easy to compute

$$S(1,q) = \sum_{k=1}^{q-1}\Big(\frac{k}{q}-\frac{1}{2}\Big)^2 = \frac{1}{12}\Big(q-3+\frac{2}{q}\Big).$$

So from this formula and (1.17) we have

$$(1.18)\qquad \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\varphi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d}\Big(\frac{q}{d}-3+\frac{2d}{q}\Big)$$

$$= \frac{\pi^2}{12}\varphi(q)\sum_{d|q}\frac{\mu(d)}{d^2} - \frac{\pi^2}{4}\frac{\varphi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d} + \frac{\pi^2}{6}\frac{\varphi(q)}{q^2}\sum_{d|q}\mu(d)$$

$$= \frac{\pi^2}{12}\frac{\varphi^2(q)}{q}\Big[\prod_{p|q}\Big(1+\frac{1}{p}\Big) - \frac{3}{q}\Big].$$

Note that q is a square-full number, $\mu(q)$ and $\varphi(q)$ are two multiplicative functions,

$$\sum_{d|q}\mu(d)\frac{\varphi^2(q/d)}{q^2/d^2}=0 \quad\text{and}\quad \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|L(1,\chi)|^2=\sum_{d|q}\sideset{}{^*}\sum_{\substack{\chi \bmod q/d\\ \chi(-1)=-1}}|L(1,\chi\chi_0)|^2.$$

By virtue of the Möbius inversion formula and (1.18) we may immediately deduce

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|L(1,\chi)|^2 &= \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod q/d\\ \chi(-1)=-1}}|L(1,\chi\chi_0)|^2\\ &= \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod q/d\\ \chi(-1)=-1}}|L(1,\chi)|^2\\ &= \sum_{d|q}\mu(d)\left\{\frac{\pi^2}{12}\frac{\varphi^2(q/d)}{q/d}\left[\prod_{p|q/d}\left(1+\frac{1}{p}\right)-\frac{3}{q/d}\right]\right\}\\ &= \frac{\pi^2}{12}\frac{\varphi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right),\end{aligned}$$

where χ_0 denotes the principal character $\bmod q$. This proves Lemma 6. □

3. Proof of the theorem

In this section, we shall complete the proof of our Theorem. Let q be an odd square-full number with $p\equiv 3 \bmod 4$ for all prime divisors p of q. Then for any integer a with $(a,q)=1$ we have $(a^2+\bar{a}^2,q)=1$. Due to the identity

$$\sideset{}{'}\sum_{u=1}^{q}\chi(u)K(u,u,2;q)=\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{u=1}^{q}\chi(u)e\Big(\frac{u(a^2+\bar{a}^2)}{q}\Big)=\tau(\chi)\sideset{}{'}\sum_{a=1}^{q}\overline{\chi}(a^2+\bar{a}^2),$$

from Lemma 1 we have

$$\begin{aligned}(1.19)\qquad &\sideset{}{'}\sum_{u=1}^{q}\sideset{}{'}\sum_{v=1}^{q}K(u,u,2;q)K(v,v,2;q)S(u\overline{v},q)\\ &=\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sideset{}{'}\sum_{u=1}^{q}\chi(u)K(u,u,2;q)\sideset{}{'}\sum_{v=1}^{q}\chi(\overline{v})K(v,v,2;q)|L(1,\chi)|^2\\ &=\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\tau(\chi)\tau(\overline{\chi})\left|\sideset{}{'}\sum_{a=1}^{q}\overline{\chi}(a^2+\bar{a}^2)\right|^2\cdot|L(1,\chi)|^2.\end{aligned}$$

If χ is not a primitive character $\bmod q$, then from Lemma 2 and the multiplicative properties of $\tau(\chi)$ we have $\tau(\chi)=0$. If χ is a primitive character $\bmod q$, then taking into account that $\tau(\overline{\chi})=\overline{\chi}(-1)\overline{\tau(\chi)}$ we have $\tau(\chi)\cdot\tau(\overline{\chi})=\overline{\chi}(-1)\tau(\chi)\cdot\overline{\tau(\chi)}=\overline{\chi}(-1)\cdot q$. Combining these identities, (1.19) and Lemma 2 we can deduce that

$$\begin{aligned}(1.20)\qquad &\sum_{u=1}^{q}{}'\sum_{v=1}^{q}{}' K(u,u,2;q)K(v,v,2;q)S(u\overline{v},q)\\ &=\frac{-1}{\pi^2}\frac{q^2}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\left|\sum_{a=1}^{q}{}'\overline{\chi}(a^2+\bar{a}^2)\right|^2\cdot|L(1,\chi)|^2,\end{aligned}$$

where $\sum^*$ denotes the summation over all primitive characters $\bmod q$.

For any primitive character $\chi \bmod q$ with $\chi(-1)=-1$, from (B) and (C) of Lemma 4 and Lemma 5 we have

$$(1.21)\qquad \left|\sum_{a=1}^{q}{}'\chi(a^2+a^{-2})\right|^2=4^{\omega(q)}\cdot q,$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q. In fact, if $q=p^\alpha$, then from (B) and (C) of Lemma 4 we know that the identity (1.21) holds. If $q=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}=p_1^{\alpha_1}\cdot q_1$ with $k\geqslant 2$ and $p_i\equiv 3 \bmod 4$ $(i=1,2,\dots k)$, then $(p_1^{\alpha_1},q_1)=1$. Let $\chi=\chi_1\chi_2$ with $\chi_1 \bmod p_1^{\alpha_1}$ and $\chi_2 \bmod q_1$. Then from Lemma 5 we have

$$(1.22)\qquad \left|\sum_{a=1}^{q}{}'\chi_1\chi_2(a^2+a^{-2})\right|=\left|\sum_{a=1}^{p_1^{\alpha_1}}{}'\chi_1(a^2+n_1a^{-2})\right|\cdot\left|\sum_{b=1}^{q_1}{}'\chi_2(b^2+n_2b^{-2})\right|,$$

where $1\equiv n_1q_1^4+n_2p_1^{4\alpha_1} \bmod q$.

It is clear that $1=\left(\frac{1}{p_1}\right)=\left(\frac{n_1}{p_1}\right)$, hence from (1.22), (B) and (C) of Lemma 4 we have

$$\begin{aligned}(1.23)\qquad \left|\sum_{a=1}^{q}{}'\chi(a^2+a^{-2})\right|&=\left|\sum_{a=1}^{q}{}'\chi_1\chi_2(a^2+a^{-2})\right|\\ &=2\cdot\sqrt{p_1^{\alpha_1}}\cdot\left|\sum_{b=1}^{q_1}{}'\chi_2(b^2+n_2b^{-2})\right|.\end{aligned}$$

Note that for any $p_i \mid q_1$ we have $p_i\equiv 3 \bmod 4$ and $1=\left(\frac{1}{p_i}\right)=\left(\frac{n_2}{p_i}\right)$, so from (1.23) by mathematical induction we may immediately deduce the identity (1.21).

Finally, combining (1.20), (1.21) and Lemma 6 we may immediately deduce the identity

$$\begin{aligned}
&{\sum_{u=1}^{q}}'{\sum_{v=1}^{q}}' K(u,u,2;q)K(v,v,2;q)S(u\overline{v},q) \\
&\quad = \frac{-1}{\pi^2}\frac{q^3}{\varphi(q)}\cdot 4^{\omega(q)}\cdot \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} |L(1,\chi)|^2 \\
&\quad = \frac{-1}{\pi^2}\frac{q^3}{\varphi(q)}\cdot 4^{\omega(q)}\cdot \frac{\pi^2}{12}\frac{\varphi^3(q)}{q^2}\prod_{p\mid q}\Bigl(1+\frac{1}{p}\Bigr) \\
&\quad = \frac{-1}{12}\cdot q\cdot \varphi^2(q)\cdot 4^{\omega(q)}\cdot \prod_{p\mid q}\Bigl(1+\frac{1}{p}\Bigr).
\end{aligned}$$

This completes the proof of our theorem. □

References

[1] *T. M. Apostol*: Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics, Springer, New York-Heidelberg-Berlin, 1976.

[2] *T. M. Apostol*: Modular Functions and Dirichlet Series in Number Theory. Graduate Texts in Mathematics, Springer, New York-Heidelberg-Berlin, 1976.

[3] *L. Carlitz*: The reciprocity theorem for Dedekind sums. Pac. J. Math. *3* (1953), 523–527.

[4] *J. B. Conrey, E. Fransen, R. Klein, C. Scott*: Mean values of Dedekind sums. J. Number Theory *56* (1996), 214–226.

[5] *T. Estermann*: On Kloostermann's sums. Mathematika, Lond. *8* (1961), 83–86.

[6] *K. Ireland, M. Rosen*: A Classical Introduction to Modern Number Theory (Rev. and expand. version of 1972). Graduate Texts in Mathematics, Springer, New York-Heidelberg-Berlin, 1982.

[7] *Y. Liu, W. Zhang*: A hybrid mean value related to the Dedekind sums and Kloosterman sums. Sci. China, Math. *53* (2010), 2543–2550; Acta Math. Sin. *27* (2011), 435–440.

[8] *W. Zhang*: On the mean values of Dedekind sums. J. Théor. Nombres Bordx. *8* (1996), 429–442.

[9] *W. Zhang*: A sum analogous to Dedekind sums and its hybrid mean value formula. Acta Arith. *107* (2003), 1–8.

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