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ESSENTIAL NORMALITY FOR CERTAIN FINITE LINEAR COMBINATIONS OF LINEAR-FRACTIONAL COMPOSITION OPERATORS ON THE HARDY SPACE H^2

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Abstract. In 1999 Nina Zorboska and in 2003 P. S. Bourdon, D. Levi, S. K. Narayan and J. H. Shapiro investigated the essentially normal composition operator C_φ, when φ is a linear-fractional self-map of $\mathbb{D}$. In this paper first, we investigate the essential normality problem for the operator T_wC_φ on the Hardy space H^2, where w is a bounded measurable function on $\partial\mathbb{D}$ which is continuous at each point of $F(\varphi)$, $\varphi \in \mathcal{S}(2)$, and T_w is the Toeplitz operator with symbol w. Then we use these results and characterize the essentially normal finite linear combinations of certain linear-fractional composition operators on H^2.

Keywords: Hardy spaces, essentially normal, composition operator, linear-fractional transformation

MSC 2010: 47B33

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ be its boundary, and $\mathrm{Hol}(\mathbb{D})$ denotes the space of all holomorphic functions on $\mathbb{D}$.

For an analytic function f on the unit disk and $0 < r < 1$, we define the dilated function f_r by $f_r(\mathrm{e}^{\mathrm{i}\theta}) = f(r\mathrm{e}^{\mathrm{i}\theta})$. It is easy to see that the functions f_r are continuous on $\partial\mathbb{D}$ for each r, hence they are in $L^p(\partial\mathbb{D}, \mathrm{d}\theta/2\pi)$, where $\mathrm{d}\theta/2\pi$ is the normalized arc length measure on the unit circle.

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D}) = H^p$ is the set of all analytic functions on the unit disk for which

$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |f_r(\mathrm{e}^{\mathrm{i}\theta})|^p \frac{\mathrm{d}\theta}{2\pi} < \infty.$$

Also we recall that $H^\infty(\mathbb{D}) = H^\infty$ is the space of all bounded analytic functions defined on $\mathbb{D}$, with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. We know that for $p \geqslant 1$, H^p is a Banach space (see, e.g., [8, p. 37]). For more information about the Hardy spaces see, for example, [7] and [8]. For $\beta \geqslant 1$, let $\mathcal{D}_\beta$ denote the reproducing kernel Hilbert space of functions analytic in the unit disk $\mathbb{D}$ and having the kernel functions $K_w(z) = (1-\overline{w}z)^{-\beta}$. The Hardy space H^2 is exactly $\mathcal{D}_1$.

For each $\psi \in L^\infty(\partial\mathbb{D})$, we define the Toeplitz operator T_ψ on H^2 by $T_\psi(f) = P(\psi f)$, where P denotes the orthogonal projection of $L^2(\partial\mathbb{D})$ onto H^2. Since an orthogonal projection has norm 1, clearly T_ψ is bounded. For any analytic self-map φ of $\mathbb{D}$, the composition operator C_φ on H^2 is defined by $C_\varphi(f) = f \circ \varphi$. It is well known (see, e.g., [8, p. 29] or [16, Theorem 1]) that the composition operators are bounded on each of the Hardy spaces H^p $(0 < p < \infty)$.

A mapping of the form

$$\varphi(z) = \frac{az+b}{cz+d} \quad (ad-bc \neq 0) \tag{1}$$

is called a linear-fractional transformation. We denote the set of those linear-fractional transformations that take the open unit disk $\mathbb{D}$ into itself by $\mathrm{LFT}(\mathbb{D})$. It is well known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are just the functions $\varphi(z) = \lambda(a-z)/(1-\bar{a}z)$, where $|\lambda| = 1$ and $|a| < 1$.

For bounded operators A and B on a Hilbert space, we use the notation $[A, B] := AB - BA$ for the commutator of A and B. Recall that an operator A is called normal if $[A, A^*] = 0$ and essentially normal if $[A, A^*]$ is compact. In 1969, H. J. Schwartz [18] showed that a composition operator on H^2 is normal if and only if it is induced by a dilation $z \to az$, where $|a| \leqslant 1$. In [21] Nina Zorboska has characterized the essentially normal composition operators induced on the Hardy space H^2 by automorphisms of the unit disk. In addition, Zorboska has shown that the composition operators induced on H^2 by linear-fractional transformations fixing no point on the unit circle are not nontrivially essentially normal. P. S. Bourdon, D. Levi, S. K. Narayan, and J. H. Shapiro in [3] have shown that a composition operator induced on H^2 by a linear-fractional self-map of the unit disk is nontrivially essentially normal if and only if it is induced by a parabolic non-automorphism. The essentially normal composition operators on other spaces have been investigated by some authors (see, e.g., [4], [12], and [13]).

If φ and ψ are linear-fractional self-maps of $\mathbb{D}$ or B_N, then $C_\varphi - C_\psi$ cannot be non-trivially compact; i.e., if the difference is compact, either C_φ and C_ψ are individually compact or $\varphi = \psi$. The fact that a difference of linear-fractional composition operators cannot be non-trivially compact on H^2 or $A^2_\alpha(\mathbb{D})$ was first obtained by

P. S. Bourdon [2] and J. Moorhouse [14] as a consequence of results on the compactness of a difference of more general composition operators in one variable. Recently there has been a great interest in studying some linear combinations of composition operators; see, for example, [9] and [11].

In this paper, we use the results of T. L. Kriete and J. L. Moorhouse [11] and T. L. Kriete, B. D. MacCluer and J. L. Moorhouse [10] in order to investigate the essential normality problem for certain finite linear combinations of linear-fractional composition operators on H^2.

2. Preliminaries

Here we collect the fundamental facts about some definitions and results which are required in the sequel.

2.1. Angular derivatives. Let φ be an analytic self-map of $\mathbb{D}$. We say that φ has a finite angular derivative at ζ on the unit circle if there is η on the unit circle such that $(\varphi(z)-\eta)/(z-\zeta)$ has a finite non-tangential limit as $z\to\zeta$. When it exists (as a finite complex number), this limit is denoted by $\varphi'(\zeta)$. By the Julia-Carathéodory Theorem (see, e.g., [7, Theorem 2.44] or [19, Chapter 4]),

$$|\varphi'(\zeta)| = d(\zeta) := \liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|},$$

where the lim inf is taken as z approaches ζ unrestrictedly in $\mathbb{D}$. Throughout this paper, let $F(\varphi)$ denote the set of all points in $\partial\mathbb{D}$ at which φ has a finite angular derivative. A necessary condition for the composition operator C_φ to act compactly on H^2 is that $F(\varphi)$ is empty; see [20] or [7, Corollarly 3.14]. This condition, however, is not sufficient unless φ is of bounded multiplicity (see [7, Corollary 3.21]).

2.2. Clark measures. Suppose that φ is an analytic self-map of $\mathbb{D}$ and α is a complex number of modulus 1. Since $\operatorname{Re}((\alpha+\varphi)/(\alpha-\varphi))$ is a positive harmonic function on $\mathbb{D}$, there exists a finite positive Borel measure μ_α on $\partial\mathbb{D}$ such that

$$\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \operatorname{Re}\Big(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\Big) = \int_{\partial\mathbb{D}} P_z \,\mathrm{d}\mu_\alpha$$

for each $z\in\mathbb{D}$, where $P_z(\mathrm{e}^{\mathrm{i}\theta}) = (1-|z|^2)/|\mathrm{e}^{\mathrm{i}\theta}-z|^2$ is the Poisson kernel at z. The measures μ_α are called the Clark measures of φ. There is a unique pair of measures μ_α^{ac} and μ_α^{s} such that $\mu_\alpha = \mu_\alpha^{\mathrm{ac}} + \mu_\alpha^{\mathrm{s}}$, where μ_α^{ac} and μ_α^{s} are the absolutely continuous and singular parts with respect to Lebesgue measure, respectively. The singular part μ_α^{s} is carried by $\varphi^{-1}(\{\alpha\})$, the set of those ζ in $\partial\mathbb{D}$ where $\varphi(\zeta)$ exists and equals α, and is itself the sum of the pure point measure

$$\mu_\alpha^{pp} = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|}\delta_\zeta, \tag{2}$$

where δ_ζ is the unit point mass measure at ζ and a continuous singular measure μ_α^{cs}, either of which can vanish. In particular, if φ is a linear-fractional non-automorphism such that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then $\mu_\alpha^{\mathrm{s}} = 0$ when $\alpha \neq \eta$ and $\mu_\eta^{\mathrm{s}} = |\varphi'(\zeta)|^{-1}\delta_\zeta$. We write $E(\varphi)$ for the closure in $\partial\mathbb{D}$ of the union of the closed supports of μ_α^{s} as α ranges over the unit circle. Therefore, by Equation (2), $F(\varphi) \subseteq E(\varphi)$. The measures μ_α were introduced as an operator-theoretic tool by D. N. Clark [5] and have been further analyzed by A. B. Aleksandrov [1], A. G. Poltoratski [15] and D. E. Sarason [17].

2.3. Cowen's adjoint formula. In [6] Carl Cowen showed that if $\varphi \in \mathrm{LFT}(\mathbb{D})$ is given by Equation (1), then

$$C_\varphi^* = T_g C_{\sigma_\varphi} T_h^*, \tag{3}$$

where $\sigma_\varphi(z) := (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ is a self-map of $\mathbb{D}$, $g(z) := (-\bar{b}z + \bar{d})^{-1}$, $h(z) := cz + d$ and $g, h \in H^\infty$. The map σ_φ is called the Krein adjoint of φ; we will write σ for σ_φ except when confusion could arise. If $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma(\eta) = \zeta$. Also, φ is an automorphism if and only if σ is, and in this case $\sigma = \varphi^{-1}$. For further details see, for example, [3].

We know that if $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$, then C_φ is compact (see, e.g., [19]). Let $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \partial\mathbb{D}$ and $\zeta_1 \neq \zeta_2$. Assume that $\varphi_1, \varphi_2 \in \mathrm{LFT}(\mathbb{D})$ are not automorphisms and that $\varphi_1(\zeta_1) = \eta_1$ and $\varphi_2(\zeta_2) = \eta_2$. Suppose that $1 \leqslant i, j \leqslant 2$ and $i \neq j$. We see that $\varphi_i \circ \sigma_j$ takes $\partial\mathbb{D}$ into $\mathbb{D}$, so $\|\varphi_i \circ \sigma_j\|_\infty < 1$ and $C_{\varphi_i \circ \sigma_j}$ is compact on H^2. Also, it is clear that $\sigma_j \circ \varphi_i$ takes $\partial\mathbb{D}$ into $\mathbb{D}$, when $\eta_j \neq \eta_i$; therefore, we have $\|\sigma_j \circ \varphi_i\|_\infty < 1$ and $C_{\sigma_j \circ \varphi_i}$ is compact on H^2. We will use these two facts frequently in this paper.

2.4. Parabolic linear-fractional self-map of $\mathbb{D}$. A map $\varphi \in \mathrm{LFT}(\mathbb{D})$ whose fixed point set, relative to the Riemann sphere, consists of a single point ζ in $\partial\mathbb{D}$ is termed parabolic. In [19, p. 3] J. H. Shapiro has shown that among the linear-fractional non-automorphisms fixing $\zeta \in \partial\mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta) = 1$; for further details see [3] and [19].

In the rest of this section, we state some useful definitions and results of [11] that we will need in the sequel.

2.5. The class $\mathcal{S}$ and $\mathcal{S}(2)$. For $\zeta \in F(\varphi)$, the first-order data of φ at ζ is given by the vector $D_1(\varphi,\zeta) := (\varphi(\zeta), \varphi'(\zeta))$. In what follows, we look at higher-order data vectors

$$D_k(\varphi,\zeta) := (\varphi(\zeta), \varphi'(\zeta), \varphi''(\zeta), \ldots, \varphi^{(k)}(\zeta))$$

at points where the corresponding derivatives make sense.

We say an analytic self-map φ of $\mathbb{D}$ has an order of contact $c > 0$ at ζ if $|\varphi(\zeta)| = 1$ and

$$\frac{1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}{|\varphi(\zeta) - \varphi(\mathrm{e}^{\mathrm{i}\theta})|^c}$$

is essentially bounded above and away from zero as $\mathrm{e}^{\mathrm{i}\theta} \to \zeta$.

We say an analytic self-map φ of $\mathbb{D}$ has a kth-order data at ζ in $F(\varphi)$ if there exist complex numbers $b_0, b_1, \ldots, b_k$ with $|b_0| = 1$ such that

$$\varphi(z) = b_0 + b_1(z-\zeta) + \ldots + b_k(z-\zeta)^k + o(|z-\zeta|^k)$$

as $z \to \zeta$ unrestrictedly in $\mathbb{D}$. In this case for any $1 \leqslant j \leqslant k$, $j!b_j$ is the non-tangential limit of $\varphi^{(j)}(z)$ at ζ (see, for example, the argument on p. 47 in [17]); we refer to this limit as $\varphi^{(j)}(\zeta)$. Note that since $|b_0| = 1$ and $\zeta \in F(\varphi)$, b_1 is the angular derivative $\varphi'(\zeta)$.

We say an analytic self-map φ of $\mathbb{D}$ has sufficient data at ζ in $\partial\mathbb{D}$ if

(i) $\zeta \in F(\varphi)$;
(ii) φ has an order of contact $2m$ at ζ for some natural number m;
(iii) φ has a $(2m)$th-order data at ζ.

Suppose that φ has a finite angular derivative at ζ. Also, let it have an analytic continuation to a neighborhood of ζ and $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$. For any α in $\partial\mathbb{D}$, consider the linear-fractional transformation $\tau_\alpha(z) := \mathrm{i}(\alpha - z)/(\alpha + z)$ which takes the unit disk onto the upper half-plane $\Omega := \{w\colon \operatorname{Im} w > 0\}$ and α to 0. Let $u := \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$. Then for w near zero, $u(w) = \sum\limits_{n=1}^{\infty} a_n w^n$. In [11, p. 2930] Kriete et al. have shown that the smallest natural number n with a_n non-real must be even. Let $n = 2m$. Also, they have proved that φ has an order of contact $2m$ at ζ. In particular, let φ be a non-automorphism linear-fractional self-map of $\mathbb{D}$ with $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Assume that for any $\alpha \in \partial\mathbb{D}$, we define the linear-fractional transformation $S_\alpha(z) := (1 + \overline{\alpha}z)/(1 - \overline{\alpha}z)$ which takes the unit disk onto the right half-plane Π and α to ∞. Set $\phi := S_\eta \circ \varphi \circ S_\zeta^{-1}$. Since $\phi(\infty) = \infty$, the function $\phi(z) = \lambda z + b$. Also, $\varphi = S_\eta^{-1} \circ (\lambda z + b) \circ S_\zeta$ and $\varphi(\mathbb{D}) \subsetneq \mathbb{D}$. Therefore, $\lambda > 0$, $\operatorname{Re} b > 0$ and $u = \tau_\eta \circ \varphi \circ \tau_\zeta^{-1} = \tau_\eta \circ S_\eta^{-1} \circ (\lambda z + b) \circ S_\zeta \circ \tau_\zeta^{-1}$. By some computations, $S_\zeta \circ \tau_\zeta^{-1}(z) = \mathrm{i}/z$ and hence $\tau_\eta \circ S_\eta^{-1} = \mathrm{i}/z$. Thus,

$$u(z) = (\mathrm{i}/z) \circ (\lambda z + b) \circ (\mathrm{i}/z) = \sum_{n=0}^{\infty} (-1)^n \frac{b^n}{(\mathrm{i})^n \lambda^{n+1}} z^{n+1}.$$

Therefore, φ has an order of contact 2 at ζ and has sufficient data at ζ. Let $\mathcal{S}$ be the class of analytic self-maps φ of $\mathbb{D}$ for which $E(\varphi)$ is a finite set (so that $E(\varphi) = F(\varphi)$) and φ has sufficient data at each point of $F(\varphi)$. We denote by $\mathcal{S}(2)$ the set of those φ in $\mathcal{S}$ which have an order of contact two at each point of $F(\varphi)$.

We write $\mathcal{L}$ for the collection of all non-automorphism linear-fractional self-maps φ of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$. It is obvious that each linear-fractional transformation ψ is determined by its second-order data $D_2(\psi, z_0)$ at each point z_0 of analyticity. Now assume that $\varphi \in \mathcal{S}(2)$ and $\zeta_0 \in F(\varphi)$. In [11, p. 2940] Kriete et al. have shown that the unique linear-fractional transformation φ_0 with $D_2(\varphi_0, \zeta_0) = D_2(\varphi, \zeta_0)$ belongs to $\mathcal{L}$.

3. Some results on essential normality of the operators $T_w C_\varphi$

The set of all bounded operators and the set of all compact operators from H^2 into itself are denoted by $B(H^2)$ and $B_0(H^2)$, respectively. We will use the notation $A \equiv B$ to indicate that the difference of two bounded operators A and B belongs to $B_0(H^2)$. In [10] Kriete et al. have shown that if $\varphi \in \mathrm{LFT}(\mathbb{D})$ is not an automorphism which satisfies $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then

$$C_\varphi^* \equiv |\varphi'(\zeta)|^{-1} C_\sigma. \tag{4}$$

In Theorem 3.1, M_w denotes the operator on $L^2 = L^2(\partial\mathbb{D})$ of multiplication by a bounded measurable function w.

Theorem 3.1 ([11], Proposition 5.19). *Suppose that $\varphi \in \mathcal{S}(2)$ with $F(\varphi) = \{\zeta_1, \ldots, \zeta_r\}$. For $i = 1, \ldots, r$, let φ_i be the unique linear-fractional transformation with $D_2(\varphi_i, \zeta_i) = D_2(\varphi, \zeta_i)$. Also assume that w is a bounded measurable function on $\partial\mathbb{D}$ which is continuous at each point of $F(\varphi)$. Then*

$$M_w C_\varphi \equiv w(\zeta_1) C_{\varphi_1} + \ldots + w(\zeta_r) C_{\varphi_r},$$

where the operators are considered as mapping H^2 to L^2.

Now we restate Theorem 3.1 in terms of Toeplitz operators.

Corollary 3.2. *Suppose that φ, $\varphi_1, \dots, \varphi_r$, $\zeta_1, \dots, \zeta_r$, w and $F(\varphi)$ are as in Theorem 3.1. Then*

$$T_w C_\varphi \equiv w(\zeta_1) C_{\varphi_1} + \dots + w(\zeta_r) C_{\varphi_r}, \tag{5}$$

where the operators are considered as mapping H^2 to H^2.

Proof. We know that $M_w C_\varphi = T_w C_\varphi + H_w C_\varphi$, where the Hankel operator H_w is the operator from H^2 into the orthogonal complement of H^2 in $L^2(\partial\mathbb{D})$ and is defined by $H_w(g) = (I - P)(wg)$ for each $g \in H^2$. By the proof of Corollary 2.2 in [10], $H_w C_\varphi$ is compact, so the result follows from Theorem 3.1. □

Let $\varphi \in \mathcal{S}(2)$ with $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$. For each $1 \leqslant i \leqslant r$, suppose that σ_i is the Krein adjoint of φ_i, where φ_i is the linear-fractional transformation related to φ and ζ_i is as Theorem 3.1. By the preceding corollary

$$(T_w C_\varphi)^* \equiv \overline{w(\zeta_1)} C^*_{\varphi_1} + \dots + \overline{w(\zeta_r)} C^*_{\varphi_r}. \tag{6}$$

Therefore, Corollary 3.2 and Equations (4), (5), and (6) imply that

$$\begin{aligned} (T_w C_\varphi)^* T_w C_\varphi &\equiv (\overline{w(\zeta_1)} C^*_{\varphi_1} + \dots + \overline{w(\zeta_r)} C^*_{\varphi_r})(w(\zeta_1) C_{\varphi_1} + \dots + w(\zeta_r) C_{\varphi_r}) \\ &\equiv (\overline{w(\zeta_1)} |\varphi'(\zeta_1)|^{-1} C_{\sigma_1} + \dots + \overline{w(\zeta_r)} |\varphi'(\zeta_r)|^{-1} C_{\sigma_r}) \\ &\quad \times (w(\zeta_1) C_{\varphi_1} + \dots + w(\zeta_r) C_{\varphi_r}) \\ &\equiv |w(\zeta_1)|^2 |\varphi'(\zeta_1)|^{-1} C_{\varphi_1 \circ \sigma_1} + \dots + |w(\zeta_r)|^2 |\varphi'(\zeta_r)|^{-1} C_{\varphi_r \circ \sigma_r}, \end{aligned} \tag{7}$$

where the last equivalence is justified by the fact that $C_{\varphi_i \circ \sigma_j} \in B_0(H^2)$ for each $1 \leqslant i, j \leqslant r$ and $i \neq j$.

Proposition 3.3. *Suppose that φ, $\varphi_1, \dots, \varphi_r$, $\zeta_1, \dots, \zeta_r$, w and $F(\varphi)$ are as in Theorem 3.1. If the restriction of φ to $F(\varphi)$ is a 1-1 function, then*

$$\begin{aligned} [T_w C_\varphi, (T_w C_\varphi)^*] &\equiv |w(\zeta_1)|^2 |\varphi'(\zeta_1)|^{-1} (C_{\sigma_1 \circ \varphi_1} - C_{\varphi_1 \circ \sigma_1}) + \dots \\ &\quad + |w(\zeta_r)|^2 |\varphi'(\zeta_r)|^{-1} (C_{\sigma_r \circ \varphi_r} - C_{\varphi_r \circ \sigma_r}). \end{aligned} \tag{8}$$

Proof. Since the restriction of φ to $F(\varphi)$ is a 1-1 function, $C_{\sigma_j \circ \varphi_i} \in B_0(H^2)$ for each $1 \leqslant i, j \leqslant r$ and $i \neq j$. Thus, as in the proof of Equation (7), we see that

$$T_w C_\varphi (T_w C_\varphi)^* \equiv |w(\zeta_1)|^2 |\varphi'(\zeta_1)|^{-1} C_{\sigma_1 \circ \varphi_1} + \dots + |w(\zeta_r)|^2 |\varphi'(\zeta_r)|^{-1} C_{\sigma_r \circ \varphi_r}.$$

The conclusion follows from the above equivalence and Equation (7). □

We infer from [10, Proposition 3.4] that $\varphi_i \circ \sigma_i$ and $\sigma_i \circ \varphi_i$ belong to $\mathcal{L}$ with the fixed points $\varphi_i(\zeta_i)$ and ζ_i, respectively. Now we present some notation used in [11], then we state a theorem that we will use frequently.

We fix $\varphi_1, \ldots, \varphi_n$ in $\mathcal{S}$. Therefore, $F := F(\varphi_1) \cup \ldots \cup F(\varphi_n)$ is a finite set. For $\zeta \in F$ and $k = 2, 4, 6, \ldots$, let

$$\mathbb{N}_k(\zeta) := \{j\colon \zeta \text{ belongs to } F(\varphi_j) \text{ and } \varphi_j \text{ has the order of contact } k \text{ at } \zeta\}.$$

Also we write $\varepsilon_k(\zeta) := \{D_k(\varphi_j, \zeta)\colon\ j \in \mathbb{N}_k(\zeta)\}$.

Theorem 3.4 ([11], Theorem 5.13)**.** *Suppose that $\varphi_1, \ldots, \varphi_n$ are in $\mathcal{S}$. Given complex numbers $c_1, \ldots, c_n$, the following are equivalent:*

(i) *$c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n}$ is compact on $\mathcal{D}_\beta$;*

(ii) *for each $\zeta \in F$, every even $k \geqslant 2$ and every d in $\varepsilon_k(\zeta)$,*

$$\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = d}} c_j = 0.$$

Proposition 3.5. *Suppose that φ, $\varphi_1, \ldots, \varphi_r$, $\zeta_1, \ldots, \zeta_r$, w and $F(\varphi)$ are as in Theorem 3.1. Let the restriction of φ to $F(\varphi)$ be a 1-1 function. Assume that $\zeta \in F(\varphi)$ is a fixed point of φ with $\varphi'(\zeta) \neq 1$. If $T_w C_\varphi$ is essentially normal, then $w(\zeta) = 0$.*

P r o o f. Without loss of generality, we can assume $\zeta_1 = \zeta$. Since the restriction of φ to $F(\varphi)$ is a 1-1 function, there are only two linear-fractional transformations $\varphi_1 \circ \sigma_1$ and $\sigma_1 \circ \varphi_1$ in Equation (8) with the same fixed point at ζ_1. By [19, p. 3], φ_1 is not a parabolic non-automorphism and Kriete et al. in [10, p. 139] have shown that in this case $\varphi_1 \circ \sigma_1 \neq \sigma_1 \circ \varphi_1$. Now apply Theorem 3.4 to $\zeta = \zeta_1$, $k = 2$ and $d = D_2(\varphi_1 \circ \sigma_1, \zeta_1)$. □

Throughout this paper, let $\varphi^{[0]}$ be the identity map on $\mathbb{D}$ and $\varphi^{[j+1]} := \varphi \circ \varphi^{[j]}$ for each $j \in \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$ and $\zeta \in F(\varphi)$, let $\varphi^{[-n]}(\{\zeta\})$ be the set of all z, where $\varphi^{[n]}(z) = \zeta$. Also, if $n = 0$, then $\varphi^{[-n]}(\{\zeta\}) := \{\zeta\}$.

Proposition 3.6. *Suppose that φ, $\varphi_1, \ldots, \varphi_r$, $\zeta_1, \ldots, \zeta_r$, w and $F(\varphi)$ are as in Theorem 3.1. Let the restriction of φ to $F(\varphi)$ be a 1-1 function. Suppose that there are $\zeta \in F(\varphi)$ and $\eta \notin F(\varphi)$ with $\varphi(\zeta) = \eta$. If $T_w C_\varphi$ is essentially normal, then $w(\zeta) = 0$ and, moreover, if for every i, $1 \leqslant i \leqslant n$, $\varphi^{[-i]}(\{\zeta\}) \cap F(\varphi) \neq \emptyset$ whenever $n \in \mathbb{N}$ and $1 \leqslant n < r$, then $w(z) = 0$ for $z \in \varphi^{[-i]}(\{\zeta\}) \cap F(\varphi)$.*

P r o o f. For convenience, let $\zeta_1 = \zeta$ and $\{\zeta_{i+1}\} = \varphi^{[-i]}(\{\zeta_1\}) \cap F(\varphi)$, where $0 < i \leqslant n$. Since the restriction of φ to $F(\varphi)$ is a 1-1 function, there is only one linear-fractional transformation $\varphi_1 \circ \sigma_1$ in Equation (8) which has a finite angular derivative at η. Hence by Theorem 3.4, $w(\zeta_1) = 0$, so one has

$$
\begin{aligned}
(9)\qquad [T_wC_\varphi, (T_wC_\varphi)^*] &\equiv |w(\zeta_2)|^2|\varphi'(\zeta_2)|^{-1}(C_{\sigma_2\circ\varphi_2} - C_{\varphi_2\circ\sigma_2}) \\
&\quad + \ldots + |w(\zeta_r)|^2|\varphi'(\zeta_r)|^{-1}(C_{\sigma_r\circ\varphi_r} - C_{\varphi_r\circ\sigma_r}) \equiv 0.
\end{aligned}
$$

Since $\varphi_2 \circ \sigma_2$ is the only linear-fractional transformation in Equation (9) with the fixed point at ζ_1, Theorem 3.4 implies that $w(\zeta_2) = 0$. Using similar arguments, the result follows. □

Proposition 3.7. *Suppose that φ, $\varphi_1, \ldots, \varphi_r$, $\zeta_1, \ldots, \zeta_r$, w and $F(\varphi)$ are as in Theorem 3.1. Let the restriction of φ to $F(\varphi)$ be a 1-1 function. Also assume that there is a smallest integer n, $1 < n \leqslant r$, such that $\varphi(\zeta_1) = \zeta_2, \ldots, \varphi(\zeta_{n-1}) = \zeta_n$ and $\varphi(\zeta_n) = \zeta_1$. If T_wC_φ is essentially normal, then $\{\varphi_i \circ \sigma_i\colon 1 \leqslant i \leqslant n\} = \{\sigma_i \circ \varphi_i\colon 1 \leqslant i \leqslant n\}$ and for each $1 \leqslant i, j \leqslant n$, $|w(\zeta_i)|^2|\varphi'(\zeta_i)|^{-1} = |w(\zeta_j)|^2|\varphi'(\zeta_j)|^{-1}$ or $w(\zeta_i) = 0$ for any $1 \leqslant i \leqslant n$.*

P r o o f. Without loss of generality, we can assume $n < r$. Let T_wC_φ be essentially normal. We infer from Equation (8) that

$$
\begin{aligned}
&[T_wC_\varphi, (T_wC_\varphi)^*] \\
&\quad \equiv (|w(\zeta_1)|^2|\varphi'(\zeta_1)|^{-1}C_{\sigma_1\circ\varphi_1} - |w(\zeta_n)|^2|\varphi'(\zeta_n)|^{-1}C_{\varphi_n\circ\sigma_n}) \\
&\qquad + (|w(\zeta_2)|^2|\varphi'(\zeta_2)|^{-1}C_{\sigma_2\circ\varphi_2} - |w(\zeta_1)|^2|\varphi'(\zeta_1)|^{-1}C_{\varphi_1\circ\sigma_1}) \\
&\qquad + \ldots + (|w(\zeta_n)|^2|\varphi'(\zeta_n)|^{-1}C_{\sigma_n\circ\varphi_n} - |w(\zeta_{n-1})|^2|\varphi'(\zeta_{n-1})|^{-1}C_{\varphi_{n-1}\circ\sigma_{n-1}}) \\
&\qquad + |w(\zeta_{n+1})|^2|\varphi'(\zeta_{n+1})|^{-1}(C_{\sigma_{n+1}\circ\varphi_{n+1}} - C_{\varphi_{n+1}\circ\sigma_{n+1}}) + \ldots \\
&\qquad + |w(\zeta_r)|^2|\varphi'(\zeta_r)|^{-1}(C_{\sigma_r\circ\varphi_r} - C_{\varphi_r\circ\sigma_r}).
\end{aligned}
$$

It is obvious that $\varphi_n \circ \sigma_n(\zeta_1) = \sigma_1 \circ \varphi_1(\zeta_1) = \zeta_1$, $\varphi_1 \circ \sigma_1(\zeta_2) = \sigma_2 \circ \varphi_2(\zeta_2) = \zeta_2$, …, and $\varphi_{n-1} \circ \sigma_{n-1}(\zeta_n) = \sigma_n \circ \varphi_n(\zeta_n) = \zeta_n$. Now we define the permutation τ on $\{1, \ldots, n\}$ by $\tau(i) = i - 1$, when $1 < i \leqslant n$ and $\tau(1) = n$. If $\{\varphi_k \circ \sigma_k\colon 1 \leqslant k \leqslant n\} = \{\sigma_k \circ \varphi_k\colon 1 \leqslant k \leqslant n\}$, then for each $1 \leqslant i, j \leqslant n$, $|w(\zeta_i)|^2|\varphi'(\zeta_i)|^{-1} = |w(\zeta_j)|^2|\varphi'(\zeta_j)|^{-1}$. This may be seen as follows. Suppose that for some $1 \leqslant i, j \leqslant n$, $|w(\zeta_i)|^2|\varphi'(\zeta_i)|^{-1} \neq |w(\zeta_j)|^2|\varphi'(\zeta_j)|^{-1}$. Hence there is $1 \leqslant j_0 \leqslant n$, where $|w(\zeta_{j_0})|^2|\varphi'(\zeta_{j_0})|^{-1} \neq |w(\zeta_{\tau(j_0)})|^2|\varphi'(\zeta_{\tau(j_0)})|^{-1}$. Since $\sigma_{j_0} \circ \varphi_{j_0}$ and $\varphi_{\tau(j_0)} \circ \sigma_{\tau(j_0)}$ are the only two linear-fractional transformations in the above equivalence with the same fixed point at ζ_{j_0}, by Theorem 3.4, $|w(\zeta_{j_0})|^2|\varphi'(\zeta_{j_0})|^{-1} = |w(\zeta_{\tau(j_0)})|^2|\varphi'(\zeta_{\tau(j_0)})|^{-1}$, so it is a contradiction. Let $w(\zeta_{i_0}) \neq 0$ for some $1 \leqslant i_0 \leqslant n$

and $\{\varphi_i \circ \sigma_i\colon 1 \leqslant i \leqslant n\} \neq \{\sigma_i \circ \varphi_i\colon 1 \leqslant i \leqslant n\}$. Then there is $1 \leqslant k_0 \leqslant n$ with $\sigma_{k_0} \circ \varphi_{k_0} \neq \varphi_{\tau(k_0)} \circ \sigma_{\tau(k_0)}$. Moreover, as we observed above, there are exactly two linear-fractional transformations $\sigma_{k_0} \circ \varphi_{k_0}$ and $\varphi_{\tau(k_0)} \circ \sigma_{\tau(k_0)}$ in the preceding equivalence with the same fixed point at ζ_{k_0}. Hence by Theorem 3.4, $w(\zeta_{k_0}) = w(\zeta_{\tau(k_0)}) = 0$. Since $\sigma_{\tau(k_0)} \circ \varphi_{\tau(k_0)}$ and $\varphi_{\tau^2(k_0)} \circ \sigma_{\tau^2(k_0)}$ are the only two linear-fractional transformations in the preceding equivalence with the same fixed point at $\zeta_{\tau(k_0)}$ and $w(\zeta_{\tau(k_0)}) = 0$, again by Theorem 3.4, $w(\zeta_{\tau^2(k_0)}) = 0$. By a similar argument, we see that for each $1 \leqslant j \leqslant n$, $w(\zeta_j) = 0$, which is a contradiction. □

For an analytic self-map φ of $\mathbb{D}$, let $\mathbb{P}_\varphi$ denote the set of $\zeta \in F(\varphi)$, where $\varphi(\zeta) = \zeta$ and $\varphi'(\zeta) = 1$. It is clear that $\mathbb{P}_\varphi$ has at most one element (see, e.g., [7, Theorem 2.48]). Let $\varphi \in \mathcal{S}(2)$ and let φ_{i_0} be the linear-fractional transformation related to φ and ζ_{i_0} as in Theorem 3.1 with $\varphi(\zeta_{i_0}) = \zeta_{i_0}$ and $\varphi'(\zeta_{i_0}) = 1$. Hence by Remark 2.6 (a) (i) in [3], $\varphi_{i_0} \circ \sigma_{i_0} = \sigma_{i_0} \circ \varphi_{i_0}$, where σ_{i_0} is the Krein adjoint of φ_{i_0}. Therefore, if the restriction of φ to $F(\varphi)$ is a 1-1 function and $\mathbb{P}_\varphi$ is a nonempty set, then Equation (8) shows that the member of $\mathbb{P}_\varphi$ has no effect on essential normality of T_wC_φ.

Theorem 3.8. *Suppose that $\varphi, \varphi_1, \ldots, \varphi_r, \zeta_1, \ldots, \zeta_r, w$ and $F(\varphi)$ are as in Theorem 3.1. Let the restriction of φ to $F(\varphi)$ be a 1-1 function. Then T_wC_φ is essentially normal if and only if for each $\zeta \in F(\varphi) - \mathbb{P}_\varphi$, $w(\zeta)$ takes one of the following:*

(i) *If ζ is the fixed point of φ and $\varphi'(\zeta) \neq 1$, then $w(\zeta) = 0$.*

(ii) *If $\varphi(\zeta) = \eta$ with $\eta \notin F(\varphi)$, then $w(\zeta) = 0$ and moreover, if for every i, $1 \leqslant i \leqslant n$, $\varphi^{[-i]}(\{\zeta\}) \cap F(\varphi) \neq \emptyset$ whenever $n \in \mathbb{N}$ and $1 \leqslant n < r$, then $w(z) = 0$ for $z \in \varphi^{[-i]}(\{\zeta\}) \cap F(\varphi)$.*

(iii) *Assume that $w(\zeta)$ is not zero in Statement (i) or (ii), i.e., there is the smallest integer n, $1 < n \leqslant r$, such that $\varphi^{[n]}(\zeta) = \zeta$. For convenience, let $h_1 = \zeta$, $h_2 = \varphi(\zeta), \ldots, h_n = \varphi^{[n-1]}(\zeta)$. For each $1 \leqslant i \leqslant n$, let ϕ_i be the linear-fractional transformation related to φ and h_i as in Theorem 3.1; also ς_i be the Krein adjoint of ϕ_i. Then $\{\phi_i \circ \varsigma_i\colon 1 \leqslant i \leqslant n\} = \{\varsigma_i \circ \phi_i\colon 1 \leqslant i \leqslant n\}$ and for each $1 \leqslant i, j \leqslant n$, $|w(h_i)|^2|\varphi'(h_i)|^{-1} = |w(h_j)|^2|\varphi'(h_j)|^{-1}$ or $w(h_i) = 0$ for any $1 \leqslant i \leqslant n$.*

P r o o f. Let T_wC_φ be essentially normal. Then by Propositions 3.5 and 3.6, Statements (i) and (ii) hold. Suppose that we cannot obtain the value of $w(\zeta)$ from Statement (i) or (ii). Since the restriction of φ to $F(\varphi)$ is a 1-1 function and $F(\varphi)$ is a finite set, there is a smallest integer n, $1 < n \leqslant r$, such that $\varphi^{[n]}(\zeta) = \zeta$, so by Proposition 3.7, the proof is complete.

Conversely, without loss of generality we can assume that $\zeta_r \in \mathbb{P}_\varphi$, there is a smallest natural number n, $1 < n < r$, with $\varphi(\zeta_1) = \zeta_2, \ldots, \varphi(\zeta_{n-1}) = \zeta_n, \varphi(\zeta_n) = \zeta_1$

and for each $i > n$ and $i \neq r$, $w(\zeta_i) = 0$. Thus, Equation (8) implies that

$$\begin{aligned}
&[T_wC_\varphi, (T_wC_\varphi)^*] \\
&\quad \equiv (|w(\zeta_1)|^2|\varphi'(\zeta_1)|^{-1}C_{\sigma_1\circ\varphi_1} - |w(\zeta_n)|^2|\varphi'(\zeta_n)|^{-1}C_{\varphi_n\circ\sigma_n}) \\
&\qquad + \ldots + (|w(\zeta_n)|^2|\varphi'(\zeta_n)|^{-1}C_{\sigma_n\circ\varphi_n} - |w(\zeta_{n-1})|^2|\varphi'(\zeta_{n-1})|^{-1}C_{\varphi_{n-1}\circ\sigma_{n-1}}) \\
&\qquad + |w(\zeta_r)|^2|\varphi'(\zeta_r)|^{-1}(C_{\sigma_r\circ\varphi_r} - C_{\varphi_r\circ\sigma_r}).
\end{aligned}$$

As we observed before, ζ_r has no effect on the essential normality of T_wC_φ. Hence by Theorem 3.4, T_wC_φ is essentially normal. □

Now for $\varphi \in \mathcal{S}(2)$, suppose that the restriction of φ to $F(\varphi)$ is not a 1-1 function. Let

$$\begin{aligned}
(10) \qquad F(\varphi) = \{&\zeta_{r_0}, \zeta_{r_0+1}, \ldots, \zeta_{r_1-1}, \zeta_{r_1}, \zeta_{r_1+1}, \ldots, \zeta_{r_{n-1}-1}, \zeta_{r_{n-1}}, \\
&\zeta_{r_{n-1}+1}, \ldots, \zeta_{r_n-1}, \zeta_{r_n}, \zeta_{r_{n+1}}, \ldots, \zeta_{r_{n+k}}\}
\end{aligned}$$

for some $n, k \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned}
(11) \qquad \varphi(\zeta_{r_0}) &= \varphi(\zeta_{r_0+1}) = \ldots = \varphi(\zeta_{r_1-1}), \ldots, \varphi(\zeta_{r_{n-1}}) \\
&= \varphi(\zeta_{r_{n-1}+1}) = \ldots = \varphi(\zeta_{r_n-1})
\end{aligned}$$

and let the restriction of φ to $\{\zeta_{r_0}, \zeta_{r_1}, \ldots, \zeta_{r_{n-1}}, \zeta_{r_n}, \zeta_{r_{n+1}}, \ldots, \zeta_{r_{n+k}}\}$ be a 1-1 function. From now on, unless otherwise stated, let $\mathbb{A}_i = \{\zeta_{r_i}, \zeta_{r_i+1}, \ldots, \zeta_{r_{i+1}-1}\}$ and $\zeta_{r_{i+1}-1} = \zeta_{r_i+|\mathbb{A}_i|-1}$ for each $0 \leqslant i \leqslant n+k$; furthermore, suppose that φ_{r_i+h} is the linear-fractional transformation related to φ and ζ_{r_i+h} as in Theorem 3.1, where $\zeta_{r_i+h} \in F(\varphi)$; also assume that σ_{r_i+h} is the Krein adjoint of φ_{r_i+h}. Let $\zeta_{r_i+h}, \zeta_{r_j+t} \in F(\varphi)$. It is obvious that $C_{\varphi_{r_i+h}\circ\sigma_{r_j+t}} \notin B_0(H^2)$ if and only if $i = j$ and $t = h$. Also, $C_{\sigma_{r_j+t}\circ\varphi_{r_i+h}} \notin B_0(H^2)$ if and only if $\varphi(\zeta_{r_i+h}) = \varphi(\zeta_{r_j+t})$. Therefore, by these facts, Equation (4) and after some patient calculations, one obtains

$$\begin{aligned}
(12) \qquad [T_wC_\varphi, (T_wC_\varphi)^*] \equiv\ & |w(\zeta_{r_0})|^2|\varphi'(\zeta_{r_0})|^{-1}C_{\sigma_{r_0}\circ\varphi_{r_0}} \\
&+ w(\zeta_{r_0})\overline{w(\zeta_{r_0+1})}|\varphi'(\zeta_{r_0+1})|^{-1}C_{\sigma_{r_0+1}\circ\varphi_{r_0}} + \ldots \\
&+ |w(\zeta_{r_1-1})|^2|\varphi'(\zeta_{r_1-1})|^{-1}C_{\sigma_{r_1-1}\circ\varphi_{r_1-1}} + \ldots \\
&+ |w(\zeta_{r_n-1})|^2|\varphi'(\zeta_{r_n-1})|^{-1}C_{\sigma_{r_n-1}\circ\varphi_{r_n-1}} + \ldots \\
&+ |w(\zeta_{r_n})|^2|\varphi'(\zeta_{r_n})|^{-1}C_{\sigma_{r_n}\circ\varphi_{r_n}} + \ldots \\
&+ |w(\zeta_{r_{n+k}})|^2|\varphi'(\zeta_{r_{n+k}})|^{-1}C_{\sigma_{r_{n+k}}\circ\varphi_{r_{n+k}}}
\end{aligned}$$

$$
\begin{aligned}
&- (|w(\zeta_{r_0})|^2|\varphi'(\zeta_{r_0})|^{-1}C_{\varphi_{r_0}\circ\sigma_{r_0}} \\
&+ |w(\zeta_{r_0+1})|^2|\varphi'(\zeta_{r_0+1})|^{-1}C_{\varphi_{r_0+1}\circ\sigma_{r_0+1}} + \ldots \\
&+ |w(\zeta_{r_n-1})|^2|\varphi'(\zeta_{r_n-1})|^{-1}C_{\varphi_{r_n-1}\circ\sigma_{r_n-1}} \\
&+ |w(\zeta_{r_n})|^2|\varphi'(\zeta_{r_n})|^{-1}C_{\varphi_{r_n}\circ\sigma_{r_n}} + \ldots \\
&+ |w(\zeta_{r_{n+k}})|^2|\varphi'(\zeta_{r_{n+k}})|^{-1}C_{\varphi_{r_{n+k}}\circ\sigma_{r_{n+k}}}).
\end{aligned}
$$

Proposition 3.9. *Suppose that φ and w are as in Theorem* 3.1 *and $F(\varphi)$ is as in Equation* (10). *In accordance with Equation* (11), *for each $0 \leqslant i < n$ we assume that $\varphi(\zeta_{r_i}) = \varphi(\zeta_{r_i+1}) = \ldots = \varphi(\zeta_{r_{i+1}-1})$. If T_wC_φ is essentially normal, then the values of $w(\zeta_{r_i}), \ldots, w(\zeta_{r_{i+1}-1})$ are all zero except at most one of them.*

P r o o f. Without loss of generality, we can assume that $w(\zeta_{r_i}) \neq 0$ and $w(\zeta_{r_i+1}) \neq 0$. Let $B = \{\sigma_{r_i} \circ \varphi_{r_i}, \sigma_{r_i+1} \circ \varphi_{r_i}, \ldots, \sigma_{r_{i+1}-1} \circ \varphi_{r_i}\}$. Every linear-fractional transformation in Equation (12) which has a finite angular derivative at ζ_{r_i} belongs to B or

$$
\{\varphi_{r_j+h} \circ \sigma_{r_j+h} \colon\ 0 \leqslant j \leqslant n+k,\ 0 \leqslant h \leqslant |\mathbb{A}_j| - 1 \text{ and } \varphi_{r_j+h}(\zeta_{r_j+h}) = \zeta_{r_i}\}.
$$

Now apply Theorem 3.4 to $k = 2$ and $d = D_2(\sigma_{r_i+1}\circ\varphi_{r_i}, \zeta_{r_i})$; hence $w(\zeta_{r_i})w(\zeta_{r_i+1}) = 0$, which is a contradiction. □

By the preceding proposition and Equation (12), we can assume that

$$
\begin{aligned}
&[T_wC_\varphi, (T_wC_\varphi)^*] \\
&\quad\equiv |w(\zeta_{r_0})|^2|\varphi'(\zeta_{r_0})|^{-1}(C_{\sigma_{r_0}\circ\varphi_{r_0}} - C_{\varphi_{r_0}\circ\sigma_{r_0}}) + \ldots \\
&\qquad+ |w(\zeta_{r_{n-1}})|^2|\varphi'(\zeta_{r_{n-1}})|^{-1}(C_{\sigma_{r_{n-1}}\circ\varphi_{r_{n-1}}} - C_{\varphi_{r_{n-1}}\circ\sigma_{r_{n-1}}}) \\
&\qquad+ |w(\zeta_{r_n})|^2|\varphi'(\zeta_{r_n})|^{-1}(C_{\sigma_{r_n}\circ\varphi_{r_n}} - C_{\varphi_{r_n}\circ\sigma_{r_n}}) + \ldots \\
&\qquad+ |w(\zeta_{r_{n+k}})|^2|\varphi'(\zeta_{r_{n+k}})|^{-1}(C_{\sigma_{r_{n+k}}\circ\varphi_{r_{n+k}}} - C_{\varphi_{r_{n+k}}\circ\sigma_{r_{n+k}}}).
\end{aligned}
$$

In the next theorem φ and w are as in Theorem 3.1 and $F(\varphi)$ is as in Equation (10). In accordance with Equation (11), for each $0 \leqslant i < n$ we assume that $\varphi(\zeta_{r_i}) = \varphi(\zeta_{r_i+1}) = \ldots = \varphi(\zeta_{r_{i+1}-1})$. Furthermore, $G(\varphi)$ in Statements (iii) and (iv) of the theorem is

$$
G(\varphi) := \{\zeta \colon\ \zeta \in F(\varphi) \text{ and } w(\zeta) \text{ is not zero in Statement (i)}\}.
$$

Theorem 3.10. *The operator T_wC_φ is essentially normal if and only if for each $\zeta \in F(\varphi) - \mathbb{P}_\varphi$, $w(\zeta)$ satisfies one of the following conditions:*

(i) *For each $0 \leqslant i < n$, the values of $w(\zeta_{r_i}), \ldots, w(\zeta_{r_{i+1}-1})$ are all zero except at most one of them.*

(ii) *If* ζ *is the fixed point of* φ *and* $\varphi'(\zeta) \neq 1$, *then* $w(\zeta) = 0$.

(iii) *If* $\varphi(\zeta) = \eta$ *for* $\eta \notin G(\varphi)$, *then* $w(\zeta) = 0$ *and moreover, if for every* j, $1 \leqslant j \leqslant m$, $\varphi^{[-j]}(\{\zeta\}) \cap G(\varphi) \neq \emptyset$ *whenever* $m \in \mathbb{N}$ *and* $1 \leqslant m < |G(\varphi)|$, *then* $w(z) = 0$ *for* $z \in \varphi^{[-j]}(\{\zeta\}) \cap G(\varphi)$.

(iv) *Suppose that* $w(\zeta)$ *is not zero in Statement* (i) *or* (ii) *or* (iii), *i.e., there is a smallest integer* n_0, $1 < n_0 \leqslant |G(\varphi)|$, *such that* $\varphi^{[n_0]}(\zeta) = \zeta$. *For convenience, assume that* $h_1 = \zeta, h_2 = \varphi(\zeta), \ldots, h_{n_0} = \varphi^{[n_0-1]}(\zeta)$. *For each* $1 \leqslant i \leqslant n_0$, *let* ϕ_i *be the linear-fractional transformation related to* φ *and* h_i *be as in Theorem* 3.1; *let* ς_i *be the Krein adjoint of* ϕ_i. *Then* $\{\phi_i \circ \varsigma_i \colon 1 \leqslant i \leqslant n_0\} = \{\varsigma_i \circ \phi_i \colon 1 \leqslant i \leqslant n_0\}$ *and for every* $1 \leqslant i, j \leqslant n_0$, $|w(h_i)|^2 |\varphi'(h_i)|^{-1} = |w(h_j)|^2 |\varphi'(h_j)|^{-1}$ *or* $w(h_i) = 0$ *for any* $1 \leqslant i \leqslant n_0$.

Proof. Let $T_w C_\varphi$ be essentially normal. Without loss of generality, by Proposition 3.9 we can assume that $w(\zeta_{r_i+h}) = 0$ when $h \neq 0$ and $0 \leqslant i \leqslant n-1$. Thus, $G(\varphi) \subseteq \{\zeta_{r_0}, \zeta_{r_1}, \ldots, \zeta_{r_{n-1}}, \zeta_{r_n}, \ldots, \zeta_{r_{n+k}}\}$. Since the restriction of φ to $G(\varphi)$ is a 1-1 function, Theorem 3.8 gives the desired conclusion.

Conversely, the conclusion follows from Theorem 3.8. □

For each $\varphi_i \in \mathcal{L}$, let σ_i be the Krein adjoint of φ_i and let $\zeta_i \in F(\varphi_i)$. In the remainder of this section, we investigate the essential normality problem for certain finite linear combinations of linear-fractional composition operators.

Proposition 3.11. *Suppose that* $r, n \in \mathbb{N}$, $1 \leqslant r \leqslant n$, *and* $c_1, \ldots, c_n \in \mathbb{C}$. *Assume that* $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$ *are pairwise distinct. Let* $F(\varphi_i) = \{\zeta_i\}$ *and* $\zeta \in \bigcap_{i=1}^{r} F(\varphi_i) - \bigcup_{i=r+1}^{n} F(\varphi_i)$. *Also for each* $1 \leqslant j \leqslant r$, *let* $\varphi_j(\zeta) \notin \{\varphi_i(\zeta) \colon 1 \leqslant i \leqslant r$ *and* $i \neq j\}$. *Furthermore, assume there is at most one integer* $i_0 \in \{1, \ldots, r\}$ *such that* $\varphi_{i_0}(\zeta) \in \bigcup_{i=1}^{n} F(\varphi_i)$. *If* $c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n}$ *is essentially normal, then the values of* $c_1, \ldots, c_r$ *are all zero except at most* c_{i_0}.

Proof. We infer from Equation (4) that

$$
\begin{aligned}
(13)\quad & [c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n}, (c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n})^*] \\
&\equiv \sum_{\varphi_j(\zeta_j) = \varphi_i(\zeta_i)} c_i \overline{c_j} |\varphi_j'(\zeta_j)|^{-1} C_{\sigma_j \circ \varphi_i} - \sum_{\zeta_j = \zeta_i} c_i \overline{c_j} |\varphi_j'(\zeta_j)|^{-1} C_{\varphi_i \circ \sigma_j} \\
&\equiv \sum_{\varphi_j(\zeta_j) = \varphi_i(\zeta_i)} c_i \overline{c_j} |\varphi_j'(\zeta_j)|^{-1} C_{\sigma_j \circ \varphi_i} - \sum_{1 \leqslant i, j \leqslant r} c_i \overline{c_j} |\varphi_j'(\zeta_j)|^{-1} C_{\varphi_i \circ \sigma_j} \\
&\quad - \sum_{\substack{\zeta_j = \zeta_i \\ i, j > r}} c_i \overline{c_j} |\varphi_j'(\zeta_j)|^{-1} C_{\varphi_i \circ \sigma_j}.
\end{aligned}
$$

For $j_0 \neq i_0$ and $1 \leqslant j_0 \leqslant r$, let $B = \{\varphi_{j_0} \circ \sigma_{j_0}\} \cup \{\varphi_i \circ \sigma_i \colon r < i \text{ and } \varphi_i(\zeta_i) = \varphi_{j_0}(\zeta_{j_0})\}$. It is clear that every linear-fractional transformation in the above equivalence which sends $\varphi_{j_0}(\zeta_{j_0})$ to $\varphi_{j_0}(\zeta_{j_0})$ belongs to B. Now apply Theorem 3.4 to $k = 2$ and $d = D_2(\varphi_{j_0} \circ \sigma_{j_0}, \varphi_{j_0}(\zeta_{j_0}))$; hence there is a finite set I, $I \subseteq \{i \colon i > r \text{ and } \varphi_i(\zeta_i) = \varphi_{j_0}(\zeta_{j_0})\}$, such that

$$|c_{j_0}|^2 |\varphi'_{j_0}(\zeta_{j_0})|^{-1} + \sum_{i \in I} |c_i|^2 |\varphi'_i(\zeta_i)|^{-1} = 0.$$

Hence $c_{j_0} = 0$, as desired. $\square$

Let $n \in \mathbb{N}$. In the next theorem for each $1 \leqslant i \leqslant n$, c_i, φ_i, ζ_i and $F(\varphi_i)$ are as in Proposition 3.11 and $F := \bigcup_{i=1}^{n} F(\varphi_i)$. Also, if for some subset $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$,

$$\bigcap_{l=1}^{m} F(\varphi_{i_l}) - \bigcup_{\substack{i \neq i_l \\ 1 \leqslant l \leqslant m}} F(\varphi_i) \neq \emptyset, \tag{14}$$

then for each $1 \leqslant l \leqslant m$, $\varphi_{i_l}(\zeta_{i_l}) \notin \{\varphi_{i_j}(\zeta_{i_j}) \colon 1 \leqslant j \leqslant m \text{ and } j \neq l\}$; moreover, there is at most one integer $j_0 \in \{1, \ldots, m\}$ such that $\varphi_{i_{j_0}}(\zeta_{i_{j_0}}) \in F$. Furthermore, G in Statement (iii) of the theorem is

$$G := \{\zeta \colon \zeta \in F(\varphi_i) \text{ and } c_i \text{ is not zero in Statement (i) for some } 1 \leqslant i \leqslant n\}.$$

Theorem 3.12. *The operator $c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n}$ is essentially normal if and only if for each $1 \leqslant j \leqslant n$ when $\zeta_j \notin \mathbb{P}_{\varphi_j}$, c_j satisfies one of the following conditions:*

(i) *Suppose that $\varphi_{r_1}(\zeta_{r_1}) = \ldots = \varphi_{r_k}(\zeta_{r_k})$ for $1 \leqslant r_1, \ldots, r_k \leqslant n$ and $\varphi_i(\zeta_i) \neq \varphi_{r_1}(\zeta_{r_1})$ when $1 \leqslant i \leqslant n$ and $i \notin \{r_1, \ldots, r_k\}$. Then the values of $c_{r_1}, \ldots, c_{r_k}$ are all zero except at most one of them.*

(ii) *If ζ_i is the fixed point of φ_i and $\varphi'_i(\zeta_i) \neq 1$, then $c_i = 0$.*

(iii) *If $\varphi_r(\zeta_r) \notin G$ when $1 \leqslant r \leqslant n$, then $c_r = 0$ and moreover, if for each j, $1 \leqslant j \leqslant k$, $\varphi_{r_1}^{-1} \circ \ldots \circ \varphi_{r_j}^{-1}(\{\zeta_r\}) \cap G \neq \emptyset$ whenever $k \in \mathbb{N}$ and $1 \leqslant r_1, \ldots, r_k \leqslant n$, then $c_{r_1} = \ldots = c_{r_k} = 0$.*

(iv) *Assume that c_i is not zero in the preceding statements, i.e., there are distinct integers $1 \leqslant r_1, \ldots, r_k \leqslant n$ such that $\{\zeta_i, \zeta_{r_1}, \ldots, \zeta_{r_k}\} \subseteq G$ and $\varphi_{r_1} \circ \ldots \circ \varphi_{r_k} \circ \varphi_i(\zeta_i) = \zeta_i$. Let $B = \{i, r_1, \ldots, r_k\}$. Then $\{\varphi_j \circ \sigma_j \colon j \in B\} = \{\sigma_j \circ \varphi_j \colon j \in B\}$ and for every $j, h \in B$, $|c_j|^2 |\varphi'_j(\zeta_j)|^{-1} = |c_h|^2 |\varphi'_h(\zeta_h)|^{-1}$, or for each $j \in B$, $c_j = 0$.*

Proof. Let $c_1C_{\varphi_1}+\ldots+c_nC_{\varphi_n}$ be essentially normal. Without loss of generality, by Proposition 3.11 and Equation (13), we can assume that there exists an integer m, $1\leqslant m\leqslant n$, such that for all distinct integers $1\leqslant i,j\leqslant m$, $F(\varphi_i)\cap F(\varphi_j)=\emptyset$ and

$$\begin{aligned}[c_1C_{\varphi_1}+\ldots+c_nC_{\varphi_n},(c_1C_{\varphi_1}+\ldots+c_nC_{\varphi_n})^*]\\ \equiv \sum_{\substack{\varphi_j(\zeta_j)=\varphi_i(\zeta_i)\\ 1\leqslant i,j\leqslant m}} c_i\overline{c_j}|\varphi_j'(\zeta_j)|^{-1}C_{\sigma_j\circ\varphi_i}-\sum_{i=1}^{m}|c_i|^2|\varphi_i'(\zeta_i)|^{-1}C_{\varphi_i\circ\sigma_i}.\end{aligned}$$

Now let $A=\{\zeta_i\colon\ 1\leqslant i\leqslant m\}$. We can rewrite

$$\begin{aligned}A=\{&\zeta_{r_0},\zeta_{r_0+1},\ldots,\zeta_{r_1-1},\zeta_{r_1},\zeta_{r_1+1},\ldots,\zeta_{r_{p-1}-1},\zeta_{r_{p-1}},\\ &\zeta_{r_{p-1}+1},\ldots,\zeta_{r_p-1},\zeta_{r_p},\zeta_{r_{p+1}},\ldots,\zeta_{r_{p+k}}\}\end{aligned}$$

for some $p,k\in\mathbb{N}\cup\{0\}$ such that

$$\varphi(\zeta_{r_0})=\varphi(\zeta_{r_0+1})=\ldots=\varphi(\zeta_{r_1-1}),\ldots,\varphi(\zeta_{r_{p-1}})=\varphi(\zeta_{r_{p-1}+1})=\ldots=\varphi(\zeta_{r_p-1})$$

and for each integer i, $0\leqslant i\leqslant k$, the value of $\varphi(\zeta_{r_{i+p}})$ is not equal to $\varphi(\zeta)$ for each $\zeta\in A-\{\zeta_{r_{i+p}}\}$. Also, there exists an integer t, $0\leqslant t\leqslant k$, such that $\varphi_{r_{i+p}}(\zeta_{r_{i+p}})=\zeta_{r_{i+p}}$ and $\varphi'_{r_{i+p}}(\zeta_{r_{i+p}})=1$ for any $t\leqslant i\leqslant k$. As we observed before, for any i, $t\leqslant i\leqslant k$, $\varphi_{r_{i+p}}\circ\sigma_{r_{i+p}}=\sigma_{r_{i+p}}\circ\varphi_{r_{i+p}}$; hence $\zeta_{r_{i+p}}$ has no effect on the essential normality of $c_1C_{\varphi_1}+\ldots+c_nC_{\varphi_n}$. Therefore, we can see that

$$\begin{aligned}&[c_1C_{\varphi_1}+\ldots+c_nC_{\varphi_n},(c_1C_{\varphi_1}+\ldots+c_nC_{\varphi_n})^*]\\ &\quad\equiv|c_{r_0}|^2|\varphi'_{r_0}(\zeta_{r_0})|^{-1}C_{\sigma_{r_0}\circ\varphi_{r_0}}+c_{r_0}\overline{c_{r_0+1}}|\varphi'_{r_0+1}(\zeta_{r_0+1})|^{-1}C_{\sigma_{r_0+1}\circ\varphi_{r_0}}+\ldots\\ &\qquad+|c_{r_1-1}|^2|\varphi'_{r_1-1}(\zeta_{r_1-1})|^{-1}C_{\sigma_{r_1-1}\circ\varphi_{r_1-1}}+\ldots\\ &\qquad+|c_{r_p-1}|^2|\varphi'_{r_p-1}(\zeta_{r_p-1})|^{-1}C_{\sigma_{r_p-1}\circ\varphi_{r_p-1}}+\ldots\\ &\qquad+|c_{r_p}|^2|\varphi'_{r_p}(\zeta_{r_p})|^{-1}C_{\sigma_{r_p}\circ\varphi_{r_p}}+\ldots+|c_{r_{p+t}}|^2|\varphi'_{r_{p+t}}(\zeta_{r_{p+t}})|^{-1}C_{\sigma_{r_{p+t}}\circ\varphi_{r_{p+t}}}\\ &\qquad-(|c_{r_0}|^2|\varphi'_{r_0}(\zeta_{r_0})|^{-1}C_{\varphi_{r_0}\circ\sigma_{r_0}}+|c_{r_0+1}|^2|\varphi'_{r_0+1}(\zeta_{r_0+1})|^{-1}C_{\varphi_{r_0+1}\circ\sigma_{r_0+1}}+\ldots\\ &\qquad+|c_{r_p-1}|^2|\varphi'_{r_p-1}(\zeta_{r_p-1})|^{-1}C_{\varphi_{r_p-1}\circ\sigma_{r_p-1}}+|c_{r_p}|^2|\varphi'_{r_p}(\zeta_{r_p})|^{-1}C_{\varphi_{r_p}\circ\sigma_{r_p}}+\ldots\\ &\qquad+|c_{r_{p+t}}|^2|\varphi'_{r_{p+t}}(\zeta_{r_{p+t}})|^{-1}C_{\varphi_{r_{p+t}}\circ\sigma_{r_{p+t}}}).\end{aligned}$$

The above equivalence is like Equation (12), so the result follows from a proof similar to that of Theorem 3.10.

Conversely, suppose that for some subset $\{i_1,\ldots,i_m\}\subseteq\{1,\ldots,n\}$, Equation (14) holds. By the hypothesis, there is at most one integer j_0, $1\leqslant j_0\leqslant m$, such that $\varphi_{i_{j_0}}(\zeta_{i_{j_0}})\in F$. Since $G\subseteq F$, Statement (iii) implies that the values of $c_{i_1},\ldots,c_{i_m}$

are all zero except at most $c_{i_{j_0}}$. Hence without loss of generality we can assume that there is a smallest natural number k, $1 < k < n$, with $\varphi_1(\zeta_1) = \zeta_2$, $\ldots$, $\varphi_{k-1}(\zeta_{k-1}) = \zeta_k$ and $\varphi_k(\zeta_k) = \zeta_1$, and for each integer i, $k+1 < i < n$, $c_i = 0$; moreover, $\varphi_{k+1}(\zeta_{k+1}) = \zeta_{k+1}$ and $\varphi'_{k+1}(\zeta_{k+1}) = 1$. Thus, Equation (13) implies that

$$\begin{aligned}
&[c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n}, (c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n})^*] \\
&\qquad \equiv \sum_{i=1}^{k+1} |c_i|^2 |\varphi'_i(\zeta_i)|^{-1} C_{\sigma_i \circ \varphi_i} - \sum_{i=1}^{k+1} |c_i|^2 |\varphi'_i(\zeta_i)|^{-1} C_{\varphi_i \circ \sigma_i} \\
&\qquad \equiv (|c_1|^2 |\varphi'_1(\zeta_1)|^{-1} C_{\sigma_1 \circ \varphi_1} - |c_k|^2 |\varphi'_k(\zeta_k)|^{-1} C_{\varphi_k \circ \sigma_k}) + \ldots \\
&\qquad\quad + (|c_k|^2 |\varphi'_k(\zeta_k)|^{-1} C_{\sigma_k \circ \varphi_k} - |c_{k-1}|^2 |\varphi'_{k-1}(\zeta_{k-1})|^{-1} C_{\varphi_{k-1} \circ \sigma_{k-1}}) \\
&\qquad\quad + |c_{k+1}|^2 |\varphi'_{k+1}(\zeta_{k+1})|^{-1} (C_{\sigma_{k+1} \circ \varphi_{k+1}} - C_{\varphi_{k+1} \circ \sigma_{k+1}}).
\end{aligned}$$

As we mentioned before, ζ_{k+1} has no effect on the essential normality of $c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n}$. Hence by Theorem 3.4, $c_1 C_{\varphi_1} + \ldots + c_n C_{\varphi_n}$ is essentially normal. □

In the following remark, we compare the results which were obtained in [3] with Theorem 3.12 when $n = 1$.

Remark 3.13. Suppose that $\varphi \in \mathrm{LFT}(\mathbb{D})$ is not an automorphism and that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Then $F(\varphi) = \{\zeta\}$ and we have:

(a) If $\zeta \neq \eta$, then by Theorem 3.12, C_φ is not essentially normal (see [3, Theorem 6.1]).
(b) If $\zeta = \eta$ and $\varphi'(\zeta) \neq 1$, then Theorem 3.12 implies that C_φ is not essentially normal (see [3, Theorem 5.2]).
(c) If $\zeta = \eta$ and $\varphi'(\zeta) = 1$, then φ is parabolic. We infer from Theorem 3.12 that C_φ is essentially normal (see [3, Theorem 4.1]).

Remark 3.14. For $1 \leqslant i \leqslant n$, let φ_i be a non-automorphism linear-fractional self-map of $\mathbb{D}$ and $B = \{i\colon 1 \leqslant i \leqslant n \text{ and } \|\varphi_i\|_\infty = 1\}$. Assume that for each $i \in B$, φ_i, ζ_i and $F(\varphi_i)$ satisfy the hypotheses of Theorem 3.12. Let for any $i \in B$, w_i be a bounded measurable function on $\partial\mathbb{D}$ which is continuous at ζ_i. Suppose that for $i \notin B$, $w_i \in L^\infty(\partial\mathbb{D})$. We know that if $\|\varphi\|_\infty < 1$, then C_φ is compact. Therefore, for $c_1, \ldots, c_n \in \mathbb{C}$, Corollary 2.2 in [10] implies that

$$c_1 T_{w_1} C_{\varphi_1} + \ldots + c_n T_{w_n} C_{\varphi_n} \equiv \sum_{i \in B} c_i w_i(\zeta_i) C_{\varphi_i}.$$

Hence by Theorem 3.12 we can characterize the essentially normal finite linear combinations of these operators on H^2.

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