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ON NONUNIFORM DICHOTOMY FOR STOCHASTIC SKEW-EVOLUTION SEMIFLOWS IN HILBERT SPACES

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Abstract. In this paper we study a general concept of nonuniform exponential dichotomy in mean square for stochastic skew-evolution semiflows in Hilbert spaces. We obtain a variant for the stochastic case of some well-known results, of the deterministic case, due to R. Datko: Uniform asymptotic stability of evolutionary processes in a Banach space, SIAM J. Math. Anal., 3(1972), 428–445. Our approach is based on the extension of some techniques used in the deterministic case for the study of asymptotic behavior of skew-evolution semiflows in Banach spaces.

Keywords: stochastic skew-evolution semiflow, nonuniform exponential dichotomy in mean square

MSC 2010: 37L55, 60H25, 93E15

1. Introduction

In the qualitative theory of evolution equations, exponential dichotomy is one the most important asymptotic properties and during the last years it was treated from various perspectives.

Several important papers on the problem of existence of stochastic semiflows for stochastic evolution equations appeared in literature and we only mention [7], [9], and [14]. For linear stochastic evolution equations with finite-dimensional noise, a stochastic semiflow was obtained by Bensoussan and Flandoli in [3].

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In [13] the authors proved the existence of perfect differentiable cocycles generated by mild solutions of a large class of semilinear stochastic evolution equations and stochastic partial differential equations.

The exponential dichotomy property in stochastic case has been discussed by many authors, and we refer here to the papers of Caraballo et al. in [5], Ateiwi A. M. in [2], and Mohammend et al. in [13]. Nonuniform stability of stochastic differential equations has been presented by the authors Buse and Barbu in [4], Da Prato and Ichikawa in [6].

Our main objective is to give a more general concept of nonuniform exponential dichotomy in mean square of stochastic skew-evolution semiflows in Hilbert spaces. Thus we obtain a variant for the stochastic case of nonuniform exponential dichotomy in mean square of some results obtained by R. Datko in [8].

Our approach is based on the extension of some techniques used in the deterministic case by many authors, and we mention here only M. Megan et al. in [11], [12] and C. Stoica and M. Megan in [15]. Some of the results for uniform asymptotic behavior in mean square of stochastic cocycles generated by stochastic differential equations was studied by D. Stoica in the paper [16].

2. Stochastic skew-evolution semiflows

Let $(\Omega, \Im, \{\Im_t\}_{t\geqslant 0}, \mathbf{P})$ be a standard filtered probability space, X a real separable Hilbert space, $\mathcal{L}(X)$ the set of all linear bounded operators from X to X and let $\Delta = \{(t,s) \in \mathbb{R}^2_+ \mid t \geqslant s \geqslant 0\}$.

Definition 2.1. A stochastic evolution semiflow on Ω is a measurable random field $\varphi : (\Delta \times \Omega, \mathcal{B}(X) \times \Im) \to (\Omega, \Im)$ satisfying the following conditions:

1. $\varphi(s,s,\omega) = \omega$,
2. $\varphi(t,s,\varphi(s,t_0,\omega)) = \varphi(t,t_0,\omega)$, $\forall (t,s),(s,t_0) \in \Delta$, $\omega \in \Omega$.

Example 2.1. Let X be a real separable Hilbert space and let Ω be the space of all continuous paths $\omega\colon \mathbb{R}_+ \to X$ such that $\omega(0) = 0$ with the compact open topology. Let $\Im_t$ for $t \geqslant 0$ be the σ-algebra generated by the set $\{\omega \to \omega(u) \in X \mid u \leqslant t\}$ and let $\Im$ be the associated Borrel σ-algebra to Ω. If $\mathbf{P}$ is a Wiener measure on Ω then $(\Omega, \Im, \{\Im_t\}_{t\geqslant 0}, \mathbf{P})$ is a filtered probability space with the Wiener motion $W(t,\omega) = \omega(t)$ for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$.

Then the mapping $\varphi\colon (\Delta \times \Omega, \mathcal{B}(X) \times \Im) \to (\Omega, \Im)$ defined by

$$\varphi(t,s,\omega) = \omega(t+s) - \omega(t), \quad \forall \omega \in \Omega$$

is a stochastic evolution semiflow on Ω. Indeed,

$$\begin{aligned}
\varphi(s,s,\omega) &= \omega(s), \quad \text{and} \\
\varphi(t,s,\varphi(s,t_0,\omega)) &= \varphi(t+s,t_0,\omega) - \varphi(t,t_0,\omega) \\
&= \omega(t+s+t_0) - \omega(t+s) - \omega(t+t_0) + \omega(t) \\
&= \varphi(t+t_0,s,\omega) - \varphi(t,s,\omega) = \varphi(t,t_0,\varphi(s,s,\omega)) = \varphi(t,t_0,\omega)
\end{aligned}$$

for all $(t,s),(s,t_0) \in \Delta$ and $\omega \in \Omega$.

Definition 2.2. A mapping $\Phi\colon (\Delta \times \Omega, \mathcal{B}(T) \times \Im \times \mathcal{B}(X)) \to \mathcal{L}(X)$ with the properties

1. $\Phi(s,s,\omega) = I$ (the identity operator on X), $\forall s \geqslant 0$, $\omega \in \Omega$;
2. $\Phi(t,t_0,\omega) = \Phi(t,s,\varphi(s,t_0,\omega))\Phi(s,t_0,\omega)$ for all $(t,s),(s,t_0) \in \Delta$ and $\omega \in \Omega$; is called a stochastic evolution cocycle on X over the stochastic evolution semiflow $\varphi\colon (\Delta \times \Omega, \mathcal{B}(X) \times \Im) \to (\Omega, \Im)$.

Definition 2.3. The mapping $\Theta\colon \Delta \times Y \to Y$ defined it by

$$\Theta(t,s,\omega,x) = (\varphi(t,s,\omega), \Phi(t,s,\omega)x)$$

where $Y = \Omega \times X$, is called the stochastic skew-evolution semiflow on Y (denoted by s.s.-e.s.), where Φ is a stochastic evolution cocycle over the stochastic evolution semiflow φ on Ω, and we denote it by $\Theta = (\varphi, \Phi)$.

Example 2.2. Let $\Theta_0(t,\omega,x) = (\theta_0(t,\omega), \varphi_0(t,\omega)x)$ by the skew-product of the metric dynamical system $\theta_0(t,.)$ on Ω, generated by the Wiener shift, introduced by Arnold L. in [1]. Then the mapping $\Theta = (\varphi, \Phi)$ defined by

$$\Theta(t,s,\omega,x) = (\varphi(t,s,\omega), \Phi(t,s,\omega)x)$$

is a stochastic skew-evolution semiflow on Y, where

$$\begin{gathered}
\varphi(t,s,\omega) = \theta_0(t-s,\omega), \\
\Phi(t,s,\omega) = \varphi_0(t-s,\omega), \quad \forall\ (t,s) \in \Delta,\ \text{and } \omega \in \Omega.
\end{gathered}$$

Consequently, the stochastic skew-evolution semiflow generalizes the notion of the clasical skew-product, considered by Arnold L. in [1].

Example 2.3. Let X be a real separable Hilbert space and let $\{W(t)\}_{t \geqslant 0}$ be an X-valued Brownian motion with a separable covariance Hilbert space H and defined on the canonical complete filtered Wiener space $(\Omega, \Im, \{\Im_t\}_{t \geqslant 0}, \mathbf{P})$ introduced in

Example 2.1. $\mathcal{L}(H, X)$ be the Banach space of all bounded linear operators from H to X. Denote by $\mathcal{L}_2(H, X) \subset \mathcal{L}(H, X)$ the Hilbert-Schmidt operators $S\colon H \to X$ with the norm $\|S\| = \Big(\sum\limits_{k=1}^{\infty} |S(f_k)|^2\Big)^{1/2}$ where $|\cdot|$ is the norm on H, and $\{f_k, k \geqslant 1\}$ is a complete orthonormal system on H.

Consider the linear stochastic evolution equation of the form

$$\text{(2.1)} \qquad \begin{cases} \mathrm{d}u(t, x, \omega) = Au(t, x, \omega)dt + Bu(t, x, \omega)\,\mathrm{d}W(t), & t > s; \\ u(s, x, \omega) = x \in X & t \leqslant s,\ \forall \omega \in \Omega \end{cases}$$

where $A\colon D(A) \subset X \to X$ is the infinitesimal operator of a strongly continuous semigroup of bounded linear operators $T(t)\colon X \to X$, $t \geqslant 0$, and $B\colon X \to \mathcal{L}_2(H, X)$ is a bounded linear operator. Note that in [10] the authors studied the uniqueness of C_0-semigroups in locally convex vector spaces.

Suppose that B can be extended to a bounded linear operator which will be also denoted by $B\colon X \to \mathcal{L}(X)$, and the series $\sum\limits_{k=1}^{\infty} \|B_k^2\|_{\mathcal{L}(X)}$ converges, where $B_k\colon X \to X$ are bounded linear operator defined by $B_k(x) = B(x)(f_k)$, $x \in X$, $k \geqslant 1$.

A mild solution of this equation generates (see [13],[7]) a stochastic skew-evolution semiflow $\Theta = (\varphi, \Phi)$ defined by

$$\Phi(t, s, \omega)x = u(t, s, x, \omega), \quad \text{for all } (t, s) \in \Delta,\ \omega \in \Omega^*, x \in X,$$

where $\Phi\colon \Delta \times \Omega \to \mathcal{L}(X)$ is a stochastic evolution cocycle over the stochastic evolution semiflow $\varphi\colon \Delta \times \Omega \to \Omega$ defined in Example 2.1, and $u(t, s, x, \omega)$ is the mild solution of equation (2.1) with the initial condition $u(s, s, x, \omega) = x$ given at the time $s \geqslant 0$, for all $\omega \in \Omega$.

3. Nonuniform exponential dichotomy in mean square

In this section we define some concepts of nonuniform dichotomy in mean square for stochastic skew-evolution semiflows in Hilbert spaces.

Let $\Theta = (\varphi, \Phi)$ be a stochastic skew-evolution semiflow on the real Hilbert space, where $\Phi\colon \Delta \times \Omega \to \mathcal{L}(X)$ is the stochastic evolution cocycle semiflow over the stochastic semiflow $\varphi\colon \Delta \times \Omega \to \Omega$.

Next we suppose that there exists a sure event $\Omega^* \in \Im$. A random variable $Z\colon (\Omega, \Im) \to (\mathbb{R}_+^*, \mathcal{B}(\mathbb{R}_+^*))$ is called tempered if there exist a positive constant k and a positive random variable $C_k(\omega)$ such that:

$$Z(\varphi(t, s, \omega)) \leqslant C_k(\omega)\mathrm{e}^{kt} \quad \text{for all } (t, s) \in \Delta$$

for ω in a sure event $\Omega^* \in \Im$.

In what follows we use the hypothesis that the phase space X is split into

$$X = X_1(\omega) \oplus X_2(\omega)$$

for all ω in a sure event $\Omega^* \in \Im$ and $\varphi(t,s,.)(\Omega^*) = \Omega^*$ for all $(t,s) \in \Delta$.

We denote by $\{\Pi_k(\omega)\}_{k=1,2}$ the family of measurable projections associated with the splitting.

The $\Im$ measurable subspaces $X_1(\omega)$ and $X_2(\omega)$ are called the stable and instable spaces, respectively. We suppose that these subspaces are invariant under the stochastic skew-evolution semiflow $\Theta = (\varphi, \Phi)$, i.e.

$$\Phi(t,s,\omega)X_k(\omega) \subset X_k(\varphi(t,s,\omega)),\ \forall (t,s) \in \Delta \text{ and } \omega \in \Omega^*,\ k = \{1,2\}.$$

Remark 3.1. The subspace $X_1(\omega)$ is finitely dimensional with a fixed non-random dimension and $X_2(\omega)$ is closed with a finite non-random codimension (see Caraballo et al. in [5], Mohamend et al. in [13]).

Definition 3.1. The family of measurable projections $\{\Pi_k(\omega)\}_{k=1,2}$ is called compatible with the stochastic skew-evolution semiflow $\Theta = (\varphi, \Phi)$ if

$$\Pi_k(\varphi(t,s,\omega))\Phi(t,s,\omega) = \Phi(t,s,\omega)\Pi_k(\omega)$$

for all $(t,s) \in \Delta$ and $\omega \in \Omega^*$.

Next we denote

$$\Phi_{\Pi_k}(t,s,\omega) = \Phi(t,s,\omega)\Pi_k(\omega), \quad \forall (t,s) \in \Delta \text{ and } \omega \in \Omega^*,\ k = \{1,2\}.$$

We observe that the Φ_{Π_k}, $k = 1,2$ are stochastic evolution cocycles and

$$\Theta_k(t,s,\omega,x) = (\varphi(t,s,\omega), \Phi_{\Pi_k}(t,s,\omega)x),\ \forall (t,s) \in \Delta,\ (\omega,x) \in Y,\ k = \{1,2\}$$

are stochastic skew-evolution semiflows on $Y = \Omega^* \times X$ for every stochastic evolution semiflow φ on Ω.

Definition 3.2. The s.s.-e.s. $\Theta = (\varphi, \Phi)$ has the exponential dichotomy in mean square if there are φ-invariant random variables $\alpha(\omega) \geqslant 0$ and $\nu(\omega) > 0$ with $\alpha(\omega) < \nu(\omega)$ and a tempered random variables $N(\omega)\colon \Omega^* \to [1,\infty]$ such that:

$$\mathbb{E}\|\Phi_{\Pi_1}(t,t_0,\omega)x\|^2 \leqslant N(\omega)\mathrm{e}^{\alpha(\omega)t}\mathrm{e}^{-\nu(\omega)(t-s)}\mathbb{E}\|\Phi_{\Pi_1}(s,t_0,\omega)x\|^2, \tag{3.1}$$

$$\mathbb{E}\|\Phi_{\Pi_2}(s,t_0,\omega)x\|^2 \leqslant N(\omega)\mathrm{e}^{\alpha(\omega)t}\mathrm{e}^{-\nu(\omega)(t-s)}\mathbb{E}\|\Phi_{\Pi_2}(t,t_0,\omega)x\|^2 \tag{3.2}$$

for all $(t,s), (s,t_0) \in \Delta$, and $(\omega,x) \in Y$, i.e. the stochastic evolution cocycle Φ_{Π_1} is exponentially stable in mean square and Φ_{Π_2} is exponentially instable in mean square.

If $\alpha = 0$ then we say that the stochastic skew-evolution semiflow $\Theta = (\varphi, \Phi)$ has the uniform exponential dichotomy in mean square.

Remark 3.2. The stochastic skew-evolution semiflow $\Theta = (\varphi, \Phi)$ has the exponential dichotomy in mean square if there are φ-invariant random variables $\alpha(\omega) \geqslant 0$ and $\nu(\omega) > 0$ with $\alpha(\omega) < \nu(\omega)$, and a tempered random variable $N(\omega)\colon\ \Omega^* \to [1, \infty]$ such that

$$\mathbb{E}\|\Phi_{\Pi_1}(t,s,\omega)x\|^2 \leqslant N(\omega)\mathrm{e}^{\alpha(\omega)t}\mathrm{e}^{-\nu(\omega)(t-s)}\mathbb{E}\|\Pi_1(\omega)x\|^2,\ \forall (t,s) \in \Delta,\ (\omega,x) \in Y,$$
$$\mathbb{E}\|\Phi_{\Pi_2}(t,s,\omega)x\|^2 \geqslant \frac{1}{N(\omega)}\mathrm{e}^{-\alpha(\omega)t}\mathrm{e}^{\nu(\omega)(t-s)}\mathbb{E}\|\Pi_2(\omega)x\|^2,\ \forall (t,s) \in \Delta,\ (\omega,x) \in Y.$$

Definition 3.3. The s.s.-e.s. $\Theta = (\varphi, \Phi)$ is called strongly measurable in mean square if for every $(t_0, \omega) \in \mathbb{R}_+ \times \Omega^*$, the mapping $t \mapsto \mathbb{E}\|\Phi(t, t_0, \omega)x\|^2$ is measurable on $[t, \infty)$.

Definition 3.4. Let $\lambda(\omega) > 0$ and $\varepsilon(\omega) \geqslant 0$ be φ-invariant random variables and $M(\omega)\colon\ \Omega^* \to [1, \infty]$ a tempered random variable. Then the s.s.-e.s. $\Theta = (\varphi, \Phi)$ is said to be

1) of exponential growth in mean square if

$$\mathbb{E}\|\Phi(t,t_0,\omega)\|^2 \leqslant M(\omega)\mathrm{e}^{\varepsilon(\omega)s}\mathrm{e}^{\lambda(\omega)(t-s)}\mathbb{E}\|\Phi(s,t_0,\omega)\|^2, \tag{3.3}$$

2) of exponential decay in mean square if

$$\mathbb{E}\|\Phi(s,t_0,\omega)x\|^2 \leqslant M(\omega)\mathrm{e}^{\varepsilon(\omega)s}\mathrm{e}^{\lambda(\omega)(t-s)}\mathbb{E}\|\Phi(t,t_0,\omega)x\|^2, \tag{3.4}$$

for all $(t,s), (s,t_0) \in \Delta$ and $(\omega,x) \in Y$.

If $\varepsilon = 0$ then we obtain the above properties in the uniform case.

Definition 3.5. Let $\beta(\omega) \geqslant 0$ be a φ-invariant random variable and $K(\omega)\colon$ $\Omega^* \to [1, \infty]$ a tempered random variable. Then the s.s.-e.s. $\Theta = (\varphi, \Phi)$ is called

1) integrally stable in mean square if

$$\int_s^t \mathbb{E}\|\Phi(\tau,t_0,\omega)x\|^2\,\mathrm{d}\tau \leqslant K(\omega)\mathrm{e}^{\beta(\omega)s}\mathbb{E}\|\Phi(s,t_0,\omega)x\|^2, \tag{3.5}$$

2) integrally instable in mean square if

$$\int_{t_0}^t \mathbb{E}\|\Phi(\tau,t_0,\omega)x\|^2\,\mathrm{d}\tau < K(\omega)\mathrm{e}^{\beta(\omega)t}\mathbb{E}\|\Phi(t,t_0,\omega)x\|^2, \tag{3.6}$$

for all $(t,t_0) \in \Delta$ and $(\omega,x) \in Y$.

Lemma 3.1. *Suppose that the s.s.-e.s. $\Theta = (\varphi, \Phi)$ has exponential growth in mean square. If it is integrally stable in mean square then there is a φ-invariant random variable $\beta(\omega) \geqslant 0$ and a tempered random variable $N(\omega)\colon \Omega^* \to [1, \infty]$ such that*

$$\mathbb{E}\|\Phi(t, t_0, \omega)x\|^2 \leqslant N(\omega)\mathrm{e}^{\varepsilon(\omega)s}\mathrm{e}^{\beta(\omega)t}\mathbb{E}\|\Phi(s, t_0, \omega)x\|^2 \tag{3.7}$$

for all $(t, s), (s, t_0) \in \Delta$, $(\omega, x) \in Y$, where $\varepsilon(\omega) \geqslant 0$ is from Definition 3.4(1).

P r o o f. When $t \geqslant s + 1$ and $s \geqslant t_0 \geqslant 0$, from Definition 3.4(1) we have

$$\begin{aligned}\mathbb{E}\|\Phi(t, t_0, \omega)x\|^2 &= \int_{t-1}^{t} \mathbb{E}\|\Phi(t, t_0, \omega)x\|^2 \,\mathrm{d}\tau \\ &\leqslant M(\omega)\mathrm{e}^{\varepsilon(\omega)t}\mathrm{e}^{\lambda(\omega)} \int_{s}^{t} \mathbb{E}\|\Phi(\tau, t_0, \omega)x\|^2 \,\mathrm{d}\tau \\ &\leqslant K(\omega)M(\omega)\mathrm{e}^{\varepsilon(\omega)t}\mathrm{e}^{\lambda(\omega)}\mathrm{e}^{\beta(\omega)s}\mathbb{E}\|\Phi(s, t_0, \omega)x\|^2\end{aligned}$$

for all $s \geqslant t_0$, and $(\omega, x) \in \Omega \times X$. For $t \in [s, s+1]$ we have the relation

$$\begin{aligned}\mathbb{E}\|\Phi(t, t_0, \omega)x\|^2 &\leqslant M(\omega)\mathrm{e}^{\varepsilon(\omega)s}\mathrm{e}^{\lambda(\omega)(t-s)}\mathbb{E}\|\Phi(s, t_0, \omega)x\|^2 \\ &\leqslant K(\omega)M(\omega)\mathrm{e}^{\lambda(\omega)}\mathrm{e}^{\varepsilon(\omega)s}\mathrm{e}^{\beta(\omega)t}\mathbb{E}\|\Phi(s, t_0, \omega)x\|^2.\end{aligned}$$

Then for $N(\omega) = K(\omega)M(\omega)\mathrm{e}^{\lambda(\omega)}$ we have the relation (3.7). □

Lemma 3.2. *Suppose that the s.s.-e.s. $\Theta = (\varphi, \Phi)$ has exponential decay in mean square. If it is integrally instable in mean square then there is a φ-invariant random variable $\beta(\omega) \geqslant 0$ and a tempered random variable $N(\omega)\colon \Omega^* \to [1, \infty]$ such that*

$$\mathbb{E}\|\Phi(s, t_0, \omega)x\|^2 \leqslant N(\omega)\mathrm{e}^{\varepsilon(\omega)s}\mathrm{e}^{\beta(\omega)t}\mathbb{E}\|\Phi(t, t_0, \omega)x\|^2 \tag{3.8}$$

for all $(t, s), (s, t_0) \in \Delta$, $(\omega, x) \in Y$, where $\varepsilon(\omega) \geqslant 0$ is from Definition 3.4(2).

P r o o f. The proof is similar to that of Lemma 3.1. □

The main result is a variant for the stochastic case of nonuniform dichotomy in mean square of the well-known theorem due to R. Datko in [8]. An analogous result has been proved by Megan and Lupa in [11] for nonuniform exponential dichotomy.

Theorem 3.1. *Let $\Theta = (\varphi, \Phi)$ be a s.s-e.s. on the Hilbert space X, strongly measurable in mean square, let $\{\Pi_k(\omega)\}_{k=1,2}$ be a family of measurable projections compatible with Θ such that Φ_{Π_1} has exponential growth in mean square and Φ_{Π_2} has exponential decay in mean square.*

If there exist φ-invariant random variables $\beta(\omega) \geqslant 0$ and $\gamma(\omega) > 0$ with $\gamma(\omega) > \beta(\omega)$ and a tempered random variable $K(\omega)\colon \Omega^* \to [1, \infty]$ such that

$$\tag{3.9} \begin{aligned}\int_s^\infty \mathrm{e}^{\gamma(\omega)(\tau-s)}\mathbb{E}\|\Phi_{\Pi_1}(\tau,t_0,\omega)x\|^2\,\mathrm{d}\tau + \int_{t_0}^s \mathrm{e}^{\gamma(\omega)(s-\tau)}\mathbb{E}\|\Phi_{\Pi_2}(\tau,t_0,\omega)x\|^2\,\mathrm{d}\tau \\ \leqslant K(\omega)\mathrm{e}^{\beta(\omega)s}(\mathbb{E}\|\Phi_{\Pi_1}(s,t_0,\omega)x\|^2 + \mathbb{E}\|\Phi_{\Pi_2}(s,t_0,\omega)x\|^2)\end{aligned}$$

for all $(t,s),(s,t_0) \in \Delta$ and $(\omega,x) \in Y$, then the stochastic skew-evolution semiflow $\Theta = (\varphi,\Phi)$ has exponential dichotomy in mean square.

Proof. Since Φ_{Π_1} has exponential growth in mean square we have that $e^{\gamma(\omega)(t-s)}\Phi_{\Pi_1}(t,s,\omega)$ has the same property. From (3.9) it follows that

$$\int_s^t \mathrm{e}^{\gamma(\omega)(\tau-t_0)}\mathbb{E}\|\Phi_{\Pi_1}(\tau,t_0,\omega)x\|^2\,\mathrm{d}\tau \leqslant K(\omega)\mathrm{e}^{\beta(\omega)s}\mathrm{e}^{\gamma(\omega)(s-t_0)}\mathbb{E}\|\Phi_{\Pi_1}(s,t_0,\omega)x\|^2$$

for all $(t,s,t_0) \in \Delta$ and $x \in \operatorname{Im}\Pi_1(\omega)$.

Therefore the stochastic cocycle $\mathrm{e}^{\gamma(\omega)(t-t_0)}\mathbb{E}\|\Phi_{\Pi_1}(t,t_0,\omega)x\|^2$ is integral stable in mean square for all $(t,s),(s,t_0) \in \Delta$, $\omega \in \Omega^*$ and by Lemma 3.1 we conclude

$$\tag{3.10} \begin{aligned}&\mathbb{E}\|\Phi_{\Pi_1}(t,t_0,\omega)x\|^2 \\ &\quad\leqslant N(\omega)\mathrm{e}^{(\varepsilon(\omega)+\beta(\omega))t}\mathrm{e}^{-(\gamma(\omega)+\varepsilon(\omega))(t-s)}\mathbb{E}\|\Phi_{\Pi_1}(s,t_0,\omega)x\|^2.\end{aligned}$$

Similarly, since Φ_{Π_2} has exponential decay in mean square, it follows that the operator $\mathrm{e}^{-\gamma(\omega)(t-s)}\Phi_{\Pi_2}(t,s,\omega)$ has also exponential decay in mean square and we obtain that

$$\int_{t_0}^s \mathrm{e}^{-\gamma(\omega)(t-s)}\mathbb{E}\|\Phi_{\Pi_2}(\tau,t_0,\omega)x\|^2\,\mathrm{d}\tau \leqslant K(\omega)\mathrm{e}^{\beta(\omega)s}\mathrm{e}^{-\gamma(\omega)(s-t_0)}\mathbb{E}\|\Phi_{\Pi_2}(s,t_0,\omega)x\|^2$$

for all $(t,s),(s,t_0) \in \Delta$ and $x \in \operatorname{Im}\Pi_2(\omega)$.

Thus Lemma 3.2 yields that there exists a constant $N(\omega) \geqslant 1$ for all $\omega \in \Omega^*$ such that

$$\tag{3.11} \mathbb{E}\|\Phi_{\Pi_2}(s,t_0,\omega)x\|^2 \leqslant N(\omega)\mathrm{e}^{(\beta(\omega)+\varepsilon(\omega))t}\mathrm{e}^{-(\gamma(\omega)+\varepsilon(\omega))(t-s)}\mathbb{E}\|\Phi_{\Pi_2}(t,t_0,\omega)x\|^2$$

for all $(t,s),(s,t_0) \in \Delta$ and $(\omega,x) \in Y$.

If we denote $\alpha(\omega) = \beta(\omega) + \varepsilon(\omega)$ and $\nu(\omega) = \gamma(\omega) + \varepsilon(\omega)$ for all $\omega \in \Omega^*$, then from the relations (3.10) and (3.11) we have that the stochastic skew-evolution semiflow $\Theta = (\varphi,\Phi)$ has exponential dichotomy in mean square. $\square$

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