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REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH CERTAIN COMMUTING CONDITION

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Abstract. In this paper, first we introduce a new notion of commuting condition that $\varphi\varphi_1 A = A\varphi_1\varphi$ between the shape operator A and the structure tensors φ and φ_1 for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Suprisingly, real hypersurfaces of type (A), that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in complex two plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfy this commuting condition. Next we consider a complete classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the commuting condition. Finally we get a characterization of Type (A) in terms of such commuting condition $\varphi\varphi_1 A = A\varphi_1\varphi$.

Keywords: real hypersurface, complex two-plane Grassmannians, Hopf hypersurface, commuting shape operator

MSC 2010: 53C50, 53C55

Introduction

We denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. Namely, $G_2(\mathbb{C}^{m+2})$ is a unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. Accordingly, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces M that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^{\perp} = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M (see [2], [3] and [4]).

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The almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a Reeb vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$), where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ and $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using the two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The Reeb vector field ξ is said to be Hopf if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a Hopf foliation of M. We say that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 2 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

On the other hand, we say that the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is isometric, when the Reeb vector field ξ on M is Killing. In [4], Berndt and Suh gave some equivalent conditions for isometric Reeb flow. Among them, we want to introduce a commuting condition between the shape operator A and the structure tensor φ, that is, $A\varphi = \varphi A$. By such a commuting condition, a characterization of real hypersurfaces of Type (A) in Theorem A was given in terms of the Reeb flow on M as follows:

Theorem B. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

In [7], Suh considered a condition that the almost contact 3-structure tensors $\{\varphi_1, \varphi_2, \varphi_3\}$ commute with the shape operator A of real hypersurface M in $G_2(\mathbb{C}^{m+2})$, and he proved that there does not exist any real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $A\varphi_\nu X = \varphi_\nu AX$, $\nu = 1, 2, 3$, for any tangent vector field X on M. In addition, he gave a characterization of real hypersurface of Type (B) under assumption that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $A\varphi_\nu X = \varphi_\nu AX$,

$\nu = 1, 2, 3$, for any tangent vector field X on T_0. Here, the distribution T_0 is defined by $T_0 = \{X \in T_pM \mid \xi \perp X\}$ (see [7]).

Summing up these statements, naturally we ask what can we say about the commuting condition between the shape operator A and the two structure tensors φ and φ_1. According to such a problem, in this paper we consider a new condition that the shape operator A commutes with two kinds of structure tensors φ and φ_1 for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ as follows:

$$(*) \qquad \varphi\varphi_1 AX = A\varphi_1\varphi X$$

for any tangent vector field X on M.

Suprisingly, by Proposition A in Section 3, we know that real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ in Theorem A satisfy the formula $(*)$. From such a point of view, we give another characterization of real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$ as follows:

Main Theorem. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the shape operator satisfies the commuting condition $(*)$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2], [3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2)\times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m}\times\mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight.

When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geqslant 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\widetilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\widetilde{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} \widetilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3} \Big\{ g(J_\nu Y, Z) J_\nu X - g(J_\nu X, Z) J_\nu Y - 2g(J_\nu X, Y) J_\nu Z \Big\} \\ & + \sum_{\nu=1}^{3} \Big\{ g(J_\nu JY, Z) J_\nu JX - g(J_\nu JX, Z) J_\nu JY \Big\}, \end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [5], [6] and [7]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \varphi X + \eta(X)N, \quad J_\nu X = \varphi_\nu X + \eta_\nu(X)N \tag{2.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (φ, ξ, η, g) induced on M in such a way that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{2.2}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in Section 1, induces an almost contact metric 3-structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{cases} \varphi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \varphi_\nu \xi_\nu = 0, \\ \varphi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \varphi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\ \varphi_\nu \varphi_{\nu+1} X = \varphi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\ \varphi_{\nu+1}\varphi_\nu X = -\varphi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1} \end{cases} \tag{2.3}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = J J_\nu$, $\nu = 1, 2, 3$ in Section 1 and (2.1), the relation between these two contact metric structures (φ, ξ, η, g) and $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{gathered} \varphi\varphi_\nu X = \varphi_\nu \varphi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ \eta_\nu(\varphi X) = \eta(\varphi_\nu X), \quad \varphi\xi_\nu = \varphi_\nu \xi. \end{gathered} \tag{2.4}$$

On the other hand, from the Kähler structure J, that is, $\widetilde{\nabla} J = 0$ and the quaternionic Kähler structure J_ν (see (1.1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX, \tag{2.5}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_\nu AX, \tag{2.6}$$

$$(\nabla_X \varphi_\nu)Y = -q_{\nu+1}(X)\varphi_{\nu+2}Y + q_{\nu+2}(X)\varphi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu \tag{2.7}$$

Summing up these formulas, we find the following

$$\begin{aligned} \nabla_X(\varphi_\nu \xi) &= \nabla_X(\varphi\xi_\nu) \\ &= (\nabla_X \varphi)\xi_\nu + \varphi(\nabla_X \xi_\nu) \\ &= q_{\nu+2}(X)\varphi_{\nu+1}\xi - q_{\nu+1}(X)\varphi_{\nu+2}\xi + \varphi_\nu \varphi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX. \end{aligned} \tag{2.8}$$

Using the above expression (1.2) for the curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes

$$
\begin{aligned}
(2.9) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi \\
+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\varphi_\nu Y - \eta_\nu(Y)\varphi_\nu X - 2g(\varphi_\nu X, Y)\xi_\nu\} \\
+ \sum_{\nu=1}^{3}\{\eta_\nu(\varphi X)\varphi_\nu\varphi Y - \eta_\nu(\varphi Y)\varphi_\nu\varphi X\} \\
+ \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\varphi Y) - \eta(Y)\eta_\nu(\varphi X)\}\xi_\nu.
\end{aligned}
$$

3. Key lemmas

Now let us assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator, that is, the shape operator A of M commutes with the structures tensors φ and φ_1 as follows:

$$(*) \qquad \varphi\varphi_1 AX = A\varphi_1\varphi X$$

for any tangent vector field X on M.

First of all, we establish one of the key lemmas as follows:

Lemma 3.1. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. If M has commuting shape operator, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

P r o o f. In order to prove our lemma, let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

From the assumption $(*)$ for $X = \xi$ and (2.2), we have

$$(3.1) \qquad \varphi_1 A\xi = \eta(\varphi_1 A\xi)\xi.$$

On the other hand, from the assumption that M is Hopf, we see that

$$(3.2) \qquad A\xi = \alpha\xi = \alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1.$$

Combining with (3.1) and (3.2), we have

$$\alpha\eta(X_0)\varphi_1 X_0 = 0,$$

because $\varphi_1\xi_1 = 0$ and the structure tensor φ_1 is skew-symmetric.

But we see that $\varphi_1 X_0$ is non-vanishing at all points of M. In fact, we obtain

$$\|\varphi_1 X_0\|^2 = g(\varphi_1 X_0, \varphi_1 X_0) = -g(\varphi_1^2 X_0, X_0) = g(X_0, X_0) = 1,$$

where we have used the equation (2.3) and the fact that X_0 is unit.

Then it follows that

$$\alpha\eta(X_0) = 0. \tag{3.3}$$

Thus we can consider the following two cases:

Case 1. $\alpha = 0$, that is, $A\xi = 0$. This case is trivial by Lemma 3.1 due to Pérez and Suh [6].

Case 2. $\alpha \neq 0$. From (3.3), we have $\eta(X_0) = 0$. This gives a contradiction.

So we complete the proof of our Lemma. □

Now, we consider another commuting condition for the shape operator A on M when the Reeb vector ξ belongs to the distribution $\mathfrak{D}^\perp$. We prove the following lemma which will be useful in the proof of Lemma 4.2 in Section 4.

Lemma 3.2. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ with $\xi \in \mathfrak{D}^\perp$. If M satisfies the following condition*

$$\varphi\varphi_1 AX = A\varphi\varphi_1 X, \quad X \in \mathfrak{D}^\perp, \tag{$**$}$$

then the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator A of M, that is, $g(A\mathfrak{D}^\perp, \mathfrak{D}) = 0$.

Proof. From now on, since $\xi \in \mathfrak{D}^\perp$, let us put $\xi = \xi_1$. Taking the covariant derivative along any direction $Y \in TM$, we have

$$\varphi AY = \nabla_Y \xi = \nabla_Y \xi_1 = q_3(Y)\xi_2 - q_2(Y)\xi_3 + \varphi_1 AY. \tag{3.4}$$

From this, taking the inner product with ξ_2 and ξ_3, we have

$$q_3(Y) = 2g(AY, \xi_3), \quad q_2(Y) = 2g(AY, \xi_2), \tag{3.5}$$

respectively.

Moreover, applying the structure tensor φ in (3.4), this equation can be written as

$$AY = \alpha\eta(Y)\xi + 2g(AY, \xi_2)\xi_2 + 2g(AY, \xi_3)\xi_3 - \varphi\varphi_1 AY, \quad Y \in TM, \tag{3.6}$$

where we have used that M is Hopf and the formulas (2.2), (2.3) and (3.5).

Putting $Y = \xi_2$ in (3.6), we get

$$\begin{aligned} A\xi_2 &= \alpha\eta(\xi_2)\xi + 2g(A\xi_2, \xi_2)\xi_2 + 2g(A\xi_2, \xi_3)\xi_3 - \varphi\varphi_1 A\xi_2 \\ &= 2g(A\xi_2, \xi_2)\xi_2 + 2g(A\xi_2, \xi_3)\xi_3 - \varphi\varphi_1 A\xi_2 \\ &= 2g(A\xi_2, \xi_2)\xi_2 + 2g(A\xi_2, \xi_3)\xi_3 - A\xi_2. \end{aligned}$$

Here from the condition $(**)$ we see that $\varphi\varphi_1 A\xi_2 = A\varphi\varphi_1\xi_2 = A\xi_2$, because $\xi_2 \in \mathfrak{D}^\perp$. Therefore the third equality in the above equation holds. Consequently, it implies

$$A\xi_2 = g(A\xi_2, \xi_2)\xi_2 + g(A\xi_2, \xi_3)\xi_3. \tag{3.7}$$

Similarly, if we consider $Y = \xi_3$ in (3.6), we get

$$A\xi_3 = g(A\xi_3, \xi_2)\xi_2 + g(A\xi_3, \xi_3)\xi_3, \tag{3.8}$$

because $\varphi\varphi_1 A\xi_3 = A\varphi\varphi_1\xi_3 = A\xi_3$.

From the two equations (3.7), (3.8) and the assumption $A\xi_1 = A\xi = \alpha\xi = \alpha\xi_1$, we have $A\xi_\nu \in \mathfrak{D}^\perp$ for any $\nu = 1, 2, 3$. So we conclude that the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator A of M, that is, $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$. This gives a complete proof of our lemma. □

Before giving the proof of our Main Theorem from the Introduction, let us check whether the shape operator A of real hypersurfaces of Type (A) or of Type (B) in Theorem A satisfies the condition $(*)$ or not.

First let us check for the case that M is locally congruent to a real hypersurface of Type (A), an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. We recall a proposition due to Berndt and Suh [3] as follows:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},\\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2, \xi_3\},\\
T_\lambda &= \{X \mid X \perp \mathbb{H}\xi,\ JX = J_1X\},\\
T_\mu &= \{X \mid X \perp \mathbb{H}\xi,\ JX = -J_1X\}
\end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector field ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Now let us check case by case whether the two sides in $(*)$ are equal to each other:

Case A-1. $X \in T_\alpha$ (i.e. $X = \xi = \xi_1$). It can easily be checked that the two sides are equal to each other.

Case A-2. $X \in T_\beta$, (i.e. $X = \xi_2$ or $X = \xi_3$). Then we put $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, where $\beta = \sqrt{2}\cot(\sqrt{2}r)$. Then by putting $X = \xi_2$ in $(*)$ we have

$$\text{Left-Hand Side} = \varphi\varphi_1 A\xi_2 = \beta\varphi\varphi_1\xi_2 = \beta\varphi\xi_3 = \beta\varphi_3\xi_1 = \beta\xi_2,$$

and

$$\text{Right-Hand Side} = A\varphi_1\varphi\xi_2 = A\varphi_1\varphi_2\xi = A\varphi_1\varphi_2\xi_1 = -A\varphi_1\xi_3 = A\xi_2 = \beta\xi_2.$$

From this we see that both sides are equal to $\beta\xi_2$. Similarly, by putting $X = \xi_3$ in $(*)$ we know that they are equal to $\beta\xi_3$.

Case A-3. $X \in T_\lambda = \{X \mid X \perp \mathbb{H}\xi,\ \varphi X = \varphi_1 X\}$. For any $X \in T_\lambda$, $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ we get

$$\varphi\varphi_1 X = \varphi^2 X = -X, \quad \varphi_1\varphi X = {\varphi_1}^2 X = -X.$$

From this we know that the formula $(*)$ is equal to $-\lambda X$.

Case A-4. $X \in T_\mu = \{X \mid X \perp \mathbb{H}\xi,\ \varphi X = -\varphi_1 X\}$. We have $\varphi\varphi_1 X = -\varphi^2 X = X$, $\varphi_1\varphi X = -{\varphi_1}^2 X = X$ for any $X \in T_\mu$. So we know that they are equal to $\mu X = 0$, because $\mu = 0$.

Hence we conclude with a remark as follows:

Remark 3.3. The shape operator A of real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ satisfies the condition $(*)$.

Second, let us check whether the shape operator A of real hypersurfaces of Type (B) satisfies the condition $(*)$. As is well known to us, a real hypersurface of Type (B) has five distinct constant principal curvatures as follows [3]:

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \operatorname{Span}\{\xi\}, \\ &T_\beta = \mathfrak{J}J\xi = \operatorname{Span}\{\xi_\nu \mid \nu = 1, 2, 3\}, \\ &T_\gamma = \mathfrak{J}\xi = \operatorname{Span}\{\varphi_\nu\xi \mid \nu = 1, 2, 3\}, \\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Here we suppose that a real hypersurface of Type (B) has the commuting shape operator A, that is, the shape operator A of M satisfies the commuting condition $\varphi\varphi_1 AX = A\varphi_1\varphi X$ for any tangent vector field X on M. Then we see that

$$\begin{aligned} \varphi\varphi_1 A\xi = A\varphi_1\varphi\xi &\Leftrightarrow \varphi\varphi_1 A\xi - A\varphi_1\varphi\xi = 0 \\ &\Leftrightarrow \varphi\varphi_1 A\xi = 0 \\ &\Leftrightarrow \alpha\varphi\varphi_1\xi = 0 \quad (\text{because } \xi \in \mathrm{T}_\alpha) \\ &\Leftrightarrow \alpha\varphi^2\xi_1 = 0 \quad (\text{by eq: } (2.4)) \\ &\Leftrightarrow -\alpha\xi_1 = 0 \quad (\text{by eq: } (2.2)) \\ &\Leftrightarrow \alpha = 0. \quad (\text{because } \xi_1\text{: unit}) \end{aligned}$$

But this case can not occur for any $r \in (0, \pi/4)$. In fact, $\alpha = -2\tan(2r)$ is non-vanishing in $(0, \pi/4)$. So we also state the following remark:

Remark 3.4. The shape operators A of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ do not satisfy the commuting condition $(*)$.

4. The proof of the Main Theorem

In this section, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator, that is, the shape operator satisfies the condition $(*)$. Then by Lemma 3.1 we consider the following two cases:

Case I: the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$,

Case II: the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$.

First, let us consider Case I, that is, $\xi \in \mathfrak{D}$.

To consider this case, we recall a one theorem by Lee and Suh [5] as follows:

Theorem C. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Then from Theorem C, we see that M is locally congruent to a real hypersurface of Type (B) under our assumption. But in Section 3 we have checked that the shape operator A of real hypersurface of Type (B) does not satisfy the condition $(*)$ (see Remark 3.4). From these facts, first we assert the following:

Theorem 4.1. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, with the commuting shape operator $\varphi\varphi_1 A = A\varphi_1\varphi$ if the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.*

Next we consider the case $\xi \in \mathfrak{D}^{\perp}$. Accordingly, we may put $\xi = \xi_1$. Then we have the following:

Lemma 4.2. *Let M be a hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ with $\xi \in \mathfrak{D}^{\perp}$. If M has commuting shape operator, that is, the shape operator A on M satisfies the condition $(*)$, then the distribution $\mathfrak{D}^{\perp}$ is invariant under the shape operator A on M.*

Proof. Since $\xi \in \mathfrak{D}^{\perp}$, let us assume $\xi = \xi_1$. Substituting $X = \xi$ in our assumption $(*)$, we have

$$\varphi\varphi_1 A\xi = 0.$$

Applying φ in the above equation, it becomes

$$\varphi_1 A\xi = \eta(\varphi_1 A\xi)\xi.$$

Taking an inner product with ξ_1, we obtain $\eta(\varphi_1 A\xi)\eta(\xi_1) = 0$. Since $\xi = \xi_1$, it means that $\eta(\varphi_1 A\xi) = 0$. So, we have

$$\varphi_1 A\xi = 0.$$

From this, we have $A\xi = \alpha\xi$ where $\alpha = g(A\xi, \xi_1) = g(A\xi, \xi)$, because $\xi = \xi_1$.

Moreover, from (2.4), we see that

$$\begin{aligned} \varphi_1\varphi X &= \varphi\varphi_1 X - \eta_1(X)\xi + \eta(X)\xi_1 \\ &= \varphi\varphi_1 X \end{aligned} \tag{4.1}$$

for any tangent vector field X on M.

Thus we can write the condition $(*)$ as

$$\varphi\varphi_1 AX = A\varphi_1\varphi X = A\varphi\varphi_1 X \tag{4.2}$$

for any tangent vector field X on M.

Now putting $X = \xi_\nu$, $\nu = 2, 3$ in (4.2), this equation can be written as

$$\varphi\varphi_1 A\xi_\nu = A\varphi\varphi_1\xi_\nu, \quad \nu = 2, 3. \tag{4.3}$$

From Lemma 3.2, we have $A\xi_\nu \in \mathfrak{D}^\perp$, $\nu = 2, 3$ under our assumption. This completes the proof of our Lemma. □

Therefore from Theorem A in the Introduction, we conclude the following:

Lemma 4.3. *Let M be a connected hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ satisfying the commuting condition $(*)$. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

As mentioned in Remark 3.3 in Section 3, the shape operator A for real hypersurfaces of Type (A) satisfies the commuting condition (*) for any tangent vector field on M. From this fact and Lemma 4.3, we arrive at the following:

Theorem 4.4. *Let M be a connected hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$ satisfying the commuting condition $(*)$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Summing up Lemma 3.1, and Theorems 4.1 and 4.4, we give a complete proof of our Main Theorem from the Introduction. □

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