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DIVERSITY IN MONOIDS

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Abstract. Let M be a (commutative cancellative) monoid. A nonunit element $q \in M$ is called almost primary if for all $a, b \in M$, $q \mid ab$ implies that there exists $k \in \mathbb{N}$ such that $q \mid a^k$ or $q \mid b^k$. We introduce a new monoid invariant, diversity, which generalizes this almost primary property. This invariant is developed and contextualized with other monoid invariants. It naturally leads to two additional properties (homogeneity and strong homogeneity) that measure how far an almost primary element is from being primary. Finally, as an application the authors consider factorizations into almost primary elements, which generalizes the established notion of factorization into primary elements.

Keywords: factorization, monoid, diversity

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1. Introduction

Throughout this paper, all monoids under consideration are commutative, and (unless otherwise stated) cancellative, and multiplicative, with identity denoted by 1. If M is a monoid, then $M^\times$ denotes the set of units (or invertible elements) of M. If $\pi \in M \setminus M^\times$, we say that π is an *atom* (or an irreducible element) of M if for all $a, b \in M$, $\pi = ab$ implies that $a \in M^\times$ or $b \in M^\times$. The set of atoms of a monoid M is denoted by $\mathcal{A}(M)$. We say that M is an *atomic* monoid if every nonunit of M can be written as a product of atoms. If $S = \{s_1, s_2, \ldots, s_k\}$ is a finite subset of M, then we denote the product of the elements of S by $\prod S := s_1 s_2 \ldots s_k$. If $A \subseteq M \setminus M^\times$, we denote the monoid generated by A to be $[A]$, and we say that A is a *generating set* of M if $M = [A]$. Thus, M is atomic if and only if $M = [\mathcal{A}(M) \cup M^\times]$. By $\mathbb{N}$, $\mathbb{N}_0$, and S_n we mean the set of natural numbers, nonnegative integers, and the group of permutations on n letters, respectively.

Let M be a monoid. We establish some terminology regarding ideal-theoretic properties of M (proofs of the following claims can be found in [6]). If A and B are (nonempty) subsets of M, then $AB := \{ab \mid a \in A, b \in B\}$ and if $x \in M$, we denote $\{x\}A$ by xA. A subset I of M is called an *ideal* of M if $IM = I$.[1] If I is an ideal of M, then $I = M$ if and only if $I \cap M^{\times} \neq \emptyset$. If $I \neq M$, we say that I is a *proper* ideal of M. We call I a *prime ideal* of M if I is a proper ideal and $M \setminus I$ is a submonoid of M (equivalently, for all $a, b \in M$ with $ab \in I$, we have $a \in I$ or $b \in I$). A proper ideal I is a *primary ideal* of M if and only if for all $a, b \in M$ with $ab \in I$, either $a \in I$ or $b^k \in I$ for some k (or, equivalently, if $ab \in I$, then either $a \in I$, or $b \in I$ or there exist $m, n \in \mathbb{N}$ such that $a^m \in I$ and $b^n \in I$). We say that I is an *almost primary* ideal if for all $a, b \in M$, $ab \in I$ implies that for some $n \in \mathbb{N}$, $a^n \in I$ or $b^n \in I$. Every prime ideal is primary, and every primary ideal is almost primary, but neither converse holds.

If I is an ideal of M, then the radical of I is

$$\sqrt{I} := \{x \in M \colon\ x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

As in the case for ideals of a ring, it can be shown that the radical of a primary ideal is prime, and that if I and J are ideals, then $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$. It is easy to see that I is almost primary if and only if $\sqrt{I}$ is a prime ideal.

If $p \in M$, then it is apparent that p is prime if and only if pM is a prime ideal of M. If $q \in M$, we say that q is a *primary element* of M if qM is a primary ideal of M, and q is an *almost primary element* of M if qM is an almost primary ideal of M.

Following Halter-Koch in [5], we say that M is a *weakly factorial monoid* (or WFM) if every nonunit element of M can be written as a product of primary elements. WFMs were named analogously after the weakly factorial domains introduced by Anderson and Mahaney in [1]. If M is a WFM, and if x is a nonunit of M, then (up to associates) there is only one factorization of x into primary elements with mutually distinct radicals (such a factorization is called a reduced factorization—see Section 4 for more details).

In this paper, we define and study a new type of monoid invariant, called diversity. If M is an atomic monoid, if we pick $x, y \in M \setminus M^{\times}$, and if we write $y = s_1^{a_1} s_2^{a_2} \ldots s_n^{a_n}$ with $s_i \in M$ and $a_i \in \mathbb{N}$, then x divides a power of y if and only if x divides a power of $s_1 s_2 \ldots s_n$. So, to measure how "far" x is from being almost primary, we can look for the largest value of n such that x divides a power of $s_1 s_2 \ldots s_n$, but not a power of any subproduct. This is what we will call the diversity of x.

Definition 1.1. Let M be a monoid.

(1) We say that $x \mid S$ (in M) if $x \in M$, S is a finite subset of M, and if there exists $t \in \mathbb{N}$ such that $x \mid (\prod S)^t$.

[1] Our notion of an ideal of a monoid is technically the concept of an s-ideal as defined in [6].

(2) We say that x strictly divides S, denoted $x \parallel S$ if $x \mid S$ but $x \nmid T$ for all $T \subsetneq S$
(3) We define the *diversity* of x, denoted $\operatorname{div}(x)$, to be

$$\operatorname{div}(x) = \sup\{|S|\colon\ S \subseteq M \ \text{ with } \ x \parallel S\}.$$

(4) We define the *diversity* of M and the *atomic diversity* of M, denoted by $\operatorname{div}(M)$ and $\operatorname{div}_{\mathrm{a}}(M)$, respectively, by

$$\operatorname{div}(M) = \sup_{x \in M} \operatorname{div}(x), \text{ and } \operatorname{div}_{\mathrm{a}}(M) = \sup_{x \in \mathcal{A}(M)} \operatorname{div}(x).$$

If x is a unit, then $x \parallel \emptyset$ (since $\prod \emptyset = 1$), and hence $\operatorname{div}(x) = 0$. Otherwise, $x \parallel S$ implies that the elements of S are pairwise nonassociate nonunits.

In Section 2, we present some preliminary results, including that, for $x \in M$, $\operatorname{div}(x) = 1$ if and only if x is almost primary (Proposition 2.3). Further, if M is atomic and if $x \in M$, we need only count sets S of atoms with $x \parallel S$ to determine $\operatorname{div}(x)$ (Corollary 2.5). We also show that $\operatorname{div}(x)$ is bounded above by both the tame degree $t(M, x)$ and $\omega(M, x)$, and that for v-Noetherian monoids (in particular, for the multiplicative monoids of Notherian or Krull domains), the diversity of every element is finite.

In Section 3, we introduce two additional properties, called "homogeneous" and "strongly homogeneous" that lie between "almost primary" and "primary" and that are also related to Definition 1.1. We show that all nonunit divisors of homogeneous (strongly homogeneous) elements are themselves homogeneous (respectively, strongly homogeneous) (Theorem 3.8)—a property that is not shared by almost primary elements. An element x is homogeneous precisely when $\sqrt{xM}$ is not only a prime ideal, but maximal amongst radicals of principal ideals (Theorem 3.8). We also show that $\operatorname{div}(x)$ is determined if x divides a set of strongly homogeneous elements (Corollary 3.10).

Finally, in Section 4, we consider factorizations of elements into almost primary elements. We find that such factorizations need not be unique; however, they are unique up to length and radical (Proposition 4.3). Factorizations into homogeneous elements are unique precisely when the homogeneous elements in question are primary (Theorem 4.5). Also, we show that every nonunit element of M can be factored into almost primary elements if and only if for every nonunit $x \in M$ with $\operatorname{div}(x) \geqslant 2$, there exist nonunit $y, z \in M$ such that $x = yz$ and $\operatorname{div}(x) = \operatorname{div}(y) + \operatorname{div}(z)$ (or, in other words, $\operatorname{div}(\cdot)\colon\ M \setminus M^{\times} \to \mathbb{N}_0^{+}$ is as close to a semigroup homomorphism as we can hope—cf. Theorem 4.4).

2. Preliminary results

Proposition 2.1. *Let M be a monoid and let $x, y \in M$. Then* $\operatorname{div}(xy) \leqslant \operatorname{div}(x) + \operatorname{div}(y)$.

Proof. Let $S \subseteq M$ with $xy \parallel S$. There exist subsets $S_x, S_y \subseteq S$ such that $x \parallel S_x$ and $y \parallel S_y$. Since $xy \mid S_x \cup S_y$, we must have $S = S_x \cup S_y$. Therefore,

$$\operatorname{div}(x) + \operatorname{div}(y) \geqslant |S_x| + |S_y| \geqslant |S_x \cup S_y| = |S|,$$

and $\operatorname{div}(xy) \leqslant \operatorname{div}(x) + \operatorname{div}(y)$. □

If x or y is a unit, we have equality in Proposition 2.1. Otherwise, equality is rare—in fact, for $x \in M \setminus M^\times$ and $n \in \mathbb{N}$, $\operatorname{div}(x^n) = \operatorname{div}(x) < n \cdot \operatorname{div}(x)$, as shown in the following lemma.

Lemma 2.2. *Suppose $S = \{s_1, s_2, \ldots, s_k\}$ and $x \in M$. Then:*
(1) *$x \mid S$ if and only if $\sqrt{s_1M} \cap \sqrt{s_2M} \cap \ldots \cap \sqrt{s_kM} \subseteq \sqrt{xM}$.*
(2) *$x \parallel S$ if $\sqrt{s_1M} \cap \sqrt{s_2M} \cap \ldots \cap \sqrt{s_kM} \subseteq \sqrt{xM}$ and if any $\sqrt{s_iM}$ is omitted for $1 \leqslant i \leqslant k$, then the intersection is no longer contained in $\sqrt{xM}$.*
(3) *If $x \parallel S$, then $\sqrt{s_iM}$ and $\sqrt{s_jM}$ are incomparable for each $i \neq j$.*
(4) *For all $m \in \mathbb{N}$, $x \parallel S$ if and only if $x^m \parallel S$. Consequently,* $\operatorname{div}(x) = \operatorname{div}(x^m)$.

Proof. We have $x \mid \{s_1, s_2, \ldots, s_k\}$ if and only if $xr = (s_1 s_2 \ldots s_k)^t$ for some $r \in M$ and $t \in \mathbb{N}$, which is equivalent to $s_1 s_2 \ldots s_k \in \sqrt{xM}$, which is also equivalent to

$$\sqrt{s_1 s_2 \ldots s_k M} = \sqrt{s_1M} \cap \sqrt{s_2M} \cap \ldots \cap \sqrt{s_kM} \subseteq \sqrt{xM}.$$

Thus, statement 1 is proved.

We see that 2 follows from 1 and the fact that $\sqrt{s_iM}$ can be omitted (for some i) if and only if $x \mid S \setminus \{s_i\}$.

We see that 3 follows directly from 2.

Finally, 4 follows from 2, the fact that $\sqrt{xM} = \sqrt{x^mM}$, and the definition of diversity. □

Proposition 2.3. *Let M be a monoid and let $x \in M$. Then:*
(1) *If $x \parallel S$, then the elements of S are pairwise nonassociate, containing no units.*
(2) $\operatorname{div}(x) = 0$ *if and only if $x \in M^\times$.*
(3) $\operatorname{div}(x) = 1$ *if and only if x is an almost primary nonunit.*
(4) *If $x \parallel S$ and $y \in S$, then neither x nor y divides $S \setminus \{y\}$.*
(5) *If $x \parallel S$ and $\{y, z\} \subseteq S$, then $x \parallel S \cup \{yz\} \setminus \{y, z\}$.*
(6) *If $x \mid R$ and $(\prod R) \mid S$, then $x \mid S$.*

P r o o f. We will prove 3; the remaining parts are straightforward.

First, suppose that x is an almost primary nonunit, and $x \parallel S$. Write $S = \{s_1, s_2, \ldots, s_k\}$. $k > 0$ since x is a nonunit. Pick $r \in M$ and $t \in \mathbb{N}$ such that $xr = (s_1 s_2 \ldots s_k)^t = s_1^t s_2^t \ldots s_k^t$. Since x is almost primary, we have $x \mid (s_i^t)^m$ for some $i, m \in \mathbb{N}$. But then $x \mid \{s_i\}$, implying that $k = 1$ and $\operatorname{div}(x) = 1$.

On the other hand, suppose that $\operatorname{div}(x) = 1$. If we have $a, b \in M$ with $x \mid ab$, then $x \mid \{a, b\}$, therefore $x \mid \{a\}$ or $x \mid \{b\}$, and hence $x \mid a^m$ or $x \mid b^m$ for some $m \in \mathbb{N}$. Therefore x is almost primary, and a nonunit since $\operatorname{div}(x) > 0$. $\square$

Theorem 2.4. *Let M be a monoid, and let A be a generating set of M. Then for all $x \in M$, $\operatorname{div}(x) = \sup\{|S|\colon\ S \subseteq A$ with $x \parallel S\}$.*

P r o o f. Set $\alpha(x) = \sup\{|S|\colon\ S \subseteq A \text{ with } x \parallel S\}$. By Definition 1.1, $\operatorname{div}(x) \geqslant \alpha(x)$, and if $x \in M^\times$, then $\operatorname{div}(x) = \alpha(x) = 0$. Suppose now that $x \notin M^\times$. Now choose $S = \{s_1, s_2, \ldots, s_k\} \subseteq M$ with $x \parallel S$. Proposition 2.3 yields that $S \cap M^\times$ is empty. For each i for $1 \leqslant i \leqslant k$, write $s_i = a_{i1} a_{i2} \ldots a_{in_i}$, where $n_i \in \mathbb{N}$ and each $a_{ij} \in A$. Then, setting $T = \{a_{ij} \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n_i\}$, we see that $x \mid T$. Therefore there exists $U \subseteq T$ such that $x \parallel U$.

We claim that for each i with $1 \leqslant i \leqslant m$, there exists some j with $1 \leqslant j \leqslant n_i$ such that $a_{ij} \in U$. To see why this claim is true, suppose (without loss of generality) that $a_{1j} \notin U$ for all $1 \leqslant j \leqslant n_1$. Then $x \mid \{a_{ij}\colon\ 2 \leqslant i \leqslant k, 1 \leqslant j \leqslant n_i\}$, implying that $x \mid \{s_2, s_3, \ldots, s_k\}$, a contradiction. Therefore $\alpha(x) \geqslant |U| \geqslant k$. By considering all such S, we have $\alpha(x) \geqslant \operatorname{div}(x)$ and hence $\operatorname{div}(x) = \alpha(x)$. $\square$

Corollary 2.5. *Let M be an atomic monoid. Then for all $x \in M$, $\operatorname{div}(x) = \sup\{|S|\colon\ S \subseteq \mathcal{A}(M)$ with $x \parallel S\}$.*

P r o o f. Since M is atomic, $M = [\mathcal{A}(M) \cup M^\times]$. We combine Theorem 2.4 and Proposition 2.3.1. $\square$

We now relate diversity to some other monoid invariants, beginning with the ω invariant introduced in [2].

Definition 2.6. Let M be a monoid. For $a, b \in M$, let $\omega(a, b)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property: for all $n \in \mathbb{N}_0$ and $a_1, a_2, \ldots, a_n \in M$, if $a = a_1 a_2 \ldots a_n$ and if $b \mid a$, then there exists a subset $\Omega \subseteq \{1, 2, \ldots, n\}$ such that $|\Omega| \leqslant N$ and $b \mid \prod\limits_{i \in \Omega} a_i$. For $b \in M$, we define

$$\omega(M, b) = \sup\{\omega(a, b)\colon\ a \in M\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Proposition 2.7. *Let M be a monoid, and let $x \in M$. Then* $\operatorname{div}(x) \leqslant \omega(M, x)$.

Proof. Let $x \parallel \{s_1, s_2, \ldots, s_k\}$. Then there exist $t \in \mathbb{N}$ and $r \in M$ such that $xr = (s_1 s_2 \ldots s_k)^t$. Therefore x divides a product of kt elements of M. If $\omega(M, x) < k$, then x would divide a proper subset of $\{s_1, s_2, \ldots, s_k\}$, a contradiction. Thus, $\omega(M, x) \geqslant k$, implying that $\operatorname{div}(x) \leqslant \omega(M, x)$. □

It should be noted that if M is a v-Noetherian monoid as defined in [3] (in particular, the multiplicative monoid of a Noetherian or Krull domain is a v-Noetherian monoid), then $\omega(M, x) < \infty$ for all $x \in M$ (cf. Lemma 3.5 of [4]). Also, if M is atomic and if π is a non-prime atom of M, then $\omega(M, \pi) \leqslant t(M, \pi)$, where $t(M, \pi)$ denotes the tame degree of M with respect to π, as defined in [3]. This proves the following corollary.

Corollary 2.8. *Let M be an atomic monoid. Then:*

(1) *For every non-prime atom $\pi \in M$,* $\operatorname{div}(\pi) \leqslant t(M, \pi)$.

(2) *If M is not factorial, then* $\operatorname{div}_{\mathrm{a}}(M) \leqslant t(M, \mathcal{A}(M))$, *where* $t(M, \mathcal{A}(M)) = \sup\{t(M, \pi) \colon \pi \in \mathcal{A}(M)\}$.

(3) *If M is a v-Noetherian monoid, then* $\operatorname{div}(x) < \infty$ *for every $x \in M$.*

Note that if π is a prime element of M, then $t(M, \pi) = 0$ and $\operatorname{div}(\pi) = 1$. To show that the hypothesis of Corollary 2.8.3 is necessary, we produce an example of a monoid with an element with infinite diversity.

Example 2.9. Let A be a countably infinite set with $A = \{x\} \cup \{y_{ij} \colon i, j \in \mathbb{N}_0, 0 \leqslant j \leqslant i\}$. Let F be the free monoid on this set, and let M be the monoid that results from reducing F modulo the following relations for all $n \in \mathbb{N}$:

$$x^{n+1} = y_{n0} y_{n1} \cdots y_{nn}.$$

Then $A = \mathcal{A}(M)$ and M is atomic and half-factorial. However, for each $n \in \mathbb{N}$, we have $x \parallel \{y_{n0}, y_{n1}, \ldots, y_{nn}\}$. Therefore $\operatorname{div}(x) = \infty$ (and M is not v-Noetherian).

The next two examples show that the diversity of a monoid is, in general, independent of the catenary degree (as defined in [3]). Also, the first example shows that we need not have equality in Proposition 2.7.

Example 2.10. Consider the following multiplicative submonoid of $\mathbb{N}$, known as the Hilbert monoid: $H = 1 + 4\mathbb{N}_0$. The atoms of H are either rational primes congruent to 1 mod 4 (these atoms are prime in H) or of the form pq where p and q are rational primes congruent to 3 mod 4 (these are the non-prime atoms of H). Given an atom pq of the latter type, it is easy to see that $\operatorname{div}(pq)$ is 2 if $p \neq q$, and 1 otherwise. Therefore $\operatorname{div}_{\mathrm{a}}(H) = 2$.

Given distinct rational primes $p_1, p_2, \ldots, p_{2n}$, each congruent to 3 mod 4, we have $p_1 p_2 \ldots p_{2n} \parallel \{p_1^2, p_2^2, \ldots, p_{2n}^2\}$, whence $\operatorname{div}(H) = \infty$. However $c(H)$, the catenary degree of H, is 2 (cf. [3]).

Further, if p is a rational prime congruent to 3 mod 4, then it is routine to check that p^2 is almost primary. If q is a rational prime other than p that is congruent to 3 mod 4, then $p^2 \mid (pq)(pq)$, but $p^2 \nmid pq$. Therefore $\omega(H, p^2) \geqslant 2 > 1 = \operatorname{div}(p^2)$.

Example 2.11. Let $M = [2, 3]$ be the additive submonoid of $\mathbb{N}_0$ generated by 2 and 3. For all $x \in M$, $x \mid \{3\}$, thus $\operatorname{div}(M) = 1$. However, $c(M) = 3$ (cf. [3]).

3. Homogeneous and strongly homogeneous elements

Diversity, as an invariant, cannot alone differentiate among prime, primary, or almost primary elements, since $\operatorname{div}(x) = 1$ for all three. However, it can differentiate between almost primary and primary elements by the following criterion: if $x, y \in M \setminus M^\times$ with y primary, then $x \mid \{y\}$ implies that $y \mid \{x\}$. However, such symmetry need not hold for almost primary elements.

Example 3.1. Let $M = \{2^a 3^b \colon\ a \in \mathbb{N},\ b \in \mathbb{N}_0\} \cup \{1\}$ be a multiplicative submonoid of $\mathbb{N}$. For any $x \in M$, $x \mid \{6\}$, whence $\operatorname{div}(M) = 1$. Also, $2 \mid \{6\}$, but $6 \nmid \{2\}$, as no power of 2 is a multiple (in $\mathbb{N}$) of 3.

The ability of y to divide $\{x\}$ whenever x divides $\{y\}$ will be of great concern to us, so we establish some notation concerning this relation. This relation was previously used by Halter-Koch for primary elements in [5].

Definition 3.2. Let M be a monoid and let $x, y \in M$. We say that y is *related* to x, denoted by $y \sim x$, if $x \mid \{y\}$.

Clearly, $y \sim x$ if and only if $\sqrt{yM} \subseteq \sqrt{xM}$. Also, it is easy to see that $\sim$ is a reflexive and transitive relation on M. As noted above, $\sim$ need not be symmetric. We wish to study nonunits for which $\sim$ is symmetric, and also to generalize this notion.

Definition 3.3. Let M be a monoid, and let $x \in M$.

(1) We say that x is *homogeneous* if $\operatorname{div}(x) = 1$ and if for all $y \in M$, $y \mid \{x\}$ implies that $x \mid \{y\}$ (or, equivalently, if $x \sim y$, then $y \sim x$).

(2) We say that x is *strongly homogeneous* if $\operatorname{div}(x) = 1$ and if for all $y \in M$ and $S \subseteq M$ with $x \in S$, we have $y \parallel S$ implies $x \mid \{y\}$.

Corollary 3.4. *Let M be a monoid. The relation $\sim$ is an equivalence relation on the set of homogeneous elements of M.*

Proposition 3.5. *Let M be a monoid and $x \in M$ a nonunit. Then the following implications hold for properties of x:*
primary $\Rightarrow$ strongly homogeneous $\Rightarrow$ homogeneous $\Rightarrow$ almost primary

Proof. Let x be primary and pick $y \in M$ and $S \subseteq M$ with $x \in S$ and $y \parallel S$. Then there exist $t \in \mathbb{N}$ and $r \in M$ with $yr = (\prod S)^t$, and so $x^t \mid yr$. As x^t is primary and $x^t \nmid r$ (or else $y \mid S \setminus \{x\}$), we see that $x^t \mid y^m$ for some $m \in \mathbb{N}$. Therefore $x \mid \{y\}$ and x is strongly homogeneous. The other implications are clear. 3.3. $\square$

We now give examples to show that none of the implications above are reversible. In Example 3.1 above, the element 6 is almost primary but not homogeneous.

Example 3.6. (*An example of a strongly homogeneous element that is not primary.*) Let $M = \mathbb{Z} \setminus \{0, -1\}$ (under multiplication). The atoms of M are of the form $\pm p$, where p is a prime natural number. Note that no atom of M is primary. To see why, if p and q are rational primes with $|p| \neq |q|$, then $p \mid (-p)q$. However, $p \nmid -p$ and p divides no power of q, implying that p is not primary.

We will now show that every atom of M is strongly homogeneous. Let $p \in M$ be an atom, and suppose that $p \mid ab$. Then, without loss of generality, we have $p \mid a$ in $\mathbb{N}$, hence $p \mid a^2$ in M, and p is almost primary.

Suppose now we have $y \in M \setminus M^\times$ with $y \parallel S$ and $p \in S$. Choose $r \in M$ and $t \in \mathbb{N}$ such that $yr = (\prod S)^t$. We have $p^t \mid yr$. As above, if $p \mid y$ in $\mathbb{N}$, then $p \mid y^2$ in M, hence $p \mid \{y\}$. However, if $p \nmid y$ in $\mathbb{N}$, then $p^t \mid r$ in $\mathbb{N}$. The only way to avoid p^t dividing r in M (and hence $y \mid S \setminus \{p\}$) is for $r = -p^t$. But then $r^2 = p^{2t}$, implying that y^2—and hence y—divides $S \setminus \{p\}$.

Therefore every atom of M is strongly homogeneous, but not primary.

Example 3.7. (*An example of an element that is homogeneous, but not strongly homogeneous.*) Consider the following multiplicative submonoid of $\mathbb{N}$:

$$M = [\{p_1p_2 \colon\ p_1, p_2 \in \mathbb{N} \text{ are distinct odd primes}\}] \cup 6\mathbb{N}.$$

First, we observe that M contains no power of any rational prime. With this, we will show that 6 is homogeneous, but not strongly homogeneous. If $6 \mid ab$ (for $a, b \in M$), then (without loss of generality) a is even, hence 6 divides a in $\mathbb{N}$. Writing $a = 6m$ ($m \in \mathbb{N}$), we see that $a^2 = 6(6m^2)$ and $6 \mid a^2$ (in M), implying that 6 is almost primary.

Also, if $y \in M \setminus M^\times$ with $y \mid \{6\}$, then pick $r \in M$, $t \in \mathbb{N}$ such that $yr = 6^t$. As above, if y is even, then $6 \mid \{y\}$. If y is odd, then, in $\mathbb{N}$, we must have $2^t \mid r$, and it must follow that, in $\mathbb{N}$, $y \mid 3^t$. However, y is then a power of 3, a contradiction. Therefore $6 \mid \{y\}$ and 6 is homogeneous.

To see why 6 is not strongly homogeneous, note that $15 \mid \{6, 35\}$, since $(6 \cdot 35)^2 = 15 \cdot (6 \cdot 490)$. However, $15 \nmid \{6\}$ (as no power of 6 is a multiple, in $\mathbb{N}$, of 5) and $15 \nmid \{35\}$ (as no power of 35 can be a multiple, in $\mathbb{N}$, of 3). Thus $15 \parallel \{6, 35\}$. However, $6 \nmid \{15\}$ (as no power of 15 is even). Therefore 6 is not strongly homogeneous.

Theorem 3.8. *Let M be a monoid, let $x \in M$. Then:*

(1) *For all $y \in M$, $x \sim y$ if and only if $\sqrt{xM} \subseteq \sqrt{yM}$.*

(2) *Suppose that x is homogeneous. For all homogeneous $y \in M$, $x \sim y$ if and only if $\sqrt{xM} = \sqrt{yM}$.*

(3) *x is homogeneous if and only if $\sqrt{xM}$ is both a prime ideal and maximal amongst radicals of proper principal ideals.*

(4) *x is strongly homogeneous if and only if $\operatorname{div}(x) = 1$ and for all $y, z \in M$, we have $y \parallel \{x, z\}$ implies $x \mid \{y\}$.*

(5) *If x is homogeneous (strongly homogeneous), then x^n is homogeneous (strongly homogeneous) for all $n \in \mathbb{N}$.*

(6) *If x is homogeneous (strongly homogeneous), then every nonunit divisor of x is homogeneous (strongly homogeneous).*

(7) *If M is atomic with $\operatorname{div}_{\mathrm{a}}(M) = 1$, then x is homogeneous if and only if x is strongly homogeneous.*

P r o o f. 1. This follows from the definitions.

2. This follows from (1) and Definition 3.3.

3. If x is homogeneous, then x is almost primary, whence $\sqrt{xM}$ is prime. If $\sqrt{xM} \subseteq \sqrt{yM}$ for some $y \in M$, then $x \in \sqrt{yM}$ implying that $y \mid \{x\}$. Thus $x \mid \{y\}$ and $\sqrt{xM} = \sqrt{yM}$. The argument is reversible.

4. Assume that there exists $z \in M$ with $z \parallel T$, $x \in T$, and $|T| \geqslant 3$. Writing $T = \{x, t_1, t_2, \ldots, t_k\}$, we see that $z \parallel \{x, t_1 t_2 \ldots t_k\}$, and thus, by hypothesis, $x \mid \{z\}$. Thus x is strongly homogeneous. The other implication is obvious.

5. The homogeneous statement follows from (3) and the fact that $\sqrt{xM} = \sqrt{x^n M}$. The strongly homogeneous statement follows from (4) via the sequence $y \parallel \{x^n, z\}$ implies $y \parallel \{x, z\}$ implies $x \mid \{y\}$ implies $x^n \mid \{y\}$.

6. Pick any nonunit z with $z \mid x$. Choose $y \in M \setminus M^\times$ with $y \mid \{z\}$, and assume that x is homogeneous. Then y divides a power of z, hence y divides a power of x and $y \mid \{x\}$. Therefore $x \mid \{y\}$, so $z \mid \{y\}$.

Now, assume that x is strongly homogeneous. Let $y \parallel S$ with $z \in S$. Set $S' = S \cup \{x\} \setminus \{z\}$. Now $y \mid S'$, and there is $T \subseteq S'$ with $y \parallel T$. If $x \notin T$, then $y \mid S \setminus \{z\}$, a contradiction. Hence $x \mid \{y\}$ and so $z \mid \{y\}$.

7. Let x be homogeneous. Pick $y \in M$ with $y \parallel S$ and $x \in S$. We observe that every irreducible dividing y must also divide a singleton subset of S, and in fact, there must be an irreducible divisor π of y such that $\pi \mid \{x\}$ (otherwise, if every irreducible divisor of y divides $S \setminus \{x\}$, then $y \mid S \setminus \{x\}$, a contradiction). As x is homogeneous, we have $x \mid \{\pi\}$, hence x divides a power of y and $x \mid \{y\}$. Therefore x is strongly homogeneous. The other implication is obvious. $\square$

Lemma 3.9. *Let M be a monoid, let $x \in M$, and suppose that $x \parallel S$ and $x \mid T$. If there exists $s \in S$ that is strongly homogeneous, then there exist $t \in T$ and a subset S' of $S \setminus \{s\}$ such that $x \parallel S' \cup \{t\}$.*

Proof. Writing $S = \{s, s_1, s_2, \ldots, s_k\}$, we see that $x \parallel S$ implies that $s \mid \{x\}$. Since $x \mid T$, we see that $s \mid T$, and (since $\operatorname{div}(s) = 1$), $s \mid \{t\}$ for some $t \in T$. Pick $a \in \mathbb{N}$ and $r \in M$ such that $sr = t^a$, and pick $\alpha \in M$ and $b \in \mathbb{N}$ with $x\alpha = (ss_1s_2 \ldots s_k)^b$. Then $x\alpha r^b = t^{ab}(s_1s_2 \ldots s_k)^b$, implying that $x \mid \{t, s_1, s_2, \ldots, s_k\}$. Thus, there exists $R \subseteq \{t, s_1, s_2, \ldots, s_k\}$ such that $x \parallel R$. We must have $t \in R$, otherwise $x \mid S \setminus \{s\}$, a contradiction. Therefore, $R = S' \cup \{t\}$ for some subset S' of S. $\square$

Corollary 3.10. *Let M be a monoid and let $x \in M$. If $x \parallel S$, and if each element of S is strongly homogeneous, then* $\operatorname{div}(x) = |S|$.

Proof. Suppose $x \parallel T$. We write $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_m\}$. Suppose that $k \leqslant m$. Repeatedly applying Lemma 3.9, we find $T' \subseteq T$ such that $|T'| \leqslant |S|$ and $x \parallel T'$. Since $x \parallel T$, we must, in fact, have $T = T'$, and therefore $m \leqslant k$, implying that $\operatorname{div}(x) = k$. $\square$

We now give an example to show that "strongly homogeneous" cannot be replaced by "homogeneous" in Lemma 3.9 or Corollary 3.10.

Example 3.11. Consider the following multiplicative subsemigroups of $\mathbb{N}$:

$$A = [2, 3, 5, 7, 2 \cdot 3 \cdot 11, 5 \cdot 7 \cdot 13], \quad \text{and} \quad B = 2 \cdot 3 \cdot 5 \cdot 7\,\mathbb{N},$$

and let $M = A \cup B$. Note that M is atomic, so for computing diversity, we need only consider subsets of $\mathcal{A}(M)$.

We first show that $\operatorname{div}(2 \cdot 3 \cdot 11) = 1$. Let $2 \cdot 3 \cdot 11 \parallel S$. Then some $s \in S \cap \mathcal{A}(M)$ is a multiple (in $\mathbb{N}$) of 11. If $s = 2 \cdot 3 \cdot 11$, then $S = \{s\}$, otherwise, $s \in B$, and hence $s = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot y$ for some $y \in \mathbb{N}$. But then $S = \{s\}$ again, since

$$(2 \cdot 3 \cdot 11)(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 5 \cdot 7 \cdot y^2) = s^2,$$

and $2 \cdot 3 \cdot 11 \mid s^2$. Therefore, $\operatorname{div}(2 \cdot 3 \cdot 11) = 1$. A similar argument shows that $\operatorname{div}(5 \cdot 7 \cdot 13) = 1$.

Further, we claim that $2 \cdot 3 \cdot 11$ is homogeneous. Let $x \parallel \{2 \cdot 3 \cdot 11\}$ for some $x \in M \setminus \{1\}$. Then $xr = (2 \cdot 3 \cdot 11)^k$ for some $r \in M$ and $k \in \mathbb{N}$. However, $(2 \cdot 3 \cdot 11)^k$ has unique factorization in M, since only atoms $2, 3, 2 \cdot 3 \cdot 11$ divide as integers, but in fact we must have k copies of $2 \cdot 3 \cdot 11$ since that is the only source of rational prime 11. Hence $x = (2 \cdot 3 \cdot 11)^j$ for some $j \in \mathbb{N}$, $2 \cdot 3 \cdot 11 \mid x$, and in particular $2 \cdot 3 \cdot 11 \mid \{x\}$. Thus, $2 \cdot 3 \cdot 11$ is homogeneous, and, similarly, so is $5 \cdot 7 \cdot 13$.

We next observe that $\operatorname{div}(2 \cdot 3 \cdot 5 \cdot 7) \geqslant 4$, as $2 \cdot 3 \cdot 5 \cdot 7 \parallel \{2, 3, 5, 7\}$. However, we also have $2 \cdot 3 \cdot 5 \cdot 7 \parallel \{2 \cdot 3 \cdot 11, 5 \cdot 7 \cdot 13\}$. Thus, the conclusion of Corollary 3.10 does not hold for M. Further, $2 \cdot 3 \cdot 5 \cdot 7$ divides none of $\{2, 2 \cdot 3 \cdot 11\}, \{3, 2 \cdot 3 \cdot 11\}, \{5, 2 \cdot 3 \cdot 11\}, \{7, 2 \cdot 3 \cdot 11\}$ and hence the conclusion of Lemma 3.9 does not hold for M.

Proposition 3.12. *Let S, T be subsets of M consisting of strongly homogeneous elements. Suppose that $x \parallel S$ and $x \parallel T$. Then $|S| = |T|$ and $\{\sqrt{sM} \colon s \in S\} = \{\sqrt{tM} \colon t \in T\}$.*

Proof. By Corollary 3.10, $|S| = \operatorname{div}(x) = |T|$. Choose $s \in S$. We have $s \mid \{x\}$ (since $x \parallel S$ and s is strongly homogeneous) and $x \mid T$. As $\operatorname{div}(s) = 1$, we have $s \mid \{t\}$ for some $t \in T$. Therefore, $\sqrt{tM} \subseteq \sqrt{sM}$, and by Theorem 3.8, $\sqrt{tM} = \sqrt{sM}$. $\square$

Corollary 3.13. *Let M be atomic. Then the following assertions are equivalent:*

(1) *Every atom of M is homogeneous.*

(2) *For every $x \in M$ and for sets $S = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and $T = \{\xi_1, \xi_2, \ldots, \xi_m\}$ of pairwise nonassociate atoms of M, if $x \parallel S$ and $x \parallel T$ then $|S| = |T|$ and for each $1 \leqslant i \leqslant n$ there exists a permutation σ such that $\sqrt{\pi_i M} = \sqrt{\xi_{\sigma(i)} M}$.*

Proof. Suppose first that every atom of M is homogeneous. Then every atom of M is strongly homogeneous by Theorem 3.8.5. The conclusion follows from Proposition 3.12.

Suppose now that the second condition holds. Choose $\pi \in \mathcal{A}(M)$. Taking $S = \{\pi\}$, the hypothesis implies that π is almost primary and thus $\operatorname{div}(x) = 1$. Now choose $y \in M \setminus M^\times$ with $y \mid \{\pi\}$. Factor y into atoms as $y = \xi_1 \ldots \xi_k$. Since $y \mid \{\xi_1, \ldots, \xi_k\}$, by hypothesis there is ξ_i with $\sqrt{\pi M} = \sqrt{\xi_i M}$ and hence $\sqrt{\pi M} \supseteq \sqrt{\xi_i M}$. By Theorem 3.8.1, $\pi \mid \{\xi_i\}$ and so $\pi \mid \{y\}$. Hence π is homogeneous. $\square$

4. Factorization into almost primary elements

Proposotion 4.1. *Let M be a monoid and let $x \in M$. Let $x = y_1 y_2 \dots y_n$ for $y_i \in M$. Then x is homogeneous (strongly homogeneous) if and only if each y_i is homogeneous (respectively, strongly homogeneous) and y_i are pairwise related (or equivalently, $\sqrt{y_i M} = \sqrt{y_j M}$ for all $1 \leqslant i \leqslant j \leqslant n$).*

Proof. If x is homogeneous (strongly homogeneous), then each y_i is homogeneous (strongly homogeneous) by Theorem 3.8.4. Also, for each $1 \leqslant i \leqslant n$, $xM \subseteq y_i M$, implying that $\sqrt{xM} \subseteq \sqrt{y_i M}$. By Theorem 3.8.1, $\sqrt{xM} = \sqrt{y_i M}$.

On the other hand, if each y_i is homogeneous and if $\sqrt{y_i M} = \sqrt{y_j M}$ for each $1 \leqslant i \leqslant j \leqslant n$, then

$$\sqrt{xM} = \sqrt{y_1 y_2 \dots y_n M} = \sqrt{y_1 M} \cap \sqrt{y_2 M} \cap \dots \cap \sqrt{y_n M} = \sqrt{y_1 M}.$$

So, $\sqrt{xM}$ is prime and maximal amongst radicals of principal ideals, whence x is homogeneous by Theorem 3.8.2.

Now, assume that each y_i is strongly homogeneous and pairwise related. Then $\sqrt{xM} = \sqrt{y_i M}$ for each i. Letting $z \in M \backslash M^{\times}$ with $z \parallel S$ and $S = \{x, s_1, s_2, \dots, s_k\}$, the fact that $\sqrt{xM} = \sqrt{y_i M}$ implies that $x \mid \{y_i\}$, whence $z \mid \{y_i, s_1, s_2, \dots, s_k\}$. So, $y_i \mid \{z\}$ for each i, implying that $x \mid \{z\}$. Therefore x is strongly homogeneous. $\square$

Definition 4.2. Let M be a monoid, let $x \in M$, and suppose that

$$x = q_1 q_2 \dots q_n,$$

where each q_i is almost primary. If $\sqrt{q_i M}$ and $\sqrt{q_j M}$ are incomparable for all $i \neq j$, we say that the above factorization is a *reduced factorization* of x into almost primary elements.

Clearly, if a nonunit element x of a monoid M can be factored into almost primary elements, then we may find a reduced factorization of x into almost primary elements, merely by consolidating almost primary divisors of x whose radicals are comparable.

In Theorem 1.5 of [5], Halter-Koch showed that reduced factorizations into primary elements are unique up to associates. In other words, if

$$q_1 q_2 \dots q_n = r_1 r_2 \dots r_m$$

are reduced factorizations of some nonunit x into primary elements, then $n = m$ and there exists $\sigma \in S_n$ such that q_i is associate to $r_{\sigma(i)}$.

If we consider factorizations into almost primary elements, then we need not have uniqueness. For example, consider Example 3.6. For distinct rational primes $p, q \in \mathbb{N}$, there are two reduced factorizations of $pq = (p)(q) = (-p)(-q)$. However, reduced factorizations into almost primary elements are unique up to length and radicals, as shown in the following.

Proposition 4.3. *Let M be a monoid, let $x \in M$, and suppose that*

$$q_1 q_2 \dots q_n = r_1 r_2 \dots r_m$$

are two reduced factorizations of x into almost primary elements. Then $\operatorname{div}(x) = n = m$ *and there exists $\sigma \in S_n$ such that $\sqrt{q_i M} = \sqrt{r_{\sigma(i)} M}$.*

Proof. As q_1 is almost primary, we have (without loss of generality) that $q_1 \mid r_1^k$ for some $k \in \mathbb{N}$. Thus, $\sqrt{r_1 M} \subseteq \sqrt{q_1 M}$. However, r_1 is almost primary, whence $r_1 \mid q_i^t$ for some i, $1 \leqslant i \leqslant n$. However, we then have $\sqrt{q_i M} \subseteq \sqrt{r_1 M} \subseteq \sqrt{q_1 M}$, forcing $i = 1$ (as our factorizations of x are reduced). Therefore $\sqrt{q_1 M} = \sqrt{r_1 M}$. Applying induction, we see that $m = n$ and that we may pair up the q's and r's by radicals.

To show that $\operatorname{div}(x) = n$, we first note that $\operatorname{div}(x) \leqslant \sum\limits_{i=1}^{n} \operatorname{div}(q_i) = n$. Also, $x \mid \{q_1, q_2, \dots, q_n\}$, and if $x \mid \{q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n\}$ for some $1 \leqslant i \leqslant n$, then $q_i \mid q_j^t$ for some $t \in \mathbb{N}$ and $j \neq i$. However, this would imply that $\sqrt{q_j M} \subseteq \sqrt{q_i M}$, a contradiction. Therefore $x \parallel \{q_1, q_2, \dots, q_n\}$ and $\operatorname{div}(x) \geqslant n$. $\square$

This next theorem, informally speaking, considers conditions that are as close as we can hope to $\operatorname{div}(\cdot)\colon M \setminus M^{\times} \to \mathbb{N} \cup \{\infty\}$ being a homomorphism of semigroups.

Theorem 4.4. *Let M be a monoid. For $x \in M \setminus M^{\times}$, the following assertions are equivalent:*

(1) *If* $\operatorname{div}(x) \geqslant 2$, *then there exist $y, z \in M \setminus M^{\times}$ with $x = yz$ and* $\operatorname{div}(x) = \operatorname{div}(y) + \operatorname{div}(z)$.

(2) *x can be written as a product of* $\operatorname{div}(x)$ *almost primary elements.*

(3) *There exists a reduced factorization of x into almost primary elements. Additionally, if M is atomic, then* $\operatorname{div_a}(M) = 1$ *if and only if* 1–3 *hold for every $x \in M \setminus M^{\times}$.*

Proof. (1 $\Rightarrow$ 2): If $\operatorname{div}(x) = 1$, there is nothing to prove. If $\operatorname{div}(x) = n \geqslant 2$, we may repeatedly apply 1 to obtain $x = y_1 y_2 \dots y_n$ with $\operatorname{div}(y_i) = 1$. It then follows that each y_i is almost primary.

(2 $\Rightarrow$ 3): Write x as a product of $\operatorname{div}(x)$ almost primary elements and, if necessary, group factors with comparable radicals together.

(3 $\Rightarrow$ 1): Suppose that $x = q_1 q_2 \dots q_n$ is such a reduced factorization and that $n \geqslant 2$. By Proposition 4.3, $\operatorname{div}(x) = n = \sum\limits_{i=1}^{n} \operatorname{div}(q_i)$.

Now suppose 1–3 hold for all $x \in M \setminus M^{\times}$. Let $x \in M$ with $\operatorname{div}(x) \geqslant 2$. By condition 1, x is reducible, hence $\operatorname{div}_{\mathrm{a}}(M) = 1$. On the other hand, suppose $\operatorname{div}_{\mathrm{a}}(M) = 1$. Then every atom of M is almost primary. Hence every $x \in M \setminus M^{\times}$ can be factored into almost primary elements, and thus has a reduced factorization into almost primary elements. □

Theorem 4.5. *Let M be a monoid. Then, the following assertions are equivalent.*

(1) *Every nonunit element of M has a reduced factorization into primary elements (i.e. M is a WFM).*

(2) *Given $x \in M \setminus M^{\times}$, there exist homogeneous $q_1, q_2, \ldots, q_n$ (each with distinct radicals) such that $x = q_1 q_2 \ldots q_n$, where this factorization into homogeneous elements is unique in the sense that if $x = q'_1 q'_2 \ldots q'_n$ is any reduced factorization of x into almost primary elements, then there exists $\sigma \in S_n$ such that q_i is associate to $q'_{\sigma(i)}$.*

Proof. $(1 \Rightarrow 2)$: This follows from Theorem 1.5 of [5].

$(2 \Rightarrow 1)$: It suffices to show that every homogeneous element is primary. Let $q \in M$ be homogeneous, and suppose that $q \mid ab$ for some $a, b \in M$. We may assume that a and b are nonunits of M (otherwise $q \mid a$ or $q \mid b$). By the hypothesis, we have reduced factorizations

$$a = a_1 a_2 \ldots a_m \quad \text{and} \quad b = b_1 b_2 \ldots b_n$$

of a and b into homogeneous elements. Since q is almost primary, $q \mid \{a_i\}$ for some i or $q \mid \{b_i\}$ for some j, i.e. either $a_i \sim q$ or $b_j \sim q$. If $q \mid \{a_i\}$ and $q \mid \{b_j\}$ for some i and j, then we are done, for q divides a power of a and a power of b. So, without loss of generality, assume that $a_1 \sim q$ and each b_j is unrelated to q. Since $\sim$ is an equivalence relation on homogeneous elements and the factorization of a is reduced, a_1 and q are unrelated to all the b_j and all the other a_i. When we reduce the factorization $a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n$ of ab, we combine related elements, so a_1 is untouched. In other words, $a_1 c_1 c_2 \ldots c_t$ is a reduced factorization of ab into almost primary elements, for some almost primary elements c_k.

Write $ab = qr$. If $r \in M^{\times}$, then by the hypothesis, q is an associate of a_1 and all the c_i are units, a contradiction. Therefore, we have a reduced factorization of $r_1 r_2 \ldots r_s$ of r into homogeneous elements. Since no more than one of the r_i can share the same radical as q, either $q r_1 r_2 \ldots r_s$ is a reduced factorization of ab into almost primary elements, or (without loss of generality) $q \sim r_1$ and $(q r_1) r_2 \ldots r_s$ is a reduced factorization of ab into almost primary elements. By the hypothesis, we have that a_1 is an associate of either q or $q r_1$. In either case $q \mid a_1$ and thus $q \mid a$.

We conclude that q is primary, and therefore M is weakly factorial. □

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