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SECOND MOMENTS OF DIRICHLET L-FUNCTIONS WEIGHTED BY KLOOSTERMAN SUMS

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Abstract. For the general modulo $q \geqslant 3$ and a general multiplicative character χ modulo q, the upper bound estimate of $|S(m,n,1,\chi,q)|$ is a very complex and difficult problem. In most cases, the Weil type bound for $|S(m,n,1,\chi,q)|$ is valid, but there are some counterexamples. Although the value distribution of $|S(m,n,1,\chi,q)|$ is very complicated, it also exhibits many good distribution properties in some number theory problems. The main purpose of this paper is using the estimate for k-th Kloosterman sums and analytic method to study the asymptotic properties of the mean square value of Dirichlet L-functions weighted by Kloosterman sums, and give an interesting mean value formula for it, which extends the result in reference of W. Zhang, Y. Yi, X. He: On the $2k$-th power mean of Dirichlet L-functions with the weight of general Kloosterman sums, Journal of Number Theory, 84 (2000), 199–213.

Keywords: general k-th Kloosterman sum, Dirichlet L-function, the mean square value, asymptotic formula

MSC 2010: 11M38, 11L05

1. Introduction

Let $q \geqslant 2$ be an integer and χ a Dirichlet character modulo q. Then for any given integers m and n, we define the general k-th Kloosterman sum $S(m,n,k,\chi;q)$ as follows:

$$S(m,n,k,\chi;q) = \sum_{a=1}^{q} \chi(a) e\Big(\frac{ma^k + n\overline{a}^k}{q}\Big),$$

where $\overline{a}$ denotes the solution b of the congruence equation $ab \equiv 1 \pmod q$. That is, b is the inverse of a modulo q, and $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$.

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The upper bound estimation of $S(m,n,1,\chi;q)$ has been studied by many authors. For example, A. Weil's important work [10] gave the upper bound estimate

$$|S(m,n,1,\chi_0;p)| \leqslant p^{\frac{1}{2}}(m,n,p)^{\frac{1}{2}},$$

where p is a prime, (m,n,p) denotes the greatest common divisor of m, n and p, χ_0 denotes the principal character mod p.

H. Salié and others proved a similar estimate for the prime power case. T. Estermann [7] gave the general conclusion:

$$|S(m,n,1,\chi_0;q)| \leqslant d(q)q^{\frac{1}{2}}(m,n,q)^{\frac{1}{2}},$$

where $d(q)$ denotes the Dirichlet divisor function.

The upper bound estimate

$$|S(m,n,1,\chi;p)| \ll (m,n,p)^{\frac{1}{2}}p^{\frac{1}{2}+\varepsilon} \tag{1.1}$$

is due principally to A. Weil's classical work [10], related results can also be found in S. Chowla [2] and A. V. Malyshev [9].

For the general modulo $q \geqslant 3$ and a general multiplicative character χ modulo q, the upper bound estimate of $|S(m,n,1,\chi,q)|$ is a very complex and difficult problem, see Lemma 12.2 and Lemma 12.3 in the book of H. Iwaniec and E. Kowalski [8]. In most cases, the Weil type bound for $|S(m,n,1,\chi,q)|$ is valid, but there are some counterexamples, see Example 5.1 in T. Cochrane and Z. Zheng's paper [4], other related results can also be found in [3], [5], and [6].

Although the value distribution of $|S(m,n,1,\chi,q)|$ is very complicated, it also presents many good distribution properties in some number theory problems. For example, Zhang Wenpeng, Yi Yuan and He Xiali [11] proved the asymptotic formula

$$\sum_{\chi\neq\chi_0} |S(m,n,1,\chi;q)|^2|L(1,\chi)|^2 = \frac{\pi^2}{6}\,\varphi^2(q)\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big) + O(q^{3/2+\varepsilon}),$$

where $\prod_{p\mid q}$ denotes the product over all prime divisors of q, ε denotes any fixed positive number.

This paper is inspired by [11]. We use the mean value theorem for Dirichlet L-functions and the analytic method to study the asymptotic properties of the mean value

$$\sum_{\chi\neq\chi_0} |S(m,n,k,\chi;p)|^2\cdot|L(1,\chi)|^{2r},$$

and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem 1. *Let p be an odd prime, $k \geqslant 2$ any fixed positive integer with $k \mid p-1$. Then for any integers r, m, n with $(mn, p) = 1$, we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |S(m, n, k, \chi; p)|^2 \cdot |L(1, \chi)|^{2r} = p^2 \sum_{n=1}^{\infty} \frac{d_r^2(n)}{n^2} + O(p^{2-1/k+\varepsilon}),$$

where $d_r(n)$ denotes the general Dirichlet divisor function, defined by the coefficients of $\zeta^r(s) = \sum\limits_{n=1}^{\infty} d_r(n)/n^s$ with $s > 1$.

For the general k-th Gauss sums $G(\chi, m, k; q) = \sum\limits_{n=1}^{q} \chi(n) e(mn^k/q)$, we can also get the following similar conclusion:

Theorem 2. *Let p be an odd prime, $k \geqslant 2$ any fixed positive integer with $k \mid p-1$. Then for any integers r, m with $(m, p) = 1$, we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |G(\chi, m, k; p)|^2 \cdot |L(1, \chi)|^{2r} = p^2 \sum_{n=1}^{\infty} \frac{d_r^2(n)}{n^2} + O(p^{2-1/k+\varepsilon}).$$

From our theorems we may immediately deduce the following two corollaries:

Corollary 1. *Let p be an odd prime, $k \geqslant 2$ any fixed positive integer with $k \mid p-1$. Then for any integers m, n with $(mn, p) = 1$, we have the asymptotic formulae*

(I) $$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |S(m, n, k, \chi; p)|^2 \cdot |L(1, \chi)|^2 = \frac{\pi^2}{6} \cdot p^2 + O(p^{2-1/k+\varepsilon});$$

(II) $$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |G(\chi, m, k; p)|^2 \cdot |L(1, \chi)|^2 = \frac{\pi^2}{6} \cdot p^2 + O(p^{2-1/k+\varepsilon}).$$

Corollary 2. *Let p be an odd prime, $k \geqslant 2$ any fixed positive integer with $k \mid p-1$. Then for any integers m, n with $(mn, p) = 1$, we have the asymptotic formulae*

(i) $$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |S(m, n, k, \chi; p)|^2 \cdot |L(1, \chi)|^4 = \frac{5\pi^4}{72} \cdot p^2 + O(p^{2-\frac{1}{k}+\varepsilon});$$

(ii) $$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} |G(\chi, m, k; p)|^2 \cdot |L(1, \chi)|^4 = \frac{5\pi^4}{72} \cdot p^2 + O(p^{2-\frac{1}{k}+\varepsilon}).$$

2. Some lemmas

In order to complete the proof of our theorems, we need the following several lemmas.

Lemma 1. *Let p be an odd prime, k any fixed positive integer with $k \mid p-1$. Then for any integers m and n, we have the estimate*

$$S(m,n,k,\chi_0;p) = \sum_{a=1}^{p-1} e\Big(\frac{ma^k + n\overline{a^k}}{p}\Big) \ll k \cdot (m,n,p)^{\frac{1}{2}} p^{\frac{1}{2}+\varepsilon}.$$

P r o o f. Let χ_1 be a character of order k mod p, that is, $\chi_1^k = \chi_0$ with k the smallest possible, let χ_0 be the principal character mod p. Then from the properties of characters mod p and the estimate (1.1) we have

$$\begin{aligned} S(m,n,k,\chi_0;p) &= \sum_{a=1}^{p-1} (1 + \chi_1(a) + \chi_1^2(a) + \ldots + \chi_1^{k-1}(a)) e\Big(\frac{ma + n\overline{a}}{p}\Big) \\ &= \sum_{r=0}^{k-1} \sum_{a=1}^{p-1} \chi_1^r(a) e\Big(\frac{ma + n\overline{a}}{p}\Big) \ll k \cdot (m,n,p)^{\frac{1}{2}} p^{\frac{1}{2}+\varepsilon}. \end{aligned}$$

This proves Lemma 1. □

Lemma 2. *Let p be an odd prime, let χ be the Dirichlet character modulo p. Then we have the estimate*

$$\sum_{r=1}^{p-1} \Big| \sum_{\chi \neq \chi_0} \chi(r) |L(1,\chi)|^{2r} \Big| = O(p^{1+\varepsilon}),$$

where χ_0 denotes the principal character modulo p, and $\varepsilon > 0$ denotes any fixed positive number.

P r o o f. See Lemma 5 of [11]. □

Lemma 3. *Let p be an odd prime, let k be any fixed positive integer with $k \mid p-1$, and let χ_0 denote the principal character modulo p. Then for any fixed positive integer r, we have the asymptotic formula*

$$\sum_{\substack{\chi \neq \chi_0 \\ \chi^{(p-1)/k} = \chi_0}} |L(1,\chi)|^{2r} = \frac{p}{k} \cdot \sum_{n=1}^{\infty} \frac{d_r^2(n)}{n^2} + O(p^{1-1/k+\varepsilon}),$$

where ε denotes any fixed positive number.

P r o o f. Let $A(y,\chi)=\sum\limits_{N<n\leqslant y}\chi(n)d_r(n)$, then from the definition of Dirichlet L-functions and Abel's identity (see Theorem 4.2 of [1]) we have

$$(1.2)\qquad |L(1,\chi)|^{2r}=\Big|\sum_{n=1}^{\infty}\frac{\chi(n)d_r(n)}{n}\Big|^2=\Big|\sum_{1\leqslant n\leqslant N}\frac{\chi(n)d_r(n)}{n}+\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,\mathrm{d}y\Big|^2$$
$$=\Big|\sum_{1\leqslant n\leqslant N}\frac{\chi(n)d_r(n)}{n}\Big|^2+\Big(\sum_{1\leqslant n\leqslant N}\frac{\chi(n)d_r(n)}{n}\Big)\Big(\int_N^{\infty}\frac{A(y,\overline{\chi})}{y^2}\,\mathrm{d}y\Big)$$
$$+\Big(\sum_{1\leqslant n\leqslant N}\frac{\overline{\chi}(n)d_r(n)}{n}\Big)\Big(\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,\mathrm{d}y\Big)+\Big|\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,\mathrm{d}y\Big|^2.$$

Let g be a primitive root of $\bmod p$. Taking $N=p^{2^r}$, note that for any integer $1\leqslant s\leqslant k-1$, if $g^{s(p-1)/k}\equiv r(s) \bmod p$ with $1<r(s)\leqslant p-1$, then $r^k(s)\equiv 1 \bmod p$, so $r(s)>p^{1/k}$ and $\overline{r(s)}>p^{1/k}$, where $r(s)\overline{r(s)}\equiv 1 \bmod p$, and from the orthogonality of the characters we have

$$(1.3)\qquad \sum_{\substack{\chi\neq\chi_0\\ \chi^{(p-1)/k}=\chi_0}}\Big|\sum_{1\leqslant n\leqslant N}\frac{\chi(n)d_r(n)}{n}\Big|^2=\sum_{\substack{\chi \bmod p\\ \chi^{(p-1)/k}=\chi_0}}\Big|\sum_{1\leqslant n\leqslant N}\frac{\chi(n)d_r(n)}{n}\Big|^2+O(p^{\varepsilon})$$
$$=\sum_{1\leqslant m\leqslant N}\sum_{1\leqslant n\leqslant N}\frac{d_r(m)d_r(n)}{mn}\sum_{\substack{\chi \bmod p\\ \chi^{(p-1)/k}=\chi_0}}\chi(m\overline{n})+O(p^{\varepsilon})$$
$$=\frac{p-1}{k}\sum_{s=0}^{k-1}\sum_{\substack{1\leqslant m\leqslant N\ 1\leqslant n\leqslant N\\ m\overline{n}\equiv g^{s(p-1)/k} \bmod p}}\frac{d_r(m)d_r(n)}{mn}+O(p^{\varepsilon})$$
$$=\frac{p-1}{k}\Big(\sum_{\substack{1\leqslant m\leqslant N\ 1\leqslant n\leqslant N\\ m\overline{n}\equiv 1 \bmod p}}\frac{d_r(m)d_r(n)}{mn}+\sum_{s=1}^{k-1}\sum_{\substack{1\leqslant m\leqslant N\ 1\leqslant n\leqslant N\\ m\overline{n}\equiv g^{s(p-1)/k} \bmod p}}\frac{d_r(m)d_r(n)}{mn}\Big)+O(p^{\varepsilon})$$
$$=\frac{p-1}{k}\sum_{n=1}^{\infty}\frac{d_r^2(n)}{n^2}+O\Big(\frac{p}{k}\sum_{s=1}^{k-1}\sum_{\substack{1\leqslant m\leqslant N\ 1\leqslant n\leqslant N\\ m\overline{n}\equiv r(s) \bmod p}}\frac{d_r(m)d_r(n)}{mn}\Big)+O(p^{\varepsilon})$$
$$=\frac{p-1}{k}\sum_{n=1}^{\infty}\frac{d_r^2(n)}{n^2}+O\Big(\frac{p}{k}\sum_{s=1}^{k-1}\sum_{1\leqslant n\leqslant N}\sum_{0\leqslant l\leqslant N/p}\frac{d_r(lp+nr(s))d_r(n)}{(lp+nr(s))n}\Big)$$
$$+O\Big(\frac{p}{k}\sum_{s=1}^{k-1}\sum_{1\leqslant m\leqslant N}\sum_{0\leqslant l\leqslant N/p}\frac{d_r(lp+m\overline{r(s)})d_r(m)}{(lp+m\overline{r(s)})m}\Big)+O(p^{\varepsilon})$$
$$=\frac{p-1}{k}\sum_{n=1}^{\infty}\frac{d_r^2(n)}{n^2}+O(p^{1-1/k+\varepsilon}).$$

From Lemma 4 of [11] we know that

$$\sum_{\chi\neq\chi_0} |A(y,\chi)|^2 \ll y^{2-4/2^r+\varepsilon}p^2. \tag{1.4}$$

Then applying the Cauchy inequality and (1.4) we can deduce that

$$\sum_{\substack{\chi\neq\chi_0\\ \chi^{(p-1)/k}=\chi_0}} \left|\Big(\sum_{1\leqslant n\leqslant N} \frac{\chi(n)d_r(n)}{n}\Big)\Big(\int_N^\infty \frac{A(y,\overline{\chi})}{y^2}\,\mathrm{d}y\Big)\right| \ll p^\varepsilon \tag{1.5}$$

and

$$\sum_{\substack{\chi\neq\chi_0\\ \chi^{(p-1)/k}=\chi_0}} \left|\int_N^\infty \frac{A(y,\chi)}{y^2}\,\mathrm{d}y\right|^2 \ll p^\varepsilon. \tag{1.6}$$

Now combining (1.2), (1.3), (1.5) and (1.6) we may immediately deduce the asymptotic formula

$$\sum_{\substack{\chi\neq\chi_0\\ \chi^{(p-1)/k}=\chi_0}} |L(1,\chi)|^{2r} = \frac{p}{k}\cdot\sum_{n=1}^\infty \frac{d_r^2(n)}{n^2} + O(p^{1-1/k+\varepsilon}).$$

This proves Lemma 3. □

3. Proof of the theorems

In this section, we shall complete the proof of our theorems. First we prove Theorem 1. From the properties of the Dirichlet characters mod$p$ we have

$$\begin{aligned}
&\sum_{\chi\neq\chi_0} |S(m,n,k,\chi;p)|^2\cdot|L(1,\chi)|^{2r} \\
&= \sum_{r=1}^{p-1}\sum_{s=1}^{p-1} e\Big(\frac{(r^k-s^k)m+(\overline{r}^k-\overline{s}^k)n}{p}\Big)\sum_{\chi\neq\chi_0}\chi(r\overline{s})|L(1,\chi)|^{2r} \\
&= \sum_{r=1}^{p-1}\sum_{s=1}^{p-1} e\Big(\frac{s^k(r^k-1)m+\overline{s}^k(\overline{r}^k-1)n}{p}\Big)\sum_{\chi\neq\chi_0}\chi(r)|L(1,\chi)|^{2r} \\
&= \varphi(p)\sum_{\substack{r=1\\ r^k\equiv 1 \bmod p}}^{p-1}\sum_{\chi\neq\chi_0}\chi(r)|L(1,\chi)|^{2r} \\
&\quad + \sum_{\substack{r=1\\ (r^k-1,p)=1}}^{p-1}\sum_{s=1}^{p-1} e\Big(\frac{s^k(r^k-1)m+\overline{s}^k(\overline{r}^k-1)n}{p}\Big)\sum_{\chi\neq\chi_0}\chi(r)|L(1,\chi)|^{2r}.
\end{aligned} \tag{1.7}$$

Using (7), Lemma 1, Lemma 2 and Lemma 3 we get

$$\begin{aligned}\sum_{\chi\neq\chi_0}|S(m,n,\chi,p)|^2|L(1,\chi)|^{2r} &= k\varphi(p)\sum_{\substack{\chi\neq\chi_0\\ \chi^{(p-1)/k}=\chi_0}}|L(1,\chi)|^{2r}\\ &\quad+O\Bigg(p^{\frac12+\varepsilon}\sum_{\substack{r=1\\ (r^k-1,p)=1}}^{p-1}(mn(r^k-1),p)^{\frac12}\Bigg|\sum_{\chi\neq\chi_0}\chi(r)|L(1,\chi)|^{2r}\Bigg|\Bigg)\\ &= p^2\cdot\sum_{n=1}^{\infty}\frac{d_r^2(n)}{n^2}+O(p^{2-1/k+\varepsilon})+O\Bigg(p^{\frac12+\varepsilon}\sum_{r=2}^{p-1}\Bigg|\sum_{\chi\neq\chi_0}\chi(r)|L(1,\chi)|^{2r}\Bigg|\Bigg)\\ &= p^2\cdot\sum_{n=1}^{\infty}\frac{d_r^2(n)}{n^2}+O(p^{2-1/k+\varepsilon})+O(p^{\frac32+\varepsilon})=p^2\cdot\sum_{n=1}^{\infty}\frac{d_r^2(n)}{n^2}+O(p^{2-1/k+\varepsilon}).\end{aligned}$$

This proves Theorem 1. □

Using the method of proving Theorem 1 we can also deduce Theorem 2. This completes the proof of our theorems. □

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