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EXISTENCE OF ONE-SIGNED SOLUTIONS OF NONLINEAR FOUR-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we are concerned with the existence of one-signed solutions of four-point boundary value problems

$$-u'' + Mu = rg(t)f(u), \quad u(0) = u(\varepsilon), \quad u(1) = u(1-\varepsilon)$$

and

$$u'' + Mu = rg(t)f(u), \quad u(0) = u(\varepsilon), \quad u(1) = u(1-\varepsilon),$$

where $\varepsilon \in (0, 1/2)$, $M \in (0, \infty)$ is a constant and $r > 0$ is a parameter, $g \in C([0,1], (0, +\infty))$, $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$. The proof of the main results is based upon bifurcation techniques.

Keywords: four-point boundary value problem, one-signed solution, bifurcation method

MSC 2010: 34B15

1. Introduction

In the past few years, the existence and multiplicity of positive solutions of nonlinear second-order Neumann problems

$$u'' + \varrho u + f(t,u) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u'(0) = u'(1) = 0, \tag{1.2}$$

where ϱ is a constant with $\varrho \in (-\infty, 0) \cup (0, \pi^2/4)$, has been studied by several authors, see Jiang and Liu [3], Sun, Li and Cheng [10]–[11]. The main tool used in [3], [10], [11] is the fixed point theorem in cones [2].

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Let $\varepsilon \in (0, 1/2)$ be given. Let us consider the existence of one-signed solutions of the nonlinear four-point boundary value problems

$$-u'' + Mu = rg(t)f(u), \quad t \in (0,1) \tag{1.3}$$

and

$$u'' + Mu = rg(t)f(u), \quad t \in (0,1), \tag{1.4}$$

subject to the boundary conditions

$$u(0) = u(\varepsilon), \quad u(1) = u(1-\varepsilon), \tag{1.5}$$

where $r > 0$ is a parameter, g and f satisfy
(H1) $g \in C([0,1], (0,+\infty))$;
(H2) $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$;
(H3) there exist $f_0, f_\infty \in (0, \infty)$ such that

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{|s|\to\infty} \frac{f(s)}{s}.$$

Obviously, if $u \in C^1[0,1]$ satisfies (1.5), then by Rolle's mean value theorem, there exist $\xi \in (0, \varepsilon)$ and $\eta \in (1-\varepsilon, 1)$ such that

$$u'(\xi) = u'(\eta) = 0.$$

Letting $\varepsilon \to 0$, then $\xi \to 0$ and $\eta \to 1$, and accordingly, (1.5) reduces to the Neumann boundary conditions (1.2).

The purpose of this paper is to prove the existence of positive solutions of (1.3) and (1.4) subject to the boundary conditions (1.5). By applying the well-known Rabinowitz's global bifurcation theorem [8], we will prove the following

Theorem 1.1. *Let $M \in (0, \infty)$ be given and* (H1)–(H3) *hold. Let λ_1 be the principal eigenvalue of the linear problem*

$$-u'' + Mu = \lambda g(t)u, \quad t \in (0,1), \quad u(0) = u(\varepsilon), \quad u(1) = u(1-\varepsilon). \tag{1.6}$$

Assume that either

$$\frac{\lambda_1}{f_\infty} < r < \frac{\lambda_1}{f_0} \tag{1.7}$$

or

$$\frac{\lambda_1}{f_0} < r < \frac{\lambda_1}{f_\infty}. \tag{1.8}$$

Then (1.3), (1.5) *has two solutions u^+ and u^- with $u^+(t) > 0$ on $[0,1]$ and $u^-(t) < 0$ on $[0,1]$.*

Theorem 1.2. *Let $M \in (0, \pi^2/4]$ be given and* (H1)–(H3) *hold. Let τ_1 be the principal eigenvalue of the linear problem*

$$u'' + Mu = \tau g(t)u, \quad t \in (0,1), \qquad u(0) = u(\varepsilon), \quad u(1) = u(1-\varepsilon). \tag{1.9}$$

Assume that either

$$\frac{\tau_1}{f_\infty} < r < \frac{\tau_1}{f_0} \tag{1.10}$$

or

$$\frac{\tau_1}{f_0} < r < \frac{\tau_1}{f_\infty}. \tag{1.11}$$

Then (1.4), (1.5) *has two solutions u^+ and u^- with $u^+(t) > 0$ on $[0,1]$ and $u^-(t) < 0$ on $[0,1]$.*

Remark 1.1. Conditions (1.7) and (1.8) are sharp to guarantee the existence of one-signed solutions. This can be seen from the following example

$$-u'' + Mu = 1 \cdot 1 \cdot (\bar{\lambda}_1 - \sigma)u, \quad u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon), \tag{1.12}$$

where $\bar{\lambda}_1$ is the principle eigenvalue of the linear problem

$$-u'' + Mu = \lambda u, \quad u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon),$$

and can be explicitly given by

$$\bar{\lambda}_1 = M.$$

Take $g(t) \equiv 1$ and $f(u) = (\bar{\lambda}_1 - \sigma)u$. Then $f_0 = f_\infty = (\bar{\lambda}_1 - \sigma)$ and (1.7), (1.8) do not hold. Though σ is allowed to approach to 0, (1.12) has no nontrivial solutions any more.

Remark 1.2. Miciano and Shivaji [7] studied the multiplicity of positive solutions of the Neumann problem

$$u'' + \lambda f(u) = 0, \quad t \in (0,1),$$
$$u'(0) = u'(1) = 0$$

by the method of time map estimation. For other results on the existence of positive solutions of singular nonlinear Neumann problems, see Rachunková, Staněk and Tvrdý [9], Sun, Cho and O'Regan [12], Chu, Sun and Chen [1], Li [5], and Li and Jiang [4].

The rest of this paper is organized as follows: In Section 2 and 3, we construct the Green functions of linear problems

$$-u'' + Mu = 0,\ t \in (0,1), \quad u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon) \tag{1.13}$$

and

$$u'' + Mu = 0,\ t \in (0,1), \quad u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon), \tag{1.14}$$

respectively. Moreover, we study the properties of these Green's functions and investigate the principal eigenvalues of the associated linear eigenvalue problem (1.6) and (1.9). Section 4 is devoted to prove our main results via Rabinowitz's global bifurcation theorem.

2. Green's function of (1.13)

In this section, we assume that

(H4) $\varepsilon \in (0, 1/2)$ and $M \in (0,\infty)$.

Let $m = \sqrt{M}$, then (H4) implies that $m > 0$.

Let ψ_1 be the unique solution of the initial value problem

$$\begin{cases} -\psi_1''(t) + M\psi_1(t) = 0, & t \in (0,1], \\ \psi_1(0) = 0, \quad \psi_1'(0) = 1. \end{cases} \tag{2.1}$$

Let φ_1 be the unique solution of the initial value problem

$$\begin{cases} -\varphi_1''(t) + M\varphi_1(t) = 0, & t \in [0,1), \\ \varphi_1(1) = 0, \quad \varphi_1'(1) = -1. \end{cases} \tag{2.2}$$

Then

$$\psi_1(t) = \frac{1}{2m}(\mathrm{e}^{mt} - \mathrm{e}^{-mt}), \quad t \in [0,1], \tag{2.3}$$

$$\varphi_1(t) = \frac{1}{2m}(\mathrm{e}^{m(1-t)} - \mathrm{e}^{-m(1-t)}), \quad t \in [0,1], \tag{2.4}$$

and it is easy to check that $\psi_1(t) > 0$ on $(0,1]$ and $\varphi_1(t) > 0$ on $[0,1)$. Moreover, ψ_1 is strictly increasing and convex on $[0,1]$, φ_1 is strictly decreasing and convex on $[0,1]$.

Lemma 2.1. *Let* (H4) *hold. Then for each* $h \in C[0,1]$, *the linear problem*

$$-u''(t) + Mu(t) = h(t), \quad t \in (0,1), \quad u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon) \tag{2.5}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 K_1(t,s)h(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{2.6}$$

where

$$\begin{aligned} K_1(t,s) = G_1(t,s) &+ G_1(\varepsilon,s)\frac{\varphi_1(1-\varepsilon)\psi_1(t) + \varphi_1(t)(\psi_1(1) - \psi_1(1-\varepsilon))}{\varrho_1} \\ &+ G_1(1-\varepsilon,s)\frac{(\varphi_1(0) - \varphi_1(\varepsilon))\psi_1(t) + \varphi_1(t)\psi_1(\varepsilon)}{\varrho_1}, \end{aligned}$$

$$G_1(t,s) = \frac{1}{\Delta_1}\begin{cases} \psi_1(s)\varphi_1(t), & 0 \leqslant s \leqslant t \leqslant 1, \\ \psi_1(t)\varphi_1(s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.7}$$

$$\Delta_1 = \psi_1'(t)\varphi_1(t) - \psi_1(t)\varphi_1'(t) = \varphi_1(0) > 0, \tag{2.8}$$

$$\varrho_1 = (\psi_1(1-\varepsilon) - \psi_1(1))(\varphi_1(\varepsilon) - \varphi_1(0)) - \psi_1(\varepsilon)\varphi_1(1-\varepsilon) > 0. \tag{2.9}$$

Moreover, if $h(t) \geqslant 0$ *and* $h(t) \not\equiv 0$ *on* $[0,1]$, *then the function* u *defined by* (2.6) *satisfies* $u(t) > 0$, $t \in [0,1]$.

P r o o f. Firstly, we show that the unique solution of (2.5) can be represented by (2.6). In fact, it follows from (2.3), (2.4) and (2.8) that the equation

$$-u''(t) + Mu(t) = 0, \quad t \in [0,1]$$

has two independent solutions ψ_1 and φ_1. Now, by the method of variation of constants, we may assume that

$$u(t) = C_1(t)\psi_1(t) + C_2(t)\varphi_1(t), \quad t \in [0,1], \tag{2.10}$$

and therefore

$$\begin{cases} \psi_1(t)C_1'(t) + \varphi_1(t)C_2'(t) = 0, \\ \psi_1'(t)C_1'(t) + \varphi_1'(t)C_2'(t) = -h(t). \end{cases} \tag{2.11}$$

It follows from (2.11) that

$$C_1'(t) = \frac{-h(t)\varphi_1(t)}{\psi_1'(t)\varphi_1(t) - \psi_1(t)\varphi_1'(t)}, \quad C_2'(t) = \frac{h(t)\psi_1(t)}{\psi_1'(t)\varphi_1(t) - \psi_1(t)\varphi_1'(t)},$$

thus,

$$C_1(t) = C_1(1) + \int_t^1 \frac{\varphi_1(s)}{\Delta_1} h(s)\,\mathrm{d}s, \quad C_2(t) = C_2(0) + \int_0^t \frac{\psi_1(s)}{\Delta_1} h(s)\,\mathrm{d}s, \tag{2.12}$$

where Δ_1 is defined as in (2.8).

Now, it follows from (2.10) and (2.12) that

$$u(t) = C_1(1)\psi_1(t) + \int_t^1 \frac{\psi_1(t)\varphi_1(s)}{\Delta_1} h(s)\,\mathrm{d}s + C_2(0)\varphi_1(t) + \int_0^t \frac{\psi_1(s)\varphi_1(t)}{\Delta_1} h(s)\,\mathrm{d}s.$$

Let $A_1 = C_1(1)$, $B_1 = C_2(0)$, then

$$u(t) = \int_0^1 G_1(t,s)h(s)\,\mathrm{d}s + A_1\psi_1(t) + B_1\varphi_1(t), \quad t \in [0,1], \tag{2.13}$$

where G_1 is defined as in (2.7). This together with (2.3) and (2.4) implies that

$$\begin{cases} u(0) = \int_0^1 G_1(0,s)h(s)\,\mathrm{d}s + A_1\psi_1(0) + B_1\varphi_1(0) = B_1\varphi_1(0), \\ u(\varepsilon) = \int_0^1 G_1(\varepsilon,s)h(s)\,\mathrm{d}s + A_1\psi_1(\varepsilon) + B_1\varphi_1(\varepsilon) \end{cases} \tag{2.14}$$

and

$$\begin{cases} u(1) = \int_0^1 G_1(1,s)h(s)\,\mathrm{d}s + A_1\psi_1(1) + B_1\varphi_1(1) = A_1\psi_1(1), \\ u(1-\varepsilon) = \int_0^1 G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s + A_1\psi_1(1-\varepsilon) + B_1\varphi_1(1-\varepsilon). \end{cases} \tag{2.15}$$

Combining (2.14), (2.15) with (1.5), it concludes that

$$\begin{cases} A_1\psi_1(\varepsilon) + B_1(\varphi_1(\varepsilon) - \varphi_1(0)) = -\int_0^1 G_1(\varepsilon,s)h(s)\,\mathrm{d}s, \\ A_1(\psi_1(1-\varepsilon) - \psi_1(1)) + B_1\varphi_1(1-\varepsilon) = -\int_0^1 G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s. \end{cases} \tag{2.16}$$

Let $C_1 := -\int_0^1 G_1(\varepsilon,s)h(s)\,\mathrm{d}s$, $C_2 := -\int_0^1 G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s$. Then (2.16) yields

$$A_1 = \frac{C_1\varphi_1(1-\varepsilon) - C_2(\varphi_1(\varepsilon) - \varphi_1(0))}{\psi_1(\varepsilon)\varphi_1(1-\varepsilon) - (\psi_1(1-\varepsilon) - \psi_1(1))(\varphi_1(\varepsilon) - \varphi_1(0))}, \tag{2.17}$$

$$B_1 = \frac{C_2\psi_1(\varepsilon) - C_1(\psi_1(1-\varepsilon) - \psi_1(1))}{\psi_1(\varepsilon)\varphi_1(1-\varepsilon) - (\psi_1(1-\varepsilon) - \psi_1(1))(\varphi_1(\varepsilon) - \varphi_1(0))}. \tag{2.18}$$

Thus, the function u can be rewritten as

$$\begin{aligned} u(t) = {} & \int_0^1 G_1(t,s)h(s)\,\mathrm{d}s + \frac{C_1\varphi_1(1-\varepsilon) - C_2(\varphi_1(\varepsilon) - \varphi_1(0))}{-\varrho_1}\psi_1(t) \\ & + \frac{C_2\psi_1(\varepsilon) - C_1(\psi_1(1-\varepsilon) - \psi_1(1))}{-\varrho_1}\varphi_1(t), \end{aligned}$$

where

$$\varrho_1 = (\psi_1(1-\varepsilon) - \psi_1(1))(\varphi_1(\varepsilon) - \varphi_1(0)) - \psi_1(\varepsilon)\varphi_1(1-\varepsilon)$$

is defined as in (2.9).

We claim that $\varrho_1 > 0$. In fact, it follows from (2.3) and (2.4) that

$$\varrho_1 = \left(\frac{1}{2m}\right)^2 [(\mathrm{e}^m - \mathrm{e}^{-m}) - (\mathrm{e}^{m(1-\varepsilon)} - \mathrm{e}^{-m(1-\varepsilon)})]^2 - \left(\frac{1}{2m}\right)^2 (\mathrm{e}^{m\varepsilon} - \mathrm{e}^{-m\varepsilon})^2. \tag{2.19}$$

The properties of the function ψ_1 imply that

$$\mathrm{e}^m - \mathrm{e}^{-m} - \mathrm{e}^{m(1-\varepsilon)} + \mathrm{e}^{-m(1-\varepsilon)} + \mathrm{e}^{m\varepsilon} - \mathrm{e}^{-m\varepsilon} > 0,$$

and so we need only to prove that

$$\mathrm{e}^m - \mathrm{e}^{-m} - \mathrm{e}^{m(1-\varepsilon)} + \mathrm{e}^{-m(1-\varepsilon)} - \mathrm{e}^{m\varepsilon} + \mathrm{e}^{-m\varepsilon} > 0.$$

Let $f(t) = \mathrm{e}^{mt} - \mathrm{e}^{-mt}$. Then

$$f'(t) = m(\mathrm{e}^{mt} + \mathrm{e}^{-mt}) > 0 \quad \text{on } [0,1].$$

Moreover, it follows from the facts

$$f(0) = 0, \quad f'(t) > 0, \quad t \in [0,1]$$

that $f(t) > 0$ for $t \in (0,1]$, and therefore

$$f''(t) = m^2(\mathrm{e}^{mt} - \mathrm{e}^{-mt}) = m^2 f(t) > 0 \quad t \in (0,1].$$

Consequently, f is strictly increasing and convex on $[0,1]$. Hence

$$f(1) - f(1-\varepsilon) > f(\varepsilon) - f(0),$$

i.e.

$$\mathrm{e}^m - \mathrm{e}^{-m} - \mathrm{e}^{m(1-\varepsilon)} + \mathrm{e}^{-m(1-\varepsilon)} - \mathrm{e}^{m\varepsilon} + \mathrm{e}^{-m\varepsilon} > 0.$$

Thus, it follows from (2.19) that $\varrho_1 > 0$.

By simple computations, we get

$$\begin{aligned} u(t) = &\int_0^1 G_1(t,s)h(s)\,\mathrm{d}s + \int_0^1 \frac{\varphi_1(1-\varepsilon)\psi_1(t)}{\varrho_1} G_1(\varepsilon,s)h(s)\,\mathrm{d}s \\ &+ \int_0^1 \frac{(\varphi_1(0) - \varphi_1(\varepsilon))\psi_1(t)}{\varrho_1} G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s \\ &+ \int_0^1 \frac{\varphi_1(t)(\psi_1(1) - \psi_1(1-\varepsilon))}{\varrho_1} G_1(\varepsilon,s)h(s)\,\mathrm{d}s \\ &+ \int_0^1 \frac{\varphi_1(t)\psi_1(\varepsilon)}{\varrho_1} G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s. \end{aligned} \tag{2.20}$$

It follows from (2.20) that (2.5) holds with $K_1(t,s)$ given by

$$\begin{aligned} K_1(t,s) = G_1(t,s) + G_1(\varepsilon,s)\frac{\varphi_1(1-\varepsilon)\psi_1(t) + \varphi_1(t)(\psi_1(1)-\psi_1(1-\varepsilon))}{\varrho_1} \\ + G_1(1-\varepsilon,s)\frac{(\varphi_1(0)-\varphi_1(\varepsilon))\psi_1(t) + \varphi_1(t)\psi_1(\varepsilon)}{\varrho_1}. \end{aligned}$$

Next, we check that the function defined by (2.6) is a solution of (2.5). From (2.20) we know that

$$\begin{aligned} u'(t) &= \frac{1}{\Delta_1}\varphi_1'(t)\int_0^t \psi_1(s)h(s)\,\mathrm{d}s + \frac{1}{\Delta_1}\psi_1'(t)\int_t^1 \varphi_1(s)h(s)\,\mathrm{d}s \\ &\quad + \psi_1'(t)\int_0^1 \frac{\varphi_1(1-\varepsilon)}{\varrho_1}G_1(\varepsilon,s)h(s)\,\mathrm{d}s \\ &\quad + \psi_1'(t)\int_0^1 \frac{(\varphi_1(0)-\varphi_1(\varepsilon))}{\varrho_1}G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s \\ &\quad + \varphi_1'(t)\int_0^1 \frac{\psi_1(1)-\psi_1(1-\varepsilon)}{\varrho_1}G_1(\varepsilon,s)h(s)\,\mathrm{d}s \\ &\quad + \varphi_1'(t)\int_0^1 \frac{\psi_1(\varepsilon)}{\varrho_1}G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s, \\ u''(t) &= \frac{1}{\Delta_1}\varphi_1''(t)\int_0^t \psi_1(s)h(s)\,\mathrm{d}s + \frac{1}{\Delta_1}\psi_1''(t)\int_t^1 \varphi_1(s)h(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\Delta_1}\varphi_1'(t)\psi_1(t)h(t) - \frac{1}{\Delta_1}\varphi_1(t)\psi_1'(t)h(t) \\ &\quad + \psi_1''(t)\int_0^1 \frac{\varphi_1(1-\varepsilon)}{\varrho_1}G_1(\varepsilon,s)h(s)\,\mathrm{d}s \\ &\quad + \psi_1''(t)\int_0^1 \frac{(\varphi_1(0)-\varphi_1(\varepsilon))}{\varrho_1}G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s \\ &\quad + \varphi_1''(t)\int_0^1 \frac{\psi_1(1)-\psi_1(1-\varepsilon)}{\varrho_1}G_1(\varepsilon,s)h(s)\,\mathrm{d}s \\ &\quad + \varphi_1''(t)\int_0^1 \frac{\psi_1(\varepsilon)}{\varrho_1}G_1(1-\varepsilon,s)h(s)\,\mathrm{d}s. \end{aligned} \tag{2.21}$$

Hence (2.20) together with (2.21) imply that

$$u'' - Mu = \frac{1}{\Delta_1}\varphi_1'(t)\psi_1(t)h(t) - \frac{1}{\Delta_1}\varphi_1(t)\psi_1'(t)h(t) = \frac{1}{\Delta_1}(-\Delta_1)h(t) = -h(t),$$

i.e.

$$-u''(t) + Mu(t) = h(t), \quad t \in [0,1].$$

Finally, we will show that the boundary conditions (1.5) can also be satisfied. It follows from (2.20) that

$$\begin{aligned}u(0)&=\frac{C_1(\psi_1(1-\varepsilon)-\psi_1(1))-C_2\psi_1(\varepsilon)}{\varrho_1}\,\varphi_1(0),\\ u(\varepsilon)&=\int_0^1 G_1(\varepsilon,s)h(s)\,\mathrm{d}s+\frac{C_2(\varphi_1(\varepsilon)-\varphi_1(0))-C_1\varphi_1(1-\varepsilon)}{\varrho_1}\,\psi_1(\varepsilon)\\ &\quad+\frac{C_1(\psi_1(1-\varepsilon)-\psi_1(1))-C_2\psi_1(\varepsilon)}{\varrho_1}\,\varphi_1(\varepsilon).\end{aligned}$$

By routine computations, we can show that $u(\varepsilon)-u(0)=0$, i.e.

$$u(\varepsilon)=u(0).$$

Similarly, we can prove that $u(1-\varepsilon)=u(1)$. $\square$

Lemma 2.2. *Let* (H4) *hold. Then the Green function K_1 defined as in Lemma* 2.1 *satisfies*

(i) $K_1\colon\ [0,1]\times[0,1]\to[0,+\infty)$ *is continuous;*

(ii) $K_1(t,s)>0$ *for each* $(t,s)\in(0,1)\times(0,1)$.

Proof. It follows from (2.7)–(2.9) that (i) and (ii) hold. $\square$

Define a cone

$$P:=\{u\in C[0,1]\colon\ u(t)\geqslant 0,\ t\in[0,1]\},\tag{2.22}$$

and a linear operator $L_1\colon\ C[0,1]\to C[0,1]$,

$$(L_1u)(t):=\int_0^1 K_1(t,s)g(s)u(s)\,\mathrm{d}s,\quad t\in[0,1].\tag{2.23}$$

Lemma 2.3. *Let* (H1) *and* (H4) *hold. Then*

(a) $L_1\colon\ C[0,1]\to C[0,1]$ *is a completely continuous linear operator;*

(b) $L_1(P\setminus\theta)\subset\operatorname{int}P$;

(c) $r(L_1)\neq 0$, *and there exists* $e_1\in C[0,1]$ *with* $e_1>0$ *on* $[0,1]$ *such that* $L_1e_1=r(L_1)e_1$. *Moreover,* $\lambda_1:=(r(L_1))^{-1}$ *is the principal eigenvalue of* (1.6).

Proof. By the standard argument, we can prove (a).

To prove (c), by the well-known Krein-Rutman Theorem [2, Theorem 19.2 and 19.3], it is enough to show that (b) is true.

For $y \in (P \setminus \theta)$, there exist $t_0 \in (0,1)$ and $\delta \in (0, \min\{t_0, 1-t_0\})$ such that

$$y(t) > 0, \quad t \in (t_0-\delta, t_0+\delta).$$

Thus, it follows from Lemma 2.2 that

$$(L_1 y)(t) = \int_0^1 K_1(t,s) r g(s) y(s)\,\mathrm{d}s \geqslant \int_{t_0-\delta}^{t_0+\delta} K_1(t,s) r g(s) y(s)\,\mathrm{d}s > 0, \quad t \in [0,1],$$

which implies that $L_1 y \in \operatorname{int} P$. □

3. Green's function of (1.14)

In this section, we assume that
(H5) $\varepsilon \in (0, 1/2)$ and $M \in (0, \pi^2/4]$.
Let $m = \sqrt{M}$, then (H5) implies that $m > 0$.
Let ψ_2 be the unique solution of the initial value problem

$$\begin{cases} \psi_2''(t) + M\psi_2(t) = 0, & t \in (0,1], \\ \psi_2(0) = 0, \quad \psi_2'(0) = 1. \end{cases} \tag{3.1}$$

Let φ_2 be the unique solution of the initial value problem

$$\begin{cases} \varphi_2''(t) + M\varphi_2(t) = 0, & t \in [0,1), \\ \varphi_2(1) = 0, \quad \varphi_2'(1) = -1. \end{cases} \tag{3.2}$$

Then it is easy to check that

$$\psi_2(t) = \frac{1}{m}\sin mt, \quad \varphi_2(t) = \frac{1}{m}\sin m(1-t), \quad t \in [0,1], \tag{3.3}$$

and $\psi_2(t) > 0$ on $(0,1]$, $\varphi_2(t) > 0$ on $[0,1)$. Moreover, ψ_2 is strictly increasing and concave on $[0,1]$, φ_2 is strictly decreasing and concave on $[0,1]$.

Lemma 3.1. *Let* (H5) *hold. Then for each* $h \in C[0,1]$, *the linear problem*

$$u''(t) + Mu(t) = h(t), \quad t \in (0,1), \quad u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon) \tag{3.4}$$

is equivalent to the integral equation

$$u(t) = \int_0^1 K_2(t,s) h(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{3.5}$$

where

$$K_2(t,s)= -G_2(t,s)+G_2(\varepsilon,s)\frac{\varphi_2(1-\varepsilon)\psi_2(t)+\varphi_2(t)(\psi_2(1)-\psi_2(1-\varepsilon))}{\varrho_2}$$
$$+G_2(1-\varepsilon,s)\frac{(\varphi_2(0)-\varphi_2(\varepsilon))\psi_2(t)+\varphi_2(t)\psi_2(\varepsilon)}{\varrho_2},$$

$$G_2(t,s)=\frac{1}{\Delta_2}\begin{cases}\psi_2(s)\varphi_2(t), & 0\leqslant s\leqslant t\leqslant 1,\\ \psi_2(t)\varphi_2(s), & 0\leqslant t\leqslant s\leqslant 1,\end{cases} \tag{3.6}$$

$$\Delta_2 := \psi_2'(t)\varphi_2(t)-\psi_2(t)\varphi_2'(t)=\varphi_2(0)=\frac{1}{m}\sin m>0, \tag{3.7}$$

$$\varrho_2 := \psi_2(\varepsilon)\varphi_2(1-\varepsilon)-(\psi_2(1-\varepsilon)-\psi_2(1))(\varphi_2(\varepsilon)-\varphi_2(0))>0. \tag{3.8}$$

Moreover, if $h(t)\geqslant 0$ and $h(t)\not\equiv 0$ on $[0,1]$, then the function u defined by (3.5) *satisfies $u(t)>0$, $t\in[0,1]$.*

P r o o f. Firstly, we show that the unique solution of (3.4) can be represented by (3.5). In fact, it follows from (3.3) and (3.7) that the equation

$$u''(t)+Mu(t)=0,\quad t\in[0,1]$$

has two independent solutions ψ_2 and φ_2. Now, by the method of variation of constants, we may assume that

$$u(t)=D_1(t)\psi_2(t)+D_2(t)\varphi_2(t),\quad t\in[0,1], \tag{3.9}$$

and therefore

$$\begin{cases}\psi_2(t)D_1'(t)+\varphi_2(t)D_2'(t)=0,\\ \psi_2'(t)D_1'(t)+\varphi_2'(t)D_2'(t)=h(t).\end{cases} \tag{3.10}$$

Then (3.10) yields

$$D_1'(t)=\frac{h(t)\varphi_2(t)}{\psi_2'(t)\varphi_2(t)-\psi_2(t)\varphi_2'(t)},\quad D_2'(t)=\frac{-h(t)\psi_2(t)}{\psi_2'(t)\varphi_2(t)-\psi_2(t)\varphi_2'(t)},$$

thus,

$$D_1(t)=D_1(1)-\int_t^1\frac{\varphi_2(s)}{\Delta_2}h(s)\,\mathrm{d}s,\quad D_2(t)=D_2(0)-\int_0^t\frac{\psi_2(s)}{\Delta_2}h(s)\,\mathrm{d}s, \tag{3.11}$$

where Δ_2 is defined as in (3.7).

Now (3.9) together with (3.11) imply that

$$u(t) = D_1(1)\psi_2(t) - \int_t^1 \frac{\psi_2(t)\varphi_2(s)}{\Delta_2} h(s)\,\mathrm{d}s + D_2(0)\varphi_2(t) - \int_0^t \frac{\psi_2(s)\varphi_2(t)}{\Delta_2} h(s)\,\mathrm{d}s.$$

Let $A_2 = D_1(1)$, $B_2 = D_2(0)$, then

$$u(t) = -\int_0^1 G_2(t,s)h(s)\,\mathrm{d}s + A_2\psi_2(t) + B_2\varphi_2(t), \tag{3.12}$$

where G_2 is defined as in (3.6). It follows from (3.3) and (3.12) that

$$\begin{cases} u(0) = -\int_0^1 G_2(0,s)h(s)\,\mathrm{d}s + A_2\psi_2(0) + B_2\varphi_2(0) = B_2\varphi_2(0), \\ u(\varepsilon) = -\int_0^1 G_2(\varepsilon,s)h(s)\,\mathrm{d}s + A_2\psi_2(\varepsilon) + B_2\varphi_2(\varepsilon) \end{cases} \tag{3.13}$$

and

$$\begin{cases} u(1) = -\int_0^1 G_2(1,s)h(s)\,\mathrm{d}s + A_2\psi_2(1) + B_2\varphi_2(1) = A_2\psi_2(1), \\ u(1-\varepsilon) = -\int_0^1 G_2(1-\varepsilon,s)h(s)\,\mathrm{d}s + A_2\psi_2(1-\varepsilon) + B_2\varphi_2(1-\varepsilon). \end{cases} \tag{3.14}$$

Then (3.13), (3.14) together with (1.5) imply that

$$\begin{cases} A_2\psi_2(\varepsilon) + B_2(\varphi_2(\varepsilon) - \varphi_2(0)) = \int_0^1 G_2(\varepsilon,s)h(s)\,\mathrm{d}s, \\ A_2(\psi_2(1-\varepsilon) - \psi_2(1)) + B_2\varphi_2(1-\varepsilon) = \int_0^1 G_2(1-\varepsilon,s)h(s)\,\mathrm{d}s. \end{cases} \tag{3.15}$$

Let

$$D_1 := \int_0^1 G_2(\varepsilon,s)h(s)\,\mathrm{d}s, \quad D_2 := \int_0^1 G_2(1-\varepsilon,s)h(s)\,\mathrm{d}s.$$

Then we know from (3.15) that

$$A_2 = \frac{D_1\varphi_2(1-\varepsilon) - D_2(\varphi_2(\varepsilon) - \varphi_2(0))}{\psi_2(\varepsilon)\varphi_2(1-\varepsilon) - (\psi_2(1-\varepsilon) - \psi_2(1))(\varphi_2(\varepsilon) - \varphi_2(0))}, \tag{3.16}$$

$$B_2 = \frac{D_2\psi_2(\varepsilon) - D_1(\psi_2(1-\varepsilon) - \psi_2(1))}{\psi_2(\varepsilon)\varphi_2(1-\varepsilon) - (\psi_2(1-\varepsilon) - \psi_2(1))(\varphi_2(\varepsilon) - \varphi_2(0))}. \tag{3.17}$$

Therefore, u can be rewritten as

$$\begin{aligned} u(t) = & -\int_0^1 G_2(t,s)h(s)\,\mathrm{d}s + \frac{D_1\varphi_2(1-\varepsilon) - D_2(\varphi_2(\varepsilon) - \varphi_2(0))}{\varrho_2}\psi_2(t) \\ & + \frac{D_2\psi_2(\varepsilon) - D_1(\psi_2(1-\varepsilon) - \psi_2(1))}{\varrho_2}\varphi_2(t), \end{aligned}$$

where

$$\varrho_2 = \psi_2(\varepsilon)\varphi_2(1-\varepsilon) - (\psi_2(1-\varepsilon) - \psi_2(1))(\varphi_2(\varepsilon) - \varphi_2(0))$$

is defined as in (3.8).

We claim that $\varrho_2 > 0$. It follows from (3.3) that

$$\varrho_2 = \Big(\frac{1}{m}\Big)^2 [\sin m\varepsilon + \sin m(1-\varepsilon) - \sin m][\sin m\varepsilon - \sin m(1-\varepsilon) + \sin m]. \tag{3.18}$$

It is easy to check that

$$\sin m\varepsilon - \sin m(1-\varepsilon) + \sin m > 0,$$

and so we need only to prove that

$$\sin m\varepsilon + \sin m(1-\varepsilon) - \sin m > 0.$$

Let $g(t) = \sin mt$. Then

$$g'(t) = m\cos mt > 0 \quad \text{on } [0,1).$$

Moreover, it follows from the facts

$$g(0) = 0, \quad g'(t) > 0, \quad t \in [0,1)$$

that $g(t) > 0$ for $t \in (0,1]$. On the other hand,

$$g''(t) = -m^2 \sin mt = -m^2 g(t) < 0 \quad \text{on } (0,1].$$

Hence, g is strictly increasing and concave on $[0,1]$, which implies that

$$\sin m\varepsilon + \sin m(1-\varepsilon) - \sin m = g(\varepsilon) + g(1-\varepsilon) - g(1) = g(\varepsilon) - (g(1) - g(1-\varepsilon)) > 0.$$

Consequently, $\varrho_2 > 0$.

In a similar manner as in the proof of Lemma 2.1, we can show that

$$\begin{aligned} K_2(t,s) = & -G_2(t,s) + G_2(\varepsilon,s)\frac{\varphi_2(1-\varepsilon)\psi_2(t) + \varphi_2(t)(\psi_2(1) - \psi_2(1-\varepsilon))}{\varrho_2} \\ & + G_2(1-\varepsilon,s)\frac{(\varphi_2(0) - \varphi_2(\varepsilon))\psi_2(t) + \varphi_2(t)\psi_2(\varepsilon)}{\varrho_2}, \end{aligned}$$

and the function u defined by (3.5) is a solution of (3.4). □

Lemma 3.2. *Let* (H5) *hold. Then the Green function* K_2 *defined as in Lemma* 3.1 *satisfies*

(i) $K_2\colon [0,1]\times[0,1]\to[0,+\infty)$ *is continuous;*

(ii) $K_2(t,s)>0$ *for each* $(t,s)\in(0,1)\times(0,1)$.

P r o o f. Obviously, K_2 is continuous on $[0,1]\times[0,1]$.

Now, we divide the proof of this Lemma into three cases.

Case 1. $\varepsilon<s<1-\varepsilon$. If $t\leqslant s$, then

$$\begin{aligned}
K_2(t,s) &= \frac{1}{\Delta_2}\frac{\varphi_2(1-\varepsilon)\psi_2(t)+(\psi_2(1)-\psi_2(1-\varepsilon))\varphi_2(t)}{\varrho_2}\cdot\psi_2(\varepsilon)\varphi_2(s)\\
&\quad+\frac{1}{\Delta_2}\frac{(\varphi_2(0)-\varphi_2(\varepsilon))\psi_2(t)+\psi_2(\varepsilon)\varphi_2(t)}{\varrho_2}\cdot\psi_2(s)\varphi_2(1-\varepsilon)\\
&\quad-\frac{1}{\Delta_2}\psi_2(t)\varphi_2(s)\\
&=\frac{1}{\varrho_2\Delta_2}[\varphi_2(1-\varepsilon)\psi_2(t)+(\psi_2(1)-\psi_2(1-\varepsilon))\varphi_2(t)]\cdot\psi_2(\varepsilon)\varphi_2(s)\\
&\quad+\frac{1}{\varrho_2\Delta_2}[(\varphi_2(0)-\varphi_2(\varepsilon))\psi_2(t)+\psi_2(\varepsilon)\varphi_2(t)]\cdot\psi_2(s)\varphi_2(1-\varepsilon)\\
&\quad-\frac{1}{\varrho_2\Delta_2}[\psi_2(\varepsilon)\varphi_2(1-\varepsilon)\psi_2(t)\varphi_2(s)\\
&\quad-(\varphi_2(\varepsilon)-\varphi_2(0))(\psi_2(1-\varepsilon)-\psi_2(1))\psi_2(t)\varphi_2(s)]\\
&\quad>0.
\end{aligned}$$

If $s\leqslant t$, then

$$\begin{aligned}
K_2(t,s) &= \frac{1}{\Delta_2}\frac{\varphi_2(1-\varepsilon)\psi_2(t)+(\psi_2(1)-\psi_2(1-\varepsilon))\varphi_2(t)}{\varrho_2}\cdot\psi_2(\varepsilon)\varphi_2(s)\\
&\quad+\frac{1}{\Delta_2}\frac{(\varphi_2(0)-\varphi_2(\varepsilon))\psi_2(t)+\psi_2(\varepsilon)\varphi_2(t)}{\varrho_2}\cdot\psi_2(s)\varphi_2(1-\varepsilon)\\
&\quad-\frac{1}{\Delta_2}\psi_2(s)\varphi_2(t)\\
&=\frac{1}{\varrho_2\Delta_2}[\varphi_2(1-\varepsilon)\psi_2(t)+(\psi_2(1)-\psi_2(1-\varepsilon))\varphi_2(t)]\cdot\psi_2(\varepsilon)\varphi_2(s)\\
&\quad+\frac{1}{\varrho_2\Delta_2}[(\varphi_2(0)-\varphi_2(\varepsilon))\psi_2(t)+\psi_2(\varepsilon)\varphi_2(t)]\cdot\psi_2(s)\varphi_2(1-\varepsilon)\\
&\quad-\frac{1}{\varrho_2\Delta_2}[\psi_2(\varepsilon)\varphi_2(1-\varepsilon)\psi_2(s)\varphi_2(t)\\
&\quad-(\varphi_2(\varepsilon)-\varphi_2(0))(\psi_2(1-\varepsilon)-\psi_2(1))\psi_2(s)\varphi_2(t)]\\
&\quad>0.
\end{aligned}$$

Case 2. $0 \leqslant s \leqslant \varepsilon$. If $t \leqslant s$, then

$$\begin{aligned}
K_2(t,s) = {} & \frac{1}{\varrho_2\Delta_2}[\varphi_2(1-\varepsilon)\psi_2(t) + (\psi_2(1)-\psi_2(1-\varepsilon))\varphi_2(t)]\cdot\psi_2(s)\varphi_2(\varepsilon) \\
& + \frac{1}{\varrho_2\Delta_2}[(\varphi_2(0)-\varphi_2(\varepsilon))\psi_2(t) + \psi_2(\varepsilon)\varphi_2(t)]\cdot\psi_2(s)\varphi_2(1-\varepsilon) \\
& - \frac{1}{\varrho_2\Delta_2}[\psi_2(\varepsilon)\varphi_2(1-\varepsilon)\psi_2(t)\varphi_2(s) \\
& - (\varphi_2(\varepsilon)-\varphi_2(0))(\psi_2(1-\varepsilon)-\psi_2(1))\psi_2(t)\varphi_2(s)],
\end{aligned}$$

this together with the facts

$$\varphi_2(t) \geqslant \varphi_2(s), \quad \psi_2(t) \leqslant \psi_2(s), \quad t \leqslant s$$

implies that

$$K_2(t,s) > 0 \quad \text{for } s \neq 0,\ t \in [0,1].$$

If $s \leqslant t$, then

$$\begin{aligned}
K_2(t,s) = {} & \frac{1}{\varrho_2\Delta_2}[\varphi_2(1-\varepsilon)\psi_2(t) + (\psi_2(1)-\psi_2(1-\varepsilon))\varphi_2(t)]\cdot\psi_2(s)\varphi_2(\varepsilon) \\
& + \frac{1}{\varrho_2\Delta_2}[(\varphi_2(0)-\varphi_2(\varepsilon))\psi_2(t) + \psi_2(\varepsilon)\varphi_2(t)]\cdot\psi_2(s)\varphi_2(1-\varepsilon) \\
& - \frac{1}{\varrho_2\Delta_2}[\psi_2(\varepsilon)\varphi_2(1-\varepsilon)\psi_2(s)\varphi_2(t) \\
& - (\varphi_2(\varepsilon)-\varphi_2(0))(\psi_2(1-\varepsilon)-\psi_2(1))\psi_2(s)\varphi_2(t)] \\
& > 0
\end{aligned}$$

for $s \neq 0$, $t \in [0,1]$.

Case 3. $1-\varepsilon \leqslant s \leqslant 1$. By similar arguments as in Case 1 and 2, we can show that

$$K_2(t,s) > 0 \quad \text{for } s \neq 1,\ t \in [0,1].$$

Consequently, $K_2(t,s) > 0$ on $(0,1)\times(0,1)$. □

Define a linear operator $L_2\colon C[0,1] \to C[0,1]$,

$$(L_2u)(t) := \int_0^1 K_2(t,s)g(s)u(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{3.19}$$

Using a similar method as in the proof of Lemma 2.3, we can prove the following

Lemma 3.3. *Let* (H5) *and* (H1) *hold. Then*

(a) $L_2\colon C[0,1] \to C[0,1]$ *is a completely continuous linear operator;*

(b) $L_2(P \setminus \theta) \subset \operatorname{int} P$*;*

(c) $r(L_2) \neq 0$, *and there exists* $e_2 \in C[0,1]$ *with* $e_2 > 0$ *on* $[0,1]$ *such that* $L_2 e_2 = r(L_2) e_2$. *Moreover,* $\tau_1 := (r(L_2))^{-1}$ *is the principal eigenvalue of* (1.9), *where* P *is defined as in* (2.22).

4. Proof of the main results

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

Let

$$E = \{u \in C^1[0,1]\colon\ u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon)\}$$

with the norm

$$\|u\| = \|u\|_\infty + \|u'\|_\infty.$$

Define an operator $L\colon\ D(L) \to Y$ by setting

$$Lu := -u'' + Mu, \quad u \in D(L),$$

where

$$D(L) = \{u \in C^2[0,1]\colon\ u(0) = u(\varepsilon),\ u(1) = u(1-\varepsilon)\}.$$

Then L is a closed operator with $L^{-1}\colon\ Y \to E$ compact.

(H3) implies that there exist two functions $\zeta, \xi \in C(\mathbb{R})$ such that

$$f(u) = f_0 u + \zeta(u), \qquad f(u) = f_\infty u + \xi(u)$$

and

$$\lim_{|u|\to 0} \frac{\zeta(u)}{u} = 0, \qquad \lim_{|u|\to\infty} \frac{\xi(u)}{u} = 0.$$

Let

$$\bar\xi(u) = \max_{0\leqslant|s|\leqslant u} |\xi(s)|,$$

then $\bar\xi$ is nondecreasing and

$$\lim_{u\to\infty} \frac{\bar\xi(u)}{u} = 0.$$

P r o o f of Theorem 1.1. The proof is similar to the proof of [6, Theorem 1.1]. We state it here for readers' convenience.

Let us consider

$$Lu - \lambda g(t) r f_0 u = \lambda g(t) r \zeta(u) \tag{4.1}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

The equation (4.1) can be converted to the equivalent equation

$$\begin{aligned} u(t) &= \int_0^1 K_1(t,s)[\lambda g(s) r f_0 u(s) + \lambda g(s) r \zeta(u(s))]\,\mathrm{d}s \\ &=: (\lambda L^{-1}[g(\cdot) r f_0 u(\cdot)] + \lambda L^{-1}[g(\cdot) r \zeta(u(\cdot))])(t). \end{aligned} \tag{4.2}$$

Furthermore, note that $\|L^{-1}[g(\cdot)\zeta(u(\cdot))]\| = o(\|u\|)$ for u near 0 in E, since

$$\begin{aligned} &\|L^{-1}[g(\cdot)\zeta(u(\cdot))]\| \\ &\quad = \max_{t\in[0,1]} \left|\int_0^1 K_1(t,s) g(s)\zeta(u(s))\,\mathrm{d}s\right| + \max_{t\in[0,1]} \left|\int_0^1 \frac{\partial K_1(t,s)}{\partial t} g(s)\zeta(u(s))\,\mathrm{d}s\right| \\ &\quad \leqslant C_0 \cdot \max_{s\in[0,1]} |g(s)| \cdot \|\zeta(u(\cdot))\|_\infty. \end{aligned}$$

In what follows, we use the terminology of Rabinowitz [8]. Let S_1^+ denote the set of functions in E which are positive on $[0,1]$ and let $S_1^- = -S_1^+$, $S_1 = S_1^- \cup S_1^+$. They are disjoint and open in E. Finally, let $\Phi_1^\pm = \mathbb{R} \times S_1^\pm$. The result of Rabinowitz [8] for the problem (4.1) can be stated as follows: For $v = \{+,-\}$, there exists a continuum $\mathscr{C}_1^v \subset \Phi_1^v$ of solutions of (4.1) joining $(\lambda_1/rf_0, 0)$ to infinity. Moreover, $\mathscr{C}_1^v \setminus \{(\lambda_1/rf_0, 0)\} \subset \Phi_1^v$.

It is clear that any solution of (4.1) of the form $(1,u)$ yields a solutions u of the four-point boundary value problem (1.3), (1.5). We will show that $\mathscr{C}_1^v$ crosses the hyperplane $\{1\}\times E$ in $\mathbb{R}\times E$. To do this, it is enough to show that $\mathscr{C}_1^v$ joins $(\lambda_1/rf_0, 0)$ to $(\lambda_1/rf_\infty, \infty)$. Let $(\mu_n, y_n) \in \mathscr{C}_1^v$ satisfy

$$\mu_n + \|y_n\| \to \infty.$$

Note that $\mu_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (4.1) for $\lambda = 0$ and $\mathscr{C}_1^v \cap (\{0\} \times E) = \emptyset$.

Case 1. $\lambda_1/f_\infty < r < \lambda_1/f_0$. In this case, we show that the interval

$$\Big(\frac{\lambda_1}{rf_\infty}, \frac{\lambda_1}{rf_0}\Big) \subseteq \{\lambda \in \mathbb{R} \colon \exists(\lambda,u) \in \mathscr{C}_1^v\}.$$

We divide the proof into two steps.

Step 1. We show that if there exists a constant $C > 0$ such that $\mu_n \subset (0, C]$, then $\mathscr{C}_1^v$ joins $(\lambda_1/rf_0, 0)$ to $(\lambda_1/rf_\infty, \infty)$.

In this case, it follows that $\|y_n\| \to \infty$. We divide the equation

$$Ly_n - \mu_n g(t) r f_\infty y_n = \mu_n g(t) r \xi(y_n) \tag{4.3}$$

by $\|y_n\|$ and set $\overline{y}_n = y_n/\|y_n\|$. Since $\overline{y}_n$ is bounded in $C^2[0,1]$, after taking a subsequence, if necessary, we have that $\overline{y}_n \to \overline{y}$ for some $\overline{y} \in E$ with $\|\overline{y}\| = 1$. Since $\bar{\xi}$ is nondecreasing, $\lim\limits_{u\to\infty} \bar{\xi}(u)/u = 0$ together with (H2) imply that

$$\lim_{n\to\infty} \frac{|\xi(y_n(t))|}{\|y_n\|} = 0,$$

since

$$\frac{|\xi(y_n(t))|}{\|y_n\|} \leqslant \frac{\bar{\xi}(|y_n(t)|)}{\|y_n\|} \leqslant \frac{\bar{\xi}(\|y_n\|_\infty)}{\|y_n\|} \leqslant \frac{\bar{\xi}(\|y_n\|)}{\|y_n\|}.$$

Thus,

$$\overline{y}(t) = \int_0^1 K_1(t,s) \bar{\mu} r g(s) f_\infty \overline{y}(s) \,\mathrm{d}s,$$

where $\bar{\mu} := \lim\limits_{n\to\infty} \mu_n$, again choosing a subsequence and relabelling, if necessary. Thus

$$L\overline{y} - \bar{\mu} g(t) r f_\infty \overline{y} = 0. \tag{4.4}$$

We claim that

$$\overline{y} \in \mathscr{C}_1^v.$$

Suppose on the contrary that $\overline{y} \notin \mathscr{C}_1^v$. Since $\overline{y} \neq 0$ is a solution of (4.4) and has no zeros, it follows that $\overline{y} \in \mathscr{C}_1^l$ for some $l \in \{+,-\}$.

By the openness of $E \setminus \mathscr{C}_1^v$, we know that there exists a neighborhood $U(\overline{y}, \varrho_0)$ such that

$$U(\overline{y}, \varrho_0) \subset E \setminus \mathscr{C}_1^v,$$

which contradicts the fact that $\overline{y}_n \to \overline{y}$ in E and $\overline{y}_n \in \mathscr{C}_1^v$. Moreover, we have from Lemma 2.3 that $\bar{\mu} r f_\infty = \lambda_1$. So

$$\bar{\mu} = \frac{\lambda_1}{rf_\infty}.$$

Therefore $\mathscr{C}_1^v$ joins $(\lambda_1/rf_0, 0)$ to $(\lambda_1/rf_\infty, \infty)$.

Step 2. We show that there exists a constant $C > 0$ such that $\mu_n \in (0, C]$ for all $n \in \mathbb{N}$.

Suppose on the contrary that there exists no C such that $\mu_n \in (0, C]$ for all $n \in \mathbb{N}$. Choosing a subsequence and relabeling, if necessary, it follows that

$$\lim_{n\to\infty} \mu_n = \infty.$$

Because (μ_n, y_n) satisfies

$$Ly_n = \mu_n g(t) r \,\frac{f(y_n)}{y_n}\, y_n, \quad 0 < t < 1,$$

from (H1) we know that there exists an interval $[\gamma, \delta] \subset [0, 1]$ such that $g(t) > 0$ for all $t \in [\gamma, \delta]$. So for $t \in [\gamma, \delta]$, $\lim_{n\to\infty} \mu_n = \infty$ yields

$$\mu_n g(t) r \,\frac{f(y_n)}{y_n} \to \infty, \quad n \to \infty.$$

By the Sturm comparison theorem, we get that y_n has at least one zero in (γ, δ) for n sufficiently large, and this contradicts the fact that y_n has no zeros in $(0, 1)$. Therefore

$$\mu_n \leqslant C$$

for some constant $C > 0$, independent of $n \in \mathbb{N}$.

Case 2. $\lambda_1/f_0 < r < \lambda_1/f_\infty$. In this case, if $(\mu_n, y_n) \in \mathscr{C}_1^v$ is such that

$$\lim_{n\to\infty} (\mu_n + \|y_n\|) = \infty$$

and

$$\lim_{n\to\infty} \mu_n = \infty,$$

then

$$\Big(\frac{\lambda_1}{rf_0}, \frac{\lambda_1}{rf_\infty}\Big) \subset \{\lambda \in (0, \infty)\colon\ (\lambda, u) \in \mathscr{C}_1^v\},$$

moreover,

$$(\{1\} \times E) \cap \mathscr{C}_1^v \neq \emptyset.$$

Assume that $\{\mu_n\}$ is bounded, applying a similar argument to that used in Step 2 of Case 1, after taking a subsequence and relabeling, if necessary, it follows that

$$(\mu_n, y_n) \to \Big(\frac{\lambda_1}{rf_\infty}, \infty\Big), \quad n \to \infty.$$

Again $\mathscr{C}_1^v$ joins $(\lambda_1/rf_0, 0)$ to $(\lambda_1/rf_\infty, \infty)$ and the result follows. $\square$

P r o o f of Theorem 1.2. Applying the similar methods as in the proof of Theorem 1.1, we can prove Theorem 1.2. □

References

[1] *J. Chu, Y. Sun, H. Chen*: Positive solutions of Neumann problems with singularities. J. Math. Anal. Appl. *337* (2008), 1267–1272.

[2] *K. Deimling*: Nonlinear Functional Analysis. Springer, Berlin, 1985.

[3] *D. Jiang, H. Liu*: Existence of positive solutions to second order Neumann boundary value problems. J. Math. Res. Expo. *20* (2000), 360–364.

[4] *X. Li, D. Jiang*: Optimal existence theory for single and multiple positive solutions to second order Neumann boundary value problems. Indian J. Pure Appl. Math. *35* (2004), 573–586.

[5] *Z. Li*: Positive solutions of singular second-order Neumann boundary value problem. Ann. Differ. Equations *21* (2005), 321–326.

[6] *R. Ma, B. Thompson*: Nodal solutions for nonlinear eigenvalue problems. Nonlinear Anal., Theory Methods Appl. *59* (2004), 707–718.

[7] *A. R. Miciano, R. Shivaji*: Multiple positive solutions for a class of semipositone Neumann two-point boundary value problems. J. Math. Anal. Appl. *178* (1993), 102–115.

[8] *P. H. Rabinowitz*: Some global results for nonlinear eigenvalue problems. J. Funct. Anal. *7* (1971), 487–513.

[9] *I. Rachůnková, S. Staněk, M. Tvrdý*: Solvability of Nonlinear Singular Problems for Ordinary Differential Equations. Hindawi Publishing Corporation, New York, 2008.

[10] *J. Sun, W. Li*: Multiple positive solutions to second-order Neumann boundary value problems. Appl. Math. Comput. *146* (2003), 187–194.

[11] *J. Sun, W. Li, S. Cheng*: Three positive solutions for second-order Neumann boundary value problems. Appl. Math. Lett. *17* (2004), 1079–1084.

[12] *Y. Sun, Y. J. Cho, D. O'Regan*: Positive solution for singular second order Neumann boundary value problems via a cone fixed point theorem. Appl. Math. Comput. *210* (2009), 80–86.

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