Applications of Mathematics

Seppo Heikkilä; Guoju Ye
Equations containing locally Henstock-Kurzweil integrable functions

Applications of Mathematics, Vol. 57 (2012), No. 6, 569–580

Persistent URL: `http://dml.cz/dmlcz/143003`

Terms of use:

© Institute of Mathematics AS CR, 2012

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

57 (2012) APPLICATIONS OF MATHEMATICS No. 6, 569–580

EQUATIONS CONTAINING LOCALLY HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS

Seppo Heikkilä, Oulu, Guoju Ye, Nanjing

(Received August 16, 2010)

Abstract. A fixed point theorem in ordered spaces and a recently proved monotone convergence theorem are applied to derive existence and comparison results for solutions of a functional integral equation of Volterra type and a functional impulsive Cauchy problem in an ordered Banach space. A novel feature is that equations contain locally Henstock-Kurzweil integrable functions.

Keywords: integrability, Henstock-Kurzweil, ordered Banach space, order cone, chain, fixed point, integral equation, Volterra, Cauchy problem, functional

MSC 2010: 28B15, 34A36, 45N05, 46B40, 47H07, 47H10,

1. Introduction

The Henstock-Kurzweil integral provides a tool for integrating highly oscillatory functions which occur in nonlinear analysis and in quantum theory. It is also easy to understand because its definition requires no measure theory. Moreover, all Bochner integrable (in real-valued case Lebesgue integrable) functions are Henstock-Kurzweil (shortly HK) integrable, but not conversely. For instance, HK integrability encloses improper integrals. The real-valued function f defined on $[0,1]$ by $f(0)=0$ and $f(t)=t^2\cos(1/t^2)$ is differentiable on $[0,1]$, and f' is HK integrable. But f' is not Lebesgue integrable on $[0,1]$. More generally, let t be called a singular point of the domain interval of a real-valued function being not Lebesgue integrable on any interval that contains t. Then (cf. [10]) there exist HK "integrable functions on an interval that admit a set of singular points with its measure as close as possible but not equal to that of the whole interval."

In this paper a fixed point theorem in the ordered normed space is applied to prove existence and comparison results for solutions of functional Volterra integral equations and mild solutions of impulsive functional Cauchy problems in a Banach space X ordered by a regular order cone. The X-valued functions in the equations considered are locally Henstock-Kurzweil integrable with respect to the independent variable, depend functionally on the unknown function, and may contain discontinuous nonlinearities.

2. Preliminaries

A closed subset X_+ of a normed space X is called an *order cone* if $X_+ + X_+ \subseteq X_+$, $X_+ \cap (-X_+) = \{0\}$ and $cX_+ \subseteq X_+$ for each $c \geqslant 0$. It is easy to see that the order relation $\leqslant$, defined by

$$x \leqslant y \quad \text{if and only if} \quad y - x \in X_+,$$

is a partial ordering in X, and that $X_+ = \{y \in X \colon 0 \leqslant y\}$. The space X, equipped with this partial ordering, is called an *ordered normed space*. The order interval $[y, z] = \{x \in X \colon y \leqslant x \leqslant z\}$ is a closed subset of X. A subset C of X is said to be a *chain* if $x \leqslant y$ or $y \leqslant x$ for all $x, y \in X$. A sequence (subset) of X is called *order bounded* if it is contained in an order interval $[y, z]$ of X. We say that an order cone X_+ of a normed space X is *normal* if there is such a constant $\gamma \geqslant 1$ that

$$0 \leqslant x \leqslant y \text{ in } X \text{ implies } \|x\| \leqslant \gamma \|y\|. \tag{2.1}$$

An order cone X_+ is called *regular* if all increasing and order bounded sequences of X_+ converge. As for the proof of the following result, see, e.g., [5, Theorems 2.2.1 and 2.4.5].

Lemma 2.1. *Let X_+ be an order cone of a Banach space X. If X_+ is regular, it is also normal. The converse holds if X is weakly sequentially complete.*

A function from a real interval $[a, b]$ to a Banach space X is *Henstock-Kurzweil (shortly HK) integrable* if there is a function $F \colon [a, b] \to X$, called a *primitive of* f, which has the following property: For every $\varepsilon > 0$, there is such a function $\delta \colon [a, b] \to (0, \infty)$ that

$$\left\| \sum_{i=1}^{m} (f(\xi_i)(t_i - t_{i-1}) - (F(t_i) - F(t_{i-1}))) \right\| < \varepsilon$$

for every partition $\{t_i\}_{i=1}^m$ of $[a, b]$ satisfying $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for every $i = 1, \ldots, m$.

If f is HK integrable on $[a,b]$, it is HK integrable on every closed subinterval $J=[c,d]$ of $[a,b]$, and $F(d)-F(c)$ is the *Henstock-Kurzweil* integral of f over J, i.e.,

$$F(d)-F(c)={}^{\mathrm{K}}\!\!\int_J f(s)\,\mathrm{d}s={}^{\mathrm{K}}\!\!\int_c^d f(s)\,\mathrm{d}s. \tag{2.2}$$

The proofs for the results of the next lemma can be found, e.g., in [12].

Lemma 2.2.

(a) *The a.e. equal functions are HK integrable and their integrals are equal if one of these functions is HK integrable.*

(b) *A Bochner integrable function* $f\colon [a,b]\to X$ *is HK integrable, and* $\int_J f(s)\,\mathrm{d}s={}^{\mathrm{K}}\!\!\int_J f(s)\,\mathrm{d}s$ *whenever* I *is a closed subinterval of* $[a,b]$.

The next result plays an important role in applications.

Lemma 2.3. *Let* X *be an ordered Banach space, and let* $f_{\pm}\colon [a,b]\to X$ *be HK integrable. If* $f_-(s)\leqslant f_+(s)$ *for a.e.* $s\in[a,b]$*, and if* J *is a closed subinterval of* $[a,b]$*, then*

$${}^{\mathrm{K}}\!\!\int_J f_-(s)\,\mathrm{d}s\leqslant{}^{\mathrm{K}}\!\!\int_I f_+(s)\,\mathrm{d}s. \tag{2.3}$$

Proof. By Lemma 2.2 (a) we may assume that $f_-(s)\leqslant f_+(s)$ for all $s\in[a,b]$. Set $f=f_+-f_-$. Then $f(s)$ belongs to the order cone X_+ of X for all $s\in[a,b]$. Let $J=[c,d]$ be a closed subinterval of $[a,b]$. The function f is HK integrable on J. To prove that ${}^{\mathrm{K}}\!\!\int_J f(s)\,\mathrm{d}s\in X_+$, notice first that ${}^{\mathrm{K}}\!\!\int_J f(s)\,\mathrm{d}s=0\in X_+$ if $c=d$. Assume next that $c<d$. According to the definition of HK integrability, we can choose for each $n\in\mathbb{N}$ a function $\delta_n\colon [c,d]\to(0,\infty)$, partitions $\{t_i^n\}_{i=1}^{m_n}$ of $[c,d]$ and points ξ_i^n so that $\xi_i^n\in[t_{i-1}^n,t_i^n]\subset(\xi_i^n-\delta(\xi_i^n),\xi_i^n+\delta(\xi_i^n))$, and that

$$\left\|{}^{\mathrm{K}}\!\!\int_I f(s)\,\mathrm{d}s-\sum_{i=1}^{m_n} f(\xi_i^n)(t_i^n-t_{i-1}^n)\right\|<\frac{1}{n}.$$

Denoting $y_n=\sum\limits_{i=1}^{m_n} f(\xi_i^n)(t_i^n-t_{i-1}^n)$, $n\in\mathbb{N}$, we obtain ${}^{\mathrm{K}}\!\!\int_J f(s)\,\mathrm{d}s=\lim\limits_{n\to\infty}y_n$. Since X_+ is closed and since $y_n\in X_+$ for every $n\in\mathbb{N}$, we have ${}^{\mathrm{K}}\!\!\int_J f(s)\,\mathrm{d}s\in X_+$. Consequently,

$$0\leqslant{}^{\mathrm{K}}\!\!\int_J f(s)\,\mathrm{d}s={}^{\mathrm{K}}\!\!\int_I f_+(s)\,\mathrm{d}s-{}^{\mathrm{K}}\!\!\int_J^t f_-(s)\,\mathrm{d}s.$$

This proves the assertion. $\square$

The next result is proved in [9, Theorem 3.1].

Lemma 2.4. *Let X be a Banach space ordered by the regular cone X_+. Assume that functions $f_n\colon [a,b] \to X$, $n \in \mathbb{N}$, and $f_\pm\colon [a,b] \to X$ are HK integrable, that the sequence $(f_n)_{n\in\mathbb{N}}$ is monotone and that $f_- \leqslant f_n \leqslant f_+$ for every $n \in \mathbb{N}$. Then there exists such a HK integrable function $f\colon [a,b] \to X$ that $f(s) = \lim_n f_n(s)$ for a.e. $s \in [a,b]$, and*

$$\lim_{n\to\infty} {}^{\mathrm{K}}\!\!\int_a^b f_n(s)\,\mathrm{d}s = {}^{\mathrm{K}}\!\!\int_a^b f(s)\,\mathrm{d}s. \tag{2.4}$$

Given a half-open real interval $[a,b)$, $-\infty < a < b \leqslant \infty$, we say that a function $f\colon [a,b) \to X$ is locally Bochner (HK) integrable if f is Bochner (respectively, HK) integrable on every closed subinterval of $[a,b)$. Denote by $L^1_{\mathrm{loc}}((a,b),E)$ the space of all strongly (Lebesgue) measurable and locally Bochner integrable functions from $[a,b)$ to X.

The following result is proved in [6, Lemma 2.4].

Proposition 2.1. *Let X be a Banach space ordered by a regular order cone X_+. Assume that $u_\pm \in L^1_{\mathrm{loc}}([a,b),X)$, and that C is a nonempty chain in the order interval $[u_-,u_+]$ of $L^1_{\mathrm{loc}}([a,b),X)$. Then C contains an increasing sequence which converges a.e. pointwise to $\sup C$ and a decreasing sequence which converges a.e. pointwise to $\inf C$.*

The following result is a consequence of [7, Theorem 1.2.1 and Proposition 1.2.1].

Theorem 2.1. *Given a partially ordered set Y and its order interval $[u_-,u_+] = \{u \in Y\colon u_- \leqslant u \leqslant u_+\}$, $u_- \leqslant u_+$, assume that $G\colon [u_-,u_+] \to [u_-,u_+]$ is an increasing mapping, and that $\sup G[C]$ and $\inf G[C]$ exist for every nonempty chain C of $[u_-,u_+]$. Then G has the least fixed point u_* and the greatest fixed point u^*, and they are increasing with respect to G.*

3. Applications to Volterra functional integral equations and to impulsive Cauchy problems

In this section we apply Theorem 2.1 to a functional integral equation of Volterra type and to a functional impulsive Cauchy problem. Throughout this section we assume that X is a Banach space ordered by a regular order cone.

3.1. Volterra equation

Consider the functional integral equation

$$u(t) = h(t,u) + {}^{\mathrm{K}}\!\!\int_a^t g(s,u(s),u)\,\mathrm{d}s, \quad t \in [a,b), \tag{3.1}$$

where $h\colon [a,b) \times L^1_{\mathrm{loc}}([a,b),X) \to X$, $g\colon [a,b) \times X \times L^1_{\mathrm{loc}}([a,b),X) \to X$, and $[a,b)$ is a half-open real interval, $-\infty < a < b \leqslant \infty$.

Definition 3.1. We say that $u \in L^1_{\mathrm{loc}}([a,b),X)$ is a *lower solution* of (3.1) if

$$u(t) \leqslant h(t,u) + {}^{\mathrm{K}}\!\!\int_a^t g(s,u(s),u)\,\mathrm{d}s \quad \text{for a.e. } t \in [a,b). \tag{3.2}$$

If the reversed inequality holds in (3.2) for a.e. $t \in [a,b)$, we say that u is an *upper solution* of (3.1). If equality holds in (3.2) for a.e. $t \in [a,b)$, we say that u is a *solution* of (3.1).

As an application of Theorem 2.1 we prove an existence and comparison result for least and greatest solutions of the equation (3.1) when h and g satisfy the following hypotheses.

(g0) $g(\cdot,u(\cdot),u)$ is locally HK integrable whenever $u\colon [a,b) \to X$ is locally Bochner integrable.

(g1) If $u,v \in L^1_{\mathrm{loc}}([a,b),X)$, and if $u(t) \leqslant v(t)$ for a.e. $t \in [a,b)$, then $g(t,u(t),u) \leqslant g(t,v(t),v)$ for a.e. $t \in [a,b)$.

(h0) $h(t,\cdot)$ is increasing for a.e. $t \in [a,b)$, and $h(\cdot,u)$ is locally Bochner integrable for every locally Bochner integrable function $u\colon [a,b) \to X$.

(lu) The equation (3.1) has a lower solution u_- and an upper solution u_+, and $u_- \leqslant u_+$.

Theorem 3.1. *If the hypotheses* (g0), (g1), (h0), *and* (lu) *are satisfied, then the equation* (3.1) *has least and greatest solutions in the order interval* $[u_-,u_+]$ *of* $L^1_{\mathrm{loc}}([a,b),X)$, *and they are increasing with respect to* h *and* g.

P r o o f. The hypothesis (g0) and [12, Theorem 7.4.1] imply that for every $u \in [u_-,u_+]$ the integral on the right-hand side of the equation

$$Gu(t) = h(t,u) + {}^{\mathrm{K}}\!\!\int_a^t g(s,u(s),u)\,\mathrm{d}s, \quad t \in [a,b), \tag{3.3}$$

is a continuous function of t, whence $Gu \in L^1_{\mathrm{loc}}([a,b),X)$. The hypotheses (g1), (h0), and (lu), and Lemma 2.3 imply that if $u,v \in [u_-,u_+]$ and $u \leqslant v$, then

$$u_-(t) \leqslant h(t,u) + {}^{\mathrm{K}}\!\!\int_a^t g(s,u(s),u)\,\mathrm{d}s \leqslant h(t,v) + {}^{\mathrm{K}}\!\!\int_a^t g(s,v(s),v)\,\mathrm{d}s \leqslant u_+(t)$$

for a.e. $t \in [a,b)$. It follows from this result and (3.3) that G is increasing, and that $G[[u_-,u_+]] \subseteq [u_-,u_+]$. According to Proposition 2.1 chains of $[u_-,u_+]$ have suprema and infima in the space $L^1_{\mathrm{loc}}([a,b),X)$. Thus G satisfies the hypotheses of Theorem 2.1, whence it has least and greatest fixed points u_* and u^*. They are also the least and greatest solutions of (3.1) in $[u_-,u_+]$. Moreover, u_* and u^* are increasing with respect to G. This result and Lemma 2.3 imply that u_* and u^* are increasing with respect to the functions h and g, which proves the last conclusion.

□

Next we consider the cases when the extremal solutions of the integral equation (3.1) can be obtained by successive approximations.

Proposition 3.1. *Assume that the hypotheses* (g0), (g1), (h0), *and* (lu) *hold.*

(a) *The successive approximations*

$$u_{n+1}(t) = h(t,u_n) + {}^{\mathrm{K}}\!\!\int_a^t g(s,u_n(s),u_n)\,\mathrm{d}s, \quad t \in [a,b),\ u_0 = u_-, \tag{3.4}$$

form an increasing sequence converging a.e. pointwise to a function $u_* \in L^1_{\mathrm{loc}}([a,b),X)$. Moreover, u_* is the least solution of (3.1) *in $[u_-,u_+]$ if $h(t,u_n) \to h(t,u_*)$ for a.e. $t \in [a,b)$ and $g(s,u_n(s),u_n) \to g(s,u_*(s),u_*)$ for a.e. $s \in [a,b)$.*

(b) *The successive approximations*

$$v_{n+1}(t) = h(t,v_n) + {}^{\mathrm{K}}\!\!\int_a^t g(s,v_n(s),v_n)\,\mathrm{d}s, \quad t \in [a,b),\ v_0 = u_+, \tag{3.5}$$

form a decreasing sequence converging a.e. pointwise to a function $u^* \in L^1_{\mathrm{loc}}([a,b),X)$. Moreover, u^* is the greatest solution of (3.1) *in $[u_-,u_+]$ if $h(t,v_n) \to h(t,u^*)$ for a.e. $t \in [a,b)$ and $g(s,v_n(s),v_n) \to g(s,u^*(s),u^*)$ for a.e. $s \in [a,b)$.*

Proof. It follows from (3.3) and (3.4) that $u_n = G^n u_-$ for each $n \in \mathbb{N}$. Since G is increasing and $u_-(s) \leqslant u_n(s) \leqslant u_+(s)$ for a.e. $s \in [a,b]$, then (u_n) is increasing and a.e. pointwise order-bounded. Because the order cone of X is regular, the a.e. pointwise limit u_* of (u_n) exists. The hypotheses of (a) and Lemma 2.4 imply that

$$h(t,u_n) \to h(t,u_*) \text{ and}$$
$$\int_a^t g(s,u_n(s),u_n)\,\mathrm{d}s \to \int_a^t g(s,u_*(s),u_*)\,\mathrm{d}s \text{ for a.e. } t \in [a,b).$$

It then follows from (3.4) as $n \to \infty$ that u_* is a solution of (3.1). By induction one can show that if u is any solution of (3.1) in $[u_-, u_+]$, then $u_n \leqslant u$ for every n. Thus $u_* = \inf\limits_n u_n \leqslant u$, so that u_* is the least solution of (3.1) in $[u_-, u_+]$.

By similar reasoning one can show that the sequence (v_n) defined in (3.5) is decreasing, equals to $(G^n u_+)$, and converges a.e. pointwise to the greatest solution u^* of (3.1) in $[u_-, u_+]$. □

The next result is an application of Theorem 3.1.

Corollary 3.1. *Let the hypotheses* (g0), (g1), (h0), *and the following hypotheses hold:*

(g2) *There exist $g_\pm \in L^1_{\mathrm{loc}}([a,b),X)$, $g_- \leqslant g_+$ such that $g_- \leqslant g(\cdot, u(\cdot), u) \leqslant g_+$ for every $u \in L^1_{\mathrm{loc}}([a,b),X)$.*

(h1) *There exist $h_\pm \in L^1_{\mathrm{loc}}([a,b),X)$, $h_- \leqslant h_+$ such that $h_- \leqslant h(\cdot, u) \leqslant h_+$ for all $u \in L^1_{\mathrm{loc}}([a,b),X)$.*

Then the integral equation (3.1) *has the least and greatest solutions, and they are increasing with respect to h and g.*

Proof. Denoting

$$u_\pm(t) = h_\pm(t) + {}^{\mathrm{K}}\!\!\int_a^t g_\pm(s)\,\mathrm{d}s, \quad t \in [a,b),$$

the hypotheses (g2) and (h1) imply that the hypothesis (lu) holds. Thus the equation (3.1) has by Theorem 3.1 the least and greatest solutions u_* and u^* in $[u_-, u_+]$, and they are increasing with respect to h and f. The hypotheses (g1), (g2), (h0), and (h1), and Lemma 2.3 imply that if $u \in L^1_{\mathrm{loc}}([a,b),X)$, then

$$u_-(t) \leqslant h(t,u) + {}^{\mathrm{K}}\!\!\int_a^t g(s,u(s),u)\,\mathrm{d}s \leqslant u_+(t) \quad \text{for a.e. } t \in [a,b).$$

Thus all the solutions of (3.1) belong to the order interval $[u_-, u_+]$, whence u_* and u^* are the least and greatest of all the solutions of (3.1). □

3.2. Cauchy problem

Consider now the functional impulsive Cauchy problem (ICP)

$$\text{(3.6)} \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t) = g(t,u(t),u) \text{ a.e. in } [a,b), \\ u(a) = x_0, \ \Delta u(\lambda) = D(\lambda, u), \ \lambda \in W, \end{cases}$$

where $g\colon [a,b)\times X\times L^1_{\rm loc}([a,b),X)\to X$, $x_0\in X$, $\Delta u(\lambda)=u(\lambda+0)-u(\lambda)$, $D\colon W\times L^1_{\rm loc}([a,b),X)\to X$, and W is a well-ordered (and hence countable) subset of (a,b).

It follows from [1, Lemma 3.1] that if $g(\cdot,u(\cdot),u)$ belongs to $L^1_{\rm loc}([a,b),X)$ whenever u is in $L^1_{\rm loc}([a,b),X)$, then problem (3.6) can be converted to the Volterra integral equation

$$u(t)=x_0+\sum_{\lambda\in W^{<t}}D(\lambda,u)+\int_a^t g(s,u(s),u)\,\mathrm{d}s,$$

where $W^{<t}=\{\lambda\in W\colon\ \lambda<t\}$, $t\in[a,b)$.

Definition 3.2. We say that $u\colon [a,b)\to X$ is a mild solution of the ICP (3.6) if $g(\cdot,u(\cdot),u)$ is locally HK integrable and satisfies the integral equation

$$u(t)=x_0+\sum_{\lambda\in W^{<t}}D(\lambda,u)+{}^{\mathrm{K}}\!\!\int_a^t g(s,u(s),u)\,\mathrm{d}s \tag{3.7}$$

where $W^{<t}=\{\lambda\in W\colon\ \lambda<t\}$, $t\in[a,b)$.

To justify Definition 3.2 notice that $[a,b)$ is a disjoint union of $C=\{a\}\cup W$ and open intervals $(\lambda,S(\lambda))$, $\lambda\in C$, where $S(\lambda)=\min\{\alpha\in C\colon\ \lambda<\alpha\}$. It follows from (3.7) by [12, Theorem 7.4.20] and by the proof of [1, Lemma 3.1], that if $u\colon [a,b)\to X$ is a mild solution of (3.6), then for every $x^*\in X^*$ there is a null-set Z in $[a,b)$, which may depend on the choice of x^*, such that

$$\begin{cases}(x^*(u))'(t)=x^*(g(t,u(t),u)) \text{ for all } t\in[a,b)\setminus Z,\\ u(a)=x_0,\ \ \Delta u(\lambda)=D(\lambda,u),\ \ \lambda\in W.\end{cases} \tag{3.8}$$

As an application of Corollary 3.1 we prove an existence and comparison result for the least and greatest mild solutions of problem (3.6).

Proposition 3.2. *Given a well-ordered subset W of (a,b), assume that $g\colon [a,b)\times X\times L^1_{\rm loc}([a,b),X)\to X$ and $D\colon W\times L^1_{\rm loc}([a,b),X)\to X$ satisfy the hypotheses* (g0)–(g2) *and*

(D0) *$D(\lambda,\cdot)$ is increasing for all $\lambda\in W$, and there exist $c_\pm\colon W\to X$ such that $c_-(\lambda)\leqslant D(\lambda,u)\leqslant c_+(\lambda)$ for all $\lambda\in W$ and $u\in L^1_{\rm loc}([a,b),X)$, and that $\sum\limits_{\lambda\in W}\|c_\pm(\lambda)\|<\infty$.*

Then the impulsive Cauchy problem (3.6) *has for every $x_0\in X$ the least and greatest mild solutions in V, and they are increasing with respect to g, D and x_0.*

P r o o f. The hypotheses given for D ensure that for each $x_0 \in X$ the relation

$$h(t,u) = x_0 + \sum_{\lambda \in W^{<t}} D(\lambda, u), \quad t \in [a,b),\ u \in L^1_{\rm loc}([a,b),X), \tag{3.9}$$

defines a mapping $h\colon [a,b) \times L^1_{\rm loc}([a,b),X) \to X$ which satisfies the hypotheses (h0), and (h1) of Corollary 3.1. Then the integral equation (3.1), which by (3.9) can be rewritten as a fixed point equation

$$u(t) = Gu(t) := x_0 + \sum_{\lambda \in W^{<t}} D(\lambda, u) + {}^{\rm K}\!\!\int_a^t g(s, u(s), u)\,{\rm d}s, \tag{3.10}$$

has by Corollary 3.1 the least and greatest solutions u_* and u^*, and they are increasing with respect to h and g. Because by Definition 3.2 the solutions of the integral equation (3.7) are mild solutions of the ICP (3.6), hence u_* and u^* are the least and greatest solutions of the (ICP) (3.6) in V, and they are increasing with respect to x_0, D, and q. □

The next result is a consequence of Proposition 3.1.

Proposition 3.3. *Assume that the hypotheses of Proposition 3.2 hold, and let G be defined by (3.10).*

(a) *The sequence $(u_n)_{n=0}^{\infty} = (G^n w_-)_{n=0}^{\infty}$ is increasing and converges a.e. pointwise to a function $u_* \in L^1_{\rm loc}([a,b),X)$. Moreover, u_* is the mild least solution of (3.6) in V if $D(\lambda, u_n) \to D(\lambda, u_*)$ for each $\lambda \in W$ and $g(s, u_n(s), u_n) \to g(s, u_*(s), u_*)$ for a.e. $s \in [a,b)$.*

(b) *The sequence $(v_n)_{n=0}^{\infty} = (G^n w_+)_{n=0}^{\infty}$ is decreasing and converges a.e. pointwise to a function $u^* \in L^1_{\rm loc}([a,b),X)$. Moreover, u^* is the greatest mild solution of (3.6) in V if $D(\lambda, v_n) \to D(\lambda, u^*)$ for each $\lambda \in W$ and $g(s, v_n(s), v_n) \to g(s, u^*(s), u^*)$ for a.e. $s \in [a,b)$.*

E x a m p l e 3.1. Let X be the Banach space l_2 of the sequences $(x_n)_{n=1}^{\infty}$ of real numbers for which $\sum\limits_{n=1}^{\infty} |x_n|^2 < \infty$, ordered componentwise and normed by $\|x\| = \Big(\sum\limits_{n=1}^{\infty} |x_n|^2\Big)^{1/2}$. The mappings $g_{\pm}\colon [0,\infty) \to l_2$, defined by $g_{\pm}(0) = (0,0,\ldots)$,

$$g_{\pm}(t) = \left(\frac{2t}{n}\cos\Big(\frac{1}{t^2}\Big) + \frac{2}{nt}\sin\Big(\frac{1}{t^2}\Big) \pm \frac{1}{n}\right)_{n=1}^{\infty}, \quad t \in (0,\infty), \tag{3.11}$$

are locally HK integrable. Thus these mappings are possible upper and lower boundaries for g in Corollary 3.1 and in Proposition 3.3 when $X = l_2$. Choosing $x_{\pm} =$

$\bigl(\pm 1/n\bigr)_{n=1}^{\infty}$, and denoting

$$\delta_k^n = \begin{cases} 1 \text{ if } k = n, \\ 0 \text{ if } k \neq n, \end{cases}$$

the solutions of the initial value problems

$$\begin{cases} w'_{\pm}(t) = g_{\pm}(t) \text{ for (a.e.) } t \in (0,\infty),\ u(0) = x_{\pm}, \\ \Delta w_{\pm}\Bigl(1 - \dfrac{1}{2k}\Bigr) = \Bigl(\pm \delta_k^n \dfrac{1}{n}\Bigr)_{n=1}^{\infty},\ k = 1,2,\ldots, \end{cases} \tag{3.12}$$

are

$$w_{\pm}(t) = \Bigl(\frac{1}{nt}\Bigl(t^2\cos\Bigl(\frac{1}{t^2}\Bigr) \pm \Bigl(t + H\Bigl(t - \frac{2n-1}{2n}\Bigr)\Bigr)\Bigr)\Bigr)_{n=1}^{\infty}. \tag{3.13}$$

In particular, the infinite system of impulsive Cauchy problems

$$\begin{cases} u'_n(t)) = \dfrac{1}{n}\Bigl(2t\cos\Bigl(\dfrac{1}{t^2}\Bigr) + \dfrac{2}{t}\sin\Bigl(\dfrac{1}{t^2}\Bigr) + g_n(u)\Bigr) \quad \text{for (a.e.) } t \in [0,\infty), \\ u_n(0)) = \dfrac{x_n}{n},\ \Delta u_n\Bigl(1 - \dfrac{1}{2n}\Bigr) = \dfrac{c_n(u)}{n}, \quad n = 1,2,\ldots, \end{cases} \tag{3.14}$$

where each $c_n, g_n\colon \mathrm{HK}_{\mathrm{loc}}((0,\infty), l_2) \to \mathbb{R}$, are increasing, $-1 \leqslant x_n, c_n(u), g_n(u) \leqslant 1$ for all $u \in \mathrm{HK}_{\mathrm{loc}}([0,\infty), l_2)$ and $n = 1,2,\ldots$, has the least and greatest solutions $u_* = (u_{*n})_{n=1}^{\infty}$ and $u^* = (u_n^*)_{n=1}^{\infty}$, and they belong to the order interval $[w_-, w_+]$, where $w_{\pm}$ are given by (3.13).

R e m a r k s 3.1. No component of the mappings $g_{\pm}$ defined in (3.11) belongs to $L^1([0,t),\mathbb{R})$ for any $t > 0$. Consequently, the mappings $g_{\pm}$ do not belong to $L^1([0,t), l_2)$ for any $t > 0$. Notice also that if g in Corollary 3.1 and in Proposition 3.3 is norm-bounded by a function of $L^1([a,t],\mathbb{R}_+)$ for every $t \in (a,b)$, then the mapping $g(\cdot, u(\cdot), u)$ belongs to $L^1([a,t), X)$ for all $t \in (a,b)$.

It follows from [9, Corollary 4.1] that the functions of an order interval $[u_-, u_+]$ of locally HK integrable functions are locally McShane integrable if one of the functions $u_{\pm}$ is locally McShane integrable. Thus the fixed points u_* and u^* in Theorem 2.1 and the solutions u_* and u^* of equations (3.1) and (3.6) considered in this section are locally McShane integrable if u_- or u_+ is locally McShane integrable. In particular, all the results of this section and Section 3 remain valid if local HK integrability is replaced by local McShane integrability.

The space of locally HK integrable functions contains also those functions $u\colon [a,b] \to X$ which are Bochner integrable on every closed subinterval $[c,d]$ of (a,b), and for which the limits of the Bochner integral $\int_c^d u(s)\,\mathrm{d}s$ when $c \to a+$ and $d \to b-$

exist (cf. [12, Theorem 3.4.5] and Remark after it). In particular, the integral in (3.1) can be replaced by the improper integral $\int_{a+}^{t}$.

The following spaces are examples of weakly sequentially complete Banach spaces which have normal order cones (cf. [7]):

1. A reflexive (e.g., a uniformly convex) Banach space ordered by a normal order cone.
2. A finite-dimensional normed space ordered by any closed cone.
3. A separable Hilbert space whose order cone is generated by an orthonormal basis.
4. A Hilbert space H with such an order cone H_+ that $(x \mid y) \geqslant 0$ for all $x, y \in H_+$.
5. A Hilbert space H whose order cone is $H_+ = \{x \in H\colon\ (x \mid \overline{e}) \geqslant c\|x\|_2\}$, where $\overline{e}$ is a unit vector of H and $c \in (0,1)$.
6. A sequence space l^p, $1 \leqslant p < \infty$, normed by the p-norm and ordered componentwise.
7. A function space $L^p(\Omega)$, $1 \leqslant p < \infty$, normed by the p-norm and ordered a.e. pointwise, where Ω is a measure space.
8. A function space $L^p([a,b],X)$, $1 \leqslant p < \infty$, ordered a.e. pointwise, where X is any of the spaces listed above.

According to Lemma 2.1 the order cones of all the above mentioned spaces are regular. In the sequence space (c_0), normed by the sup-norm the componentwise ordering is induced by the cone of all nonnegative sequences. This cone is also regular.

As for other results on non-absolute integral equations and impulsive differential equations in Banach spaces, see, e.g., [2], [3], [4], [11], [13], [14], [15], [16], [17]. Compared with these papers a novelty of the results of Section 4 is that the existence results for suprema and infima of chains in the space of locally Henstock-Kurzweil integrable functions derived in Section 3 allow us to apply fixed point results in ordered spaces. Similar methods are used in [8] in the case when Volterra integral equations and impulsive differential equations contain locally Henstock-Lebesgue integrable functions.

References

[1] *S. Carl, S. Heikkilä*: On discontinuous implicit and explicit abstract impulsive boundary value problems. Nonlinear Anal., Theory Methods Appl. *41* (2000), 701–723.

[2] *M. Federson, M. Bianconi*: Linear Fredholm integral equations and the integral of Kurzweil. J. Appl. Anal. *8* (2002), 83–110.

[3] *M. Federson, Š. Schwabik*: Generalized ordinary differential equations approach to impulsive retarded functional differential equations. Differ. Integral Equ. *19* (2006), 1201–1234.

[4] *M. Federson, P. Táboas*: Impulsive retarded differential equations in Banach spaces via Bochner-Lebesgue and Henstock integrals. Nonlinear Anal., Theory Methods Appl. *50* (2002), 389–407.
[5] *D. Guo, Y. J. Cho, J. Zhu*: Partial Ordering Methods in Nonlinear Problems. Nova Science Publishers, Inc., New York, 2004.
[6] *S. Heikkilä, S. Kumpulainen, M. Kumpulainen*: On improper integrals and differential equations in ordered Banach spaces. J. Math. Anal. Appl. *319* (2006), 579–603.
[7] *S. Heikkilä, V. Lakshmikantham*: Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations. Marcel Dekker, Inc., New York, 1994.
[8] *S. Heikkilä, S. Seikkala*: On non-absolute functional Volterra integral equations and impulsive differential equations in ordered Banach spaces. Electron. J. Differ. Equ., paper No. 103 (2008), 1–11.
[9] *S. Heikkilä, G. Ye*: Convergence and comparison results for Henstock-Kurzweil and McShane integrable vector-valued functions. Southeast Asian Bull. Math. *35* (2011), 407–418.
[10] *J. Lu, P.-Y. Lee*: On singularity of Henstock integrable functions. Real Anal. Exch. *25* (2000), 795–797.
[11] *B.-R. Satco*: Nonlinear Volterra integral equations in Henstock integrability setting. Electron. J. Differ. Equ., paper No. 39 (2008), 1–9.
[12] *Š. Schwabik, G. Ye*: Topics in Banach Space Integration. World Scientific, Hackensack, 2005.
[13] *A. Sikorska-Nowak*: On the existence of solutions of nonlinear integral equations in Banach spaces and Henstock-Kurzweil integrals. Ann. Pol. Math. *83* (2004), 257–267.
[14] *A. Sikorska-Nowak*: Existence theory for integrodifferential equations and Henstock-Kurzweil integral on Banach spaces. J. Appl. Math., Article ID31572 (2007), 1–12.
[15] *A. Sikorska-Nowak*: Existence of solutions of nonlinear integral equations in Banach spaces and Henstock-Kurzweil integrals. Ann. Soc. Math. Pol., Ser. I, Commentat. Math. *47* (2007), 227–238.
[16] *A. Sikorska-Nowak*: Nonlinear integrodifferential equations of mixed type in Banach spaces. Int. J. Math. Math. Sci., Article ID65947 (2007), 1–14.
[17] *A. Sikorska-Nowak*: Nonlinear integral equations in Banach spaces and Henstock-Kurzweil-Pettis integrals. Dyn. Syst. Appl. *17* (2008), 97–107.

Authors' addresses: *S. Heikkilä*, Department of Mathematical Sciences, University of Oulu, Box 3000, FIN-90014, Finland, e-mail: `sheikki@cc.oulu.fi`; *Guoju Ye*, College of Science, Hohai University, Nanjing 210098, P. R. China, e-mail: `yegj@hhu.edu.cn`.