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ON THE INTERSECTION OF TWO DISTINCT k-GENERALIZED FIBONACCI SEQUENCES

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Abstract. Let $k \geqslant 2$ and define $F^{(k)} := (F_n^{(k)})_{n\geqslant 0}$, the k-generalized Fibonacci sequence whose terms satisfy the recurrence relation $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \ldots + F_{n-k}^{(k)}$, with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$. The sequences $F := F^{(2)}$ and $T := F^{(3)}$ are the known Fibonacci and Tribonacci sequences, respectively. In 2005, Noe and Post made a conjecture related to the possible solutions of the Diophantine equation $F_n^{(k)} = F_m^{(l)}$. In this note, we use transcendental tools to provide a general method for finding the intersections $F^{(k)} \cap F^{(m)}$ which gives evidence supporting the Noe-Post conjecture. In particular, we prove that $F \cap T = \{0, 1, 2, 13\}$.

Keywords: k-generalized Fibonacci numbers, linear forms in logarithms, reduction method

MSC 2010: 11B39, 11D61, 11J86

1. Introduction

Several problems in number theory are actually questions about the intersection of two known sequences (or sets). Before giving examples, let us recall some terminology: let $F := (F_n)_{n\geqslant 0}$ be the *Fibonacci sequence*, $\mathbb{P} := \{p\colon p \text{ prime}\}$, $\mathcal{P} := \{y^t\colon y, t \in \mathbb{Z},\ t > 1\}$ (the perfect powers), $\mathcal{F} := \{n!\colon n \in \mathbb{Z},\ n \geqslant 0\}$, $\mathcal{R} := \{a(10^n - 1)/9\colon 1 \leqslant a \leqslant 9,\ n \in \mathbb{Z},\ n > 0\}$ (the *repdigits* or *unidigital numbers*). Below, we cite some results about the intersection of these sets:

- ▷ Erdös and Selfridge [8] proved that $\mathcal{F} \cap \mathcal{P} = \{1\}$.
- ▷ In 2000, Luca [25] proved that $F \cap \mathcal{R} = \{0, 1, 2, 3, 5, 8, 55\}$.
- ▷ Luca [26] also proved that $F \cap \mathcal{F} = \{1, 2\}$.
- ▷ In 2003, Bugeaud et al [4] showed that $F \cap \mathcal{P} = \{0, 1, 8, 144\}$ (see [28] for a generalization).

▷ Let $(a_n)_{n\geqslant 1}$ be the tower given by $a_1 = 1$ and $a_n = n^{a_{n-1}}$, for $n \geqslant 2$. Luca and the author [27] proved that $\{a_1 + \ldots + a_n \colon n \geqslant 1\} \cap \mathcal{P} = \{1\}$.

However, some related questions are still open problems, as for instance the sets $\mathbb{P} \cap F$ and $\mathbb{P} \cap \mathcal{R}$ are unknown.

Let $k \geqslant 2$ and denote $F^{(k)} := (F_n^{(k)})_{n\geqslant 0}$, the *k-generalized Fibonacci sequence* whose terms satisfy the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \ldots + F_{n-k}^{(k)}, \tag{1.1}$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$.

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called *k-step Fibonacci numbers*, the *Fibonacci k-sequence*, or *k-bonacci numbers*. Clearly, for $k = 2$ we obtain the well-known Fibonacci numbers and for $k = 3$, Tribonacci numbers.

Recall that Tribonacci numbers have a long history. For the first time, they were studied in 1914 by Agronomoff [1] and subsequently by many others. The name Tribonacci was coined in 1963 by Feinberg [9]. The basic properties of Tribonacci numbers can be found in [18], [24], [36], [38]. For recent papers, we refer the reader to [3], [19], [20], [33] and to the collection [21], [22], [23].

Recently, Alekseyev [2] described how to compute the intersection of two Lucas sequences including the sequences of Fibonacci, Pell, Lucas and Lucas-Pell numbers. In general, we refer the reader to [34], [35], [37] for results on the intersection of two recurrence sequences.

In a very recent paper, Togbé and the author [29] proved that only finitely many terms of a linear recurrence sequence whose characteristic polynomial has a dominant root can be repdigits. As an application, since the characteristic polynomial of the recurrence in (1.1), namely $x^k - x^{k-1} - \ldots - x - 1$, has just one root α such that $|\alpha| > 1$ (see for instance [39]), hence $F^{(k)} \cap \mathcal{R}$ is a finite set, for all $k \geqslant 2$. See also the article [32] for some results on the set $F^{(k)} \cap \mathbb{P}$ and a conjecture on the intersection $F^{(k)} \cap F^{(m)}$. We point out that this last intersection is, to the best of our knowledge, not known even in the easiest case $(k, m) = (2, 3)$, that is, for numbers that are both Fibonacci and Tribonacci. A possible way to find this intersection is to look at the Fibonacci and Tribonacci sequences modulo p^t, where p is a prime number. We refer the reader to [5], [13], [16], [17] for results of this nature. However, this approach seems to be hard to work in practice. This observation prompted the author to look for a more interesting and constructive approach which could be useful in the general case.

It is important to notice that Mignotte (see [31]) showed that if $(u_n)_{n\geqslant 0}$ and $(v_n)_{n\geqslant 0}$ are two linearly recurrence sequences then, under some weak technical assumptions, the equation

$$u_n = v_m$$

has only finitely many solutions in positive integers m, n. Moreover, all such solutions are effectively computable. Therefore, it seems reasonable to think that $F^{(k)} \cap F^{(m)}$ is a finite set for all $k \neq m$.

The goal of this paper is to apply transcendental tools to provide a method for studying the intersection $F^{(k)} \cap F^{(m)}$, for integers $2 \leqslant k < m$ and determine completely this set for $(k, m) = (2, 3)$ (confirming the expectation). More precisely, our result is the following.

Theorem 1. *The only solution of the Diophantine equation*

$$F_n = T_m \tag{1.2}$$

in positive integer numbers m and n with $n > 3$, is $(n, m) = (7, 6)$. Hence, $F \cap T = \{0, 1, 2, 13\}$.

We organize this paper as follows. In Section 2, we will recall some useful properties such as a result of Matveev on linear forms in three logarithms and the reduction method of Baker-Davenport that we will use in the proof of Theorem 1. In Section 3, we first use Baker's method to obtain a bound for n, then we completely prove Theorem 1 by means of the Baker-Davenport reduction method.

2. Auxiliary results

We recall the well-known Binet's formula:

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \quad \text{for all } n \geqslant 0, \tag{2.1}$$

where $\varphi = (1 + \sqrt{5})/2$. It is almost unnecessary to stress that this is a very helpful formula which moreover allows to deduce that

$$\varphi^{n-2} < F_n < \varphi^{n-1} \text{ for all } n \geqslant 1.$$

In 1982, Spickerman [36] found the following "Binet-style" formula for the Tribonacci sequence:

$$T_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1} \quad \text{for all } n \geqslant 0, \tag{2.2}$$

where α, β, γ are the roots of $x^3 - x^2 - x - 1 = 0$. Explicitly, we have

$$\begin{aligned}
\alpha &= \frac{1}{3} + \frac{1}{3}\bigl(19 - 3\sqrt{33}\bigr)^{1/3} + \frac{1}{3}\bigl(19 + 3\sqrt{33}\bigr)^{1/3},\\
\beta &= \frac{1}{3} - \frac{1}{6}\bigl(1 + \mathrm{i}\sqrt{3}\bigr)\bigl(19 - 3\sqrt{33}\bigr)^{1/3} - \frac{1}{6}\bigl(1 - \mathrm{i}\sqrt{3}\bigr)\bigl(19 + 3\sqrt{33}\bigr)^{1/3},\\
\gamma &= \frac{1}{3} - \frac{1}{6}\bigl(1 - \mathrm{i}\sqrt{3}\bigr)\bigl(19 - 3\sqrt{33}\bigr)^{1/3} - \frac{1}{6}\bigl(1 + \mathrm{i}\sqrt{3}\bigr)\bigl(19 + 3\sqrt{33}\bigr)^{1/3}.
\end{aligned}$$

Another interesting formula due to Spickermann is

$$T_n = \mathrm{Round}\Bigl[\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n\Bigr],$$

where, as usual, Round$[x]$ is the nearest integer to x.

Since $\alpha^{-2} < \alpha/(\alpha-\beta)(\alpha-\gamma) = 0.33622\ldots < \alpha$, the above identity yields the bounds

$$\alpha^{n-3} < T_n < \alpha^{n+2} \quad \text{for all } n \geqslant 1.$$

The Fibonacci and Tribonacci numbers can also be computed using the generating functions

$$\frac{z}{1-z-z^2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + 34z^8 + \ldots, \tag{2.3}$$

$$\frac{z}{1-z-z^2-z^3} = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + 81z^8 + \ldots \tag{2.4}$$

In order to prove Theorem 1, we will use a lower bound for a linear form in three logarithms *à la Baker* and such a bound was given by the following result of Matveev [30].

Lemma 1. *Let $\alpha_1, \alpha_2, \alpha_3$ be real algebraic numbers and let b_1, b_2, b_3 be nonzero rational numbers. Define*

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ and let A_1, A_2, A_3 be positive real numbers which satisfy

$$A_j \geqslant \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \textit{for } j = 1, 2, 3.$$

Assume that

$$B \geqslant \max\bigl\{1, \max\{|b_j| A_j/A_1;\ 1 \leqslant j \leqslant 3\}\bigr\}.$$

Define also

$$C = 6750000 \cdot \mathrm{e}^4 (20.2 + \log(3^{5.5} D^2 \log(\mathrm{e}D))).$$

If $\Lambda \neq 0$, *then*

$$\log|\Lambda| \geqslant -CD^2 A_1 A_2 A_3 \log(1.5\mathrm{e}DB \log(\mathrm{e}D)).$$

As usual, in the above statement, the *logarithmic height* of an s-degree algebraic number α is defined as

$$h(\alpha) = \frac{1}{s}\left(\log|a| + \sum_{j=1}^{s} \log\max\{1, |\alpha^{(j)}|\}\right),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$), $(\alpha^{(j)})_{1\leqslant j\leqslant s}$ are the conjugates of α and, as usual, the absolute value of the complex number $z = a + b\mathrm{i}$ is $|z| = \sqrt{a^2 + b^2}$.

After finding an upper bound on n which is generally too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethö [6]. For a real number x, we use $\|x\| = \min\{|x - n| \colon n \in \mathbb{N}\} = |x - \mathrm{Round}[x]|$ for the distance from x to the nearest integer.

Lemma 2. *Suppose that* M *is a positive integer. Let* p/q *be a convergent of the continued fraction expansion of* γ *such that* $q > 6M$ *and let* $\varepsilon = \|\mu q\| - M\|\gamma q\|$, *where* μ *is a real number. If* $\varepsilon > 0$, *then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m, n *with*

$$\frac{\log(Aq/\varepsilon)}{\log B} \leqslant m < M.$$

See Lemma 5, a) in [6]. Now, we are ready to deal with the proofs of our results.

3. The proof of Theorem 1

3.1. Finding a bound on n. By Binet's formulae (2.1) and (2.2) we get

$$\frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} = \frac{\alpha^m}{\alpha'} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'}.$$

Let us denote by α', β', γ' the values of $Q(x) = -x^2 + 4x - 1$ at $x = \alpha, \beta, \gamma$, respectively. By (2.2) and equation (1.2), we have

$$\frac{\varphi^n}{\sqrt{5}} - \frac{\alpha^m}{\alpha'} = \frac{(-1)^n \varphi^{-n}}{\sqrt{5}} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'}, \ m, n \geqslant 1.$$

More precisely,

$$\left|\frac{\varphi^n}{\sqrt{5}} - \frac{\alpha^m}{\alpha'}\right| \leqslant \left|\frac{\varphi^{-1}}{\sqrt{5}}\right| + 2\left|\frac{\beta}{\beta'}\right| < 0.67 \quad \text{for any } m, n \geqslant 1 \tag{3.1}$$

where in the last inequality we have used $|\beta| = |\gamma| = 0.73735\ldots$ and $|\beta'| = |\gamma'| = 3.84631\ldots$.

Define

$$\Lambda = \Lambda(m, n) = m \log \alpha - n \log \varphi + \log\left(\frac{\sqrt{5}}{\alpha'}\right).$$

Then

$$\Lambda = \log\left(\frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'}\right),$$

which yields

$$|e^\Lambda - 1| = \left|\frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'} - 1\right|.$$

On the other hand, from (3.1) we get

$$\left|\varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'}\right| < 0.67 \cdot \sqrt{5} < 1.5.$$

Hence

$$|e^\Lambda - 1| = \frac{1}{\varphi^n}\left|\varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'}\right| < \frac{1.5}{\varphi^n}.$$

Since $\varphi = 1.61803\ldots$, we have $1.5/\varphi^n < \varphi^{-n+1}$ and then

$$|e^\Lambda - 1| < \varphi^{-n+1}. \tag{3.2}$$

We claim that $\Lambda \neq 0$. In fact, towards a contradiction, suppose that $\Lambda = 0$ and thus $\alpha^m \sqrt{5}/\alpha' = \varphi^n$. Therefore α^{2m}/α'^2 is a quadratic algebraic number. However $\alpha^{2m}/\alpha'^2 \in \mathbb{Q}(\alpha)$ which is absurd, because α is a 3-degree algebraic number.

If $\Lambda > 0$, then $\Lambda < \mathrm{e}^{\Lambda} - 1 < \varphi^{-n+2}$ (see (3.2)). If $\Lambda < 0$, then $1 - \mathrm{e}^{-|\Lambda|} = |\mathrm{e}^{\Lambda} - 1| < \varphi^{-n+2}$. Thus, for $\Lambda < 0$, we get

$$|\Lambda| < \mathrm{e}^{|\Lambda|} - 1 < \frac{\varphi^{-n+1}}{1 - \varphi^{-n+1}} < \varphi^{-n+2},$$

where we have used the fact that $1 - \varphi^{-n+1} > 1/\varphi$ for all $n > 3$.

Hence, we have $|\Lambda| < \varphi^{-n+2}$ for any $\Lambda \neq 0$, which yields

$$\log |\Lambda| < -(n-2) \log \varphi. \tag{3.3}$$

Now, we will apply Lemma 1. Take

$$\alpha_1 = \alpha,\ \alpha_2 = \varphi,\ \alpha_3 = \sqrt{5}/\alpha',\ b_1 = m,\ b_2 = -n,\ b_3 = 1.$$

Then $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\alpha, \varphi)$, $D = 6$ and $C < 1.2 \cdot 10^{10}$.

It is easy to verify that $1/\alpha'$ is a root of $44x^3 - 2x - 1$ and that $\sqrt{5}/\alpha'$ is a root of $1936x^6 - 880x^4 + 100x^2 - 125$. Since $\sqrt{5}/\alpha'$ is a 6-degree algebraic number, its minimal polynomial (over $\mathbb{Z}$) is $1936x^6 - 880x^4 + 100x^2 - 125$. Using direct calculation, we verify that the absolute value of every root of the minimal polynomial is less than 1. Hence $h(\alpha_3) < (\log 1936)/6 < 1.262$. Next, we have $h(\alpha_1) = (\log \alpha)/3 = 0.204$ and $h(\alpha_2) = (\log \varphi)/2 < 0.241$. We then take $A_1 = 1.22,\ A_2 = 1.45$ and $A_3 = 7.58$. Since (1.2) implies $n > m$, we have

$$\max\bigl\{1, \max\{|b_j| A_j / A_1;\ 1 \leqslant j \leqslant 3\}\bigr\} = \max\{m, 1.2n\} = 1.2n =: B.$$

Hence, Lemma 1 yields

$$\log |\Lambda| > -6.8 \cdot 10^{12} \log(82n). \tag{3.4}$$

Combining the estimates (3.3) and (3.4), we get

$$6.8 \cdot 10^{12} \log(82n) > (n-2) \log \varphi,$$

and this inequality implies $n < 6 \cdot 10^{14}$ and, by the trivial estimate $m < n$, we have $m < 6 \cdot 10^{14}$. In order to improve the estimates, we use the bounds on F_n and T_m together with Equation (1.2) to obtain $\alpha^{m-3} < T_m = F_n < \varphi^{n-1}$, which yields

$m < 0.8n + 2.2$. Hence, $m < 4.8 \cdot 10^{14}$. Similarly, $\varphi^{n-2} < F_n = T_m < \alpha^{m+2}$ yields $n < 1.3m + 4.6$.

3.2. Reducing the bound. The next goal is to reduce the bound on m. For that, let us suppose, without loss of generality, that $\Lambda > 0$ (the other case can be handled in a similar way by considering $0 < \Lambda' = -\Lambda$).

We know that $0 < \Lambda < \varphi^{-n+2}$ and therefore

$$0 < m \log \alpha - n \log \varphi + \log \left(\frac{\sqrt{5}}{\alpha'} \right) < \varphi^{-m+2}.$$

Dividing by $\log \varphi$, we get

$$0 < m\widehat{\gamma} - n + \mu < 5.45 \cdot \varphi^{-m}, \tag{3.5}$$

with $\widehat{\gamma} = \log \alpha / \log \varphi$ and $\mu = \log(\sqrt{5}/\alpha')/\log \varphi$.

Surely $\widehat{\gamma}$ is an irrational number (actually, this number is transcendental by the Gelfond-Schneider theorem: if α and β are algebraic numbers with $\alpha \neq 0$ or 1, and β is irrational, then α^β is transcendental). So, let us denote by p_n/q_n the nth convergent of its continued fraction.

In order to reduce our bound on m, we will use Lemma 2. For that, taking $M = 4.8 \cdot 10^{14}$, we have that

$$\frac{p_{33}}{q_{33}} = \frac{53739149317980067}{42436582738078750},$$

and then $q_{33} > 6M$. Moreover, we get

$$\|\mu q_{33}\| - M\|\widehat{\gamma} q_{33}\| > 0.028 =: \varepsilon.$$

Thus all the hypotheses of Lemma 2 are satisfied and we take $A = 5.45$ and $B = \varphi$. It follows from Lemma 2 that there is no solution of the inequality in (3.5) (and then for the Diophantine equation (1.2)) in the range

$$\left[\left\lfloor \frac{\log(Aq_{33}/\varepsilon)}{\log B} \right\rfloor + 1, M \right] = [91, 4.8 \cdot 10^{14}].$$

Therefore $m \leqslant 90$ and then $n \leqslant 120$. To conclude, we use the formulas in (2.3) and (2.4) together with the Mathematica command

```
Intersection[CoefficientList[Series[x/(1 − x − x^2), x, 0, 120], x],
CoefficientList[Series[x/(1 − x − x^2 − x^3), x, 0, 90], x]]
```

to find the possible solutions. Fastly, Mathematica returns us the set $\{0, 1, 2, 13\}$ as its answer. This completes the proof.

4. Final remarks and a conjecture

We point out that the method in proof of Theorem 1 is quite general and that it can be used to work on the intersection of two arbitrary k-generalized Fibonacci sequences. In fact, in a similar fashion, we found the set $F^{(k)} \cap F^{(m)}$ for $4 \leqslant k < m \leqslant 10$. These cases suggest that the following statement (which is Conjecture 1 in [32]) should be true.

Conjecture 1. *Let $k < m$ be positive integer numbers. Then*

$$F^{(k)} \cap F^{(m)} = \begin{cases} \{0,1,2,13\}, & \text{if } (k,m)=(2,3), \\ \{0,1,2,4,504\}, & \text{if } (k,m)=(3,7), \\ \{0,1,2,8\}, & \text{if } k=2 \text{ and } m>3, \\ \{0,1,2,\ldots,2^{k-1}\}, & \text{otherwise.} \end{cases}$$

When working on these cases it may be helpful that the polynomials $\psi_k(x) := x^k - x^{k-1} - \ldots - x - 1$ are irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1-2^{-k})$ and 2 (as seen in [39]). Also, in a recent paper, G. Dresden [7, Theorem 1] gave a simplified "Binet-like" formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2+(k+1)(\alpha_i - 2)}\, \alpha_i^{n-1}$$

for $\alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x) = 0$. There are many other ways of representing these k-generalized Fibonacci numbers, as can be seen in [10], [11], [12], [14], [15]. Also, it was proved in [7, Theorem 2] that

$$F_n^{(k)} = \text{Round}\Big[\frac{\alpha - 1}{2+(k+1)(\alpha-2)}\, \alpha^{n-1}\Big],$$

where α is the dominant root of $\psi_k(x)$.

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