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CONTAINERS AND WIDE DIAMETERS OF $P_3(G)$

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Abstract. The P_3 intersection graph of a graph G has for vertices all the induced paths of order 3 in G. Two vertices in $P_3(G)$ are adjacent if the corresponding paths in G are not disjoint.

A w-container between two different vertices u and v in a graph G is a set of w internally vertex disjoint paths between u and v. The length of a container is the length of the longest path in it. The w-wide diameter of G is the minimum number l such that there is a w-container of length at most l between any pair of different vertices u and v in G.

Interconnection networks are usually modeled by graphs. The w-wide diameter provides a measure of the maximum communication delay between any two nodes when up to $w-1$ nodes fail. Therefore, the wide diameter constitutes a measure of network fault tolerance.

In this paper we construct containers in $P_3(G)$ and apply the results obtained to the study of their connectivity and wide diameters.

Keywords: P_3 intersection graph, connectivity, container, wide diameter

MSC 2010: 05C40, 05C76, 05C99

1. Introduction

Interconnection networks are usually modeled by graphs whose vertices represent nodes and whose edges are associated with the communication links between nodes. Then, a path between two vertices in the graph represents a possible communication route between the corresponding nodes in the network. Moreover, the path length gives a measure of the communication delay experienced when communicating through the route it represents. Thus, the diameter of the graph, defined as the maximum distance between any two different nodes, represents the maximum communication delay between any two nodes of the network. As a consequence, when finding interconnection network models it is important for the diameter to be as small as possible.

A good interconnection network model should also remain communicating, and with reasonable efficiency, in the presence of faults. The graph connectivity measures the maximum number of faults that a network can tolerate to remain communicating. The efficiency of the communication under a given number of faults is quantified by the wide diameter, which measures the increase in the maximum communication delay between any two nodes in the networks.

Many graphs presenting good properties as interconnection network models can be obtained using graph operators [5]. In particular, the line graph has been widely used to obtain good interconnection network models. The path graph operator [1] has also been shown to produce good interconnection network models. In fact, the path graph operator can yield graphs whose degree and diameter are similar to those of a line graph but use less edges [4]. In this paper we explore the use of the P_3 intersection graph operator [6] as a possible way to build new interconnection network models.

2. Definitions and notation

Let $G = (V, E)$ be a simple graph (i.e. without loops or multiple edges) with $n = |V|$ vertices and $m = |E|$ edges. Let u and v be any two vertices in G. A path of length l between u and v is a sequence $u = x_0, x_1, \ldots, x_l = v$ such that $(x_i, x_{i+1}) \in E$ for every $i = 0, \ldots, l-1$. If $u = v$ the path is a cycle. A graph is connected if there is a path between any two different vertices. The distance between u and v in G, denoted as $d_G(u, v)$, is the length of a shortest path between u and v. When the graph G is clear from the context, we will abbreviate $d_G(u, v)$ to $d(u, v)$. If the graph G is connected, the diameter of G, denoted as $\mathrm{diam}(G)$, is the maximum distance between any two vertices in G. If G has a cycle, the girth of G, denoted as $\mathrm{girth}(G)$, is the length of a shortest cycle in G. We refer the reader to [2] for graph theory concepts not presented in this section.

Let us assume that the graph G is connected. Then the vertex connectivity $\kappa(G)$ of G is defined as the minimum cardinality of a set of vertices S such that $G - S$ is either disconnected or trivial (i.e. consists of one isolated vertex). As a consequence of Menger's Theorem, $\kappa(G)$ is the minimum number of internally vertex disjoint paths between any two different vertices in G. Analogously, the edge connectivity $\kappa'(G)$ of G is defined as the minimum cardinality of a set of edges whose deletion from G results in a disconnected graph. Using Menger's Theorem again, $\kappa'(G)$ is the minimum number of edge disjoint paths between any two different vertices in G. Since two edge disjoint paths must be internally vertex disjoint, then $\kappa(G) \leqslant \kappa'(G) \leqslant \delta(G)$. A graph G is maximally connected if $\kappa(G) = \kappa'(G) = \delta(G)$.

The following concepts were introduced by Du et al. in [3] and later studied by several authors. Let u and v be a pair of vertices of G, $u \neq v$. A container between

u and v is a set $C(u,v)$ of internally vertex disjoint paths between u and v. The length of the container $C(u,v)$, denoted as $l(C(u,v))$, is the length of the longest path in $C(u,v)$. The cardinality of $C(u,v)$, denoted as $w(C(u,v))$, is the width of the container $C(u,v)$. If $C(u,v)$ has width w, we will denote it as $C_w(u,v)$. Notice that the maximum integer w such that there exists a container of width w between every pair of different vertices is the vertex connectivity $\kappa(G)$.

The w-wide distance between any two different vertices u and v is $d_w(u,v)$ defined as the minimum length of a container of width w between u and v. The w-wide diameter of a connected graph G is denoted by $D_w(G)$ and defined as

$$D_w(G) = \max\{d_w(u,v)\colon\ u,v \in V\}.$$

Thus, $D_w(G)$ is the minimum number l such that there exists a container of width w and length at most l between any pair of different vertices u and v. Notice that $D_w(G) = \infty$ if $w > \kappa(G)$, so it is only interesting to study $D_w(G)$ when $1 \leqslant w \leqslant \kappa(G)$.

The P_3 intersection graph is a graph operator introduced by Menon and Vijaykumar in [6]. For a given graph G, the P_3 intersection graph of G, denoted as $P_3(G)$, has for vertices all the induced paths of order 3 in G. Two different vertices of $P_3(G)$ are adjacent if the corresponding paths in G intersect.

Notice that the P_3 intersection graph operator differs from the path graph operator introduced by Broersma and Hoede [1] and widely studied throughout the literature. Indeed, for a positive integer k, the k-path graph of a graph G has for vertices the set of all paths of length k. Two vertices are adjacent whenever the intersection of the corresponding paths forms a path of length $k-1$ in G and their union forms either a cycle or a path of length $k+1$ in G. Therefore, since every induced path of order 3 is also a path of length 2, the set of vertices of the graph $P_3(G)$ is a subset of the set of vertices of the 2-path graph of G. Also, for every graph G, the graph $P_3(G)$ is an induced subgraph of the intersection graph of all paths of length 3 in G, denoted as $\mathrm{Int}(P_3,G)$. Other results on $P_3(G)$ intersection graphs can be found in [7]. For a survey on graph operators we refer the reader to [8].

Figure 1 illustrates the differences between the above mentioned operators.

Notice that every path of length 2 in G forms a vertex in the 2-path graph of G and in the intersection graph $\mathrm{Int}(P_3,G)$. However, the paths $v_1v_2v_3$, $v_1v_3v_2$ and $v_2v_1v_3$ do not form vertices in the graph $P_3(G)$ because they are not induced paths since v_1,v_2,v_3,v_1 is a cycle in G. Also, the vertices $v_1v_3v_4$ and $v_2v_3v_4$ are adjacent in $P_3(G)$ and in the intersection graph $\mathrm{Int}(P_3,G)$. However, they are not adjacent in the 2-path graph of G because the union of the paths $v_1v_3v_4$ and $v_2v_3v_4$ in G does not form a path or a cycle in G.

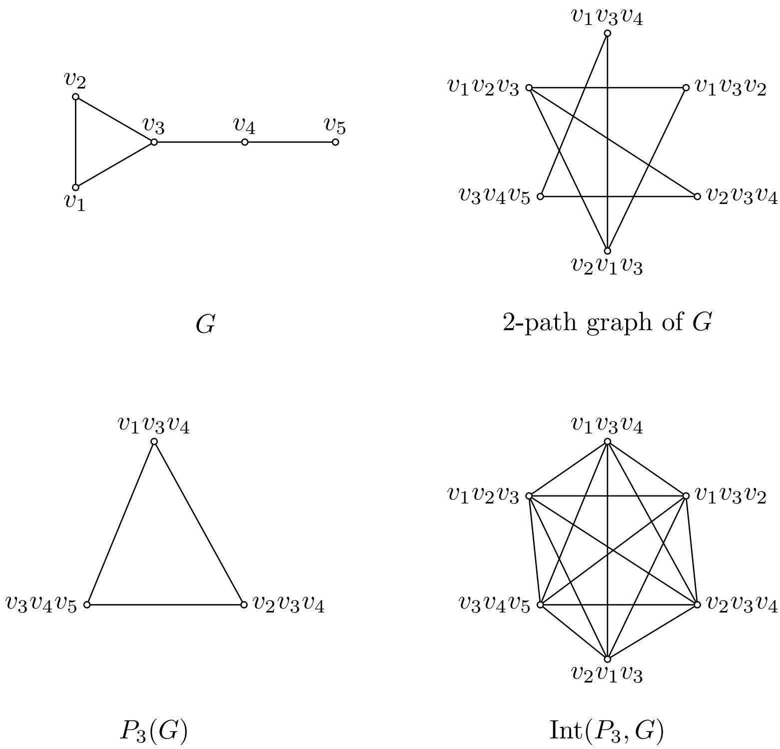

Figure 1: A graph G, its 2-path graph, $P_3(G)$ and $\mathrm{Int}(P_3, G)$.

In this paper we study containers in $P_3(G)$ and apply the results obtained to the study of connectivity and wide diameters.

3. Containers and connectivity

We recall that the P_3 intersection path graph of a given graph G is the graph $P_3(G)$ that has for vertices the induced paths of order 3 in G. Two distinct vertices in $P_3(G)$ are adjacent if the corresponding paths intersect. Throughout this paper, if a_1, a_2, a_3 is an induced path of order 3 in G then the corresponding vertex in $P_3(G)$ will be denoted as $a_1a_2a_3$.

Notice that if G is a connected graph of order at most 5, then $P_3(G)$ is either empty or complete. Furthermore, if G is the result of removing an arbitrary edge e from a complete graph, that is, $G = K_n - \{e\}$ for some integer $n \geqslant 3$ and some edge $e \in E(K_n)$, then $P_3(G) = K_{n-2}$. Indeed, let us assume the vertices of G labeled as $\{v_1, \ldots, v_n\}$. Without loss of generality, we can assume e to be the edge (v_1, v_2). Then $P_3(G)$ has $n-2$ vertices, namely, $v_1v_iv_2$ with $i = 3, \ldots, n$. Besides, there is an edge between any two vertices, because they all share the same endpoints. As a consequence, $P_3(G) = K_{n-2}$.

Lemma 3.1. *Let G be a graph of order n different from $K_n - \{e\}$. If there exists a container of width w between any two distinct vertices in G, then there exists a container of width w between any two distinct vertices in $P_3(G)$.*

Proof. Let us assume there exists a container of width w between any two distinct vertices in G. Let us also assume that $u_1u_2u_3$ and $v_1v_2v_3$ are two distinct vertices in $P_3(G)$. We distinguish two cases and give a construction of a $C_w(u_1u_2u_3, v_1v_2v_3)$ in $P_3(G)$ for each case.

Case 1. Suppose there exist $u_i \in \{u_1, u_2, u_3\}$ and $v_j \in \{v_1, v_2, v_3\}$ such that u_i is not adjacent to v_j. Let $u_i, a_1, a_2, a_3, \ldots, a_{k-1}, v_j \in C_w(u_i, v_j)$. If it is an induced path then $u_1u_2u_3, u_ia_1a_2, \ldots, a_{k-2}a_{k-1}v_j, v_1v_2v_3$ is a path joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$. If it is not an induced path, then there exists an induced path $u_i, b_1, b_2, \ldots, b_m, v_j$ where the internal vertices satisfy $b_1, b_2, \ldots, b_m \in \{a_1, a_2, \ldots, a_{k-1}\}$. Since all paths in $C_w(u_i, v_j)$ are internally vertex disjoint, the new paths in $P_3(G)$ joining $u_1u_2u_3$ and $v_1v_2v_3$ obtained by the previous procedure are also internally vertex disjoint.

Case 2. Suppose that each $u_i \in \{u_1, u_2, u_3\}$ is adjacent to a vertex $v_j \in \{v_1, v_2, v_3\}$. In particular, let us assume that u_1 is adjacent to u_p for $v_p \in \{v_1, v_2, v_3\}$ and let us consider a container $C_w(u_1, v_p)$. Then we proceed with each path of length at least 2 in $C_w(u_1, v_p)$ as we did in Case 1 and obtain a set of internally vertex disjoint paths in $P_3(G)$ joining $u_1u_2u_3$ and $v_1v_2v_3$. However, since u_1 is adjacent to u_p, there is a possibility that the path u_1, v_p is in $C_w(u_1, v_p)$, so the previous process will only lead to $w - 1$ internally disjoint paths joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$. In that case we can add the path $u_1u_2u_3, u_2u_1v_p, v_1v_2v_3$ to the $w - 1$ paths previously obtained. Clearly, this leads to a $C_w(u_1u_2u_3, v_1v_2v_3)$ in $P_3(G)$. □

Lemma 3.2. *Let G be a graph of order n different from $K_n - \{e\}$. Then $\kappa(G) \leqslant \kappa(P_3(G)) \leqslant 9\kappa(G)$.*

Proof. Let G be a connected graph with vertex connectivity $\kappa(G)$. From the above lemma we conclude that $\kappa(G) \leqslant \kappa(P_3(G))$.

Let $u_1u_2u_3$ and $v_1v_2v_3$ be any two non adjacent vertices in $P_3(G)$. According to Menger's theorem, there exists $\kappa(G)$ internally vertex disjoint paths between u_i and v_j for $i, j \in \{1, 2, 3\}$. Since there are 9 pairs of vertices u_i and v_j for $i, j \in \{1, 2, 3\}$, the set of all those paths has cardinality $9\kappa(G)$. Since every path joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ is associated with a path between a pair of vertices u_i and v_j for $i, j \in \{1, 2, 3\}$ in G, having more than $9\kappa(G)$ paths between an arbitrary pair of different vertices $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ implies having more than $\kappa(G)$ between every pair of different vertices u_i and v_j for $i, j \in \{1, 2, 3\}$ in G. $\square$

Lemma 3.3. *For any connected graph G, $\Delta(P_3(G)) \leqslant (9\Delta^2(G) - 17\Delta(G) + 6)/2$.*

Proof. Let $u_1u_2u_3$ be a vertex in $P_3(G)$. Then $u_1u_2u_3$ has the maximum degree in $P_3(G)$ when u_1, u_2, u_3 and all their neighbors in G have degree $\Delta(G)$. Notice that the number of induced paths of order 3 will be maximum if G is K_3-free. Let u_{ij} denotes the neighbors of u_i and u_{ijk} denotes the neighbors of u_{ij}. The induced paths of order 3 containing u_1 are u_1, u_{1i}, u_{1ij}; u_{1i}, u_1, u_{1j}; u_{1i}, u_1, u_2, where $i, j, k \in \{1, 2, \ldots, \Delta(G)-1\}$ which is $(\Delta(G)-1)(\Delta(G)-1)+(\Delta(G)-1)(\Delta(G)-2)/2+(\Delta(G) - 1)$ in number. Similarly for u_3. The induced paths of order 3 containing u_2 are u_2, u_{2i}, u_{2ij}; u_{2k}, u_2, u_{2i}; u_{2i}, u_2, u_1; u_{2i}, u_2, u_3 where $i, k \in \{1, 2, \ldots, \Delta(G) - 2\}$; $j \in \{1, 2, \ldots, \Delta(G) - 1\}$. Therefore, the total number of induced paths of order 3 containing either u_1, u_2 or u_3 is $(\Delta(G) - 2)(\Delta(G) - 1) + (\Delta(G) - 2)(\Delta(G) - 3)/2 + 2(\Delta(G) - 2)$. The mentioned induced paths of order 3 correspond to all the neighbors of $u_1u_2u_3$ in $P_3(G)$. Thus, the degree of $u_1u_2u_3$ is $d(u_1u_2u_3) \leqslant 2[(\Delta(G)-1)(\Delta(G)-2)/2 + (\Delta(G) - 1)(\Delta(G) - 1) + (\Delta(G) - 1)] + (\Delta(G) - 2)(\Delta(G) - 3)/2 + (\Delta(G) - 2)(\Delta(G) - 1) + 2(\Delta(G) - 2) \leqslant (9\Delta^2(G) - 17\Delta(G) + 6)/2$. $\square$

Notice that here exists infinitely many graphs attaining the above upper bound, as the following theorem proves.

Theorem 3.4. *If G is a d-regular graph with* $\operatorname{girth}(G) \geqslant 6$, *then $P_3(G)$ is a d'-regular graph where $d' = (9d^2 - 17d + 6)/2$. Moreover, $P_3(G)$ is maximally edge connected.*

Proof. Let G be a d-regular graph. Let $u_1u_2u_3$ be any vertex in $P_3(G)$. Let the neighbors of u_i be denoted by u_{ij}, $j = 1, \ldots, d$ and the neighbors of u_{ij} be denoted as u_{ijk}, $k = 1, \ldots, d$. Since $\operatorname{girth}(G) \geqslant 6$, the vertices u_1, u_2 and u_3 do not have common neighbors. Similarly, the neighbohoods of $u_{li}, l = \{1, 3\}$ and u_{2j} for $i \in \{1, 2, \ldots, d-1\}$, $j \in \{1, 2, \ldots, d-2\}$ are mutually disjoint. Notice that since $\operatorname{girth}(G) \geqslant 6$, u_{1i} and u_{3j} do not have common neighbors. Then, the only vertices in $P_3(G)$ which are adjacent to $u_1u_2u_3$ are the following ones.

The vertices corresponding to the induced paths of order 3 in G that contain the vertex u_1 are $u_1u_{1i}u_{1ij}$ and $u_{3i}u_1u_{1k}$, $i \neq k$ where $i, j, k \in \{1, 2, \ldots, d-1\}$. Similarly for the induced paths of order 3 that contain the vertex u_3. Thus, there are $2[(d-1)^2+(d-1)(d-2)/2]$ vertices in $P_3(G)$ adjacent to $u_1u_2u_3$. Now, the vertices in $P_3(G)$ corresponding to the induced paths of length 3 in G that include the vertex u_2 are in the form $u_2u_{2i}u_{2ij}$ or $u_{2i}u_2u_{2k}$; $i \neq k$ where $i, k \in \{1, 2, \ldots, d-2\}$ and $j \in \{1, 2, d-1\}$, so there is a total of $(d-2)(d-1)+(d-2)(d-3)/2$ such vertices. There also exist vertices in $P_3(G)$ corresponding to the induced paths of order 3 in G which have common vertices u_1 and u_2 which are of the form $u_{1,i}$, u_1, u_2 and u_1, u_2, $u_{2,j}$, $i \in \{1,2,\ldots,d-1\}$ and $j \in \{1,2,\ldots,d-2\}$. Similarly, there exist $2[(d-1)+(d-2)]$ vertices in $P_3(G)$ corresponding to the induced paths of order 3 having vertices u_2 and u_3 in common. Thus, the total number of vertices adjacent to $u_1u_2u_3$ is $2[(d-1)^2+(d-1)(d-2)/2]+(d-2)(d-1)+(d-2)(d-3)/2+2[(d-1)+(d-2)] = (9d^2-17d+6)/2$.

Next we prove that $P_3(G)$ is maximally edge connected. That is, $\kappa(P_3(G)) = d'$. To this end, we prove that between any two adjacent vertices $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ there exist d' edge disjoint paths.

Case 1. Let us assume that u_1,u_2,u_3 and v_1,v_2,v_3 are two induced paths of order 3 with two vertices in common. Notice that the case $u_1 = v_1$, $u_3 = v_3$ and $u_2 \neq v_2$ cannot arise because $\mathrm{girth}(G) > 6$. Therefore, without loss of generality, let us assume $u_1 = v_1$ and $u_2 = v_2$, and $u_3 \neq v_3$.

Consider the following d' edge disjoint paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$.

$u_1u_2u_3,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_3u_2v_3,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_{1x}u_1u_{1y},\ v_1v_2v_3;\ x,y \in \{1,2,\ldots,d\}, x \neq y,\ u_{1x} \neq u_2,\ u_{1y} \neq u_2$.

$u_1u_2u_3,\ u_1u_{1x}u_{1xy},\ v_1v_2v_3;\ x,y \in \{1,2,\ldots,d\},\ u_{1x} \neq u_2$.

$u_1u_2u_3,\ u_{2x}u_2u_{2y},\ v_1v_2v_3;\ x,y \in \{1,2,\ldots,d\}, x \neq y,\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ u_{2x} \neq v_3$.

$u_1u_2u_3,\ u_2u_{2x}u_{2xy},\ v_1v_2v_3;\ x,y \in \{1,2,\ldots,d\},\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ u_{2x} \neq v_3,\ u_{2xy} \neq u_2$.

$u_1u_2u_3,\ u_1u_2u_{2x},\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ u_{2x} \neq v_3$.

$u_1u_2u_3,\ u_{1x}u_1u_2,\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{1x} \neq u_2$.

$u_1u_2u_3,\ u_3u_2u_{2x},\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ u_{2x} \neq v_3$.

$u_1u_2u_3,\ v_3u_2u_{2x},\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ u_{2x} \neq v_3$.

$u_1u_2u_3,\ u_2u_3u_{3x},\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{3x} \neq u_2$.

$u_1u_2u_3,\ u_2v_3v_{3x},\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ v_{3x} \neq u_2$.

$u_1u_2u_3,\ u_{3x}u_3u_{3y},\ u_3u_2u_{2x},\ u_{2x}u_2u_{2y},\ u_{2y}u_2v_3,\ v_{3x}u_3v_{3y},\ v_1v_2v_3;\ x,y \in \{1,2,\ldots,d\}$, $x \neq y,\ u_{3x} \neq u_2,\ u_{3y} \neq u_2,\ v_{3x} \neq u_2,\ v_{3y} \neq u_2,\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ u_{2x} \neq v_3$.

$u_1u_2u_3,\ u_3u_{3x}u_{3xy},\ u_3u_2v_3,\ v_3v_{3x}v_{3y},\ v_1v_2v_3;\ x,y \in \{1,2,\ldots,d\}, x \neq y,\ u_{3x} \neq u_2$, $v_{3x} \neq u_2$.

Case 2. Let the induced 3-paths u_1,u_2,u_3 and v_1,v_2,v_3 have a vertex in common. We distinguish the following cases:

Case 2a. Let $u_1 = v_1$. This case is equivalent to $u_3 = v_3$, $u_1 = v_3$ or $u_3 = v_1$.

Then, the d' edge disjoint paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ are the following ones:

$u_1u_2u_3,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_2u_1v_2,\ v_1v_2v_3$.

$u_1u_2u_3$, $u_{1x}u_1u_{1y}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}, x \neq y$, $u_{1x} \neq u_2$, $u_{1x} \neq v_2$.
$u_1u_2u_3$, $u_1u_{1x}u_{1xy}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $u_{1x} \neq u_2$, $u_{1x} \neq v_2$, $u_{1xy} \neq u_1$.
$u_1u_2u_3$, $u_1u_2u_{2x}$, $v_1v_2v_3$; $x \in \{1, 2, \ldots, d\}$, $u_{2x} \neq u_1$, $u_{2x} \neq u_3$.
$u_1u_2u_3$, $u_1v_2v_{2x}$, $v_1v_2v_3$; $x \in \{1, 2, \ldots, d\}$, $v_{2x} \neq v_1$, $v_{2x} \neq v_3$.
$u_1u_2u_3$, $u_{2x}u_2u_{2y}$, $u_2u_1v_2$, $v_{2x}v_2v_{2y}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $x \neq y$, $u_{2x} \neq u_1$, $u_{2x} \neq u_3$, $v_{2x} \neq v_1$, $v_{2x} \neq v_3$
$u_1u_2u_3$, $u_2u_{2x}v_{2xy}$, $u_2u_1v_2$, $v_2v_{2x}v_{2xy}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}, x \neq y$, $u_{2x} \neq u_1$, $v_{2x} \neq v_1$.
$u_1u_2u_3$, $u_{3x}u_3u_{3y}$, $u_3u_2u_{2x}$, $u_{2x}u_2u_{2y}$, $u_2u_1v_2$, $v_{2x}v_2v_{2y}$, $v_3v_2v_{2y}$, $v_{3x}v_3v_{3y}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $x \neq y$, $u_{2x} \neq u_1$, $u_{2x} \neq u_3$, $u_{3x} \neq u_2$, $v_{2x} \neq v_1$, $v_{2x} \neq v_3$, $v_{3x} \neq v_2$, $v_{2y} \neq v_3$.
$u_1u_2u_3$, $u_3u_{3x}u_{3xy}$, $u_{3x}u_3u_2$, $u_2u_1v_2$, $v_2v_3v_{3x}$, $v_3v_{3x}v_{3xy}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $x \neq y$, $u_{3x} \neq u_2$, $u_{3x} \neq v_2$.
$u_1u_2u_3$, $u_{3x}u_3u_2$, $u_{2x}u_2u_1$, $u_1v_2v_{2x}$, $v_2v_3v_{3x}$, $v_1v_2v_3$; $x \in \{1, 2, \ldots, d\}$, $u_{3x} \neq u_2$, $u_{2x} \neq u_1$, $u_{2x} \neq u_3$, $v_{3x} \neq v_2$, $v_{2x} \neq v_1$, $v_{2x} \neq v_3$
$u_1u_2u_3$, $u_3u_2u_{2x}$, $u_2v_1v_2$, $v_3v_2v_{2x}$, $v_1v_2v_3$; $x \in \{1, 2, \ldots, d\}$, $u_{2x} \neq u_1$, $u_{2x} \neq u_3$, $v_{2x} \neq v_1$, $v_{2x} \neq v_3$.

Case 2b. Let $u_1 = v_2$. This case is equivalent to $u_3 = v_2$, $v_1 = u_2$ or $v_3 = u_2$. Then, the d' edge disjoint paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ are the following ones:
$u_1u_2u_3$, $v_1v_2v_3$.
$u_1u_2u_3$, $u_2u_1v_1$, $v_1v_2v_3$.
$u_1u_2u_3$, $u_2u_1v_3$, $v_1v_2v_3$.
$u_1u_2u_3$, $u_{1x}u_1u_{1y}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $x \neq y$, $u_{1x} \neq u_2$, $u_{1x} \neq v_1$, $u_{1x} \neq v_3$, $u_{1y} \neq u_2$, $u_{1y} \neq v_1$, $u_{1y} \neq v_3$.
$u_1u_2u_3$, $u_1u_{1x}u_{1xy}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $u_{1x} \neq u_2$, $u_{1x} \neq v_1$, $u_{1x} \neq v_3$.
$u_1u_2u_3$, $u_1u_2u_{2x}$, $v_1v_2v_3$; $x \in \{1, 2, \ldots, d\}$, $u_{2x} \neq u_1$, $u_{2x} \neq u_3$.
$u_1u_2u_3$, $u_1v_1v_{1x}$, $v_1v_2v_3$; $x \in \{1, 2, \ldots, d\}$, $v_{1x} \neq u_1$.
$u_1u_2u_3$, $u_1v_3v_{3x}$, $v_1v_2v_3$; $x \in \{1, 2, \ldots, d\}$, $v_{3x} \neq u_1$.
$u_1u_2u_3$, $u_{2x}u_2u_{2y}$, $u_2u_1v_1$, $v_{1x}v_1v_{1y}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $x \neq y$, $u_{2x} \neq u_1$, $u_{2y} \neq u_1$, $u_{2x} \neq u_3$, $u_{2y} \neq u_3$, $v_{1x} \neq v_2$, $v_{1y} \neq v_2$.
$u_1u_2u_3$, $u_2u_{2x}u_{2xy}$, $u_2u_1v_1$, $v_1v_{1x}v_{1xy}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}, x \neq y$, $u_{2x} \neq u_1$, $u_{2x} \neq u_3$,$v_{1x} \neq v_2$.
$u_1u_2u_3$, $u_{3x}u_3u_{3y}$, $u_{3x}u_3u_2$, $u_2u_1v_1$, $v_{1x}v_1v_{1y}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $x \neq y$, $u_{3x} \neq u_2$, $u_{3y} \neq u_2$, $v_{1x} \neq u_1$, $v_{1y} \neq u_1$.
$u_1u_2u_3$, $u_3u_{3x}u_{3xy}$, $u_{3x}u_3u_2$, $u_2u_1v_3$, $v_3v_{3x}v_{3xy}$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $u_{3x} \neq u_2$, $v_{3x} \neq v_2$.
$u_1u_2u_3$, $u_1v_1v_{1x}$, $v_{1x}v_1v_{1y}$, $v_{1x}v_1v_2$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $v_{1x} \neq v_{1y} \neq v_2$.
$u_1u_2u_3$, $u_1v_3v_{3x}$, $v_{3x}v_3v_{3y}$, $v_{3x}v_1v_2$, $v_1v_2v_3$; $x, y \in \{1, 2, \ldots, d\}$, $v_{3x} \neq v_{3y} \neq v_2$.

$u_1u_2u_3,\ u_{3x}u_3u_2,\ u_{2x}u_2u_1,\ u_1v_1v_{1x},\ u_1v_3v_{3x},\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{3x} \neq u_2,\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ v_{3x} \neq v_2,\ v_{2x} \notin \{v_1, v_3, u_2\}$.

$u_1u_2u_3,\ u_3u_2u_{2x},\ u_2u_1v_3,\ v_3v_2v_{2x},\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{2x} \neq u_1,\ u_{2x} \neq u_3,\ v_{2x} \notin \{v_1, v_3, u_2\}$.

Case 2c. Let $u_2 = v_2$. Then, the d' edge disjoint paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ are the following ones:

$u_1u_2u_3,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_1u_2v_1,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_1u_2v_3,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_3u_2v_1,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_3u_2v_3,\ v_1v_2v_3$.

$u_1u_2u_3,\ u_{1x}u_1u_2,\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{1x} \neq u_2$.

$u_1u_2u_3,\ u_{3x}u_3u_2,\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ u_{3x} \neq u_2$.

$u_1u_2u_3,\ v_{1x}v_1u_2,\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ v_{1x} \neq u_2$.

$u_1u_2u_3,\ v_{3x}v_3u_2,\ v_1v_2v_3;\ x \in \{1,2,\ldots,d\},\ v_{3x} \neq u_2$.

$u_1u_2u_3,\ u_{2x}u_2u_{2y},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ x \neq y,\ u_{2x} \notin \{u_1, u_3, v_1, v_3\},\ u_{2y} \notin \{u_1, u_3, v_1, v_3\}$.

$u_1u_2u_3,\ u_2u_{2x}u_{2xy},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ u_{2x} \notin \{u_1, u_3, v_1, v_3\},\ u_{2xy} \neq u_2$.

$u_1u_2u_3,\ u_1u_{1x}u_{1xy},\ u_2u_1u_{1x},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ u_{1x} \neq u_2$.

$u_1u_2u_3,\ u_3u_{3x}u_{3xy},\ u_2u_3u_{3x},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ u_{3x} \neq u_2$.

$u_1u_2u_3,\ u_2v_1v_{1x},\ v_1v_{1x}v_{1xy},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ v_{1x} \neq u_2$.

$u_1u_2u_3,\ u_2v_3v_{3x},\ v_3v_{3x}v_{3xy},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ v_{3x} \neq u_2$.

$u_1u_2u_3,\ u_{1x}u_1u_{1y},\ u_1u_2v_1,\ v_{1x}v_1v_{1y},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ x \neq y,\ u_{1x} \neq u_2,\ u_{1y} \neq u_2,\ v_{1x} \neq u_2,\ v_{1y} \neq u_2$.

$u_1u_2u_3,\ u_{3x}u_3u_{3y},\ u_3u_2v_3,\ v_{3x}v_3v_{3y},\ v_1v_2v_3;\ x, y \in \{1,2,\ldots,d\},\ x \neq y,\ u_{3x} \neq u_2,\ u_{3y} \neq u_2,\ v_{3x} \neq u_2,\ v_{3y} \neq u_2$. □

4. Containers and wide diameters

Theorem 4.1. *Let G be a connected graph with vertex connectivity $\kappa(G)$. For every integer w with $1 < w \leqslant \kappa(G)$,*

$$D_w(P_3(G)) \leqslant \lceil D_w(G)/2 \rceil + 1.$$

Proof. Let $D_w(G) = l$. Then there exists a container of width w and length at most l between any two vertices u and v in G. Let $u_1u_2u_3$ and $v_1v_2v_3$ be any two vertices in $P_3(G)$. Then by Lemma 2.1, there exists a container $C_w(u_1u_2u_3, v_1v_2v_3)$ corresponding to the container $C_w(u_i, v_j)$. If $C_w(u_i, v_j)$ has length l, then the length of $C_w(u_1u_2u_3, v_1v_2v_3)$ is also l. Let $u_i, a_1, a_2, \ldots, a_{l-1}, v_j$ be a path of length l in $C_w(u_i, v_j)$. This corresponds to a path of maximum length in $C_w(u_1u_2u_3, v_1v_2v_3)$. Clearly the maximum length occurs when $u_i, a_1, a_2, \ldots, a_{l-1}, v_j$ in $C_w(u_i, v_j)$ is an induced path. Then the path $u_1u_2u_3, u_ia_1a_2, a_2a_3a_4, a_4a_5a_6, \ldots, a_{k-2}a_{k-1}v_j, v_1v_2v_3$ is in $C_w(u_1u_2u_3, v_1v_2v_3)$ and has length $\lceil (l+1)/2 \rceil$ if l is even and $\lceil l/2 \rceil + 1$ if l is odd. It only remains to observe that if l is even, then $\lceil (l+1)/2 \rceil = \lceil l/2 \rceil + 1$. $\square$

Theorem 4.2. *Let G be a connected graph and let w be an integer such that $\kappa(P_3(G)) \geqslant w > \kappa(G)$. Then*

$$D_w(P_3(G)) \leqslant \max\{D_\beta(G),\ 1 \leqslant \beta \leqslant \kappa(G)\}.$$

Proof. Let $u_1u_2u_3$ and $v_1v_2v_3$ be any two vertices in $P_3(G)$. Then there exist $\kappa(P_3(G))$ paths between them. Without loss of generality, choose $u_1u_2u_3$ and $v_1v_2v_3$ in such a way that there exists a container of length $l = D_w(P_3(G))$ between them. Then there is a path of length l between u_i and v_j, $u_i \in \{u_1u_2u_3\}$ and $v_j \in \{v_1v_2v_3\}$. In that path, there is an induced path of length say, l', joining the vertices u_i and v_j. Then $d_\beta(u,v) \geqslant l'$. Thus $l \leqslant l' \leqslant d_\beta(u,v) \leqslant D_\beta(G)$ for any $1 \leqslant \beta \leqslant \kappa(G)$. $\square$

5. Future research

This paper explores the possible use of the $P_3(G)$ intersection graph in the design of interconnection networks. While the results are auspicious, many questions remain open. The following ones are just a sample:

1. Theorem 3.4 shows that if a graph G satisfies certain conditions, then $P_3(G)$ is regular and maximally edge connected. It would be interesting to find families of graphs G satisfying the conditions in the theorem for which the diameter of $P_3(G)$ is relatively small in comparison with its order and degree. That would yield a family of graphs that has good properties for the degree/diameter problem [2].
2. In Section 4 we presented upper bounds for the w-wide diameters of $P_3(G)$ in terms of the corresponding w-wide diameters of G. However, it would also be interesting to know upper bounds for $D_w(P_3(G))$ in terms of the diameter of $P_3(G)$.
3. The P_3 intersection graph of a graph G has for vertices all induced paths of order 3 and the 2-path graph of G has for vertices all paths of length 2. That is the set of vertices is the same if G has girth at least 4. In this case it will be interesting to compare the properties of the P_3 intersection graph of a graph G

with those of the 2-path graph of G and determine under what circumstances a P_3 intersection graph or a 2-path graph would constitute a more suitable interconnection network model.

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