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CONVERGENCE MODEL OF INTEREST RATES OF CKLS TYPE

Zuzana Zíková and Beáta Stehlíková

This paper deals with convergence model of interest rates, which explains the evolution of interest rate in connection with the adoption of Euro currency. Its dynamics is described by two stochastic differential equations – the domestic and the European short rate. Bond prices are then solutions to partial differential equations. For the special case with constant volatilities closed form solutions for bond prices are known. Substituting its constant volatilities by instantaneous volatilities we obtain an approximation of the solution for a more general model. We compute the order of accuracy for this approximation, propose an algorithm for calibration of the model and we test it on the simulated and real market data.

1. INTRODUCTION

Term structure of interest rates (also called yield curve) explains the relation between the time to maturity of a discount bond and its present price. A discount bond is a security which pays its holder a unit amount of money at specified time T (called maturity). Let $P(t,T)$ be the price of a discount bond with maturity T at time t. It is connected with the corresponding interest rate $R(t,T)$, which defines term structure of interest rates, by formulae

$$P(t,T) = e^{-R(t,T)(T-t)}, \quad \text{i.e.} \quad R(t,T) = -\frac{\ln(P(t,T))}{T-t}. \tag{1}$$

Short rate (or instantaneous interest rate) is an interest rate for infinitesimally short time. It represents the beginning of the yield curve: $r(t) = \lim_{T \to t^+} R(t,T)$. In practice it is approximated by an interest rate with short maturity. For a more detailed introduction to interest rates see for example [1, 8, 12].

Figure 1 shows an example of a short rate evolution and of a term structure at a given day. It is worth noting that before adopting the euro, the interest rates in the country are influenced by the rates in the eurozone. We illustrate this with Figure 2, where we show the Slovak and eurozone instantaneous interest rates in the last quarter before Slovakia adopted the euro currency. This feature is a base of so called convergence
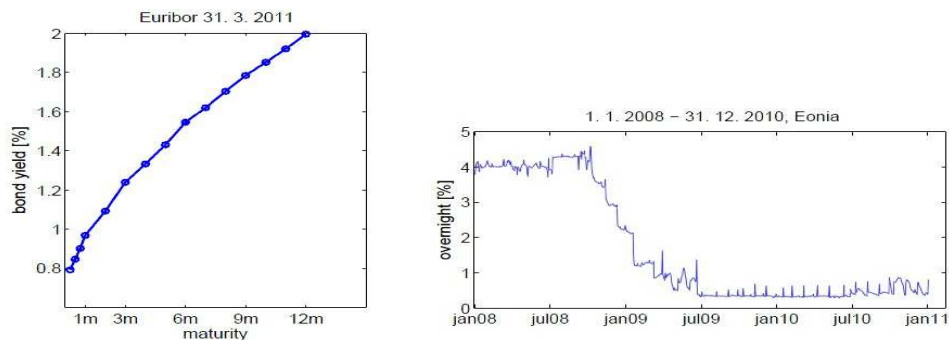
Fig. 1. Euribor – term structure od interest rate (left),
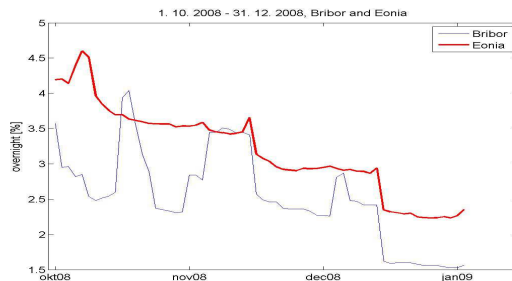Eonia – short rate (right).

Fig. 2. Instantaneous interest rate – Bribor and Eonia, last quarter
2008.

models of interest rate, which we study in this paper.

The paper is organized as follows: Section 2 is introductory and deals with one-factor and two-factor short rate models in general. Section 3 describes convergence models as a special class of two-factor models. In particular, we present a generalization of the known models which we study in the following sections. The closed form of the bond pricing equation is not known and hence in section 4 we propose an approximation formula for the domestic bond price and derive its accuracy. In sections 5-9 we deal with calibration algorithm – its description, simulation study and application to real market data. In the last section we give a some concluding remarks.

2. SHORT RATE MODELS

Short rate models are formulated by stochastic differential equation (SDE) for a variable X:

$$\mathrm{d}X = \mu(X,t)\mathrm{d}t + \sigma(X,t)\mathrm{d}W,$$

which defines the short rate $r = r(X)$. Here W is a Wiener process, function $\mu(X, t)$ is the trend or drift part and the volatility $\sigma(X, t)$ represents fluctuations around the drift. Choosing different drift $\mu(X, t)$ and volatility $\sigma(X, t)$ leads to various one-factor (where X is scalar) and multi-factor models (where X is vector). In short rate models, the bond prices are given as solutions to the partial differential equation. We address this question for specific classes of models in following sections.

2.1. One-factor models

In one-factor models the evolution of the short rate is given by one SDE:

$$\mathrm{d}r = \mu(r, t)\,\mathrm{d}t + \sigma(r, t)\,\mathrm{d}W. \tag{2}$$

The model can be set in two ways:

- using SDE in the real (i.e. observed) probability measure and specifying so called market price of risk,

- using SDE in the risk-neutral probability measure.

The volatilities are the same in both measures and for the drift function holds:

$$\text{(risk-neutral drift)} = \text{(real drift)} - \text{(market price of risk)} \times \text{(volatility)}, \tag{3}$$

see [8]. Considering the equation (2) in the real measure and market price of risk equal to $\lambda(r, t)$, the bond price $P(r, t, T)$ is a solution to the PDE (see [1]):

$$\frac{\partial P}{\partial t} + \mu(r, t)\frac{\partial P}{\partial r} + \frac{\sigma^2(r, t)}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0, \qquad \text{for} \quad r > 0, \quad t \in (0, T), \tag{4}$$

satisfying the terminal condition $P(r, T, T) = 1$.

We present some well known models and we refer the reader to [1, 11] for more detailed treatment. The common feature of these models, which will be useful later, is their linear drift in the risk-neutral measure:

$$\mathrm{d}r = (b_1 + b_2 r)\,\mathrm{d}t + \sigma(r, t)\,\mathrm{d}w. \tag{5}$$

The simplest model is Vasicek model which has a mean-reverting drift $\mu(r, t) = \kappa(\theta - r)$ and constant volatility $\sigma(r, t) = \sigma$ in the real measure, where $\kappa, \theta, \sigma > 0$ are constants. Assuming constant market price of risk λ we obtain the risk-neutral form (5) with $b_1 = \kappa\theta - \lambda\sigma$, $b_2 = -\kappa$, $\sigma(r, t) = \sigma$. The PDE (4) for the bond price $P(\tau, r)$ with maturity $\tau = T - t$ has a solution in the form:

$$P(r, \tau) = e^{A(\tau) - rD(\tau)}, \tag{6}$$

where functions $D(\tau), A(\tau)$ can be expressed as follows: (see, e.g. [1, 5, 8])

$$
\begin{aligned}
D(\tau) &= \frac{-1 + e^{b_2\tau}}{b_2}, \\
A(\tau) &= \left(\frac{-1 + e^{b_2\tau}}{b_2} - \tau\right) \times \left(\frac{-b_1}{b_2} - \frac{\sigma^2}{2b_2^2}\right) + \frac{\sigma^2}{4b_2^3}\left(1 - e^{b_2\tau}\right)^2.
\end{aligned}
\tag{7}
$$

Cox–Ingersoll–Ross (CIR) model also assumes the mean-reverting drift in the real measure, but the volatility is taken to be $\sigma(r,t) = \sigma\sqrt{r}$. If $\lambda(r,t) = \lambda\sqrt{r}$, we again obtain the risk-neutral process (5), this time with $b_1 = \kappa\theta, b_2 = -\kappa - \lambda\sigma, \sigma(r,t) = \sigma\sqrt{r}$. A solution of the PDE (4) again takes the form (6). Functions $A(\tau)$ and $D(\tau)$ are given by (see, e.g. [1, 5, 8])

$$A(\tau) = -\frac{2b_1}{\sigma_e^2}\ln\frac{2\theta e^{(\theta-b_2)\frac{\tau}{2}}}{(\theta - b_2)(e^{\theta\tau} - 1) + 2\theta}, \qquad D(\tau) = \frac{-2\left(e^{\theta\tau} - 1\right)}{(\theta - b_2)(e^{\theta\tau} - 1) + 2\theta}, \qquad (8)$$

where $\theta = \sqrt{b_2^2 + 2\sigma_e^2}$.

A convenient property of Vasicek and CIR models is the existence of explicit solutions to the bond pricing equation. However, their choice of volatility is not confirmed by analysis of real data. In their pioneering paper [2] Chan-Karolyi-Longstaff-Sanders (CKLS) considered a model with $\sigma(r,t) = \sigma r^\gamma$ with $\gamma \geq 0$. Most of the previously considered models, including Vasicek and CIR, were rejected when tested as restrictions to this more general model. The CKLS model does not admit a closed form expression for bond prices. Approximate analytical solutions and their accuracy were studied in [3], [11], [10].

2.2. Two-factor models

Let us consider a model defined by the following system of SDEs:

$$\begin{aligned}
dr &= \mu_r(r, x, t)\, dt + \sigma_r(r, x, t)\, dW_1, \\
dx &= \mu_x(r, x, t)\, dt + \sigma_x(r, x, t)\, dW_2,
\end{aligned} \qquad (9)$$

where $\rho \in (-1, 1)$ is the correlation between the increments of Wiener processes W_1 and W_2, i.e. $Cov(dW_1, dW_2) = \rho\, dt$. Process x is a random process, which is connected with instantaneous rate. It can be a long-term interest rate, a short-term interest rate in another country, etc. Relations between real and risk-neutral parameters are analogous as in the one-factor case:

(risk-neutral drift function)$_r$ = (real drift function)$_r$ − $\lambda_r(r, x, t) \times$ (volatility)$_r$,

(risk-neutral drift function)$_x$ = (real drift function)$_x$ − $\lambda_x(r, x, t) \times$ (volatility)$_x$,

where λ_r, λ_x are market prices of risk of the short rate and the factor x respectively.

If the short rate satisfies SDE (9) in the real measure and market prices of risk are $\lambda_r(r, x, t), \lambda_x(r, x, t)$, then the bond price P satisfies the following PDE (assuming that the factor x is positive):

$$\frac{\partial P}{\partial t} + (\mu_r(r, x, t) - \lambda_r(r, x, t)\sigma_r(r, x, t))\frac{\partial P}{\partial r} + (\mu_x(r, x, t) - \lambda_x(r, x, t)\sigma_x(r, x, t))\frac{\partial P}{\partial x}$$

$$+ \frac{\sigma_r(r, x, t)^2}{2}\frac{\partial^2 P}{\partial r} + \frac{\sigma_x(r, x, t)^2}{2}\frac{\partial^2 P}{\partial x} + \rho\sigma_r(r, x, t)\sigma_x(r, x, t)\frac{\partial^2 P}{\partial r\partial x} - rP = 0$$

for $r, x > 0$, $t \in (0, T)$ and the terminal condition $P(r, x, T) = 1$ for $r, x > 0$. The PDE is derived using Itô lemma and construction of riskless portfolio, see, e.g. [5, 1].

3. CONVERGENCE MODELS

Convergence models form a special class of two-factor models. A convergence model is used to model the entry of observed country into the monetary union (EMU). It describes the behavior of two short-term interest rates, the domestic one and the instantaneous short rate for EMU countries. European short rate is modeled using a one-factor model. It is assumed to have an influence on the evolution of the domestic short rate and hence it enters the SDE for its evolution. This kind of model was proposed for the first time in [4]. The model is based on Vasicek model, the volatilities of the short rates are constant. Analogical model of Cox–Ingersoll–Ross type, where the volatilities are proportional to the square root of the short rate, was considered in [6] and [7]. In the following sections we describe these two models and show how they price the bonds. Then we present a generalization with nonlinear volatility, which is analogous to the volatility in one-factor CKLS model.

3.1. Convergence model of Vasicek type

The first convergence model was proposed in the paper [4] by Corzo and Schwartz in the real probability measure:

$$
\begin{aligned}
\mathrm{d}r_d &= (a + b\,(r_e - r_d))\,\mathrm{d}t + \sigma_d\,\mathrm{d}W_d, \\
\mathrm{d}r_e &= (c\,(d - r_e))\,\mathrm{d}t + \sigma_e\,\mathrm{d}W_e,
\end{aligned}
\tag{10}
$$

where $Cov(dW_1, dW_2) = \rho\,\mathrm{d}t$. They considered constant market prices of risk, i.e. $\lambda_d(r_d, r_e, \tau) = \lambda_d$ and $\lambda_e(r_d, r_e, \tau) = \lambda_e$. Hence for the European interest rate we have one-factor Vasicek model and we can easily price European bonds. Coefficient $b > 0$ expresses the power of attracting the domestic short rate to the European one with the possibility of deviation determined by the coefficient a. Rewriting the model into risk-neutral measure we obtain:

$$
\begin{aligned}
\mathrm{d}r_d &= (a + b\,(r_e - r_d) - \lambda_d\sigma_d)\,\mathrm{d}t + \sigma_d\,\mathrm{d}W_d, \\
\mathrm{d}r_e &= (c\,(d - r_e) - \lambda_e\sigma_e)\,\mathrm{d}t + \sigma_e\,\mathrm{d}W_e,
\end{aligned}
\tag{11}
$$

where $Cov[dW_d, dW_e] = \rho\,\mathrm{d}t$. We consider a more general model in risk-neutral measure, in which the risk-neutral drift of the domestic short rate is given by a general linear function of variables r_d, r_e and the risk-neutral drift of the European short rate is a general linear function of r_e. It means that the evolution of the domestic and the European short rates is given by:

$$
\begin{aligned}
\mathrm{d}r_d &= (a_1 + a_2 r_d + a_3 r_e)\,\mathrm{d}t + \sigma_d\,\mathrm{d}W_d, \\
\mathrm{d}r_e &= (b_1 + b_2 r_e)\,\mathrm{d}t + \sigma_e\,\mathrm{d}W_e,
\end{aligned}
\tag{12}
$$

where $Cov[dW_d, dW_e] = \rho\,\mathrm{d}t$. Note that the system (12) corresponds to the system (11) with $a_1 = a - \lambda_d\sigma_d$, $a_2 = -b$, $a_3 = b$, $b_1 = cd - \lambda_e\sigma_e$, $b_2 = -c$. Price $P(r_d, r_e, \tau)$ of a bond with time to maturity $\tau = T - t$ then satisfies the PDE

$$
\begin{aligned}
&-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P}{\partial r_e} \\
&+ \frac{\sigma_d^2}{2}\frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_e^2}{2}\frac{\partial^2 P}{\partial r_e^2} + \rho\sigma_d\sigma_e\frac{\partial^2 P}{\partial r_d \partial r_e} - r_d P = 0,
\end{aligned}
\tag{13}
$$

for $r_d, r_e > 0$, $\tau \in (0, T)$ and the initial condition $P(r_d, r_e, 0) = 1$ for $r_d, r_e > 0$. Its solution can be found in the same way as in the original paper [4]. Assuming the solution in the form

$$P(r_d, r_e, \tau) = e^{A(\tau) - D(\tau)r_d - U(\tau)r_e}, \tag{14}$$

and setting it into the equation (13) we obtain the system of ordinary differential equations (ODEs):

$$\begin{aligned}
\dot{D}(\tau) &= 1 + a_2 D(\tau), \\
\dot{U}(\tau) &= a_3 D(\tau) + b_2 U(\tau), \\
\dot{A}(\tau) &= -a_1 D(\tau) - b_1 U(\tau) + \frac{\sigma_d^2 D^2(\tau)}{2} + \frac{\sigma_e^2 U^2(\tau)}{2} + \rho \sigma_d \sigma_e D(\tau) U(\tau)
\end{aligned} \tag{15}$$

with initial conditions $A(0) = D(0) = U(0) = 0$. The solution of this system is given by:

$$\begin{aligned}
D(\tau) &= \frac{-1 + e^{a_2 \tau}}{a_2}, \\
U(\tau) &= \frac{a_3 \left(a_2 - a_2 e^{b_2 \tau} + b_2 \left(-1 + e^{a_2 \tau} \right) \right)}{a_2 \left(a_2 - b_2 \right) b_2}, \\
A(\tau) &= \int_0^\tau -a_1 D(s) - b_1 U(s) + \frac{\sigma_d^2 D^2(s)}{2} + \frac{\sigma_e^2 U^2(s)}{2} + \rho \sigma_d \sigma_e D(s) U(s) \mathrm{d}s.
\end{aligned} \tag{16}$$

Note that the function $A(\tau)$ can be easily written in the closed form without an integral. We leave it in this form for the sake of brevity. Furthermore, we consider only the case when $a_2 \neq b_2$. If $a_2 = b_2$, then $U(\tau)$ has another form, but it is a very special case and we will not consider it further.

3.2. Convegence model of CIR type

Firstly we formulate the convergence model of CIR type (i. e. the volatilities are proportional to the square root of the short rates) in the real measure.

$$\begin{aligned}
\mathrm{d}r_d &= (a + b (r_e - r_d)) \, \mathrm{d}t + \sigma_d \sqrt{r_d} \, \mathrm{d}W_d, \\
\mathrm{d}r_e &= (c (d - r_e)) \, \mathrm{d}t + \sigma_e \sqrt{r_e} \, \mathrm{d}W_e,
\end{aligned} \tag{17}$$

where $Cov[dW_d, dW_e] = \rho \, \mathrm{d}t$. If we assume the market prices of risk equal to $\lambda_e \sqrt{r_e}$, $\lambda_d \sqrt{r_d}$ we obtain risk neutral processes of the form:

$$\begin{aligned}
\mathrm{d}r_d &= (a_1 + a_2 r_d + a_3 r_e) \, \mathrm{d}t + \sigma_d \sqrt{r_d} \, \mathrm{d}W_d, \\
\mathrm{d}r_e &= (b_1 + b_2 r_e) \, \mathrm{d}t + \sigma_e \sqrt{r_e} \, \mathrm{d}W_e,
\end{aligned} \tag{18}$$

where $Cov[dW_d, dW_e] = \rho \, \mathrm{d}t$. In what follows we consider this general risk- neutral formulation (18).

The European short rate is described by one-factor CIR model, so we are able to price European bonds using an explicit formula. Price of domestic bond $P(r_d, r_e, \tau)$ with maturity τ satisfies the PDE

$$
\begin{aligned}
-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P}{\partial r_e} \\
+\frac{\sigma_d^2 r_d^2}{2}\frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_e^2 r_e^2}{2}\frac{\partial^2 P}{\partial r_e^2} + \rho \sigma_d \sqrt{r_d}\sigma_e \sqrt{r_e}\frac{\partial^2 P}{\partial r_d \partial r_e} - r_d P = 0,
\end{aligned}
\tag{19}
$$

for $r_d, r_e > 0, \tau \in (0, T)$ with the initial condition $P(r_d, r_e, 0) = 1$ for $r_d, r_e > 0$. It was shown in [6] (in a slightly different parametrization of the model) that solution in the form (14) exists only when $\rho = 0$. In this case we obtain system of ODEs

$$
\begin{aligned}
\dot{D}(\tau) &= 1 + a_2 D(\tau) - \frac{\sigma_d^2 D^2(\tau)}{2}, \\
\dot{U}(\tau) &= a_3 D(\tau) + b_2 U(\tau) - \frac{\sigma_e^2 U^2(\tau)}{2}, \\
\dot{A}(\tau) &= -a_1 D(\tau) - b_1 U(\tau),
\end{aligned}
\tag{20}
$$

with initial conditions $A(0) = D(0) = U(0) = 0$, which can be solved numerically.

3.3. Convergence model of CKLS type

We consider a model in which risk-neutral drift of the European short rate r_e is a linear function of r_e, risk-neutral drift of the domestic short rate r_d is a linear function of r_d and r_e and volatilities take the form $\sigma_e r_e^{\gamma_e}$ and $\sigma_d r_d^{\gamma_d}$, i.e.

$$
\begin{aligned}
\mathrm{d}r_d &= (a_1 + a_2 r_d + a_3 r_e)\,\mathrm{d}t + \sigma_d r_d^{\gamma_d} dW_d, \\
\mathrm{d}r_e &= (b_1 + b_2 r_e)\,\mathrm{d}t + \sigma_e r_e^{\gamma_e} dW_e,
\end{aligned}
\tag{21}
$$

where $Cov[dW_d, dW_e] = \rho\,\mathrm{d}t$. Parameters $a_1, a_2, a_3, b_1, b_2 \in \mathbb{R}, \sigma_d, \sigma_e > 0, \gamma_d, \gamma_e \geq 0$ are given constants and $\rho \in (-1, 1)$ is a constant correlation between the increments of Wiener processes $\mathrm{d}W_d$ a $\mathrm{d}W_e$. We will refer to this model as *two-factor convergence model of Chan-Karolyi-Longstaff-Sanders (CKLS) type*. The domestic bond price $P(r_d, r_e, \tau)$ with the maturity τ satisfies PDE:

$$
\begin{aligned}
-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P}{\partial r_e} \\
+\frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_e^2 r_e^{2\gamma_e}}{2}\frac{\partial^2 P}{\partial r_e^2} + \rho \sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\frac{\partial^2 P}{\partial r_d \partial r_e} - r_d P = 0,
\end{aligned}
\tag{22}
$$

for $r_d, r_e > 0, \tau \in (0, T)$, with initial condition $P(r_d, r_e, 0) = 1$ for $r_d, r_e > 0$. Unlike for Vasicek and uncorrelated CIR model, in this case it is not possible to find solution in the separable form (14). For this reason, we are looking for an approximative solution.

4. APPROXIMATION OF THE DOMESTIC BOND PRICE SOLUTION

The bond prices in the CKLS type convergence model are not known in a closed form. This is already the case for the European bonds, i.e. one-factor CKLS model. We use the

approximation from [10]. In this approximation we consider one-factor Vasicek model
with the same risk-neutral drift and we set current volatility σr^γ instead of constant
volatility into the closed form formula for the bond prices. We obtain

$$\ln P_e^{ap}(\tau, r) = \left(\frac{b_1}{b_2} + \frac{\sigma^2 r^{2\gamma}}{2b_2^2}\right) \left(\frac{1 - e^{b_2\tau}}{b_2} + \tau\right) + \frac{\sigma^2 r^{2\gamma}}{4b_2^3} \left(1 - e^{b_2\tau}\right)^2 + \frac{1 - e^{b_2\tau}}{b_2} r. \quad (23)$$

We use this approach to propose an approximation for the domestic bond prices. We
consider the domestic bond prices in Vasicek convergence model with the same risk-
-neutral drift and we set $\sigma_d r_d^{\gamma_d}$ instead of σ_d and $\sigma_e r_e^{\gamma_e}$ instead of σ_e into (16). Hence,
we have

$$\ln P^{ap} = A - D r_d - U r_e \quad (24)$$

where

$$D(\tau) = \frac{-1 + e^{a_2\tau}}{a_2},$$

$$U(\tau) = \frac{a_3\left(a_2 - a_2 e^{b_2\tau} + b_2\left(-1 + e^{a_2\tau}\right)\right)}{a_2\left(a_2 - b_2\right) b_2},$$

$$A(\tau) = \int_0^\tau -a_1 D(s) - b_1 U(s) + \frac{\sigma_d^2 r_d^{2\gamma_d} D^2(s)}{2} + \frac{\sigma_e^2 r_e^{2\gamma_e} U^2(s)}{2}$$
$$+ \rho \sigma_d r_d^{\gamma_d} \sigma_e r_e^{\gamma_e} D(s) U(s) \mathrm{d}s.$$

4.1. Accuracy of the approximation for CIR model with zero correlation

In CIR convergence model the domestic bond price $P^{CIR,\rho=0}$ has a separable form (14)
and functions A, D, U are characterized by a system of ODEs (20). This enables us to
compute Taylor expansion of its logarithm around $\tau = 0$. We can compare it with the
expansion of proposed approximation $\ln P^{CIR,\rho=0,ap}$ (computed either using its closed
form expression (24) or the system of ODEs (16) for Vasicek convergence model). More
detailed computation can be found in [14]. In this way we obtain the accuracy of the
approximation for the CIR model with zero correlation:

$$\ln P^{CIR,\rho=0,ap} - \ln P^{CIR,\rho=0} = \frac{1}{24}\left(-a_2\sigma_d^2 r_d - a_1\sigma_d^2 - a_3\sigma_d^2 r_e\right)\tau^4 + o(\tau^4) \quad (25)$$

for $\tau \to 0^+$.

4.2. Numerical results for CIR model with zero correlation

Let us consider real measure parameters: $a = 0$, $b = 2$, $\sigma_d = 0.03$, $c = 0.2$, $d = 0.01$,
$\sigma_e = 0.01$ and market price of risk $\lambda_d = -0.25$, $\lambda_e = -0.1$. In the risk-neutral setting
(18) we have $a_1 = a - \lambda_d\sigma_d = 0.0075$, $a_2 = -b = -2$, $a_3 = b = 2$, $b_1 = cd - \lambda_e\sigma_e = 0.003$,
$b_2 = -c = -0.2$, $\sigma_d = 0.03$, $\sigma_e = 0.01$. With the initial values for the short rates $r_d =$
1.7% a $r_e = 1\%$ we generate the evolution of domestic and European short rates using
Euler–Maruyama discretization. In the Figure 3 we see that this choice of parameters
leads to a realistic behaviour of interest rates. In Table 1 we compare the exact interest
rate (i.e. numerical solution of the system (7)) and the approximative interest rate

given by (24). We observe very small differences. Note that the Euribor market data are quoted with the accuracy 10^{-3}. Choosing other days, with other combination of r_d, r_e, leads to very similar results. The difference between exact and approximative interest rate remains nearly the same.
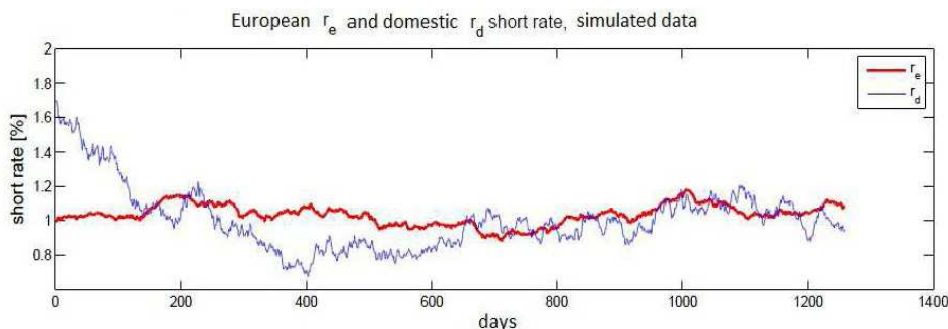

Fig. 3. Simulation of European and domestic short rate for 1200 days.

Mat. [year]	Exact yield [%]	Approx. yield [%]	Diff. [%]	Mat. [year]	Exact yield [%]	Approx. yield [%]	Diff. [%]
$\frac{1}{4}$	1.63257	1.63256	7.1E-006	$\frac{1}{4}$	1.08249	1.08250	-8.2E-006
$\frac{1}{2}$	1.58685	1.58684	1.4E-005	$\frac{1}{2}$	1.15994	1.15996	-1.7E-005
$\frac{3}{4}$	1.55614	1.55614	4.8E-006	$\frac{3}{4}$	1.21963	1.21964	-7.0E-006
1	1.53593	1.53592	1.1E-005	1	1.26669	1.26671	-1.6E-005
5	1.56154	1.56155	-5.0E-006	5	1.53685	1.53691	-6.2E-005
10	1.65315	1.65323	-8.3E-005	10	1.65113	1.65127	-1.4E-004
20	1.74696	1.74722	-2.5E-004	20	1.74855	1.74884	-2.9E-004
30	1.78751	1.78787	-3.7E-004	30	1.78879	1.78918	-3.9E-004

Tab. 1. Exact and approximative domestic yield for 1st (left) observed day, $r_d = 1.7\%$, $r_e = 1\%$ and for 252nd (right) observed day, $r_d = 1.75\%$, $r_e = 1.06\%$.

4.3. Accuracy of the approximation in general CKLS model

Aim of this section is to derive the order of accuracy of the proposed approximation in the general case. We use analogous method as in [10] and [11] for one-factor models and in [6] to study the influence of correlation ρ on bond prices in the convergence CIR model.

Let $f^{ex} = \ln P^{ex}$ be the logarithm of the exact price P^{ex} of the domestic bond in two factor convergence model of CKLS type. It satisfies the PDE (22). Let $f^{ap} = \ln P^{ap}$ be the logarithm of the approximative price P^{ap} for the domestic bond price given by (24). By setting f^{ap} to the left-hand side of (22) we obtain non-zero right-hand side, which

we denote as $h(r_d, r_e, \tau)$. We expand it into Taylor expansion and obtain that

$$h(r_d, r_e, \tau) = k_3(r_d, r_e)\tau^3 + k_4(r_d, r_e)\tau^4 + o(\tau^4), \tag{26}$$

for $\tau \to 0^+$, where

$$k_3(r_d, r_e) = \frac{1}{6}\sigma_d^2 \gamma_d r_d^{2\gamma_d - 2} \left(2a_1 r_d + 2a_2 r_d^2 + 2a_3 r_d r_e - r_d^{2\gamma_d}\sigma_d^2 + 2\gamma_d r_d^{2\gamma_d}\sigma_d^2 \right),$$

$$\begin{aligned}
k_4(r_d, r_e) =& \frac{1}{48}\frac{1}{r_e^2}r_d^{-2+\gamma_d}\sigma_d \Big(12a_2^2\gamma_d r_d^{2+\gamma_d}r_e^2\sigma_d - 16\gamma_d r_d^{1+3\gamma_d}r_e^2\sigma_d^3 + 6a_3b_1\gamma_e r_d^2 r_e^{1+\gamma_e}\rho\sigma_e \\
&+ 6a_3b_2\gamma_e r_d^2 r_e^{2+\gamma_e}\rho\sigma_e + 6a_3^2\gamma_d r_d r_e^{3+\gamma_e}\rho\sigma_e - 3a_3\gamma_d r_d^{2\gamma_d}r_e^{2+\gamma_e}\rho\sigma_d^2\sigma_e \\
&+ 3a_3\gamma_d^2 r_d^{2\gamma_d}r_e^{2+\gamma_e}\rho\sigma_d^2\sigma_e + 6a_3\gamma_d\gamma_e r_d^{1+\gamma_d}r_e^{1+2\gamma_e}\rho^2\sigma_d\sigma_e^2 - 3a_3\gamma_e r_d^2 r_e^{3\gamma_e}\rho\sigma_e^3 \\
&+ 3a_3\gamma_e^2 r_d^2 r_e^{3\gamma_e}\rho\sigma_e^3 + 6a_1\gamma_d r_d r_e^2 \left(2a_2 r_d^{\gamma_d}\sigma_d + a_3 r_e^{\gamma_e}\rho\sigma_e \right) \\
&+ 6a_2\gamma_d r_e^2 \left((-1+2\gamma_d) r_d^{3\gamma_d}\sigma_d^3 + a_3 r_d \left(2r_d^{\gamma_d}r_e\sigma_d + r_d r_e^{\gamma_e}\rho\sigma_e \right) \right) \Big).
\end{aligned}$$

We define function $g(\tau, r_d, r_e) := f^{ap} - f^{ex} = \ln P^{ap} - \ln P^{ex}$ as a difference between logarithm of the approximation and the exact price. Using the PDEs satisfied by f^{ex} and f^{ap} we obtain the following PDE for the function g:

$$\begin{aligned}
&-\frac{\partial g}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial g}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial g}{\partial r_e} + \frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\left[\left(\frac{\partial g}{\partial r_d}\right)^2 + \frac{\partial^2 g}{\partial r_d^2} \right] \\
&\quad + \frac{\sigma_e^2 r_e^{2\gamma_e}}{2}\left[\left(\frac{\partial g}{\partial r_e}\right)^2 + \frac{\partial^2 g}{\partial r_d^2} \right] + \rho\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\left(\frac{\partial g}{\partial r_d}\frac{\partial g}{\partial r_e} + \frac{\partial^2 g}{\partial r_d \partial r_e} \right) \\
&= h(r_d, r_e, \tau) + \frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\left[\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_d} \right] + \frac{\sigma_e^2 r_e^{2\gamma_e}}{2}\left[\left(\frac{\partial f^{ex}}{\partial r_e}\right)^2 - \frac{\partial f^{ap}}{\partial r_e}\frac{\partial f^{ex}}{\partial r_e} \right] \\
&\quad + \rho\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\left[2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_e} \right].
\end{aligned} \tag{27}$$

Suppose that $g(r_d, r_e, \tau) = \sum_{k=\omega}^{\infty} c_k(r_d, r_e)\tau^k$. For $\tau = 0$ is both the exact and approximative bond price equal to one, so $f^{ex}(r_d, r_e, 0) = f^{ap}(r_d, r_e, 0) = 0$. It means that $\omega > 0$ and on the left hand side of the equation (27) the term with the lowest order is $c_\omega\omega\tau^{\omega-1}$. Now we investigate the order of the right hand side of the equation.

We know that $f^{ex}(r_d, r_e, 0) = 0$. It means that $f^{ex} = O(\tau)$ and also partial derivation $\frac{\partial f^{ex}}{\partial r_d}$ and $\frac{\partial f^{ex}}{\partial r_e}$ are of the order $O(\tau)$. From the approximation formula (24) we can see that $\frac{\partial f^{ap}}{\partial r_d} = O(\tau)$, $\frac{\partial f^{ap}}{\partial r_e} = O(\tau^2)$. Since $h(r_d, r_e, \tau) = O(\tau^3)$, the right hand side of the equation (27) is at least of the order τ^2. The left hand side of the equation (27) is of the order $\tau^{\omega-1}$ and hence $\omega - 1 \geq 2$, i.e. $\omega \geq 3$. It means that

$$f^{ap}(r_d, r_e, \tau) - f^{ex}(r_d, r_e, \tau) = O(\tau^3).$$

Using this expression we can improve estimation of the derivative $\frac{\partial f^{ex}}{\partial r_e}$ as follows: $\frac{\partial f^{ex}}{\partial r_e} = \frac{\partial f^{ap}}{\partial r_e} + O(\tau^3) = O(\tau^2) + O(\tau^3) = O(\tau^2)$. We also estimate the terms on the right hand

side in the equation (27):

$$\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_d} = \frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_d}\right) = O(\tau).O(\tau^3) = O(\tau^4), \qquad (28)$$

$$\left(\frac{\partial f^{ex}}{\partial r_e}\right)^2 - \frac{\partial f^{ap}}{\partial r_e}\frac{\partial f^{ex}}{\partial r_e} = \frac{\partial f^{ex}}{\partial r_e}\left(\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_e}\right) = O(\tau^2).O(\tau^3) = O(\tau^5), \qquad (29)$$

$$2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_e} = \frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_e}\right)$$
$$+ \frac{\partial f^{ex}}{\partial r_e}\left(\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_d}\right) = O(\tau).O(\tau^3) + O(\tau^2).O(\tau^3) = O(\tau^4) + O(\tau^5) = O(\tau^4). \qquad (30)$$

Since $h(r_d, r_e, \tau) = O(\tau^3)$, the right hand side of the equation (27) is $O(\tau^3)$ and the coefficient at τ^3 is the coefficient of the function $h(r_d, r_e, \tau)$ at τ^3, i.e. $k_3(r_d, r_e)$. It means that $\omega = 4$ and comparing the coefficients at τ^3 on the left and right-hand side of (27) we obtain $-4c_4(r_d, r_e) = k_3(r_d, r_e)$, i.e. $c_4(r_d, r_e) = -\frac{1}{4}k_3(r_d, r_e)$. Hence we have proved the following theorem.

Theorem 4.1. Let $P^{ex}(r_d, r_e, \tau)$ be the price of the domestic bond in two-factor CKLS convergence model, i.e. satisfying equation (22) and let P^{ap} be the approximative solution defined by (24). Then

$$\ln P^{ap}(r_d, r_e, \tau) - \ln P^{ex}(r_d, r_e, \tau) = c_4(r_d, r_e)\tau^4 + o(\tau^4)$$

for $\tau \to 0^+$, where coefficient c_4 is given by

$$c_4(r_d, r_e) = -\frac{1}{24}\sigma_d^2 \gamma_d r_d^{2\gamma_d - 2}\left(2a_1 r_d + 2a_2 r_d^2 + 2a_3 r_d r_e - r_d^{2\gamma_d}\sigma_d^2 + 2\gamma_d r_d^{2\gamma_d}\sigma_d^2\right). \qquad (31)$$

Note that if we substitute $\gamma_d = \frac{1}{2}$ and $\rho = 0$ into Theorem 4.1, we obtain the formula (25) for CIR model derived earlier in (25).

4.4. Improvement of the approximation

In some cases it is possible to improve an approximation by calculating more terms in Taylor expansion of the function $g = \ln P^{ap} - \ln P^{ex}$. It is so also in this case. Using that $f^{ap} - f^{ex} = O(\tau^4)$, we are able to improve estimates (28) and (30) and to deduce that also the coefficient at τ^4 on the right hand side of equation (27) comes only from the function h. Hence it is equal to $k_4(r_d, r_e)$, which is given by (27). Comparing coefficients at τ^4 on the left and right hand side of (27) we obtain:

$$-5c_5 + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial c_4}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial c_4}{\partial r_e}$$
$$+ \frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\frac{\partial^2 c_4}{\partial r_d^2} + \frac{\sigma_e^2 r_e^{2\gamma_e}}{2}\frac{\partial^2 c_4}{\partial r_e^2} + 4\rho\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\frac{\partial^2 c_4}{\partial r_d \partial r_e} = k_4,$$

which enables us to express c_5 using already known quantities.

Let us define an approximation $\ln P^{ap2}$ by:

$$\ln P^{ap2}(r_d, r_e, \tau) = \ln P^{ap} - c_4(r_d, r_e)\tau^4 - c_5(r_d, r_e)\tau^5.$$

Then $\ln P^{ap2} - \ln P^{ex} = O(\tau^6)$ and therefore the new approximation $\ln P^{ap2}$ is of the order $O(\tau^6)$.

5. THE CALIBRATION ALGORITHM

Model calibration is certainly not a problem with a straightforward solution. In general, calibration methods can focus on statistical analysis of time series of instantanous interest rate, match of theoretical and market yield curves, or combine these two approaches. An example of statistical analysis is the paper [2], where the form of the volatility in one-factor model is determined using the generalized method of moments applied to time series of short rate. An example of comparison of theoretical and estimated yield curves is in paper [13]. The existence of explicit formulae for bond prices in one-factor CIR model allowed the calibration of parameters in this way. The combination of these approaches can be found for example in paper [4] about Vasicek convergence model. All parameters, which can be estimated from the time series of domestic and European short rates, are estimated in this way. The remaining market prices of risk are then estimated using the yield curves.

However, using this combined approach, most parameters are estimated from the time series of the short rates. Information from the time series of interest rates with other maturities (which contain several times more data) is used only to estimate the market price of risk.

In [6] the author considers a CIR convergence model and uses a modification of Ait–Sahalia's approximation of densities to estimate short rate parameters. Market price of risk is estimated from the yield curves. However, the author claims, that by changing some already estimated short rate parameters it is possible to obtain a significant improvement of objective function. Therefore, our aim was to propose such a calibration method that would use the information from term structures to estimate all parameters. Such an approach requires an efficient calculation of bond prices. This is achieved by using an approximative analytical formula.

5.1. Formulation of the optimization problems

We consider the convergence model CKLS type in risk-neutral measure given by equation (21). Firstly let us define the following notation

- P_d^{ap}, P_e^{ap} are approximations of the price of domestic and European bonds,

- R_d, R_e are actual yields observed on the market,

- $\tau_d = (\tau_d^1, \ldots \tau_d^{m_d})$, $\tau_e = (\tau_e^1, \ldots \tau_e^{m_e})$ are maturities of domestic and European yields,

- the data are observed during n_d, resp. n_e days,

- index i corresponds to days and index j corresponds to maturities.

We consider estimation of the parameters of European interest rates as a separate problem. The relationship between European and domestic interest rates is not a mutual influence of two variables, but the European rates influence the domestic ones. Hence the estimated European parameters of the model can not be dependent on the choice of country for which we consider the convergence model and on the domestic interest rates in this country. This approach was also used in [6]. The calibration model is therefore divided into two parts:

1. Estimation of European parameters, which is based on minimizing function

$$F_e(b_1, b_2, \sigma_e, \gamma_e) = \frac{1}{m_e n_e} \sum_{i=1}^{n_e} \sum_{j=1}^{m_e} w_e(i, j) \Big(-\frac{\ln P_e^{ap}(i, j)}{\tau_e(j)} - R_e(i, j) \Big)^2.$$

2. Estimation of domestic parameters, which is based on minimizing function

$$F_d(a_1, a_2, a_3, \sigma_d, \rho, \gamma_e) = \frac{1}{m_d n_d} \sum_{i=1}^{n_d} \sum_{j=1}^{m_d} w_d(i, j) \Big(-\frac{\ln P_d^{ap}(i, j)}{\tau_d(j)} - R_d(i, j) \Big)^2,$$

where in the computation of P_d^{ap} we use the values b_1, b_2, σ_e obtained in the first step.

Functions w_e, w_d express weights. As in [13] we choose $w_e(i, j) = \tau_e(j)^2$ a $w_d(i, j) = \tau_d(j)^2$. However, the proposed algorithm can be adapted also for a different choice of weights. For our choice of weights we have the following objective functions:

$$F_e(b_1, b_2, \sigma_e, \gamma_e) = \frac{1}{m_e n_e} \sum_{i=1}^{n_e} \sum_{j=1}^{m_e} \Big(\ln P_e^{ap}(i, j) + R_e(i, j) \tau_e(j) \Big)^2, \tag{32}$$

$$F_d(a_1, a_2, a_3, \sigma_d, \rho, \gamma_d) = \frac{1}{m_d n_d} \sum_{i=1}^{n_d} \sum_{j=1}^{m_d} \Big(\ln P_d^{ap}(i, j) + R_d(i, j) \tau_d(j) \Big)^2. \tag{33}$$

6. THE ALGORITHM FOR ESTIMATING PARAMETERS IN CIR MODEL WITH ZERO CORRELATION

Our first goal is to estimate parameters in the convergence CIR model with zero correlation. In this case we can calculate the exact yield curve by solving system of ODEs (20). We can therefore simulate the exact data and monitor the accuracy of our calibration and its individual steps.

6.1. Simulated data

We choose the same parameters as in the section 4.2. We generate the domestic and European short rates for $n_e = n_d = 1260$ days, i.e. 5 years (252 days per year). Furthermore, we generate European (using the explicit formula) and domestic (numerically solving the system of ODEs) yields for maturities $\tau_e = \tau_d = \big(\frac{1}{12}, \frac{2}{12}, \dots \frac{12}{12} \big)$, i.e. $m_e = m_d = 12$.

6.2. Estimation of the European parameters

This step is an estimate of the CIR model parameters. Estimation method is taken from [10]. For a given value of the power γ_e ($\frac{1}{2}$ in this case) the estimation of the other three parameters can be reduced to a one-dimensional problem. The remaining two parameters can be expressed from the first-order conditions and substituted into the objective function, which then becomes a function of one parameter. The objective function is then optimized with respect to this parameter.

We note (see again [10]) that if we estimate a model with different γ_e than the true value (in particular, if we estimate a Vasicek model), the estimate of the risk-neutral drift does not change much. This feature was an inspiration for estimation of domestic parameters, described in the next section.

6.3. Estimation of the domestic parameters

Step 1: Estimation of the risk-neutral drift
Based on the results for the one-factor model we try to estimate the risk-neutral drift of domestic interest rates as risk-neutral drift for Vasicek convergence model. When doing so, we add upper index vas to all parameters and objective function i.e. F_d^{vas}, a_1^{vas}, a_2^{vas}, etc., to emphasize that we are estimating Vasicek model. We omit this index when using them in the subsequent steps.

The first step is to estimate the one-factor Vasicek model for European data, for which we use algorithm from the [10]. Then, to estimate the domestic parameters, we optimize the function F_d^{vas} in the form:

$$F_d^{vas}(a_1^{vas}, a_2^{vas}, a_3^{vas}, \sigma_d^{vas}) = \sum_{i=1}^{n_d} \sum_{j=1}^{m_d} \Big(\ln P_d^{vas}(i,j) + R_d(i,j)\tau_d(j) \Big)^2. \tag{34}$$

Recall that in Vasicek model

$$\ln P_d^{vas}(r_d, r_e, \tau) = A(\tau) - D(\tau)r_d - U(\tau)r_e, \tag{35}$$

hence the term in F_d^{vas} corresponding to the particular jth maturity $\tau_d(j)$ is

$$\sum_{i=1}^{n_d} \Big(A(\tau_d(j)) - D(\tau_d(j))r_d(i) - U(\tau_d(j))r_e(i) + R_d(i,j)\tau_d(j) \Big)^2. \tag{36}$$

Since j is fixed, values $A(\tau_d(j))$, $-D(\tau_d(j))$, $-U(\tau_d(j))$ are constants. Sum (36), which should be small (to minimize sum over j), then resembles the linear regression

$$- R_d(i,j)\tau_d(j) \sim c_{0j} + c_{1j}r_d(i) + c_{2j}r_e(i) \qquad \text{for} \quad i = 1, \dots n. \tag{37}$$

For each j we estimate this linear regression and write results into matrix

$$C = \begin{bmatrix} c_{01} & c_{11} & c_{21} \\ c_{02} & c_{12} & c_{22} \\ \vdots & \vdots & \vdots \\ c_{0m_d} & c_{1m_d} & c_{2m_d} \end{bmatrix}.$$

Comparing (35) and (37) we see that

$$c_{0j} \sim A(\tau_d(j)), \qquad c_{1j} \sim -D(\tau_d(j)), \qquad c_{2j} \sim -U(\tau_d(j)). \tag{38}$$

We determine the parameters of the functions A, D, U to obtain a good match of the terms in (38).

- Function D depends only on the parameter a_2^{vas}. We solve one-dimensional optimization problem

$$G_1(a_2^{vas}) = \sum_{j=1}^{m_d} \Big(-D(\tau_d(j), a_2^{vas}) - c_{1j}\Big)^2 \to \min_{a_2^{vas}} \tag{39}$$

and we obtain the estimate of the parameter a_2^{vas}.

- The function U depends on the parameters a_3^{vas}, b_1^{vas}. Parameter b_1^{vas} is already estimated from European interest rates. Hence we have a one-dimensional optimization problem again:

$$G_2(a_2^{vas}) = \sum_{j=1}^{m_d} \Big(-U(\tau_d(j), a_3^{vas}) - c_{2j}\Big)^2 \to \min_{a_3^{vas}} \tag{40}$$

and by solving it we obtain the estimate of a_3^{vas}.

- Function A depends on all parameters a_1^{vas}, a_2^{vas}, a_3^{vas}, b_1^{vas}, b_2^{vas}, σ_d^{vas}, σ_e^{vas}, but all parameters except a_1^{vas}, σ_d^{vas} are already estimated. Note that A is a linear function of the parameters a_1^{vas} and $(\sigma_d^{vas})^2$. Therefore the optimal solution of the problem

$$G_3(a_1^{vas}, (\sigma_d^{vas})^2) = \sum_{j=1}^{m_d} \Big(A(\tau_d(j), a_1^{vas}, (\sigma_d^{vas})^2) - c_{0j}\Big)^2 \to \min_{a_1^{vas}, (\sigma_d^{vas})^2} \tag{41}$$

can be calculated explicitly from the first order optimality conditions by solving system of two linear equations. However, we observed (for several sets of generated data) that these estimates are unstable because the system matrix is bad conditioned, with the condition number between 10^{18} and 10^{21}. It turned out that a better approach is to use only the first order condition from the derivative with respect to $(\sigma_d^{vas})^2$. Hence we proceed as follows.

The function $\ln P^{ap}$ is expressed in the form:

$$\ln P^{ap} = A(\tau) + D(\tau)r_d + U(\tau)r_e = c_0(r_d, r_e, \tau) + c_1(r_d, r_e, \tau)(\sigma_d^{vas})^2, \tag{42}$$

where the coefficients $c_0(r_d, r_e, \tau)$, $c_1(r_d, r_e, \tau)$ do not depend on $(\sigma_d^{vas})^2$ and can be expressed explicitly. For given values of remaining parameters, the optimal value of $(\sigma_d^{vas})^2$ is calculated.

Thus for each value a_1^{vas} we have the corresponding optimal value $(\sigma_d^{vas})^2$ and we can formulate a one-dimensional optimization problem:

$$G_4(a_1^{vas}) = \sum_{j=1}^{m_d} \Big(A(\tau_d(j), a_1^{vas}) - c_{0j} \Big)^2 \to \min_{a_1^{vas}}. \qquad (43)$$

This procedure produces stable results.

Step 2: Estimation of the volatility
So far we have estimated the parameters b_1, b_2, σ_e for European interest rates and parameters a_1, a_2, a_3 from the drift of the domestic interest rate. Substituting all these parameters into the objective function F_d it remains a function one parameter σ_d and it is easy to find its optimal value.

Step 3: Final modification of the parameters
In the first two steps we have sequentially estimated all the domestic parameters. However, this does not guarantee that we have achieved the global minimum of the objective function. Hence, we try to improve them by optimizing the function F_d with respect to all of them together. The current estimated values (which are expected to be close to optimum) were taken as starting values and the optimization was performed one more time with respect to all parameters.

6.4. Simulation analysis

We have implemented a numerical experiment in which we generated 1000 sets of domestic and European short rates and yield curves. We have used the same parameters in risk-neutral measure as in sections 4.2 and 6.1. The initial values were generated from uniform distribution on the interval [0.02, 0.04] for the domestic and from the interval [0.005, 0.025] for the European short rate.

Our aim was to check the accuracy of the proposed estimation algorithm, as well as to see the usefullness of the step 3 described above since it requires much more time that the previous steps.

Based on the results, we decide to stop the estimation after the second step. The estimation of the drift is very precise. Less precision is achieved at estimating volatility, but it is still satisfactory. Table 2 shows what does this precision mean for the estimated yield curves. We again recall that the market Euribor rates are quoted with three decimal places. We conclude that using our algorithm the yield curves are estimated with a high precision. The detailed descriptive statistics of the European estimates and of the domestic estimates after Step 2 and after Step 3 can be found in [14].

7. GENERALIZATION FOR CKLS MODEL WITH ZERO CORRELATION AND THE KNOWN γ_E, γ_D

The generalization for CKLS model with zero correlation and the known γ_e, γ_d is straightforward. Firstly we estimate European parameters. For given value of γ_e estimation of other three parameters can be reduced to one-dimensional problem, as it was mentioned in 6.2. Outcome of this optimization is estimation of parameters b_1, b_2, σ_e.

6 months	differences in domestic yields after 2nd step in [%]	differences in domestic yields final estimates in [%]
minimum	1.19E-08	3.83E-10
maximum	4.17E-06	2.15E-06
median	4.74E-07	1.15E-07
mean value	7.01E-07	2.58E-07
standard deviation	6.85E-07	3.85E-07

12 months	differences in domestic yields after 2nd step in [%]	differences in domestic yields final estimates in [%]
minimum	4.87E-09	1.54E-11
maximum	9.78E-06	1.67E-06
median	1.06E-06	6.56E-08
mean value	1.66E-06	1.70E-07
standard deviation	1.70E-06	2.58E-07

Tab. 2. Descriptive statistics for domestic estimates after 2nd step and final estimates for maturities 6 and 12 months.

Secondly we estimate domestic parameters. Estimation of risk-neutral drift remains the same as in section 6.3, because it is the estimate of the Vasicek convergence model. Estimation of volatility is realized as the minimization of the objective function over the parameter σ_d in the same way as in section 6.3. The only change is the calculation of the objective function where instead of $\gamma_d = \frac{1}{2}$ we consider another γ_d. Based on the simulation results for the CIR model we omit final four-dimensional parameter optimization with respect to the parameters a_1, a_2, a_3, σ_d.

8. ESTIMATION OF CORRELATION ρ A PARAMETERS γ_E, γ_D

To estimate the power γ_e we can use the procedure from [10]. The estimation described above is performed over a range of γ_e. Based on the objective function, the optimal γ_e is chosen. However, trying this approach to estimate γ_d and ρ did not work. We tried to find an explanation why these strategies fail in [14], where a more detailed treatment can be found.

The approximation error and the dependence on the correlation ρ and the power γ_d is numerically about the same order. Probably, it is the reason why we can not distinguish them and therefore we can not determine their correct values. However, the errors of the consecutive steps accumulate. We refer the reader to [13] for more numerical and analytical results on this question.

9. CALIBRATION OF THE MODEL USING THE REAL MARKET DATA

We have used Bribor (Bratislava Interbank Offered Rate) and Euribor (Euro Interbank Offered Rate) data from the last three months before the Slovak Republic enter the monetary union (1. 10. 2008 – 31. 12. 2008, i. e. $n_e = n_d = 62$). As the domestic short rate we use overnight Bribor, as the European short rate we use Eonia (Euro OverNight Index Average). The yields are considered for the same set of maturities in both domestic and European case. We take $\tau_e = \tau_d = (\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{6}{12}, \frac{9}{12}, \frac{12}{12})$, i. e. $m_e = m_d = 6$.

Using these data we have estimated the convergence CIR model (18) with zero correlation, i. e. $\rho = 0$. The estimates of the parameters are summarized in Table 3.

parameter	b_1	b_2	σ_e	a_1	a_2	a_3	σ_d
estimated value	0.0227	0.5000	1.1427	0.0879	-8.2052	7.3827	5.0000

Tab. 3. Estimated parameters of the CIR convergence model (18) with
zero correlation.

We compare the exact and the estimated yield curves for several selected days. In the Figure 4 we show European (left) and domestic (right) yield curves for 1st, 31st and 61st day.
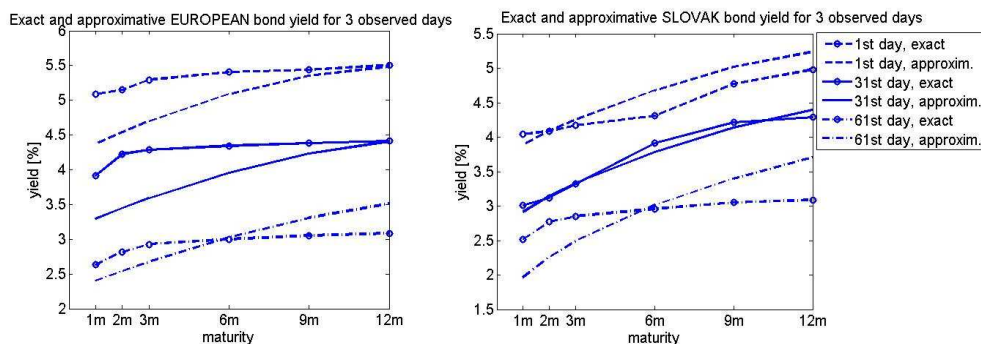

Fig. 4. Estimated and real term structures for three observed days –
European (left), Slovak (right).

In the Table 4 we numerically compare exact and estimated yields for one typical day. We observe much lower accuracy for the European rates. Hence an important task is to

Mat. [year]	Market yield	Estim. yield	Diff.	Rel. diff.	Mat. [year]	Market yield	Estim. yield	Diff.	Rel. diff.
$\frac{1}{12}$	3.9140	3.2953	0.6187	0.158	$\frac{1}{12}$	3.0100	2.9174	0.0926	0.031
$\frac{2}{12}$	4.2260	3.4461	0.7799	0.185	$\frac{2}{12}$	3.1200	3.1490	-0.0290	0.009
$\frac{3}{12}$	4.2860	3.5891	0.6969	0.163	$\frac{3}{12}$	3.3200	3.3336	-0.0136	0.004
$\frac{6}{12}$	4.3450	3.9604	0.3846	0.089	$\frac{6}{12}$	3.9100	3.7805	0.1295	0.033
$\frac{9}{12}$	4.3810	4.2327	0.1483	0.034	$\frac{9}{12}$	4.2100	4.1358	0.0742	0.018
$\frac{12}{12}$	4.4120	4.4065	0.0055	0.001	$\frac{12}{12}$	4.2900	4.3957	-0.1057	0.025

Tab. 4. Accuracy of the estimation of European (left) and domestic
(right) yield curves.

improve estimation of the European data, since its results are used in finding an estimate of domestic parameters. If we would have chosen a different model for European interest rates, it might have also improved the estimation of domestic yield curves.

10. CONCLUSION

The paper deals with an approximation of the bond price in convergence model of the CKLS type on the basis of the solution of the Vasicek model. We have numerically tested the proposed approximation on the CIR model with a zero correlation, for which the exact solution is known and also analytically derived the accuracy for the general CKLS model. The difference of logarithms of the exact solution and proposed approximation is of the order $O(\tau^4)$.

We have suggested an algorithm for calibrating the model using our approximation formula. Testing our proposed algorithm has been performed on the simulated data for the CIR model with zero correlation, where the exact solution is known. The accuracy for the domestic term structure in the percentage points is 10^{-6} till 10^{-7}. Finally we have implemented this method using real market data, especially Bribor (Bratislava Interbank Offered Rate) from the last three month period (1. 10. 2008 – 31. 12. 2008) before entering the monetary union.

Our next aim is to propose alternative model for estimation European interest rates and thus improve not only fit of the European data but also the domestic ones, because estimated European parameters enter the estimation of domestic parameters as already known constants.

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Zuzana Zíková, Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava. Slovak Republic.

    e-mail: zuzana.zikova@gmail.com

Beáta Stehlíková, Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava. Slovak Republic.

    e-mail: stehlikova@pc2.iam.fmph.uniba.sk