Guimei Luo

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MIXED COMPLEMENTARITY PROBLEMS FOR ROBUST OPTIMIZATION EQUILIBRIUM IN BIMATRIX GAME*

Guimei Luo, Guangzhou

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Abstract. In this paper, we investigate the bimatrix game using the robust optimization approach, in which each player may neither exactly estimate his opponent's strategies nor evaluate his own cost matrix accurately while he may estimate a bounded uncertain set. We obtain computationally tractable robust formulations which turn to be linear programming problems and then solving a robust optimization equilibrium can be converted to solving a mixed complementarity problem under the $l_1 \cap l_\infty$-norm. Some numerical results are presented to illustrate the behavior of the robust optimization equilibrium.

Keywords: robust optimization equilibrium, bimatrix game, $l_1 \cap l_\infty$-norm, mixed complementarity problem

MSC 2010: 91A05, 90C05, 90C46

1. Introduction

In [18], [19], Nash modeled each player as rational and wanting to maximize his expected payoff with respect to the probability distributions given by the mixed strategies. Moreover, Nash proved that each game of the aforementioned type has an equilibrium in mixed strategies. However, in real-world game-theoretic situations, players are often uncertain of some aspects of the structure of the game, such as payoff functions. Harsanyi [14] considered the case with uncertain payoff functions, an extension of Nash's framework, and modeled these incomplete information games as what was called "Bayesian" games. In that model, the uncertain payoffs were treated as expectation. Some contributions to the literature have relaxed the common prior and common knowledge assumptions of Harsanyi's model, for example, see [15], [17] etc. Aghassi and Bertsimas in [1] proposed a distribution-free, robust

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optimization model for incomplete-information games. Using a worst-case approach to uncertainty, in the absence of probability distributions, [1] proved the existence of an equilibrium in robust finite games with bounded uncertain payoffs and no private information. At the same time, [1] provided an expression for equilibrium set when payoffs are bounded polyhedral uncertain.

Robust optimization is a technique for handling optimization problems with uncertain parameters, in which those uncertain parameters are assumed to belong to the so-called uncertain sets, and then the objective function is minimized (or maximized) by taking into account the worst possible case. Initial results on robust optimization were given by Soyster in [20]. Twenty years later, Ben-Tal and Nemirovski [2], [3], [4] and independently of them, El Ghaoui [9], [10] renewed the discussion of optimization under uncertainty. They investigated ellipsoidal models of uncertainty, which for the robust LP (linear programming) case are less conservative than the column-wise model proposed by Soyster in [20]. However, such a robust counterpart is more computationally demanding than that of the corresponding nominal problem. Bertsimas and Sim [6], [7] offered an alternative model of symmetric uncertainty, under which the robust counterpart is tractable preserving the computational complexity of the nominal problem. Subsequently, Chen et al. [8] refined the framework of [7] to asymmetric situations.

In our work, to capture the essence of the underlying random variables, we consider the robust optimization equilibrium for a bimatrix game from the cardinality of an asymmetrically uncertain set under the $l_1 \cap l_\infty$-norm, in which each player attempts to minimize his own cost with either each player's own cost matrix or his opponent's uncertain strategies. In this situation, the model turns to be a bimatrix game:

$$\text{player one} \quad \min_{\mathbf{y}\in\mathcal{Y}} \{\mathbf{y}^{\mathrm{T}}\tilde{\mathbf{A}}\mathbf{z}\}_{\tilde{\mathbf{A}}\in D_A,\, \mathbf{z}\in\mathcal{Z}} \tag{1.1}$$

and

$$\text{player two} \quad \min_{\mathbf{z}\in\mathcal{Z}} \{\mathbf{y}^{\mathrm{T}}\tilde{\mathbf{B}}\mathbf{z}\}_{\tilde{\mathbf{B}}\in D_B,\, \mathbf{y}\in\mathcal{Y}}, \tag{1.2}$$

where $\mathcal{Y} := \{\mathbf{y} \in \mathbb{R}^n \colon \mathbf{y} \geqslant 0, \mathbf{e}_n^{\mathrm{T}}\mathbf{y} = 1\}$ and $\mathcal{Z} := \{\mathbf{z} \in \mathbb{R}^m \colon \mathbf{z} \geqslant 0, \mathbf{e}_m^{\mathrm{T}}\mathbf{z} = 1\}$ denote the mixed strategy sets for players one and two, respectively. Furthermore, we assume that D_A, D_B are bounded and $\mathbf{z}$, $\mathbf{y}$ can be estimated on bounded sets $\mathcal{Z}^{\mathcal{U}}$ and $\mathcal{Y}^{\mathcal{U}}$ in (1.1) and (1.2), respectively. Following [1], the robust counterparts of problems (1.1) and (1.2) can be stated as

$$\text{player one} \quad \min_{\mathbf{y}\in\mathcal{Y}} \max_{\tilde{\mathbf{A}}\in D_A,\, \tilde{\mathbf{z}}\in\mathcal{Z}^{\mathcal{U}}} \mathbf{y}^{\mathrm{T}}\tilde{\mathbf{A}}\tilde{\mathbf{z}} \tag{1.3}$$

and

$$\text{(1.4)} \qquad \text{player two} \quad \min_{\mathbf{z}\in\mathcal{Z}} \max_{\tilde{\mathbf{B}}\in D_B,\, \tilde{\mathbf{y}}\in\mathcal{Y}^{\mathcal{U}}} \tilde{\mathbf{y}}^{\mathrm{T}}\tilde{\mathbf{B}}\mathbf{z}.$$

A pair of strategies $(\mathbf{y},\mathbf{z})$ is called a *shape robust optimization equilibrium* for problems (1.1) and (1.2) if $\mathbf{y}$ optimizes (1.3) and $\mathbf{z}$ optimizes (1.4) simultaneously. In fact, how to deal with an uncertain set plays an important role in the solution to this model. Hayashi et al. [13] introduced a concept of the robust Nash equilibrium, proved the existence of the robust Nash equilibrium and studied the bimatrix game under standard ellipsoid uncertainty. Taking into account the cardinality of an uncertain set, Luo et al. [16] considered a symmetric case where the uncertain set includes a standard ellipsoid, a flat ellipsoid and ellipsoidal cylinders as well as under the l_2-norm, which implied that their robust counterparts were SOCPs (second order cone programming) and then solving the corresponding robust optimization equilibrium turned to be solving SOCCPs (second order cone complementarity problems). In our work, we study problems (1.1) and (1.2) under the $l_1\cap l_\infty$-norm and obtain that the corresponding robust counterparts turn to be LPs. Then the robust optimization equilibrium can be converted to an MCP (mixed complementarity problem) that can be efficiently solved by existing methods [11]:

$$\text{(1.5)} \qquad \mathbb{R}^{\varsigma}_{+} \ni \mathbf{G}\mathbf{x}+\mathbf{q} \perp \mathbf{H}\mathbf{x}+\mathbf{r} \in \mathbb{R}^{\varsigma}_{+}, \quad \mathbf{C}\mathbf{x}=\mathbf{d},$$

where $\mathbf{x}\in\mathbb{R}^{\varsigma+\tau}$, constant matrices $\mathbf{G},\mathbf{H}\in\mathbb{R}^{\varsigma\times(\varsigma+\tau)}$, $\mathbf{q},\mathbf{r}\in\mathbb{R}^{\varsigma}$, $\mathbf{C}\in\mathbb{R}^{\tau\times(\varsigma+\tau)}$ and $\mathbf{d}\in\mathbb{R}^{\tau}$. On one hand, under ellipsoidal uncertainties, that is, selecting the l_2-norm, the robust counterparts of LPs turn to be SOCPs [5], hence solving the robust optimization equilibria for player one and two in this case turns to be solving SOCCPs. On the other hand, when the norm of the uncertain set is selected to be the $l_1\cap l_\infty$-norm, the robust counterparts retain the computational complexity, that is, the robust counterparts of LPs are still LPs and then solving the robust optimization equilibria for players one and two in this case turns to be solving MCPs. As SOCCPs are numerically harder to solve than MCPs, our model reduces the computational complexity as compared with SOCCPs.

The paper is organized as follows. Section 2 considers the robust optimization equilibrium when each player can estimate his own cost matrix exactly while his opponent's strategies are uncertain. Section 3 considers the case where each player can evaluate his opponent's strategy exactly but is uncertain of his own cost matrix. Section 4 presents some numerical results.

2. Uncertainty in the opponent's strategy

In this section, we focus on the case where each player knows his own cost matrix exactly but is uncertain of his opponent's strategy. Furthermore, each player can estimate the opponent's strategy set by $\mathcal{Z}^{\mathcal{U}}$ and $\mathcal{Y}^{\mathcal{U}}$, respectively. In this case, problems (1.3) and (1.4) can be written as

$$\text{(2.1)} \qquad \text{player one} \quad \min_{\mathbf{y}} \max_{\tilde{\mathbf{z}} \in \mathcal{Z}^{\mathcal{U}}} \mathbf{y}^{\mathrm{T}} A \tilde{\mathbf{z}} \quad \text{s.t. } \mathbf{y} \geqslant 0,\ \mathbf{e}_n^{\mathrm{T}} \mathbf{y} = 1$$

and

$$\text{(2.2)} \qquad \text{player two} \quad \min_{\mathbf{z}} \max_{\tilde{\mathbf{y}} \in \mathcal{Y}^{\mathcal{U}}} \tilde{\mathbf{y}}^{\mathrm{T}} \mathbf{B} \mathbf{z} \quad \text{s.t. } \mathbf{z} \geqslant 0,\ \mathbf{e}_m^{\mathrm{T}} \mathbf{z} = 1.$$

We first consider (2.1). For each $\tilde{\mathbf{z}} \in \mathcal{Z}^{\mathcal{U}}$, we let

$$\text{(2.3)} \qquad \tilde{\mathbf{z}} = \mathbf{z} + \sum_{k=1}^{\mathcal{N}_1} \Delta \mathbf{z}^k d_k,$$

where $\mathbf{z}$ is the nominal value of the data, $\Delta \mathbf{z}^k$ are known directions of data perturbation, d_k are primitive uncertainties. $\mathcal{N}_1$ may be small, modeling situations involving a small collection of primitive uncertainties, or large, potentially as large as the number of entries in the data. In the former case, the elements of $\tilde{\mathbf{z}}$ are strongly dependent, while in the latter the elements of $\tilde{\mathbf{z}}$ are weakly dependent or even independent (when $\mathcal{N}_1$ is equal to the number of data entries). Let $d_k^1 = \max\{0, d_k\}$ and $d_k^2 = \max\{0, -d_k\}$, $k = 1, \ldots, \mathcal{N}_1$. It is clear that $\mathbf{d} = \mathbf{d}^1 - \mathbf{d}^2 = (d_1, d_2, \ldots, d_{\mathcal{N}_1})^{\mathrm{T}}$. Under these assumptions, the asymmetric uncertain set $\mathcal{Z}^{\mathcal{U}}$ can be expressed as

$$\text{(2.4)} \quad \mathcal{Z}^{\mathcal{U}} = \{\tilde{\mathbf{z}} \in \mathbb{R}^m \colon \exists\, \mathbf{d}^1, \mathbf{d}^2 \in \mathbb{R}^{\mathcal{N}_1} \colon \tilde{\mathbf{z}} = \mathbf{z} + \Delta \mathbf{Z}(\mathbf{d}^1 - \mathbf{d}^2),\ \mathbf{z} + \Delta \mathbf{Z}(\mathbf{d}^1 - \mathbf{d}^2) \geqslant 0, \\ \mathbf{e}_m^{\mathrm{T}} \Delta \mathbf{Z}(\mathbf{d}^1 - \mathbf{d}^2) = 0,\ \|\mathbf{P}_1^{-1} \mathbf{d}^1 + \mathbf{Q}_1^{-1} \mathbf{d}^2\| \leqslant \Upsilon,\ \mathbf{d}^1, \mathbf{d}^2 \geqslant 0\},$$

where $\Delta \mathbf{Z} = (\Delta \mathbf{z}^1, \Delta \mathbf{z}^2, \ldots, \Delta \mathbf{z}^{\mathcal{N}_1}) \in \mathbb{R}^{m \times \mathcal{N}_1}$ is the matrix of directions of data perturbation, and $\mathbf{P}_1 = \operatorname{diag}(p_1^1, \ldots, p_{\mathcal{N}_1}^1)$, $\mathbf{Q}_1 = \operatorname{diag}(q_1^1, \ldots, q_{\mathcal{N}_1}^1)$ with $p_k^1, q_k^1 > 0$, $k = 1, \ldots, \mathcal{N}_1$ are forward and backward deviation matrices related to the random variable d_k, and Υ is a parameter controlling the tradeoff between robustness and optimality. The conditions $\mathbf{e}_m^{\mathrm{T}} \Delta \mathbf{Z}(\mathbf{d}^1 - \mathbf{d}^2) = 0$ and $\mathbf{z} + \Delta \mathbf{Z}(\mathbf{d}^1 - \mathbf{d}^2) \geqslant 0$ ensure $\tilde{\mathbf{z}}$ to be a mixed strategy. The norm $\|\cdot\|$ is a vector norm satisfying

$$\text{(2.5)} \qquad \|\mathbf{u}\| = \|\!|\mathbf{u}|\!\|$$

and its dual norm $\|\cdot\|^*$ is given by

$$\|\mathbf{s}\|^* = \max_{\|\mathbf{x}\|\leqslant 1} \mathbf{s}^{\mathrm{T}}\mathbf{x}. \tag{2.6}$$

Lemma 2.1 (Chen, Sim, and Sun [8]). *Let*

$$\pi^* = \max\{\mathbf{a}^{\mathrm{T}}\mathbf{v} + \mathbf{b}^{\mathrm{T}}\mathbf{w}\colon\ \|\mathbf{v}+\mathbf{w}\| \leqslant \Omega,\ \mathbf{v},\mathbf{w} \geqslant 0\}.$$

Then $\Omega\|\mathbf{t}\|^* = \pi^*$, *where* $\mathbf{t} = (t_1,\dots,t_{\mathcal{N}_1})^{\mathrm{T}}$ *with* $t_j = \max\{a_j, b_j, 0\}$, $j \in \mathcal{N}_1$.

Lemma 2.2 (Chen, Sim, and Sun [8]). *If the norm* $\|\cdot\|$ *satisfies* (2.5) *and* (2.6), *then*

(1) *for all* $\mathbf{v}$, $\mathbf{w}$ *such that* $|\mathbf{v}| \leqslant |\mathbf{w}|$ *we have* $\|\mathbf{v}\|^* \leqslant \|\mathbf{w}\|^*$, *and*
(2) $\|\mathbf{t}\|^* \geqslant \|\mathbf{t}\|_2\ \forall\,\mathbf{t}$.

Using Lemmas 2.1 and 2.2, Luo et al. obtained an equivalent optimization formulation for problem (2.1). This is given in the following lemma.

Lemma 2.3 (Luo and Li [16]). *Let* $\mathcal{Z}^{\mathcal{U}}$ *be given by* (2.4). *Then problem* (2.1) *is equivalent to the following optimization problem over* $(\mathbf{y},\theta,\alpha,\mathbf{r},\gamma,\mathbf{f}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\mathcal{N}_1} \times \mathbb{R} \times \mathbb{R}^m$:

$$\begin{aligned}
\min\quad & \theta \\
\text{s.t.}\quad & \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{z} + \mathbf{z}^{\mathrm{T}}\mathbf{f} + \Upsilon\gamma \leqslant \theta, \\
& \|\mathbf{r}\|^* \leqslant \gamma, \\
& \mathbf{r} \geqslant \mathbf{P}_1(\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{f} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m\alpha), \\
& \mathbf{r} \geqslant -\mathbf{Q}_1(\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{f} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m\alpha), \\
& \mathbf{e}_n^{\mathrm{T}}\mathbf{y} = 1,\ \mathbf{y} \geqslant 0,\ \mathbf{f} \geqslant 0.
\end{aligned} \tag{2.7}$$

Analogously, $\mathcal{Y}^{\mathcal{U}}$ can be expressed as

$$\begin{aligned}
\mathcal{Y}^{\mathcal{U}} = \{\tilde{\mathbf{y}} \in \mathbb{R}^n\colon\ \exists\,\mathbf{h}^1,\mathbf{h}^2 \in \mathbb{R}^{\mathcal{N}_2}\colon\ & \tilde{\mathbf{y}} = \mathbf{y} + \Delta\mathbf{Y}(\mathbf{h}^1 - \mathbf{h}^2), \\
& \mathbf{y} + \Delta\mathbf{Y}(\mathbf{h}^1 - \mathbf{h}^2) \geqslant 0,\ \mathbf{e}_n^{\mathrm{T}}\Delta\mathbf{Y}(\mathbf{h}^1 - \mathbf{h}^2) = 0, \\
& \|\mathbf{P}_2^{-1}\mathbf{h}^1 + \mathbf{Q}_2^{-1}\mathbf{h}^2\| \leqslant \Omega,\ \mathbf{h}^1,\mathbf{h}^2 \geqslant 0\},
\end{aligned} \tag{2.8}$$

where $\Delta\mathbf{Y} = (\Delta\mathbf{y}^1, \Delta\mathbf{y}^2, \dots, \Delta\mathbf{y}^{\mathcal{N}_2}) \in \mathbb{R}^{n\times\mathcal{N}_2}$, $\mathbf{h}^{i'} = (h_1^{i'}, h_2^{i'}, \dots, h_{\mathcal{N}_2}^{i'})^{\mathrm{T}} \in \mathbb{R}^{\mathcal{N}_2}$, $i' = 1,2$ is the matrix of directions of data perturbation, $\mathbf{P}_2 = \operatorname{diag}(p_1^2,\dots,p_{\mathcal{N}_2}^2)$, $\mathbf{Q}_2 = \operatorname{diag}(q_1^2,\dots,q_{\mathcal{N}_2}^2)$ satisfying $p_l^2, q_l^2 > 0$, $l = 1,\dots,\mathcal{N}_2$, are forward and backward deviation matrices related to the random variable h_l, and Ω is a parameter controlling the tradeoff between robustness and optimality. Then problem (2.2) is equivalent to the following optimization problem over $(\mathbf{z}, \eta, \mathbf{g}, \mathbf{s}, \zeta) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\mathcal{N}_2} \times \mathbb{R}$:

$$
\begin{aligned}
\text{(2.9)} \qquad \min \quad & \mathbf{y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \mathbf{y}^{\mathrm{T}}\mathbf{g} + \Omega\zeta \\
\text{s.t.} \quad & \|\mathbf{s}\|^* \leqslant \zeta, \\
& \mathbf{s} \geqslant \mathbf{P}_2(\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{g} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{e}_n\eta), \\
& \mathbf{s} \geqslant -\mathbf{Q}_2(\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{g} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{e}_n\eta), \\
& \mathbf{e}_m^{\mathrm{T}}\mathbf{z} = 1,\ \mathbf{z} \geqslant 0,\ \mathbf{g} \geqslant 0.
\end{aligned}
$$

To obtain tractable formulations for problems (2.7) and (2.9) under the $l_1 \cap l_\infty$-norm, we need to investigate the dual norm of the $l_1 \cap l_\infty$-norm. To this end, Bertsimas et al. [5] defined a different norm, called the D-norm. Specifically, for $\mathbf{x} = (x_1, \ldots, x_\nu)^{\mathrm{T}} \in \mathbb{R}^\nu$ and $p \in [1, \nu]$, the D-norm is defined by

$$
\text{(2.10)} \qquad |||\mathbf{x}|||_p = \max_{\{S \cup \{t\}\colon\, S \subseteq N,\, |S| \leqslant \lfloor p \rfloor,\, t \in N \setminus S\}} \Big\{ \sum_{j' \in S} |x_{j'}| + (p - \lfloor p \rfloor)|x_t| \Big\},
$$

where N denotes the set of indices j', $j' = 1, \ldots, \nu$ with $x_{j'}$ subject to parameter uncertainty. The following result can be easily obtained from [5].

Lemma 2.4.

(a) *The dual norm of the norm* $|||\cdot|||_p$ *is given by*

$$
\text{(2.11)} \qquad |||\mathbf{s}|||_p^* = \max\Big\{\frac{1}{p}\|\mathbf{s}\|_1, \|\mathbf{s}\|_\infty\Big\}.
$$

(b) *The inequality* $|||\mathbf{x}|||_p \leqslant \gamma$ *with* $\mathbf{x} \geqslant 0$ *is equivalent to*

$$
\text{(2.12)} \qquad p\theta + \sum_{j'=1}^{\nu} t_{j'} \leqslant \gamma,\ t_{j'} + \theta \geqslant |x_{j'}|,\ t_{j'} \geqslant 0,\ \forall\, j' = 1, \ldots, \nu,\ \theta \geqslant 0.
$$

Consider the case where $\mathcal{Y}^{\mathcal{U}}$ and $\mathcal{Z}^{\mathcal{U}}$ are bounded uncertain sets under the $l_1 \cap l_\infty$-norm. In other words, the norm in (2.4) is given by (2.11) with $p = \gamma$. Due to Lemma 2.11 (a) and the fact that the dual norm of the dual norm is the original norm, the dual norm in (2.7) is given by (2.10). Then, by Lemma 2.4 (b), the constraint

$$
\|\mathbf{r}\|^* \leqslant \gamma
$$

in (2.7) under the $l_1 \cap l_\infty$-norm is equivalent to

$$\begin{gathered}\Upsilon\delta + \sum_{k=1}^{\mathcal{N}_1} w_k \leqslant \gamma,\ w_k + \delta \geqslant |r_k|,\ \forall\, k = 1, \ldots, \mathcal{N}_1,\\ \Delta \geqslant 0,\ \mathbf{w} = (w_1, w_2, \ldots, w_{\mathcal{N}_1})^{\mathrm{T}} \in \mathbb{R}_+^{\mathcal{N}_1}.\end{gathered}$$

Therefore, problem (2.7) under the $l_1 \cap l_\infty$-norm can be expressed as the following optimization problem over $(\mathbf{y}, \alpha, \mathbf{r}, \gamma, \mathbf{f}, \mathbf{w}, \delta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{\mathcal{N}_1} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{\mathcal{N}_1} \times \mathbb{R}$:

$$\begin{aligned} \min\quad & \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{z} + \mathbf{z}^{\mathrm{T}}\mathbf{f} + \Upsilon\gamma \\ \text{s.t.}\quad & \Upsilon\delta + \mathbf{e}_{\mathcal{N}_1}^{\mathrm{T}}\mathbf{w} \leqslant \gamma, \\ & \mathbf{r} \leqslant \mathbf{w} + \mathbf{e}_{\mathcal{N}_1}\delta, \\ & -\mathbf{r} \leqslant \mathbf{w} + \mathbf{e}_{\mathcal{N}_1}\delta, \\ & \mathbf{r} \geqslant \mathbf{P}_1(\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{f} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m\alpha), \\ & \mathbf{r} \geqslant -\mathbf{Q}_1(\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{f} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m\alpha), \\ & \mathbf{e}_n^{\mathrm{T}}\mathbf{y} = 1,\ \mathbf{y} \geqslant 0,\ \mathbf{f} \geqslant 0,\ \mathbf{w} \geqslant 0,\ \delta \geqslant 0. \end{aligned} \tag{2.13}$$

Problem (2.13) is an LP whose KKT conditions are

$$\begin{gathered} \mathbb{R}_+^m \ni \mathbf{f} \perp \mathbf{z} + \Delta\mathbf{Z}\mathbf{P}_1\mathbf{u}_3 - \Delta\mathbf{Z}\mathbf{Q}_1\mathbf{u}_4 \in \mathbb{R}_+^m, \\ \mathbb{R}_+^n \ni \mathbf{y} \perp \mathbf{A}\mathbf{z} + \mathbf{A}\Delta\mathbf{Z}\mathbf{P}_1\mathbf{u}_3 - \mathbf{A}\Delta\mathbf{Z}\mathbf{Q}_1\mathbf{u}_4 + \mathbf{e}_n\xi_1 \in \mathbb{R}_+^n, \\ \mathbb{R}_+^{\mathcal{N}_1} \ni \mathbf{r} - \mathbf{P}_1(\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{f} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m\alpha) \perp \mathbf{u}_3 \in \mathbb{R}_+^{\mathcal{N}_1}, \\ \mathbb{R}_+^{\mathcal{N}_1} \ni \mathbf{r} + \mathbf{Q}_1(\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{f} + \Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m\alpha) \perp \mathbf{u}_4 \in \mathbb{R}_+^{\mathcal{N}_1}, \\ \mathbb{R}_+ \ni \gamma - \Upsilon\delta - \mathbf{e}_{\mathcal{N}_1}^{\mathrm{T}}\mathbf{w} \perp \xi_2 \in \mathbb{R}_+, \\ \mathbb{R}_+ \ni \delta \perp \Upsilon\xi_2 - \mathbf{e}_{\mathcal{N}_1}^{\mathrm{T}}\mathbf{u}_1 - \mathbf{e}_{\mathcal{N}_1}^{\mathrm{T}}\mathbf{u}_2 \in \mathbb{R}_+, \\ \mathbb{R}_+^{\mathcal{N}_1} \ni \mathbf{w} + \mathbf{e}_{\mathcal{N}_1}\delta - \mathbf{r} \perp \mathbf{u}_1 \in \mathbb{R}_+^{\mathcal{N}_1}, \\ \mathbb{R}_+^{\mathcal{N}_1} \ni \mathbf{w} + \mathbf{e}_{\mathcal{N}_1}\delta + \mathbf{r} \perp \mathbf{u}_2 \in \mathbb{R}_+^{\mathcal{N}_1}, \\ \mathbb{R}_+^{\mathcal{N}_1} \ni \mathbf{w} \perp \mathbf{e}_{\mathcal{N}_1}\xi_2 - \mathbf{u}_1 - \mathbf{u}_2 \in \mathbb{R}_+^{\mathcal{N}_1}, \\ \mathbf{e}_m^{\mathrm{T}}\Delta\mathbf{Z}\mathbf{P}_1\mathbf{u}_3 - \mathbf{e}_m^{\mathrm{T}}\Delta\mathbf{Z}\mathbf{Q}_1\mathbf{u}_4 = 0, \\ \Upsilon - \xi_2 = 0, \quad \mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3 - \mathbf{u}_4 = 0, \quad \mathbf{e}_n^{\mathrm{T}}\mathbf{y} = 1, \end{gathered} \tag{2.14}$$

where $\xi_1, \xi_2 \in \mathbb{R}$, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbb{R}^{\mathcal{N}_1}$ are Lagrangian multipliers.

Similarly, problem (2.9) under the $l_1 \cap l_\infty$-norm can be expressed as the following optimization problem over $(\mathbf{z}, \beta, \mathbf{s}, \zeta, \mathbf{g}, \mathbf{t}, \sigma) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{\mathcal{N}_1} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{\mathcal{N}_1} \times \mathbb{R}$:

$$
\begin{aligned}
\min \quad & \mathbf{y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \mathbf{y}^{\mathrm{T}}\mathbf{g} + \Omega\zeta \\
\text{s.t.} \quad & \Omega\sigma + \mathbf{e}_{\mathcal{N}_2}^{\mathrm{T}}\mathbf{t} \leqslant \zeta, \\
& \mathbf{s} \leqslant \mathbf{t} + \mathbf{e}_{\mathcal{N}_2}\sigma, \\
& -\mathbf{s} \leqslant \mathbf{t} + \mathbf{e}_{\mathcal{N}_2}\sigma, \\
& \mathbf{s} \geqslant \mathbf{P}_2(\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{g} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{e}_n\beta), \\
& \mathbf{s} \geqslant -\mathbf{Q}_2(\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{g} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{e}_n\beta), \\
& \mathbf{e}_m^{\mathrm{T}}\mathbf{z} = 1,\ \mathbf{z} \geqslant 0,\ \mathbf{g} \geqslant 0,\ \mathbf{t} \geqslant 0,\ \sigma \geqslant 0.
\end{aligned}
\tag{2.15}
$$

Problem (2.15) is an LP whose KKT conditions are

$$
\begin{gathered}
\mathbb{R}_+^n \ni \mathbf{g} \perp \mathbf{y} + \Delta\mathbf{Y}\mathbf{P}_2\mathbf{v}_3 - \Delta\mathbf{Y}\mathbf{Q}_2\mathbf{v}_4 \in \mathbb{R}_+^n, \\
\mathbb{R}_+^m \ni \mathbf{z} \perp \mathbf{B}^{\mathrm{T}}\mathbf{y} + \mathbf{B}^{\mathrm{T}}\Delta\mathbf{Y}\mathbf{P}_2\mathbf{v}_3 - \mathbf{B}^{\mathrm{T}}\Delta\mathbf{Y}\mathbf{Q}_2\mathbf{v}_4 + \mathbf{e}_m\lambda_1 \in \mathbb{R}_+^m, \\
\mathbb{R}_+^{\mathcal{N}_2} \ni \mathbf{s} - \mathbf{P}_2(\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{g} + \Delta\mathbf{Y}^{\mathrm{T}}\mathbf{e}_n\beta) \perp \mathbf{v}_3 \in \mathbb{R}_+^{\mathcal{N}_2}, \\
\mathbb{R}_+^{\mathcal{N}_2} \ni \mathbf{s} + \mathbf{Q}_2(\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + \boldsymbol{\Delta}\mathbf{Y}^{\mathrm{T}}\mathbf{g} + \boldsymbol{\Delta}\mathbf{Y}^{\mathrm{T}}\mathbf{e}_{\mathbf{n}}\beta) \perp \mathbf{v}_{\mathbf{4}} \in \mathbb{R}_+^{\mathcal{N}_{\mathbf{2}}}, \\
\mathbb{R}_+ \ni \zeta - \Omega\sigma - \mathbf{e}_{\mathcal{N}_2}^{\mathrm{T}}\mathbf{t} \perp \lambda_2 \in \mathbb{R}_+, \quad \mathbb{R}_+ \ni \sigma \perp \Omega\lambda_2 - \mathbf{e}_{\mathcal{N}_2}^{\mathrm{T}}\mathbf{v}_1 - \mathbf{e}_{\mathcal{N}_2}^{\mathrm{T}}\mathbf{v}_2 \in \mathbb{R}_+, \\
\mathbb{R}_+^{\mathcal{N}_2} \ni \mathbf{t} + \mathbf{e}_{\mathcal{N}_2}\sigma - \mathbf{s} \perp \mathbf{v}_1 \in \mathbb{R}_+^{\mathcal{N}_2}, \quad \mathbb{R}_+^{\mathcal{N}_2} \ni \mathbf{t} + \mathbf{e}_{\mathcal{N}_2}\sigma + \mathbf{s} \perp \mathbf{v}_2 \in \mathbb{R}_+^{\mathcal{N}_2}, \\
\mathbb{R}_+^{\mathcal{N}_2} \ni \mathbf{t} \perp \mathbf{e}_{\mathcal{N}_2}\lambda_2 - \mathbf{v}_1 - \mathbf{v}_2 \in \mathbb{R}_+^{\mathcal{N}_2}, \quad \mathbf{e}_n^{\mathrm{T}}\Delta\mathbf{Y}\mathbf{P}_2\mathbf{v}_3 - \mathbf{e}_n^{\mathrm{T}}\Delta\mathbf{Y}\mathbf{Q}_2\mathbf{v}_4 = 0, \\
\Omega - \lambda_2 = 0, \quad \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 - \mathbf{v}_4 = 0, \quad \mathbf{e}_m^{\mathrm{T}}\mathbf{z} = 1,
\end{gathered}
\tag{2.16}
$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, $\mathbf{v_1}, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^{\mathcal{N}_2}$ are Lagrangian multipliers.

Let $\mathbf{x} = (\mathbf{x}_1^{\mathrm{T}}, \mathbf{x}_2^{\mathrm{T}})^{\mathrm{T}}$, where

$$
\mathbf{x}_1 = (\mathbf{y}^{\mathrm{T}}, \alpha, \mathbf{r}^{\mathrm{T}}, \gamma, \mathbf{f}^{\mathrm{T}}, \mathbf{w}^{\mathrm{T}}, \delta, \xi_1, \xi_2, \mathbf{u}_1^{\mathrm{T}}, \mathbf{u}_2^{\mathrm{T}}, \mathbf{u}_3^{\mathrm{T}}, \mathbf{u}_4^{\mathrm{T}})^{\mathrm{T}}
$$

and

$$
\mathbf{x}_2 = (\mathbf{z}^{\mathrm{T}}, \beta, \mathbf{s}^{\mathrm{T}}, \zeta, \mathbf{g}^{\mathrm{T}}, \mathbf{t}^{\mathrm{T}}, \sigma, \lambda_1, \lambda_2, \mathbf{v}_1^{\mathrm{T}}, \mathbf{v}_2^{\mathrm{T}}, \mathbf{v}_3^{\mathrm{T}}, \mathbf{v}_4^{\mathrm{T}}).
$$

Let $\mathbf{d} = (0, \Upsilon, 1, 0, 0, \Omega, 1, 0)^{\mathrm{T}}$, and let

$$
\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & 0 & 0 & 0 \\ \mathbf{G}_2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{G}_3 & 0 \\ 0 & 0 & \mathbf{G}_4 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & \mathbf{H}_1 & 0 & 0 \\ 0 & \mathbf{H}_2 & \mathbf{H}_3 & 0 \\ \mathbf{H}_4 & 0 & 0 & \mathbf{H}_5 \\ 0 & 0 & 0 & \mathbf{H}_6 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & \mathbf{C}_1 & 0 & 0 \\ \mathbf{C}_2 & \mathbf{C}_3 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{C}_4 \\ 0 & 0 & \mathbf{C}_5 & \mathbf{C}_6 \end{pmatrix},
$$

where

$$\mathbf{G}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -\mathbf{e}_{\mathcal{N}_1}^{\mathrm{T}} & -\Upsilon \\ 0 & 0 & -\mathbf{I}_{\mathcal{N}_1} & 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & \mathbf{e}_{\mathcal{N}_1} \\ 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & \mathbf{e}_{\mathcal{N}_1} \\ G_1^1 & G_1^2 & \mathbf{I}_{\mathcal{N}_1} & 0 & G_1^3 & 0 & 0 \\ G_1^4 & G_1^5 & \mathbf{I}_{\mathcal{N}_1} & 0 & G_1^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{G}_2 = \begin{pmatrix} \mathbf{I}_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_m & 0 & 0 \end{pmatrix}, \qquad \mathbf{G}_3 = \begin{pmatrix} \mathbf{I}_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_n & 0 & 0 \end{pmatrix},$$

$$\mathbf{G}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -\mathbf{e}_{\mathcal{N}_2}^{\mathrm{T}} & -\Omega \\ 0 & 0 & -\mathbf{I}_{\mathcal{N}_2} & 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & \mathbf{e}_{\mathcal{N}_2} \\ 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & \mathbf{e}_{\mathcal{N}_2} \\ G_1^7 & G_2^1 & \mathbf{I}_{\mathcal{N}_2} & 0 & G_2^2 & 0 & 0 \\ G_1^8 & G_2^3 & \mathbf{I}_{\mathcal{N}_2} & 0 & G_2^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{H}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_1} \\ 0 & \mathbf{e}_{\mathcal{N}_1} & -\mathbf{I}_{\mathcal{N}_1} & -\mathbf{I}_{\mathcal{N}_1} & 0 & 0 \\ 0 & \Upsilon & -\mathbf{e}_{\mathcal{N}_1}^{\mathrm{T}} & -\mathbf{e}_{\mathcal{N}_1}^{\mathrm{T}} & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_2 = \begin{pmatrix} \mathbf{e}_n & 0 & 0 & 0 & -{G_1^1}^{\mathrm{T}} & -{G_1^4}^{\mathrm{T}} \\ 0 & 0 & 0 & 0 & \Delta\mathbf{Z}\mathbf{P}_1 & -\Delta\mathbf{Z}\mathbf{Q}_1 \end{pmatrix}, \quad \mathbf{H}_3 = \begin{pmatrix} \mathbf{A} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{I}_m & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_4 = \begin{pmatrix} \mathbf{B}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{I}_n & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{H}_5 = \begin{pmatrix} \mathbf{e}_m & 0 & 0 & 0 & -{G_1^7}^{\mathrm{T}} & -{G_1^8}^{\mathrm{T}} \\ 0 & 0 & 0 & 0 & \Delta\mathbf{Y}\mathbf{P}_2 & -\Delta\mathbf{Y}\mathbf{Q}_2 \end{pmatrix},$$

$$\mathbf{H}_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{\mathcal{N}_2} \\ 0 & \mathbf{e}_{\mathcal{N}_2} & -\mathbf{I}_{\mathcal{N}_2} & -\mathbf{I}_{\mathcal{N}_2} & 0 & 0 \\ 0 & \Omega & -\mathbf{e}_{\mathcal{N}_2}^{\mathrm{T}} & -\mathbf{e}_{\mathcal{N}_2}^{\mathrm{T}} & 0 & 0 \end{pmatrix},$$

$$\mathbf{C}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -G_1^{2\,\mathrm{T}} & -G_1^{5\,\mathrm{T}} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} \mathbf{e}_n^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{C}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{\mathcal{N}_1} & -\mathbf{I}_{\mathcal{N}_1} & -\mathbf{I}_{\mathcal{N}_1} & -\mathbf{I}_{\mathcal{N}_1} \end{pmatrix},$$

$$\mathbf{C}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & -G_2^{1\,\mathrm{T}} & -G_2^{3\,\mathrm{T}} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{C}_5 = \begin{pmatrix} \mathbf{e}_m^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{\mathcal{N}_2} & -\mathbf{I}_{\mathcal{N}_2} & -\mathbf{I}_{\mathcal{N}_2} & -\mathbf{I}_{\mathcal{N}_2} \end{pmatrix}$$

with $G_1^1 = -\mathbf{P}_1\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, $G_1^2 = -\mathbf{P}_1\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m$, $G_1^3 = -\mathbf{P}_1\Delta\mathbf{Z}^{\mathrm{T}}$, $G_1^4 = \mathbf{Q}_1\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, $G_1^5 = \mathbf{Q}_1\Delta\mathbf{Z}^{\mathrm{T}}\mathbf{e}_m$, $G_1^6 = \mathbf{Q}_1\Delta\mathbf{Z}^{\mathrm{T}}$, $G_1^7 = -\mathbf{P}_2\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}$, $G_1^8 = \mathbf{Q}_2\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{B}$, $G_2^1 = -\mathbf{P}_2\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{e}_n$, $G_2^2 = -\mathbf{P}_2\Delta\mathbf{Y}^{\mathrm{T}}$, $G_2^3 = \mathbf{Q}_2\Delta\mathbf{Y}^{\mathrm{T}}\mathbf{e}_n$, $G_2^4 = \mathbf{Q}_2\Delta\mathbf{Y}^{\mathrm{T}}$.

Consequently, the problem to find $(\mathbf{y},\mathbf{z})$ satisfying problems (2.1) and (2.2) simultaneously can be formulated as the problem to find $(\mathbf{y},\mathbf{z})$ satisfying the KKT conditions (2.14) and (2.16) simultaneously. The latter can be further stated as an MCP (1.5), where $\varsigma = 5(\mathcal{N}_1+\mathcal{N}_2)+2(m+n)+4$, $\tau = \mathcal{N}_1+\mathcal{N}_2+6$. Therefore, we have the following result.

Theorem 2.1. *Let $\mathcal{Z}^{\mathcal{U}}$ and $\mathcal{Y}^{\mathcal{U}}$ be given by* (2.4) *and* (2.8)*, respectively. Then a robust optimization equilibrium for problems* (2.1) *and* (2.2) *under the $l_1\cap l_\infty$-norm can be formulated as an MCP as above.*

3. Uncertainty in cost matrix

In this section we derive the optimization formulations for problems (1.3) and (1.4) in which each player's cost matrix is uncertain. In this case, problems (1.3) and (1.4) can be written as

(3.1) player one $\displaystyle\min_{\mathbf{y}} \max_{\tilde{\mathbf{A}}\in D_A} \mathbf{y}^{\mathrm{T}}\tilde{A}\mathbf{z}$ s.t. $\mathbf{y} \geqslant 0,\ \mathbf{e}_n^{\mathrm{T}}\mathbf{y} = 1$

and

(3.2) player two $\displaystyle\min_{\mathbf{z}} \max_{\tilde{\mathbf{B}}\in D_B} \mathbf{y}^{\mathrm{T}}\tilde{\mathbf{B}}\mathbf{z}$ s.t. $\mathbf{z} \geqslant 0,\ \mathbf{e}_m^{\mathrm{T}}\mathbf{z} = 1.$

We consider the case where D_A and D_B are constraint-wise uncertain sets, that is, $\tilde{\mathbf{A}} \in D_A = \prod_{j=1}^{m} D_A^j$ and $\tilde{\mathbf{B}} \in D_B = \prod_{i=1}^{n} D_B^i$. Similarly to (2.3), we let $\mathbf{v}_j$ be primitive uncertainties with $\mathbf{v}_j = (v_{j1}, \dots, v_{j,L_j})^{\mathrm{T}}$. Then the asymmetric uncertain set can be described as

$$(3.3)\ D_{\mathbf{A}}^j := \left\{ \mathbf{A}_j^c + \sum_{l_j=1}^{L_j} \Delta \mathbf{r}_j^{l_j} (v_j^{1,l_j} - v_j^{2,l_j}) \colon \|\bar{\mathbf{M}}_j^{-1} \mathbf{v}_j^1 + \bar{\mathbf{N}}_j^{-1} \mathbf{v}_j^2\| \leqslant \Upsilon_j,\ \mathbf{v}_j^1, \mathbf{v}_j^2 \geqslant 0 \right\}, \quad j = 1, \dots, m,$$

where $\bar{\mathbf{M}}_j = \operatorname{diag}(\overline{m}_j^1, \dots, \overline{m}_j^{L_j})$, $\bar{\mathbf{N}}_j = \operatorname{diag}(\bar{n}_j^1, \dots, \bar{n}_j^{L_j})$ with $\overline{m}_j^{l_j}, \bar{n}_j^{l_j} > 0$, $l_j = 1, \dots, L_j$, $j = 1, \dots, m$, $\mathbf{A}_j^c$ is the nominal value for the jth column of the matrix $\tilde{\mathbf{A}}$, $\Delta \mathbf{r}_j^{l_j} \in \mathbb{R}^{n \times 1}$, $(l_j = 1, \dots, L_j)$ are the directions of the data perturbation, $\mathbf{v}_j = \mathbf{v}_j^1 - \mathbf{v}_j^2$ with $\mathbf{v}_j^1 = (v_j^{1,1}, \dots, v_j^{1,L_j})^{\mathrm{T}}$, $\mathbf{v}_j^2 = (v_j^{2,1}, \dots, v_j^{2,L_j})^{\mathrm{T}}$, $v_j^{1,l_j} = \max\{0, v_{j,l_j}\}$, and $v_j^{2,l_j} = \max\{0, -v_{j,l_j}\}$. Similarly,

$$(3.4)\ D_{\mathbf{B}}^i := \left\{ \mathbf{B}_i^r + \sum_{k_i=1}^{K_i} \Delta \mathbf{s}_i^{k_i} (u_i^{1,k_i} - u_i^{2,k_i}) \colon \|\bar{\mathbf{P}}_i^{-1} \mathbf{u}_i^1 + \bar{\mathbf{Q}}_i^{-1} \mathbf{u}_i^2\| \leqslant \Omega_i,\ \mathbf{u}_i^1, \mathbf{u}_{\mathbf{i}}^{\mathbf{2}} \geqslant \mathbf{0} \right\}, \quad i = 1, \dots, n,$$

where $\bar{\mathbf{P}}_i = \operatorname{diag}(\bar{p}_i^1, \dots, \bar{p}_i^{K_i})$, $\bar{\mathbf{Q}}_i = \operatorname{diag}(\bar{q}_i^1, \dots, \bar{q}_i^{K_i})$ with $\bar{p}_i^{k_i}, \bar{q}_i^{k_i} > 0$, $k_i = 1, \dots, K_i$, $i = 1, \dots, n$, $\mathbf{B}_i^r$ is the nominal value for the ith row of the matrix $\tilde{\mathbf{B}}$ and $\Delta \mathbf{s}_i^{k_i} \in \mathbb{R}^{1 \times m}$. With this notation, problems (3.1) and (3.2) can be stated as

$$(3.5) \qquad \text{player one} \quad \begin{aligned} \min_{\mathbf{y}, \mathbf{f}_j, \varrho_j} \ & \mathbf{y}^{\mathrm{T}} \mathbf{A} \mathbf{z} + \sum_{j=1}^{m} \Upsilon_j z_j \varrho_j \\ \text{s.t. } & \|\mathbf{f}_j\|^* \leqslant \varrho_j,\ j = 1, \dots, m, \\ & \mathbf{f}_j \geqslant \bar{\mathbf{M}}_j \Delta \mathbf{R}_j^{\mathrm{T}} \mathbf{y},\ j = 1, \dots, m, \\ & \mathbf{f}_j \geqslant -\bar{\mathbf{N}}_j \Delta \mathbf{R}_j^{\mathrm{T}} \mathbf{y},\ j = 1, \dots, m, \\ & \mathbf{y} \geqslant 0,\ \mathbf{e}_n^{\mathrm{T}} \mathbf{y} = 1, \end{aligned}$$

where $\Delta \mathbf{R}_j = (\Delta \mathbf{r}_j^1, \Delta \mathbf{r}_j^2, \dots, \Delta \mathbf{r}_j^{L_j}) \in \mathbb{R}^{n \times L_j}$, and

$$(3.6) \qquad \text{player two} \quad \begin{aligned} \min_{\mathbf{z}, \mathbf{g}_i, \sigma_i} \ & \mathbf{y}^{\mathrm{T}} \mathbf{B} \mathbf{z} + \sum_{i=1}^{n} \Omega_i y_i \sigma_i \\ \text{s.t. } & \|\mathbf{g}_i\|^* \leqslant \sigma_i,\ i = 1, \dots, n, \\ & \mathbf{g}_i \geqslant \bar{\mathbf{P}}_i \Delta \mathbf{S}_i^{\mathrm{T}} \mathbf{z},\ i = 1, \dots, n, \\ & \mathbf{g}_i \geqslant -\bar{\mathbf{Q}}_i \Delta \mathbf{S}_i^{\mathrm{T}} \mathbf{z},\ i = 1, \dots, n, \\ & \mathbf{z} \geqslant 0,\ \mathbf{e}_m^{\mathrm{T}} \mathbf{z} = 1, \end{aligned}$$

where $\Delta \mathbf{S}_i = ({\Delta \mathbf{s}_i^1}^{\mathrm{T}}, {\Delta \mathbf{s}_i^2}^{\mathrm{T}}, \dots, {\Delta \mathbf{s}_i^{K_i}}^{\mathrm{T}}) \in \mathbb{R}^{m \times K_i}$.

Next we investigate the KKT conditions for problems (3.1) and (3.2), where D_A and D_B are bounded uncertain sets under the $l_1 \cap l_\infty$-norm. In other words, the norm in expression (3.3) is given by (2.11) with $p = \Gamma_j$. Similarly to the analysis in Section 2, the constraints

$$\|\mathbf{f}_j\|^* \leqslant \varrho_j, \quad j = 1, \dots, m, \tag{3.7}$$

in (3.5) under $l_1 \cap l_\infty$-norm are equivalent to

$$\begin{gathered}\Upsilon_j \theta_j + \sum_{l_j=1}^{L_j} w_j^{l_j} \leqslant \varrho_j, \quad f_j^{l_j} \leqslant w_j^{l_j} + \theta_j, \quad -f_j^{l_j} \leqslant w_j^{l_j} + \theta_j, \ \forall\, l_j = 1, \dots, L_j, \\ \theta_j \geqslant 0, \quad \mathbf{w}_j = (w_j^1, w_j^2, \dots, w_j^{L_j})^{\mathrm{T}} \in \mathbb{R}_+^{L_j}, \quad j = 1, \dots, m.\end{gathered} \tag{3.8}$$

Let

$$\begin{gathered}\Upsilon = \operatorname{diag}(\Upsilon_j), \quad \varrho = (\varrho_1, \varrho_2, \dots, \varrho_m)^{\mathrm{T}}, \quad \theta = (\theta_1, \theta_2, \dots, \theta_m)^{\mathrm{T}}, \\ \mathbf{R_M} = (\Delta\mathbf{R}_1 \bar{\mathbf{M}}_1, \Delta\mathbf{R}_2 \bar{\mathbf{M}}_2, \dots, \Delta\mathbf{R}_m \bar{\mathbf{M}}_m) \in \mathbb{R}^{n \times L}, \\ \mathbf{R_N} = (\Delta\mathbf{R}_1 \bar{\mathbf{N}}_1, \Delta\mathbf{R}_2 \bar{\mathbf{N}}_2, \dots, \Delta\mathbf{R}_m \bar{\mathbf{N}}_m) \in \mathbb{R}^{n \times L}, \\ \bar{\mathbf{L}} = \begin{pmatrix} \mathbf{e}_{L_1}^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & \mathbf{e}_{L_2}^{\mathrm{T}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{e}_{L_m}^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{m \times L}, \\ \mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_m \end{pmatrix} \in \mathbb{R}^L \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_m \end{pmatrix} \in \mathbb{R}^L.\end{gathered}$$

Then (3.8) can be written as

$$\Upsilon\theta + \bar{\mathbf{L}}\mathbf{w} \leqslant \varrho, \quad \mathbf{f} \leqslant \mathbf{w} + \bar{\mathbf{L}}^{\mathrm{T}}\theta, \quad -\mathbf{f} \leqslant \mathbf{w} + \bar{\mathbf{L}}^{\mathrm{T}}\theta, \quad \theta \in \mathbb{R}_+^m, \quad \mathbf{w} \in \mathbb{R}_+^L.$$

Therefore, problem (3.5) under the $l_1 \cap l_\infty$-norm can be expressed as the following optimization problem over $(\mathbf{y}, \varrho, \mathbf{f}, \theta, \mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^L \times \mathbb{R}^m \times \mathbb{R}^L$:

$$\begin{aligned} \min \quad & \mathbf{y}^{\mathrm{T}} \mathbf{A} \mathbf{z} + (\Upsilon \mathbf{z})^{\mathrm{T}} \varrho \\ \text{s.t.} \quad & \mathbf{f} \leqslant \mathbf{w} + \bar{\mathbf{L}}^{\mathrm{T}}\theta, \ -\mathbf{f} \leqslant \mathbf{w} + \bar{\mathbf{L}}^{\mathrm{T}}\theta, \\ & \mathbf{f} \geqslant \mathbf{R_M}^{\mathrm{T}}\mathbf{y}, \ \mathbf{f} \geqslant -\mathbf{R_N}^{\mathrm{T}}\mathbf{y}, \\ & \Upsilon\theta + \bar{\mathbf{L}}\mathbf{w} \leqslant \varrho, \ \mathbf{e}_n^{\mathrm{T}}\mathbf{y} = 1, \\ & \mathbf{y} \in \mathbb{R}_+^n, \ \theta \in \mathbb{R}_+^m, \ \mathbf{w} \in \mathbb{R}_+^L. \end{aligned} \tag{3.9}$$

Problem (3.9) is an LP whose KKT conditions are

$$
\begin{gathered}
\mathbb{R}_+^n \ni \mathbf{y} \perp \mathbf{A}\mathbf{z} + \mathbf{R}_\mathbf{M}\mathbf{s_4} - \mathbf{R}_\mathbf{N}\mathbf{s_5} + \mathbf{e_n}\xi \in \mathbb{R}_+^{\mathbf{n}},\\
\mathbb{R}_+^L \ni \mathbf{w} \perp \bar{\mathbf{L}}^{\mathrm{T}}\mathbf{s}_1 - \mathbf{s}_2 - \mathbf{s}_3 \in \mathbb{R}_+^L,\\
\mathbb{R}_+^L \ni \mathbf{w} - \mathbf{f} + \bar{\mathbf{L}}^{\mathrm{T}}\theta \perp \mathbf{s}_2 \in \mathbb{R}_+^L, \quad \mathbb{R}_+^L \ni \mathbf{w} + \mathbf{f} + \bar{\mathbf{L}}^{\mathrm{T}}\theta \perp \mathbf{s}_3 \in \mathbb{R}_+^L,\\
\mathbb{R}_+^L \ni \mathbf{f} - \mathbf{R_M}^{\mathrm{T}}\mathbf{y} \perp \mathbf{s}_4 \in \mathbb{R}_+^L, \quad \mathbb{R}_+^L \ni \mathbf{f} + \mathbf{R_N}^{\mathrm{T}}\mathbf{y} \perp \mathbf{s}_5 \in \mathbb{R}_+^L,\\
\mathbb{R}_+^m \ni \varrho - \Upsilon\theta - \bar{\mathbf{L}}\mathbf{w} \perp \mathbf{s}_1 \in \mathbb{R}_+^m, \quad \mathbb{R}_+^m \ni \theta \perp \Upsilon\mathbf{s}_1 - \bar{\mathbf{L}}\mathbf{s}_2 - \bar{\mathbf{L}}\mathbf{s}_3 \in \mathbb{R}_+^m,\\
\Upsilon\mathbf{z} - \mathbf{s}_1 = 0, \quad \mathbf{s}_2 - \mathbf{s}_3 - \mathbf{s}_4 - \mathbf{s}_5 = 0, \quad \mathbf{e}_n^{\mathrm{T}}\mathbf{y} = 1,
\end{gathered} \tag{3.10}
$$

where $\xi \in \mathbb{R}$, $\mathbf{s}_1 \in \mathbb{R}^m$, $\mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5 \in \mathbb{R}^L$ are Lagrangian multipliers.

Similarly, let

$$
\begin{gathered}
\Omega = \mathrm{diag}(\Omega_i), \quad \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)^{\mathrm{T}}, \quad \eta = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}},\\
\mathbf{v}_i = (v_i^1, v_i^2, \ldots, v_i^{K_i})^{\mathrm{T}} \in \mathbb{R}_+^{K_i},\ i = 1, \ldots, n,\\
\mathbf{S_P} = (\Delta\mathbf{S}_1\bar{\mathbf{P}}_1, \Delta\mathbf{S}_2\bar{\mathbf{P}}_2, \ldots, \Delta\mathbf{S}_n\bar{\mathbf{P}}_n) \in \mathbb{R}^{m\times K},\\
\mathbf{S_Q} = (\Delta\mathbf{S}_1\bar{\mathbf{Q}}_1, \Delta\mathbf{S_2}\bar{\mathbf{Q}}_\mathbf{2}, \ldots, \boldsymbol{\Delta}\mathbf{S_n}\bar{\mathbf{Q}}_\mathbf{n}) \in \mathbb{R}^{\mathbf{m}\times\mathbf{K}},
\end{gathered}
$$

$$
\bar{\mathbf{K}} = \begin{pmatrix} \mathbf{e}_{K_1}^{\mathrm{T}} & 0 & \ldots & 0\\ 0 & \mathbf{e}_{K_2}^{\mathrm{T}} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \mathbf{e}_{K_n}^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{n\times K},\ \mathbf{g} = \begin{pmatrix} \mathbf{g}_1\\ \mathbf{g_2}\\ \vdots\\ \mathbf{g_n} \end{pmatrix} \in \mathbb{R}^K \text{ and } \mathbf{v} = \begin{pmatrix} \mathbf{v}_1\\ \mathbf{v}_2\\ \vdots\\ \mathbf{v}_n \end{pmatrix} \in \mathbb{R}_+^K.
$$

Then problem (3.6) under the $l_1 \cap l_\infty$-norm can be expressed as the following optimization problem over $(\mathbf{z}, \sigma, \mathbf{g}, \eta, \mathbf{v}) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^K \times \mathbb{R}^n \times \mathbb{R}^K$:

$$
\begin{aligned}
\min \quad & \mathbf{y}^{\mathrm{T}}\mathbf{B}\mathbf{z} + (\Omega\mathbf{y})^{\mathrm{T}}\sigma\\
\text{s.t.} \quad & \mathbf{g} \leqslant \mathbf{v} + \bar{\mathbf{K}}^{\mathrm{T}}\eta,\ -\mathbf{g} \leqslant \mathbf{v} + \bar{\mathbf{K}}^{\mathrm{T}}\eta,\ \mathbf{g} \geqslant \mathbf{S_P}^{\mathrm{T}}\mathbf{z}, \quad \mathbf{g} \geqslant -\mathbf{S_Q}^{\mathrm{T}}\mathbf{z},\\
& \Omega\eta + \bar{\mathbf{K}}\mathbf{v} \leqslant \varrho, \quad \mathbf{e}_m^{\mathrm{T}}\mathbf{z} = 1,\\
& \mathbf{z} \in \mathbb{R}_+^m, \quad \eta \in \mathbb{R}_+^n, \quad \mathbf{v} \in \mathbb{R}_+^K.
\end{aligned} \tag{3.11}
$$

Problem (3.11) is an LP whose KKT conditions are

$$
\begin{gathered}
\mathbb{R}_+^m \ni \mathbf{z} \perp \mathbf{B}^{\mathrm{T}}\mathbf{y} + \mathbf{S_P}\mathbf{t}_4 - \mathbf{S_Q}\mathbf{t}_5 + \mathbf{e}_m\lambda \in \mathbb{R}_+^m,\\
\mathbb{R}_+^K \ni \mathbf{v} \perp \bar{\mathbf{K}}^{\mathrm{T}}\mathbf{t_1} - \mathbf{t_2} - \mathbf{t_3} \in \mathbb{R}_+^{\mathbf{K}},\\
\mathbb{R}_+^K \ni \mathbf{v} - \mathbf{g} + \bar{\mathbf{K}}^{\mathrm{T}}\eta \perp \mathbf{t}_2 \in \mathbb{R}_+^K, \quad \mathbb{R}_+^K \ni \mathbf{v} + \mathbf{g} + \bar{\mathbf{K}}^{\mathrm{T}}\eta \perp \mathbf{t}_3 \in \mathbb{R}_+^K,\\
\mathbb{R}_+^K \ni \mathbf{g} - \mathbf{S_P}^{\mathrm{T}}\mathbf{z} \perp \mathbf{t_4} \in \mathbb{R}_+^{\mathbf{K}}, \quad \mathbb{R}_+^{\mathbf{K}} \ni \mathbf{g} + \mathbf{S_Q}^{\mathrm{T}}\mathbf{z} \perp \mathbf{t_5} \in \mathbb{R}_+^{\mathbf{K}},\\
\mathbb{R}_+^n \ni \sigma - \Omega\eta - \bar{\mathbf{K}}\mathbf{v} \perp \mathbf{t}_1 \in \mathbb{R}_+^n, \quad \mathbb{R}_+^n \ni \eta \perp \Omega\mathbf{t}_1 - \bar{\mathbf{K}}\mathbf{t}_2 - \bar{\mathbf{K}}\mathbf{t}_3 \in \mathbb{R}_+^n,\\
\Omega\mathbf{y} - \mathbf{t}_1 = 0, \quad \mathbf{t}_2 - \mathbf{t}_3 - \mathbf{t}_4 - \mathbf{t}_5 = 0, \quad \mathbf{e}_m^{\mathrm{T}}\mathbf{z} = 1,
\end{gathered} \tag{3.12}
$$

where $\lambda \in \mathbb{R}$, $\mathbf{t}_1 \in \mathbb{R}^n$, $\mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5 \in \mathbb{R}^K$ are Lagrangian multipliers.

Combining (3.10) and (3.12), we obtain an MCP (1.5) with

$$\mathbf{x} = (\mathbf{y}^{\mathrm{T}}, \varrho, \mathbf{f}^{\mathrm{T}}, \theta, \mathbf{w}^{\mathrm{T}}, \mathbf{s}_1^{\mathrm{T}}, \mathbf{s}_2^{\mathrm{T}}, \mathbf{s}_3^{\mathrm{T}}, \mathbf{s}_4^{\mathrm{T}}, \mathbf{s}_5^{\mathrm{T}}, \xi, \mathbf{z}^{\mathrm{T}}, \sigma, \mathbf{g}^{\mathrm{T}}, \eta, \mathbf{v}^{\mathrm{T}}, \mathbf{t}_1^{\mathrm{T}}, \mathbf{t}_2^{\mathrm{T}}, \mathbf{t}_3^{\mathrm{T}}, \mathbf{t}_4^{\mathrm{T}}, \mathbf{t}_5^{\mathrm{T}}, \lambda)^{\mathrm{T}},$$

$$K = K_1 + \ldots + K_n,\ L = L_1 + \ldots + L_m,\ \varsigma = 3m + 3n + 5L + 5K,$$
$$\tau = m + n + L + K + 2,$$

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & 0 & 0 & 0 \\ \mathbf{G}_2 & 0 & \mathbf{G}_3 & 0 \\ 0 & 0 & \mathbf{G}_4 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & \mathbf{H}_1 & 0 & 0 \\ \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 & \mathbf{H}_5 \\ 0 & 0 & 0 & \mathbf{H}_6 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3 & 0 \\ \mathbf{C}_4 & 0 & \mathbf{C}_5 & \mathbf{C}_6 \end{pmatrix}$$

where

$$\mathbf{G}_1 = \begin{pmatrix} 0 & \mathbf{I}_m & 0 & -\Upsilon & -\bar{\mathbf{L}} \\ 0 & 0 & -\mathbf{I}_L & \bar{\mathbf{L}}^{\mathrm{T}} & \mathbf{I}_L \\ 0 & 0 & \mathbf{I}_L & \bar{\mathbf{L}}^{\mathrm{T}} & \mathbf{I}_L \\ -\mathbf{R}_M^{\mathrm{T}} & 0 & \mathbf{I}_L & 0 & 0 \\ \mathbf{R}_N^{\mathrm{T}} & 0 & \mathbf{I}_L & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_m & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_L \end{pmatrix}, \quad \mathbf{G}_4 = \begin{pmatrix} 0 & \mathbf{I}_n & 0 & -\Omega & -\bar{\mathbf{K}} \\ 0 & 0 & -\mathbf{I}_K & \bar{\mathbf{K}}^{\mathrm{T}} & \mathbf{I}_K \\ 0 & 0 & \mathbf{I}_K & \bar{\mathbf{K}}^{\mathrm{T}} & \mathbf{I}_K \\ -\mathbf{S}_U^{\mathrm{T}} & 0 & \mathbf{I}_K & 0 & 0 \\ \mathbf{S}_W^{\mathrm{T}} & 0 & \mathbf{I}_K & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_n & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_K \end{pmatrix},$$

$$\mathbf{G}_2 = \begin{pmatrix} \mathbf{I}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{G}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mathbf{I}_m & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_1 = \begin{pmatrix} \mathbf{I}_m & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_L & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_L & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_L & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_L & 0 \\ \Upsilon & -\bar{\mathbf{L}} & -\bar{\mathbf{L}} & 0 & 0 & 0 \\ \bar{\mathbf{L}}^{\mathrm{T}} & -\mathbf{I}_{\mathbf{L}} & -\mathbf{I}_{\mathbf{L}} & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{H}_6 = \begin{pmatrix} \mathbf{I}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_K & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_K & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_K & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_K & 0 \\ \Omega & -\bar{\mathbf{K}} & -\bar{\mathbf{K}} & 0 & 0 & 0 \\ \bar{\mathbf{K}}^{\mathrm{T}} & -\mathbf{I}_K & -\mathbf{I}_K & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mathbf{B}^{\mathrm{T}} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{H}_3 = \begin{pmatrix} 0 & 0 & 0 & \mathbf{R}_M & -\mathbf{R}_N & \mathbf{e}_n \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}_4 = \begin{pmatrix} \mathbf{A} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{H}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{S}_P & -\mathbf{S}_Q & \mathbf{e}_m \end{pmatrix},$$

$$\mathbf{C}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{I}_m & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_L & -\mathbf{I}_L & -\mathbf{I}_L & -\mathbf{I}_L & 0 \end{pmatrix},$$

$$\mathbf{C}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{I}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_K & -\mathbf{I}_K & -\mathbf{I}_K & -\mathbf{I}_K & 0 \end{pmatrix},$$

where $\mathbf{C}_1, \mathbf{C}_5$ are matrices whose first elements are $\mathbf{e}_n^{\mathrm{T}}$, $\mathbf{e}_m^{\mathrm{T}}$, respectively and the other elements are all zeros, $\mathbf{C_3}, \mathbf{C}_4$ are matrices whose elements on the second line and the first column are Υ, Ω, respectively and the other elements are all zeros, and $\mathbf{d} = (1, 0^{\mathrm{T}}, 0^{\mathrm{T}}, 1, 0^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}}$.

Theorem 3.1. *Let D_A and D_B be given by* (3.3) *and* (3.4)*, respectively. Then a robust optimization equilibrium for problems* (3.1) *and* (3.2) *under the $l_1 \cap l_\infty$-norm can be formulated as an MCP given above.*

4. Numerical experiments

In the previous sections, we have shown that some robust optimization equilibrium problems for bimatrix games can be formulated as MCPs. In this section, we present some numerical results for robust optimization equilibrium. We only consider the case where the two players' cost matrices are uncertain and $L_j = K_i = 3$ for all $i, j = 1, 2, 3$. While doing numerical experiments, we adopt the algorithm in [12] to solve the MCP. Consider the bimatrix game with cost matrices

$$\mathbf{A} = \begin{pmatrix} -16 & 20 & 10 \\ 11 & -9 & 40 \\ -15 & -10 & -27 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -14 & -40 & -18 \\ -11 & 10 & 50 \\ 36 & 16 & 40 \end{pmatrix}.$$

We select

$$\bar{\mathbf{M}}_1 = \bar{\mathbf{N}}_2 = \bar{\mathbf{N}}_3 = 2\mathbf{I}_3,\ \bar{\mathbf{N}}_1 = 3\mathbf{I_3},\ \bar{\mathbf{M}}_\mathbf{2} = \mathbf{4I_3},\ \bar{\mathbf{M}}_\mathbf{3} = \mathbf{I_3},$$
$$\bar{\mathbf{P}}_1 = 4\mathbf{I}_3,\ \bar{\mathbf{Q}}_1 = \bar{\mathbf{Q}}_2 = 3\mathbf{I}_3,\ \bar{\mathbf{P}}_2 = \bar{\mathbf{Q}}_3 = 2\mathbf{I}_3,\ \bar{\mathbf{P}}_3 = \mathbf{I}_3,$$

$$\Delta\mathbf{R}_1 = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Delta\mathbf{R}_2 = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}, \quad \Delta\mathbf{R}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Delta\mathbf{S}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \Delta\mathbf{S}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \quad \text{and} \quad \Delta\mathbf{S}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

in (3.3) and (3.4), respectively. In practical applications, the above data such as L_j, K_i, $\bar{\mathbf{M}}_j$, $\bar{\mathbf{N}}_j$, $\bar{\mathbf{P}}_i$, $\bar{\mathbf{Q}}_i$, $i, j = 1, 2, 3$, $\Delta\mathbf{R}_j$ and $\Delta\mathbf{S}_i$ are obtained by statistics or sampling or other techniques. Denote $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3)$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

Robust optimization equilibria for various Υ and Ω are listed in Tabs. 1 and 2. The meaning of the columns in Tabs. 1 and 2 is listed below:

- Υ/Ω: parameter controlling the tradeoff between robustness and optimality. The meaning $\Upsilon = 0.1$ is that $\Upsilon_1 = \Upsilon_2 = \Upsilon_3 = 0.1$ and so is Ω.
- $\overline{y}_r/\overline{z}_r$: robust optimization equilibrium.
- $\overline{y}_r^{\mathrm{T}} A \overline{z}_r / \overline{y}_r^{\mathrm{T}} B \overline{z}_r$: cost value of robust optimization equilibrium.

Υ	Ω	$\overline{y}_r$	$\overline{z}_r$	$\overline{y}_r^{\mathrm{T}} A \overline{z}_r$	$\overline{y}_r^{\mathrm{T}} B \overline{z}_r$
0.1	0.5	(0, 0.1, 0.9)	(0, 1, 0)	−9.9	15.4
0.5	0.1	(0, 0.3208, 0.6792)	(0, 1, 0)	−9.6792	14.0752
0.5	0.5	(0, 0.3332, 0.6668)	(0, 1, 0)	−9.6668	14.0008
1	1	(0, 0.3333, 0.6667)	(0.1, 0.9, 0)	−9.6667	14.0002
5	5	(0.3459, 0.024, 0.6301)	(0.109, 0.5623, 0.3287)	−5.5188	6.5722

Table 1. Robust optimization equilibrium with matrix symmetric uncertainty.

Υ	Ω	$\overline{y}_r$	$\overline{z}_r$	$\overline{y}_r^{\mathrm{T}} A \overline{z}_r$	$\overline{y}_r^{\mathrm{T}} B \overline{z}_r$
0.1	0.5	(0, 0, 1)	(0, 1, 0)	−10	16
0.5	0.1	(0, 1, 0)	(0.0001, 0.4718, 0.528)	−18.9755	28.6724
0.5	0.5	(0.0215, 0.4734, 0.5051)	(0.4714, 0.5285, 0.0001)	−5.9723	12.2980
1	1	(0, 0.0134, 0.9866)	(0, 1, 0)	−9.9866	15.9196
5	5	(0.3010, 0.3028, 0.3963)	(0.4078, 0.592, 0.0002)	−3.4245	1.1650

Table 2. Robust optimization equilibrium with matrix asymmetric uncertainty.

Tabs. 1 and 2 show the results with symmetric and asymmetric uncertain sets, respectively under the $l_1 \cap l_\infty$-norm, that is, $\bar{\mathbf{M}}_j = \bar{\mathbf{N}}_j = \bar{\mathbf{P}}_i = \bar{\mathbf{Q}}_i = \mathbf{I}_3$ for all $i, j = 1, 2, 3$. From Tab. 1, we see that the cost value for each player varies slowly though Υ and Ω vary from 0.1 to 1. Compared with the symmetric situation in Tab. 1, Tab. 2 indicates that when Υ equals 0.1 and Ω equals 0.5 or both Υ and Ω are equal to 1, the cost values in the two tables are close to each other while when Υ and Ω vary simultaneously, the cost value for each player varies distinctly in Tab. 2. From the two tables, we see that

(a) the deviation matrices $\bar{\mathbf{M}}_j$, $\bar{\mathbf{N}}_j$, $\bar{\mathbf{P}}_i$, $\bar{\mathbf{Q}}_i$ in the asymmetric uncertain set play an important role in controlling the robustness and optimality,

(b) compared with the symmetric case, the costs for the two players in the asymmetric case do not always increase as the parameters increase. This phenomenon indicates that the forward and backward deviation matrices may have important influence on the results. At the same time, the results show that as one of the cost goes up then the other declines. From this point, the numerical experiments show that the results are relatively reasonable and our model is feasible, and

(c) the parameters Υ and Ω and the deviation matrices play an important role in controlling the robustness and optimality. However, how to choose an appropriate parameter or deviation matrix is a significantly hard work.

References

[1] *M. Aghassi, D. Bertsimas*: Robust game theory. Math. Program. *107* (2006), 231–273.

[2] *A. Ben-Tal, A. Nemirovski*: Robust convex optimization. Math. Oper. Res. *23* (1998), 769–805.

[3] *A. Ben-Tal, A. Nemirovski*: Robust solutions of uncertain linear programs. Oper. Res. Lett. *25* (1999), 1–13.

[4] *A. Ben-Tal, A. Nemirovski*: Robust solutions of linear programming problems contaminated with uncertain data. Math. Program. *88* (2000), 411–424.

[5] *D. Bertsimas, D. Pachamanova, M. Sim*: Robust linear optimization under general norms. Oper. Res. Lett. *32* (2004), 510–516.

[6] *D. Bertsimas, M. Sim*: The price of robustness. Oper. Res. *52* (2004), 35–53.

[7] *D. Bertsimas, M. Sim*: Tractable approximations to robust conic optimization problems. Math. Program. *107* (2006), 5–36.

[8] *X. Chen, M. Sim, P. Sun*: A robust optimization perspective of stochastic programming. Oper. Res. *55* (2007), 1058–1071.

[9] *L. El Ghaoui, F. Oustry, H. Lebret*: Robust solutions to least-squares problems with uncertain data. SIAM J. Matrix Anal. Appl. *18* (1997), 1035–1064.

[10] *L. El Ghaoui, F. Oustry, H. Lebret*: Robust solutions to uncertain semidefinite programs. SIAM J. Optim. *9* (1998), 33–52.

[11] *F. Facchinei, J. S. Pang*: Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. I. Springer, New York, 2003.

[12] *S. Hayashi, N. Yamashita, M. Fukushima*: A combined smoothing and regularization method for monotone second-order cone complementarity problems. SIAM J. Optim. *15* (2005), 593–615.

[13] *S. Hayashi, N. Yamashita, M. Fukushima*: Robust Nash equilibria and second-order cone complementarity problems. J. Nonlinear. Convex Anal. *6* (2005), 283–296.

[14] *J. C. Harsanyi*: Games with incomplete information played by "Bayesian" playes, Part II. Manage. Sci. *14* (1968), 320–334.

[15] *B. Holmström, R. Myerson*: Efficient and durable decision rules with incomplete information. Econometrica *51* (1983), 1799–1820.

[16] *G. M. Luo, D. H. Li*: Robust optimization equilibrium with deviation measures. Pac. J. Optim. *5* (2009), 427–441.
[17] *J. Mertens, S. Zamir*: Formualation of Bayesian analysis for games with incomplete information. Int. J. Game Theory *14* (1985), 1–29.
[18] *J. F. Nash jun.*: Equilibrium points in n-person games. Proc. Natl. Acad. Sci. USA *36* (1950), 48–49.
[19] *J. Nash*: Non-cooperative games. Ann. Math. *54* (1951), 286–295.
[20] *A. L. Soyster*: Convex programming with set-inclusive constraints and applications to inexact linear programming. Oper. Res. *21* (1973), 1154–1157.

Author's address: *G. Luo*, Department of Applied Mathematics, Guangdong University of Finance, Guangzhou, Guangdong, 510521, P. R. China, e-mail: `gmluo@sina.cn`.