Josef Dalík

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THE INVERTIBILITY OF THE ISOPARAMETRIC MAPPINGS FOR TRIANGULAR QUADRATIC LAGRANGE FINITE ELEMENTS*

Josef Dalík, Brno

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Abstract. A reference triangular quadratic Lagrange finite element consists of a right triangle $\hat{K}$ with unit legs S_1, S_2, a local space $\hat{\mathcal{L}}$ of quadratic polynomials on $\hat{K}$ and of parameters relating the values in the vertices and midpoints of sides of $\hat{K}$ to every function from $\hat{\mathcal{L}}$. Any isoparametric triangular quadratic Lagrange finite element is determined by an invertible isoparametric mapping $\mathcal{F}_h = (F_1, F_2) \in \hat{\mathcal{L}} \times \hat{\mathcal{L}}$. We explicitly describe such invertible isoparametric mappings $\mathcal{F}_h$ for which the images $\mathcal{F}_h(S_1)$, $\mathcal{F}_h(S_2)$ of the segments S_1, S_2 are segments, too. In this way we extend the well-known result going back to W. B. Jordan, 1970, characterizing those invertible isoparametric mappings whose restrictions to the segments S_1 and S_2 are linear.

Keywords: isoparametric triangular quadratic Lagrange finite element, invertible isoparametric mapping

MSC 2010: 65N30, 65N50

1. Introduction

In this section, we define triangular isoparametric quadratic Lagrange finite elements, motivate our main result and explain it. We define

$$\hat{K} = \{[\xi, \eta]\colon\ 0 \leqslant \xi \leqslant 1 \text{ and } 0 \leqslant \eta \leqslant 1 - \xi\},$$

$\hat{a}^1 = [0,1]$, $\hat{a}^2 = [0,1/2]$, $\hat{a}^3 = o \equiv [0,0]$, $\hat{a}^4 = [1/2,0]$, $\hat{a}^5 = [1,0]$, $\hat{a}^6 = [1/2,1/2]$, $S_1 = \overline{\hat{a}^1\hat{a}^3}$, $S_2 = \overline{\hat{a}^3\hat{a}^5}$, $S_3 = \overline{\hat{a}^5\hat{a}^1}$ and denote by $\mathrm{int}(\hat{K})$ the *interior* of $\hat{K}$.

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A *reference triangular quadratic Lagrange finite element* $\hat{\mathcal{K}}$ consists of

a) the *right triangle with unit legs* $\hat{K}$,
b) the *local space* $\hat{\mathcal{L}}$ of restrictions of polynomials of degree two or less to the triangle $\hat{K}$,
c) the "set of parameters" relating the values $\hat{p}(\hat{a}^i)$ for $i = 1, \ldots, 6$ to each $\hat{p} \in \hat{\mathcal{L}}$. (These *parameters* determine $\hat{p}$ uniquely.)

For arbitrary points $a^1, \ldots, a^6$ in $\mathbb{R}^2$, we put $h = \max(|a^1a^3|, |a^3a^5|, |a^5a^1|)$ and define the *isoparametric mapping* $\mathcal{F}_h = (F_1, F_2) \in \hat{\mathcal{L}} \times \hat{\mathcal{L}}$ by

$$\mathcal{F}_h(\hat{a}^i) = a^i \quad \text{for } i = 1, \ldots, 6.$$

If $\mathcal{F}_h$ is invertible, then we denote by $\mathcal{G}_h$ the inverse of $\mathcal{F}_h$, see Fig. 1.

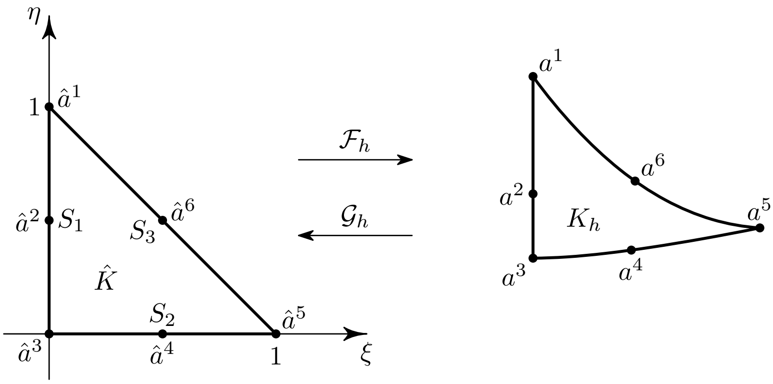

Figure 1. The mappings $\mathcal{F}_h$ and $\mathcal{G}_h$.

An *isoparametric triangular quadratic Lagrange finite element* $\mathcal{K}_h$ is determined by an invertible isoparametric mapping $\mathcal{F}_h$. It consists of

(a) the *curved triangle* $K_h = \mathcal{F}_h(\hat{K})$,
(b) the *local space* $\mathcal{L}_h$ of functions

$$p_h(x, y) = \hat{p}(\mathcal{G}_h(x, y)) \quad \text{for all } \hat{p} \in \hat{\mathcal{L}},$$

(c) the "set of parameters" relating the values $p_h(a^1), \ldots, p_h(a^6)$ to any $p_h \in \mathcal{L}_h$. (These *parameters* determine p_h in $\mathcal{L}_h$ uniquely.)

Most often, finite elements of this type are used for an accurate approximation of piecewise smooth curved boundaries of domains of boundary-value problems for partial differential equations. For example, if we approximate the weak solution u of the problem

$$-\Delta u = f \text{ in } \Omega \subseteq \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega$$

with piecewise smooth boundary $\partial\Omega$ by the linear finite-element approximation u_h related to a triangulation $\mathcal{T}_h$ with discretization step h, then the order of the H^1-norm of the error $u-u_h$ is $O(h)$. The use of quadratic finite elements on the triangles from $\mathcal{T}_h$ increases this order to $O(h^{3/2})$. To obtain an error of a higher order, $\partial\Omega$ has to be approximated more exactly. The triangular isoparametric quadratic Lagrange finite element is a standard tool for this purpose and its use leads to the H^1-norm of error of the optimal order $O(h^2)$. In this application, two sides of the curved triangle K_h remain straight; we denote them by $\overline{a^1a^3}$ and $\overline{a^3a^5}$ as in Fig. 2.

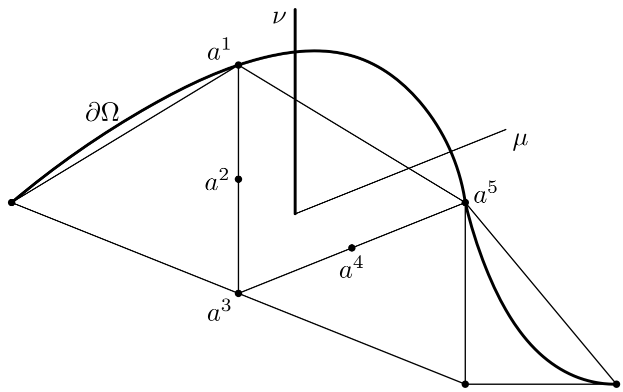

Figure 2. Illustration of the Jordan result.

The famous result by Jordan [3], see also Strang, Fix [10], Mitchell, Wait [9], and Křížek, Neittaanmäki [6], says that, under the assumptions $a^2 = (a^1 + a^3)/2$, $a^4 = (a^3+a^5)/2$, the isoparametric mapping $\mathcal{F}_h$ is invertible if and only if the point a^6 is situated between the rays μ and ν of the points $x = (a^2 + a^4)/2 + v(a^5 - a^3)$ with $v \geqslant 0$ and $x = (a^2+a^4)/2+u(a^1-a^3)$ with $u \geqslant 0$, respectively, see Fig. 2. It is natural to ask what happens when the points a^2, a^4 change their positions. This question is especially interesting in the case $a^2 \in \overline{a^1a^3}$, $a^4 \in \overline{a^3a^5}$ because of simplicity of its implementation.

In this paper, we work with points $a^1,\ldots,a^5$ such that a^1, a^3, a^5 do not appear on one straight line exclusively. We describe the *admissible set*

$$\mathcal{A}d = \mathcal{A}d(a^1,\ldots,a^5) \equiv \{a^6\colon \text{ the mapping } \mathcal{F}_h \text{ is invertible}\}$$

in the case $a^2 \in \overline{a^1a^3}$, $a^4 \in \overline{a^3a^5}$

In Section 2, we characterize injective restrictions of isoparametric mappings to the sides S_1, S_2, S_3 and injective isoparametric mappings (on $\hat{K}$) by means of their Jacobians. In Section 3, we study the invertibility of $\mathcal{F}_h$ under the assumption that there exist coordinates u, v, U, V such that $a^2 = a^3+u(a^1-a^3)$, $a^4 = a^3+v(a^5-a^3)$, and $a^6 = a^3 + U(a^1 - a^3) + V(a^5 - a^3)$. According to Jordan [3], $\mathcal{A}d$ is equal to the *Jordan admissible set*

$$\mathcal{A}d_J = \{a^6 \colon\ 1/4 \leqslant U,\ 1/4 \leqslant V\} \tag{1.1}$$

in the case $u = 1/2 = v$. In Section 4, we prove that the admissible set is non-empty if and only if $1/4 \leqslant u \leqslant 3/4$, $1/4 \leqslant v \leqslant 3/4$ and describe it in the following explicit way. If $1 < u + v$ and either $u = 3/4$ or $v = 3/4$, then

$$\mathcal{A}d = \{a^6 \in \mathcal{A}d_J \colon\ U(4v-1) + V(4u-1) \geqslant 3(u+v) - 5/2\}.$$

If $1 < u + v$, $u < 3/4$ and $v < 3/4$, then $\mathcal{A}d = \mathcal{A}d_J - \mathcal{T}$, where $\mathcal{T}$ is the curved triangle bounded by the rays μ, ν and by the negatively oriented arc $t_U t_V$ of the ellipse with centre $s = [u/2 + 1/8, v/2 + 1/8]$ touching the line μ, ν at the point $t_U = [1/4, (12u+8v-9)/4/(4u-1)]$, $t_V = [(8u+12v-9)/4/(4v-1), 1/4]$, respectively. Fig. 3 illustrates the admissible set in the case $u = 2/3$, $v = 1/2$.

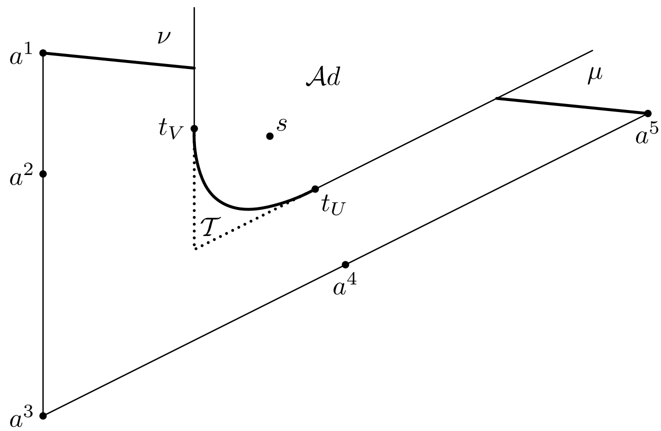

Figure 3. The admissible set.

Further, in the case $1/2 < u + v \leqslant 1$ we have

$$\mathcal{A}d = \mathcal{A}d_J$$

and, finally, if $u = 1/4 = v$, then

$$\mathcal{A}d = \mathcal{A}d_J \cup \{a^6 \colon\ U < 1/4,\ V < 1/4,\ (4U-1)(4V-1) > 1\}.$$

Although the isoparametric finite elements are widely used and also analysed, see Section 4.3 from Ciarlet [2] for example, our knowledge of the invertible isoparametric mappings is very poor. These mappings have been characterized for bilinear finite elements in Strang-Fix [10] and for pyramidal and prismatic finite elements in Knabner, Summ [5]. In Knabner, Korotov, Summ [4], an algorithm checking the positivity of the Jacobian of the isoparametric mapping for trilinear finite elements on hexahedra has been presented. Inverse isoparametric mappings have been studied in Yuan, Huang, Yang, Pian [11] and Lautersztajn-S, Samuelsson [7] by the tools of differential geometry and continuously invertible mappings, in Meisters, Olech [8] by topological methods. In Barrett [1], some general necessary conditions for the invertibility of isoparametric mappings can be found. Results of this kind should give the developers of programming systems based on the finite element methods criteria guaranteeing correct implementations of the isoparametric finite elements.

2. Abstract invertible isoparametric mappings

We characterize invertible pairs of quadratic polynomials on the interval $\langle 0,1\rangle$ in Lemma 1 and discuss an important special case in Corollary 1. In Lemma 2 we prove that an isoparametric mapping $\mathcal{F}_h$ is invertible (on $\hat{K}$) if and only if the Jacobian of $\mathcal{F}_h$ is non-zero in $\operatorname{int}(\hat{K})$ and the restriction of $\mathcal{F}_h$ to each of the sides S_1, S_2, S_3 is an injection.

Lemma 1. *For arbitrary points $a, b, c \in \mathbb{R}^2$, the mapping $\mathcal{F} = (F_1, F_2)\colon \langle 0,1\rangle \to \mathbb{R}^2$ such that F_1, F_2 are quadratic polynomials satisfying*

$$\mathcal{F}(0) = a, \quad \mathcal{F}(1/2) = b, \quad \mathcal{F}(1) = c$$

is an injection if and only if $a \neq c$ and

(2.1) $$b = a + u(c-a) \text{ for some } u \in \mathbb{R} \Longrightarrow 1/4 \leqslant u \leqslant 3/4.$$

Remark. The condition (2.1) says that $\mathcal{F}$ is an injection if and only if the point b is situated anywhere in $\mathbb{R}^2$ except the two thick open rays illustrated in Fig. 4.

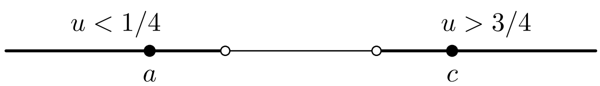

Figure 4. Graphical illustration of Lemma 1.

Proof. Of course, we have

$$\mathcal{F}(t) = a + 2(b-a)t + (a - 2b + c)(2t^2 - t)$$

for $t \in \langle 0,1\rangle$. This mapping is not an injection if and only if

$$\exists\, t_1, t_2\colon\ 0 \leqslant t_1 < t_2 \leqslant 1 \text{ and}$$
$$\mathcal{F}(t_1) - \mathcal{F}(t_2) = (t_1 - t_2)[2(b-a) + (a - 2b + c)(2(t_1 + t_2) - 1)] = 0.$$

Putting $t^* = (t_1 + t_2)/2$, this condition is equivalent to

$$\exists\, t^* \in (0,1)\colon\ (4 - 8t^*)b = (3 - 4t^*)a + (1 - 4t^*)c.$$

Hence, $\mathcal{F}$ is an injection if and only if

$$\forall\, t^* \in (0,1)\colon\ (4 - 8t^*)b \neq (3 - 4t^*)a + (1 - 4t^*)c. \tag{2.2}$$

Setting $t^* = 1/2$, we obtain $a \neq c$. In the case $t^* \neq 1/2$, (2.2) gives us

$$\forall\, t^* \in (0,1/2) \cup (1/2,1)\colon\ b \neq a + u(c-a) \text{ for } u = (1 - 4t^*)/(4 - 8t^*).$$

It is easy to see that $t^* \in (0,1/2) \iff u \in (-\infty, 1/4)$ and $t^* \in (1/2,1) \iff u \in (3/4,\infty)$, so that (2.1) follows immediately. $\square$

Corollary 1. *Let f_0, f_1, f_2 be real numbers and P a quadratic polynomial on the interval $\langle 0,1\rangle$ such that $P(0) = f_0$, $P(1/2) = f_1$, $P(1) = f_2$. Then the following statements* a), b) *are true.*

a) *P is an injection if and only if $f_0 \neq f_2$ and*

$$\frac{1}{4}\min(3f_0 + f_2, f_0 + 3f_2) \leqslant f_1 \leqslant \frac{1}{4}\max(3f_0 + f_2, f_0 + 3f_2).$$

b) *P attains its absolute minimum [maximum] on $\langle 0,1\rangle$ in a unique point $m \in (0,1)$ if and only if*

$$f_1 < \frac{1}{4}\min(3f_0 + f_2, f_0 + 3f_2) \quad \Big[f_1 > \frac{1}{4}\max(3f_0 + f_2, f_0 + 3f_2)\Big].$$

Then

$$P(m) = f_1 - \frac{(f_0 - f_2)^2}{8(f_0 - 2f_1 + f_2)}.$$

P r o o f. a) In the case $a = [0, f_0]$, $b = [0, f_1]$, $c = [0, f_2]$, Lemma 1 says that P is an injection if and only if $f_0 \neq f_2$ and

$$f_1 = f_0 + u(f_2 - f_0) \quad \text{for some } u\colon\ 1/4 \leqslant u \leqslant 3/4.$$

This is equivalent to the fact that f_1 is situated between $f_0+(f_2-f_0)/4 = (3f_0+f_2)/4$ and $f_0+3(f_2-f_0)/4 = (f_0+3f_2)/4$.

b) It follows by a) that P attains a proper local minimum [maximum] in a point $m \in (0,1)$ if and only if

$$f_1 < \frac{1}{4}\min(f_0+3f_2, 3f_0+f_2) \quad \Big[f_1 > \frac{1}{4}\max(f_0+3f_2, 3f_0+f_2)\Big].$$

As $P(t) = f_0 + 2(f_1-f_0)t + (f_0-2f_1+f_2)(2t^2-t)$ and $P'(t) = -3f_0+4f_1-f_2+4t(f_0-2f_1+f_2)$, we have $P'(m)=0$ if and only if

$$m = \frac{1}{4} + \frac{f_0-f_1}{2(f_0-2f_1+f_2)}.$$

Then

$$P(m) = f_1 - \frac{(f_2-f_0)^2}{8(f_0-2f_1+f_2)}.$$

□

We denote by $J = J(\xi,\eta)$ the Jacobian of the isoparametric mapping $\mathcal{F}_h$.

Lemma 2. *Let us consider such points $a^1,\dots,a^6$ that the restriction of the isoparametric mapping $\mathcal{F}_h$ to each of the segments S_1, S_2, S_3 is an injection. Then $\mathcal{F}_h$ is invertible if and only if*

$$J(\xi,\eta) \neq 0 \quad \textit{for all} \quad [\xi,\eta] \in \operatorname{int}(\hat{K}).$$

Proof. Let us put $\mathcal{F}_h = (F_1, F_2)$ and

$$M_1(\xi,\eta) = \begin{bmatrix} \dfrac{\partial F_1}{\partial \xi} & \dfrac{\partial F_1}{\partial \eta} \\ \dfrac{\partial F_2}{\partial \xi} & \dfrac{\partial F_2}{\partial \eta} \end{bmatrix}(\xi,\eta), \quad M_2 = \begin{bmatrix} \dfrac{1}{2}\dfrac{\partial^2 F_1}{\partial \xi^2} & \dfrac{\partial^2 F_1}{\partial \xi \partial \eta} & \dfrac{1}{2}\dfrac{\partial^2 F_1}{\partial \eta^2} \\ \dfrac{1}{2}\dfrac{\partial^2 F_2}{\partial \xi^2} & \dfrac{\partial^2 F_2}{\partial \xi \partial \eta} & \dfrac{1}{2}\dfrac{\partial^2 F_2}{\partial \eta^2} \end{bmatrix}.$$

We first prove the following useful property of quadratic polynomials.

a) If a, b are different points from $\hat{K}$, then

$$\mathcal{F}_h(b) - \mathcal{F}_h(a) = M_1(c)(b-a)$$

for $c = \frac{1}{2}(a+b)$: due to the Taylor Theorem, we have

$$\mathcal{F}_h(b) = \mathcal{F}_h(a) + M_1(a)(b-a) + M_2 \begin{bmatrix} (b_1-a_1)^2 \\ (b_1-a_1)(b_2-a_2) \\ (b_2-a_2)^2 \end{bmatrix} \tag{2.3}$$

and

$$\mathcal{F}_h(a) = \mathcal{F}_h(b) + M_1(b)(a-b) + M_2 \begin{bmatrix} (a_1-b_1)^2 \\ (a_1-b_1)(a_2-b_2) \\ (a_2-b_2)^2 \end{bmatrix}. \tag{2.4}$$

If we subtract (2.4) from (2.3), divide the difference by two and use the facts that the entries of M_1 are linear and those of M_2 are constant, we get a).

Let there exist different points a, b in $\hat{K}$ such that $\mathcal{F}_h(a) = \mathcal{F}_h(b)$. Due to our assumptions, a, b do not belong to the same side of $\hat{K}$. But then $c = \frac{1}{2}(a+b) \in \operatorname{int}(\hat{K})$ and $M_1(c)(b-a)$ is the zero point o by a). Hence, $J(c) = \det(M_1(c)) = 0$.

Conversely, let $\mathcal{F}_h$ be invertible. For every point $c \in \operatorname{int}(\hat{K})$ and for arbitrary different $a, b \in \hat{K}$ such that $c = \frac{1}{2}(a+b)$, we have $\overline{ab} \subseteq \hat{K}$. Then $o \neq \mathcal{F}_h(b) - \mathcal{F}_h(a) = M_1(c)(b-a)$ by a) and we conclude that $J(c) \neq 0$. $\square$

The inverse implication of Lemma 2 is a consequence of the abstract Theorem 1 from [8]. Lemma 2 and the continuity of the Jacobian J give us the following statement.

Corollary 2. *For arbitrary points $a^1, \ldots, a^6$, the isoparametric mapping $\mathcal{F}_h$ is invertible if and only if the restriction of $\mathcal{F}_h$ to each of the segments S_1, S_2, S_3 is an injection and J is positive on* $\operatorname{int}(\hat{K})$ *or J is negative on* $\operatorname{int}(\hat{K})$.

3. Special invertible isoparametric mappings

In this section, we consider points $a^1, \ldots, a^6$ such that $a^2 \in \overline{a^1a^3}$ and $a^4 \in \overline{a^3a^5}$. We say that u, v, U, V are *coordinates* whenever

$$\begin{gathered} a^2 = a^3 + u(a^1 - a^3), \;\; a^4 = a^3 + v(a^5 - a^3), \\ a^6 = a^3 + U(a^1 - a^3) + V(a^5 - a^3), \end{gathered} \tag{3.1}$$

and identify the point a^6 with the ordered pair $[U, V]$. For arbitrary points a, b, c, we put

$$D(abc) = \begin{vmatrix} a_1 - c_1 & a_2 - c_2 \\ b_1 - c_1 & b_2 - c_2 \end{vmatrix}.$$

We prove that $J = D(a^1a^3a^5)\tilde{J}$ and express the coefficients of the polynomial $\tilde{J}(\xi, \eta)$ in terms of the coordinates in Proposition 1. Led by Corollary 2, we characterize those coordinates for which $\tilde{J}$ is negative in $\operatorname{int}(\hat{K})$ in Lemma 3. We find necessary and sufficient conditions guaranteeing non-negativity of $\tilde{J}$ on $\partial\hat{K}$ in Lemma 7 and prove that, under these conditions, $\tilde{J}$ is positive in $\operatorname{int}(\hat{K})$ in Lemma 8.

The following direct consequences of Lemma 1 give us necessary conditions for the admissible set to be non-empty.

Corollary 3. *Let $\mathcal{F}_h$ be an isoparametric mapping related to the points $a^1, \ldots, a^6$ with coordinates u, v, U, V. Then*

a) $\mathcal{F}_h|_{S_1}$ *is an injection* $\Longleftrightarrow 1/4 \leqslant u \leqslant 3/4$.
b) $\mathcal{F}_h|_{S_2}$ *is an injection* $\Longleftrightarrow 1/4 \leqslant v \leqslant 3/4$.
c) $\mathcal{F}_h|_{S_3}$ *is an injection* $\Longleftrightarrow [U + V = 1 \Longrightarrow 1/4 \leqslant U \leqslant 3/4]$.

Proposition 1. *Let $\mathcal{F}_h$ be an isoparametric mapping related to the points $a^1, \ldots, a^6$ with coordinates u, v, U, V.*

Then $J(\xi,\eta) = D(a^1a^3a^5)\tilde{J}(\xi,\eta)$, $\tilde{J}(\xi,\eta) = a + b\xi + c\eta + d\xi^2 + e\xi\eta + f\eta^2$ and

$$\begin{aligned}
a &= (4u-1)(4v-1),\\
b &= 4[(4v-1)(U-u) - (4u-1)(2v-1)],\\
c &= 4[(4u-1)(V-v) - (4v-1)(2u-1)],\\
d &= 16(u-U)(2v-1),\\
e &= 16(2u-1)(2v-1),\\
f &= 16(v-V)(2u-1).
\end{aligned}$$

If the ordered triple (a^1, a^3, a^5) is oriented positively, then $J(\xi,\eta)$ and $\tilde{J}(\xi,\eta)$ have the same sign.

P r o o f. It is easy to see that the polynomials

$$\begin{aligned}
\hat{L}_1(\xi,\eta) &= \eta(2\eta-1),\\
\hat{L}_2(\xi,\eta) &= 4\eta(1-\xi-\eta),\\
\hat{L}_3(\xi,\eta) &= (1-\xi-\eta)(1-2\xi-2\eta),\\
\hat{L}_4(\xi,\eta) &= 4\xi(1-\xi-\eta),\\
\hat{L}_5(\xi,\eta) &= \xi(2\xi-1),\\
\hat{L}_6(\xi,\eta) &= 4\xi\eta
\end{aligned}$$

create the Lagrange basis in $\hat{\mathcal{L}}$ related to the points $\hat{a}^1, \ldots, \hat{a}^6$. If we put $\mathcal{F}_h(\xi,\eta) = \hat{L}_1(\xi,\eta)a^1 + \ldots + \hat{L}_6(\xi,\eta)a^6$, express a^2, a^4, a^6 by (3.1), compute and simplify the Jacobian, then we obtain $J(\xi,\eta) = D(a^1a^3a^5)\tilde{J}(\xi,\eta)$ as well as the above form of $\tilde{J}$. The last statement is well known. $\square$

We have characterized the injectivity of $\mathcal{F}_h|_{S_i}$ for $i = 1, 2, 3$ in Corollary 3. Then, due to Corollary 2, $\mathcal{F}_h$ is invertible whenever $\tilde{J} > 0$ in $\operatorname{int}(\hat{K})$ or $\tilde{J} < 0$ in $\operatorname{int}(\hat{K})$. We investigate the second case in Lemma 3.

Definition. For arbitrary coordinates u, v, U, V, we put $J_0 = \tilde{J}(\hat{a}^1)$, $J_1 = \tilde{J}(\hat{a}^2)$, $J_2 = \tilde{J}(\hat{a}^3)$.

Due to Proposition 1, we have

$$\begin{aligned} J_0 &= (3 - 4u)(4V - 1), \\ J_1 &= 2v + 2V - 1, \\ J_2 &= (4u - 1)(4v - 1). \end{aligned}$$

Lemma 3. *Let the points $a^1, \dots, a^6$ with coordinates u, v, U, V satisfy $1/4 \leqslant u \leqslant 3/4$, $1/4 \leqslant v \leqslant 3/4$. Then $\tilde{J} < 0$ in $\operatorname{int}(\hat{K})$ if and only if*

$$u = 1/4 = v,\ U < 1/4,\ V < 1/4,\ \text{and } (4U - 1)(4V - 1) > 1.$$

P r o o f. We prove the following statements a)–c).

a) $\tilde{J}|_{S_1} \leqslant 0 \Longleftrightarrow [v = 1/4$ and $V \leqslant 1/4]$: Let us assume that

$$J_0 \leqslant 0,\ J_1 \leqslant 0,\ J_2 \leqslant 0.$$

Then, as $u, v \in \langle 1/4, 3/4\rangle$, $J_2 \leqslant 0 \iff J_2 = 0 \Longleftrightarrow u = 1/4$ or $v = 1/4$. Further, $\tilde{J} \leqslant 0$ on S_1 if and only if $\tilde{J}$ has not an absolute maximum at any unique point inside of S_1. This is equivalent to

$$J_1 \leqslant \frac{1}{4}\max(3J_0 + J_2, J_0 + 3J_2) = \frac{1}{4}J_0$$

due to Corollary 1 b). By evaluating J_1 and J_0, we obtain

$$0 \leqslant 1 + 4u - 8v + 4V - 16uV.$$

If we admit $v > 1/4$, then $u = 1/4$ and this inequality simplifies to $0 \leqslant 2 - 8v$. Hence, $v = 1/4$ and, in this case, $J_1 \leqslant 0$ means $V \leqslant 1/4$.

b) $\tilde{J}|_{S_2} \leqslant 0 \Longleftrightarrow [u = 1/4$ and $U \leqslant 1/4]$: This statement can be proved in the same way as a).

The following statement c) characterizes the points $a^6 = [U, V]$ such that $\tilde{J} < 0$ in $\operatorname{int}(\hat{K})$ under the assumption $\tilde{J}|_{S_1 \cup S_2} \leqslant 0$.

c) Let us assume that $u = 1/4 = v$, $U \leqslant 1/4$ and $V \leqslant 1/4$. Then $\tilde{J} < 0$ in $\operatorname{int}(\hat{K})$ if and only if

$$(4U - 1)(4V - 1) > 1.$$

In the case $u = 1/4 = v$, Proposition 1 gives us

$$\frac{1}{2}\tilde{J}(\xi,\eta) = 2\xi\eta\big(1 - \sqrt{(1-4U)(1-4V)}\big) - \big(\xi\sqrt{1-4U} - \eta\sqrt{1-4V}\big)^2.$$

It is easy to see that $\tilde{J} < 0$ in $\operatorname{int}(\hat{K})$ if and only if $1 - \sqrt{(1-4U)(1-4V)} < 0$ and this is equivalent to $(4U-1)(4V-1) > 1$. $\square$

Now, let us study the more common case $\tilde{J} > 0$ in $\operatorname{int}(\hat{K})$.

Lemma 4. *Let $\mathcal{F}_h$ be an isoparametric mapping related to the points $a^1, \ldots, a^6$ with coordinates u, v, U, V such that $1/4 \leqslant u \leqslant 3/4$ and $1/4 \leqslant v \leqslant 3/4$. Then the following statements* a), b) *are valid.*

a) $\tilde{J}|_{S_1} \geqslant 0 \Longleftrightarrow 1/4 \leqslant V$.

b) $\tilde{J}|_{S_2} \geqslant 0 \Longleftrightarrow 1/4 \leqslant U$.

P r o o f. a) Let us assume that $\tilde{J}|_{S_1} \geqslant 0$. Then $J_0 \geqslant 0$ and $J_2 \geqslant 0$ give us

$$(3.2) \qquad (3-4u)(4V-1) \geqslant 0 \quad \text{and} \quad (4u-1)(4v-1) \geqslant 0.$$

If we assume $u < 3/4$, then $1/4 \leqslant V$ due to (3.2). In the case $u = 3/4$ we have

$$J_0 = 0, \quad J_1 = 2v + 2V - 1, \quad J_2 = 2(4v-1).$$

If, moreover, $v = 1/4$, then $J_2 = 0$ and

$$\tilde{J}|_{S_1} \geqslant 0 \Longleftrightarrow J_1 \geqslant 0 \Longleftrightarrow 1/4 \leqslant V.$$

In the case $v > 1/4$ we have $J_0 = 0 < J_2$ and Corollary 1 b) gives us

$$\tilde{J}|_{S_1} \geqslant 0 \Longleftrightarrow J_1 \geqslant \frac{1}{4}\min(3J_0 + J_2, J_0 + 3J_2) = \frac{1}{4}J_2 \Longleftrightarrow \frac{1}{4} \leqslant V.$$

Conversely, let us assume that $1/4 \leqslant V$ for some $u, v \in \langle 1/4, 3/4\rangle$. Then $J_0 \geqslant 0$, $J_2 \geqslant 0$ and we prove that $\tilde{J}|_{S_1}$ does never attain its absolute minimum at any unique inner point of S_1. Due to Corollary 1 b), it is sufficient to verify the following implications a1), a2).

a1) $J_2 \leqslant J_0 \Longrightarrow J_1 \geqslant \frac{1}{4}(J_0 + 3J_2)$: $J_2 \leqslant J_0$ is equivalent to

$$(3.3) \qquad (4u-1)(4v-1) \leqslant (3-4u)(4V-1)$$

and $J_1 \geqslant \frac{1}{4}(J_0 + 3J_2)$ is equivalent to

$$(3.4) \qquad (12u-5)(4v-1) \leqslant (4u-1)(4V-1).$$

If $u = 3/4$, then $v = 1/4$ by (3.3), so that (3.4) is equivalent to the valid statement $1/4 \leqslant V$. Let us now assume that $1/4 \leqslant u < 3/4$. By (3.3), we obtain

$$\frac{(4u-1)(4v-1)}{3-4u} \leqslant 4V - 1 \tag{3.5}$$

and we prove the inequality

$$(12u-5)(4v-1) \leqslant \frac{(4u-1)^2(4v-1)}{3-4u}. \tag{3.6}$$

If $4v - 1 = 0$, then (3.6) is true. If $4v - 1 > 0$, then (3.6) is equivalent to

$$(12u-5)(3-4u) - (4u-1)^2 = -(8u-4)^2 \leqslant 0.$$

The statements (3.6) and (3.5) give us (3.4) immediately.

The implication

a2) $J_0 < J_2 \Longrightarrow J_1 \geqslant \frac{1}{4}(3J_0 + J_2)$

can be verified by the same procedure as a1).

The proof of the statement b) is an analogy of the proof of a). □

The use of the following expressions facilitates the study of the sign of $\tilde{J}$ on the side S_3.

Definition. For arbitrary coordinates u, v, U, V, we put

$$\begin{aligned}
\omega(U,V) &= (4u-1)(4V-1) + (4v-1)(4U-1) - 8(u+v-1),\\
\varrho(U,V) &= \omega(U,V) - 2(3-4v)(4U-1),\\
\sigma(U,V) &= \omega(U,V) - 2(3-4u)(4V-1),\\
F(U,V) &= 4(3-4u)(3-4v)(4U-1)(4V-1) - \omega(U,V)^2.
\end{aligned}$$

After the obvious Lemma 5, the role of the above-defined expressions is apparent from Lemma 6.

Lemma 5. *For arbitrary coordinates u, v, U, V, we have*

$$[\varrho(U,V) = 0 \text{ and } \sigma(U,V) = 0] \Longrightarrow F(U,V) = 0.$$

Lemma 6. *Let $\mathcal{F}_h$ be an isoparametric mapping related to the points $a^1, \ldots, a^6$ with coordinates u, v, U, V. Then $\tilde{J}|_{S_3}$ has an absolute minimum in a unique inner point m of S_3 if and only if*

$$\varrho(U,V) < 0 \quad \text{and} \quad \sigma(U,V) < 0.$$

In this case

$$\tilde{J}(m) = -\frac{F(U,V)}{2[\varrho(U,V)+\sigma(U,V)]}.$$

P r o o f. Let us put $f(\eta) = \tilde{J}(1-\eta,\eta)$ for $\eta \in \langle 0,1\rangle$. As

$$\begin{aligned} f(0) &= (3-4v)(4U-1),\\ f(1/2) &= 1-2(u+v-U-V),\\ f(1) &= (3-4u)(4V-1) \end{aligned}$$

by Proposition 1, we can see that

$$\begin{aligned} f(1/2) - \frac{3}{4}f(0) - \frac{1}{4}f(1) &= \frac{1}{4}\,\varrho(U,V),\\ f(1/2) - \frac{1}{4}f(0) - \frac{3}{4}f(1) &= \frac{1}{4}\,\sigma(U,V),\\ f(0) - 2f(1/2) + f(1) &= -\frac{1}{4}\,[\varrho(U,V)+\sigma(U,V)]. \end{aligned}$$

These relations and Corollary 1 b) give us Lemma 6. □

Now, we can characterize the non-negativity of $\tilde{J}$ on $\partial\hat{K}$.

Lemma 7. *Let $\mathcal{F}_h$ be an isoparametric mapping related to the points $a^1,\ldots,a^6$ with coordinates u, v, U, V such that $1/4 \leqslant u \leqslant 3/4$ and $1/4 \leqslant v \leqslant 3/4$. Then $\tilde{J} \geqslant 0$ on $\partial\hat{K}$ if and only if $1/4 \leqslant U$, $1/4 \leqslant V$ and*

$$[\varrho(U,V) < 0 \ \text{and}\ \sigma(U,V) < 0] \Longrightarrow F(U,V) \geqslant 0. \tag{3.7}$$

P r o o f. Due to Lemma 4, $\tilde{J}|_{S_1\cup S_2} \geqslant 0$ if and only if $1/4 \leqslant U$ and $1/4 \leqslant V$. In this case we have $\tilde{J}(\hat{a}^5) \geqslant 0$ and $\tilde{J}(\hat{a}^1) \geqslant 0$. Now, $\tilde{J}|_{S_3} \geqslant 0$ if and only if the absolute minimum of $\tilde{J}$ at a unique inner point of S_3 is non-negative whenever it exists. This condition is equivalent to (3.7) due to Lemma 6. □

Lemma 8. *Let the coordinates u, v, U, V of the points $a^1,\ldots,a^6$ satisfy $1/4 \leqslant u \leqslant 3/4$, $1/4 \leqslant v \leqslant 3/4$, $1/4 \leqslant U$, and $1/4 \leqslant V$. Then*

$$\tilde{J} \geqslant 0 \ \text{on}\ \partial\hat{K} \Longrightarrow \tilde{J} > 0 \ \text{in}\ \operatorname{int}(\hat{K}).$$

P r o o f. Assume that $\tilde{J} \geqslant 0$ on $\partial\hat{K}$. We show that $\tilde{J} > 0$ in $\operatorname{int}(\hat{K})$ directly or we prove that $\tilde{J}$ attains its global minimum at the points from $\partial\hat{K}$ only. We repeatedly use the formulas

$$\frac{\partial^2\tilde{J}}{\partial\xi^2} = 32(u-U)(2v-1), \quad \frac{\partial^2\tilde{J}}{\partial\eta^2} = 32(v-V)(2u-1).$$

a) Let $\partial^2\tilde{J}/\partial\xi^2 > 0$. Then $2v-1 \neq 0$ and, as

$$\begin{aligned}
\frac{\partial\tilde{J}}{\partial\xi}(0,0) &= 4[(4v-1)(U-u)-(4u-1)(2v-1)],\\
\frac{\partial\tilde{J}}{\partial\xi}(1,0) &= 4[(3-4v)(U-u)-(4U-1)(2v-1)],\\
\frac{\partial\tilde{J}}{\partial\xi}(0,1) &= 4[(4v-1)(U-u)-(3-4u)(2v-1)],
\end{aligned}$$

$2v-1>0$ gives us $u>U$ and $(\partial\tilde{J}/\partial\xi)(0,0)<0$, $(\partial\tilde{J}/\partial\xi)(1,0)\leqslant 0$, $(\partial\tilde{J}/\partial\xi)(0,1)<0$. These inequalities and the linearity of $\partial\tilde{J}/\partial\xi$ lead to $\partial\tilde{J}/\partial\xi<0$ in $\operatorname{int}(\hat{K})$. Analogously, if $2v-1<0$, then $u<U$ and we obtain $(\partial\tilde{J}/\partial\xi)(0,0)>0$, $(\partial\tilde{J}/\partial\xi)(1,0)>0$, $(\partial\tilde{J}/\partial\xi)(0,1)\geqslant 0$, so that $\partial\tilde{J}/\partial\xi>0$ in $\operatorname{int}(\hat{K})$.

b) Let $\partial^2\tilde{J}/\partial\eta^2>0$. Then $2u-1\neq 0$ and we verify the implications

$$\begin{aligned}
2u-1>0 &\Longrightarrow \frac{\partial\tilde{J}}{\partial\eta}<0 \quad \text{in } \operatorname{int}(\hat{K}),\\
2u-1<0 &\Longrightarrow \frac{\partial\tilde{J}}{\partial\eta}>0 \quad \text{in } \operatorname{int}(\hat{K})
\end{aligned}$$

by the same arguments as in the case a).

c) If $\partial^2\tilde{J}/\partial\xi^2<0$ or $\partial^2\tilde{J}/\partial\eta^2<0$, then it is obvious that $\tilde{J}$ attains the absolute minimum on $\hat{K}$ at points from $\partial\hat{K}$ only.

d) Let $\partial^2\tilde{J}/\partial\xi^2=0=\partial^2\tilde{J}/\partial\eta^2$. With respect to the assumptions, it is sufficient to consider the following cases i)–iv).

i) If $u=1/2=v$, then $\tilde{J}=1+(4U-2)\xi+(4V-2)\eta$ and, as $4U-2\geqslant -1$ and $4V-2\geqslant -1$, we obtain $\tilde{J}>0$ in $\operatorname{int}(\hat{K})$.

ii) If $u=1/2=U$, $v\neq 1/2$, then $\tilde{J}=4v-1-4(2v-1)\xi+4(V-v)\eta$, so that $\partial\tilde{J}/\partial\xi=-4(2v-1)\neq 0$.

iii) The case $u\neq 1/2$, $v=1/2=V$ is an analogy of ii).

iv) If $u=U\neq 1/2$, $v=V\neq 1/2$, then

$$\begin{aligned}
\tilde{J} = (4u-1)(4v-1) &- 4(4u-1)(2v-1)\xi\\
&- 4(4v-1)(2u-1)\eta+16(2u-1)(2v-1)\xi\eta
\end{aligned}$$

and we obtain $\partial\tilde{J}/\partial\xi=4(2v-1)[1-4u+4(2u-1)\eta]$. The value of $b(\eta)=1-4u+4(2u-1)\eta$ is $1-4u\leqslant 0$ for $\eta=0$ and $4u-3\leqslant 0$ for $\eta=1$. As $1-4u<0$ or $4u-3<0$, $b(\eta)$ is negative in $\operatorname{int}(\hat{K})$ and, consequently, $\partial\tilde{J}/\partial\xi$ is non-zero in $\operatorname{int}(\hat{K})$. $\square$

4. The admissible set

We repeat that, due to (1.1), $\mathcal{A}d(a^1,\dots,a^5)=\mathcal{A}d_J=\{a^6\colon\ 1/4\leqslant U,\ 1/4\leqslant V\}$ for all points $a^1,\dots,a^5$ with the coordinates $u=1/2=v$. In Theorem 1, we describe the admissible set for all points $a^1,\dots,a^5$ with any given coordinates u, v.

Definition. For arbitrary points $a^1,\dots,a^5$ with coordinates u, v, we denote by φ the curve $F(U,V)=0$, by μ, ν, ϱ_0, σ_0 the straight line $U=1/4$, $V=1/4$, $\varrho(U,V)=0$, $\sigma(U,V)=0$ respectively and we put $r=[1/4,1/4]$, $s=[(4u+1)/8,(4v+1)/8]$.

Theorem 1. *For arbitrary points $a^1,\dots,a^5$ with coordinates u, v, the admissible set $\mathcal{A}d=\mathcal{A}d(a^1,\dots,a^5)$ is non-empty if and only if*

$$1/4\leqslant u\leqslant 3/4\quad\text{and}\quad 1/4\leqslant v\leqslant 3/4. \tag{4.1}$$

If (4.1) is true, then $\mathcal{A}d$ attains the following forms a)–d).

a) *If $1<u+v$ and either $u=3/4$ or $v=3/4$, then*

$$\mathcal{A}d=\{a^6\in\mathcal{A}d_J\colon\ U(4v-1)+V(4u-1)\geqslant 3(u+v)-5/2\}.$$

b) *If $1<u+v$, $u<3/4$ and $v<3/4$, then the curve φ is an ellipse with centre s touching the straight line μ and ν at the point*

$$t_U=\Big[\frac14,\frac{12u+8v-9}{4(4u-1)}\Big]\quad\text{and}\quad t_V=\Big[\frac{8u+12v-9}{4(4v-1)},\frac14\Big]$$

respectively. If we denote by $\mathcal{T}$ the "curved triangle" bounded by the segments $\overline{t_Vr}$, $\overline{rt_U}$ and by the negatively oriented arc t_Ut_V *of φ, then*

$$\mathcal{A}d=\mathcal{A}d_J-\mathcal{T}.$$

c) *If $1/2<u+v\leqslant 1$, then*

$$\mathcal{A}d=\mathcal{A}d_J.$$

d) *If $u=1/4=v$, then*

$$\mathcal{A}d=\mathcal{A}d_J\cup\{a^6\colon\ U<1/4,\ V<1/4,\ \text{and}\ (4U-1)(4V-1)>1\}.$$

P r o o f. Let us choose the indexes in such a way that the ordered triple (a^1, a^3, a^5) is oriented positively. If $a^6 \in \mathcal{A}d(a^1, \ldots, a^5)$, then the isoparametric mapping $\mathcal{F}_h$ is an injection. We obtain (4.1) by Corollary 3 a), 3 b).

Conversely, let the conditions (4.1) be valid. Then $\mathcal{F}_h|_{S_1}$ and $\mathcal{F}_h|_{S_2}$ are injections by Corollary 3 a), 3 b). According to Corollary 2, it remains to describe the set of points a^6 such that $\mathcal{F}_h|_{S_3}$ is an injection and either $\tilde{J} > 0$ in $\mathrm{int}(\hat{K})$ or $\tilde{J} < 0$ in $\mathrm{int}(\hat{K})$. By Lemma 3, $\tilde{J} < 0$ concerns the case d) only. Lemma 4 says that $\tilde{J}|_{S_1 \cup S_2} \geqslant 0$ if and only if $a^6 \in \mathcal{A}d_J$. Then $\mathcal{F}_h|_{S_3}$ is an injection by Corollary 3 c). For all points $a^6 \in \mathcal{A}d_J$ satisfying (3.7), we have $\tilde{J}|_{\partial\hat{K}} \geqslant 0$ by Lemma 7 and we obtain $\tilde{J} > 0$ in $\mathrm{int}(\hat{K})$ by Lemma 8. Then $\mathcal{F}_h$ is invertible due to Corollary 2. Hence, we prove the statements a)–c) whenever we characterize the points a^6 from $\mathcal{A}d_J$ with property (3.7).

Proof of a). If $u = 3/4$ or $v = 3/4$, then

$$\omega(U,V) = \max\{\varrho(U,V), \sigma(U,V)\} \quad \text{and} \quad F(U,V) = -\omega(U,V)^2,$$

so that

$$[\varrho(U,V) < 0 \text{ and } \sigma(U,V) < 0] \Longrightarrow F(U,V) < 0.$$

Hence, implication (3.7) is valid if and only if

$$\varrho(U,V) \geqslant 0 \quad \text{or} \quad \sigma(U,V) \geqslant 0$$

and this is equivalent to $\omega(U,V) \geqslant 0$.

Proof of b). Let us assume that $1 < u + v$, $u < 3/4$ and $v < 3/4$. We can see that

$$\begin{aligned} F_{UU} &\equiv \frac{\partial^2 F}{\partial U^2} = -32(4v-1)^2, \\ F_{UV} &\equiv \frac{\partial^2 F}{\partial U \partial V} = 32[2(3-4u)(3-4v) - (4u-1)(4v-1)], \\ F_{VV} &\equiv \frac{\partial^2 H}{\partial V^2} = -32(4u-1)^2 \end{aligned}$$

and

$$F_{UU}F_{VV} - (F_{UV})^2 = 32^3(3-4u)(3-4v)(u+v-1).$$

It is a matter of routine to show that s is the stationary point of F and

$$F(s) = 8(3-4u)(3-4v)(u+v-1).$$

As $F_{UU} < 0$, $F_{UU}F_{VV} - (F_{UV})^2 > 0$ and $F(s) > 0$, s is the proper local maximum of F and the curve φ is an ellipse with centre s. Moreover, we can show easily that

μ, ν is the tangent line of φ at the point t_U, t_V respectively and $\varrho_0 = \overline{xt_U}$, $\sigma_0 = \overline{xt_V}$ for

$$x = \Big[\frac{1}{2} + \frac{(u-v)(u-1/2)}{8uv-5u-5v+3}, \frac{1}{2} + \frac{(v-u)(v-1/2)}{8uv-5u-5v+3}\Big].$$

But then $x \in \varphi$ due to Lemma 5. As $\varrho(r) < 0$ and $\sigma(r) < 0$, the set of points $a^6 \in \mathcal{A}d_J$ satisfying

$$\varrho(U,V) < 0 \quad \text{and} \quad \sigma(U,V) < 0$$

is equal to the quadrilateral $\overline{xt_V rt_U}$. Hence, implication (3.7) is not valid in a point $a^6 \in \mathcal{A}d_J$ if and only if

$$a^6 \in \overline{xt_V rt_U} \quad \text{and} \quad F(U,V) < 0.$$

These are exactly the points from the curved triangle $\mathcal{T}$ (see Fig. 5).

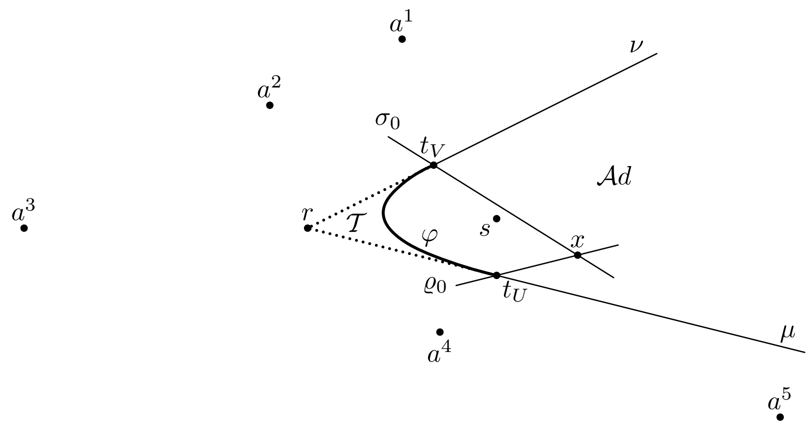

Figure 5. Notation from the proof of b).

Proof of c). If $u + v \leqslant 1$ and $a^6 \in \mathcal{A}d_J$, then $\omega(U,V) \geqslant 0$ and we prove implication (3.7) in the following steps i), ii).

i) If $(3-4u)(4V-1) \geqslant (3-4v)(4U-1)$, then the assumption

$$\varrho(U,V) = \omega(U,V) - 2(3-4v)(4U-1) < 0$$

together with $\omega(U,V) \geqslant 0$ give us

$$\begin{aligned} F(U,V) &= 4(3-4u)(3-4v)(4U-1)(4V-1) - \omega(U,V)^2 \\ &> 4(3-4u)(3-4v)(4U-1)(4V-1) - [2(3-4v)(4U-1)]^2 \\ &= 4(3-4v)(4U-1)[(3-4u)(4V-1) - (3-4v)(4U-1)] \geqslant 0. \end{aligned}$$

ii) If $(3-4v)(4U-1) > (3-4u)(4V-1)$, then we prove

$$\sigma(U,V) < 0 \Longrightarrow F(U,V) > 0$$

by means of the same procedure as in i).

Proof of d). Let us assume that $u = 1/4 = v$. In this case, we have proved that the mapping $\mathcal{F}_h$ is invertible for all points $a^6 \in \mathcal{A}d_J$ in c). Lemma 3 tells us that $\tilde{J} < 0$ in $\operatorname{int}(\hat{K})$ for all points a^6 such that

$$U < 1/4, \quad V < 1/4, \quad \text{and} \quad (4U-1)(4V-1) > 1.$$

Then $\tilde{J}|_{S_i}$, $i = 1, 2, 3$, are injections by Corollary 3 and $\mathcal{F}_h$ is invertible by Corollary 2. $\square$

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Author's address: *J. Dalík*, Brno University of Technology, Faculty of Civil Engineering, Department of Mathematics, Veveří 95, 602 00 Brno, Czech Republic, e-mail: `dalik.j@fce.vutbr.cz`.