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PERIODIC SOLUTIONS FOR SECOND ORDER HAMILTONIAN SYSTEMS*

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Abstract. By using the least action principle and minimax methods in critical point theory, some existence theorems for periodic solutions of second order Hamiltonian systems are obtained.

Keywords: periodic solutions, minimax methods, second order Hamiltonian systems

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1. Introduction

Consider periodic solutions of the Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T), \end{cases} \tag{1.1}$$

where $T > 0$, $F\colon\ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for all $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| \leqslant a(|x|)b(t), \quad |\nabla F(t, x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$.

The corresponding functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T F(t, u(t))\,\mathrm{d}t \tag{1.2}$$

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is continuously differentiable and weakly lower semicontinuous on H_T^1, and

$$\langle \varphi'(u), v\rangle = \int_0^T [(\dot u(t), \dot v(t)) + (\nabla F(t, u(t)), v(t))] \,\mathrm{d}t$$

for any $u, v \in H_T^1$, where

$$H_T^1 = \{u\colon [0,T] \to \mathbb{R}^n\colon \\ u \text{ is absolutely continuous, } u(0) = u(T),\ \dot u \in L^2([0,T], \mathbb{R}^n)\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \Big(\int_0^T [|\dot u(t)|^2 + |u(t)|^2] \,\mathrm{d}t\Big)^{1/2}$$

for $u \in H_T^1$. It is well known that the solutions of problem (1.1) correspond to the critical points of φ.

It has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods; we refer the readers to [1]–[15] and the references therein. Particularly, when the nonlinearity $\nabla F(t,x)$ is bounded, that is, there exists $g(t) \in L^1([0,T], \mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \leqslant g(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$, and that

$$\int_0^T F(t,x) \,\mathrm{d}t \to \pm\infty \quad \text{as } |x| \to \infty,$$

Mawhin and Willem [3] proved that problem (1.1) admits a periodic solution. After that, Tang [9] generalized these results to the sublinear case. In detail, he assumed that the nonlinearity $\nabla F(t,x)$ satisfies the following conditions:

$$|\nabla F(t,x)| \leqslant p(t)|x|^\alpha + q(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and a.e. } t \in [0,T], \tag{1.3}$$

and

$$|x|^{-2\alpha} \int_0^T F(t,x) \,\mathrm{d}t \to \pm\infty \quad \text{as } |x| \to \infty, \tag{1.4}$$

where $p(t), q(t) \in L^1([0,T], \mathbb{R}^+)$ and $\alpha \in [0,1)$. Under these conditions, periodic solutions of problem (1.1) have been obtained. Subsequently, Zhao in [14], [15] considered problem (1.1) when $\nabla F(t,x)$ was linear. He completed the results in [9] corresponding to $\alpha = 1$. As pointed out in [16], there are functions not satisfying the conditions in [9], [14], [15], thus the authors improved the conditions in [9], [14], [15] and obtained some new results.

Recently, Zhang and Wang [17] used a control function $h(|x|)$ instead of $|x|^\alpha$ in (1.3) and (1.4) and got some new results which improved many existed works. More precisely, they obtained the following main results.

Theorem A ([17]). *Suppose that F satisfies assumption* (A) *and the following conditions:*

(H1) *There exist constants $C_0 > 0$, $K_1 > 0$, $K_2 > 0$, $\alpha \in [0,1)$ and a nonnegative function $h \in C([0,\infty),[0,\infty))$ with the properties*
   (i) $h(s) \leqslant h(t) \quad \forall\, s \leqslant t,\ s,t \in [0,\infty)$,
   (ii) $h(s+t) \leqslant C_0(h(s)+h(t)) \quad \forall\, s,t \in [0,\infty)$,
   (iii) $0 \leqslant h(s) \leqslant K_1 s^\alpha + K_2 \quad \forall\, s,t \in [0,\infty)$,
   (iv) $h(s) \to \infty \quad$ *as* $s \to \infty$.
   Moreover, there exist $f, g \in L^1([0,T],\mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \leqslant f(t)h(|x|) + g(t) \quad \text{for all } x \in \mathbb{R}^n \text{ and a.e. } t \in [0,T]. \tag{1.5}$$

(H2) *There exists a function $h \in C([0,\infty),[0,\infty))$ which satisfies the conditions* (i)–(iv) *and*

$$h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t \to \infty \quad \text{as } |x| \to \infty. \tag{1.6}$$

Then (1.1) *has at least one solution which minimizes the functional φ in H_T^1.*

Theorem B ([17]). *Suppose that* (H1), *assumption* (A) *and the following condition hold:*

(H3) $h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t \to -\infty \quad$ *as* $|x| \to \infty$.

Then (1.1) *has at least one solution in H_T^1.*

Motivated by the ideas of [16], [17], we will use weaker conditions instead of (H2) and (H3). Here are our main results.

Theorem 1.1. *Suppose that F satisfies* (H1), (A) *and the following condition:*

(H2)′ *There exists a function $h \in C([0,\infty),[0,\infty))$ which satisfies the conditions* (i)–(iv) *and*

$$\liminf_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t > \frac{T^2C_0^2}{8\pi^2}\int_0^T f^2(t)\,\mathrm{d}t. \tag{1.7}$$

Then (1.1) *has at least one solution which minimizes the functional φ in H_T^1.*

Theorem 1.2. *Suppose that F satisfies* (H1), (A) *and the following condition:*

(H3)′ $\limsup_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t < -\frac{3}{8}(T^2C_0^2/\pi^2)\int_0^T f^2(t)\,\mathrm{d}t.$

Then (1.1) *has at least one solution in H_T^1.*

Theorem 1.3. *Suppose that F satisfies* (H1) *with $\alpha = 1$, assumption* (A) *and that*

$$\int_0^T f(t)\,\mathrm{d}t < \frac{12}{TK_1C_0} \tag{1.8}$$

and

$$\liminf_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t > \frac{3T^2C_0^2\int_0^T f^2(t)\,\mathrm{d}t}{2\pi^2(12 - K_1C_0T\int_0^T f(t)\,\mathrm{d}t)}. \tag{1.9}$$

Then (1.1) *has at least one solution which minimizes the functional φ in H_T^1.*

Theorem 1.4. *Suppose that F satisfies* (H1) *with $\alpha = 1$, assumption* (A), (1.8), *and the following condition:*

(H4) $\limsup_{|x|\to\infty} h^{-2}(|x|)\displaystyle\int_0^T F(t,x)\,\mathrm{d}t < -\frac{216T^2C_0^2\int_0^T f^2(t)\,\mathrm{d}t}{\pi^2(24 - K_1C_0T\int_0^T f(t)\,\mathrm{d}t)^2}.$

Then (1.1) *has at least one solution in H_T^1.*

2. Proofs of theorems

For $u \in H_T^1$, let $\bar{u} = T^{-1} \int_0^T u(t)\,\mathrm{d}t$ and $\tilde{u} = u(t) - \bar{u}$, then one has

$$\|\tilde{u}\|_\infty^2 \leqslant \frac{T}{12} \int_0^T |\dot{u}(t)|^2\,\mathrm{d}t \quad \text{(Sobolev's inequality)},$$

and

$$\|\tilde{u}\|_{L^2}^2 \leqslant \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2\,\mathrm{d}t \quad \text{(Wirtinger's inequality)},$$

where $\|\tilde{u}\|_\infty := \max\limits_{0\leqslant t\leqslant T} |\tilde{u}(t)|$. For the sake of convenience, we denote

$$M_1 = \Big(\int_0^T f^2(t)\,\mathrm{d}t\Big)^{1/2}, \quad M_2 = \int_0^T f(t)\,\mathrm{d}t, \quad M_3 = \int_0^T g(t)\,\mathrm{d}t.$$

Proof of Theorem 1.1. Due to (1.7), we can choose $a_1 > T^2/4\pi^2$ such that

$$\liminf_{|x|\to\infty} h^{-2}(|x|) \int_0^T F(t,x)\,\mathrm{d}t > \frac{a_1 C_0^2}{2} \int_0^T f^2(t)\,\mathrm{d}t. \tag{2.1}$$

It follows from (H1), Sobolev's inequality and Wirtinger's inequality that

$$\begin{aligned}
&\Big|\int_0^T [F(t,u(t)) - F(t,\bar{u})]\,\mathrm{d}t\Big| \\
&\quad = \Big|\int_0^T \int_0^1 (\nabla F(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t))\,\mathrm{d}s\,\mathrm{d}t\Big| \\
&\quad \leqslant \int_0^T \int_0^1 f(t) h(|\bar{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \int_0^1 g(t)|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \leqslant \int_0^T \int_0^1 f(t) h(|\bar{u}| + |\tilde{u}(t)|)|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \int_0^1 g(t)|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \leqslant \int_0^T \int_0^1 C_0 f(t)(h(|\bar{u}|) + h(|\tilde{u}(t)|))|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t + M_3\|\tilde{u}\|_\infty \\
&\quad \leqslant C_0 h(|\bar{u}|)\Big(\int_0^T f^2(t)\Big)^{1/2}\Big(\int_0^T |\tilde{u}(t)|^2\,\mathrm{d}t\Big)^{1/2} \\
&\qquad + C_0 \int_0^T f(t) h(|\tilde{u}(t)|)|\tilde{u}(t)|\,\mathrm{d}t + M_3\|\tilde{u}\|_\infty \\
&\quad \leqslant C_0 M_1 h(|\bar{u}|)\|\tilde{u}\|_{L^2} + C_0 \int_0^T f(t)(K_1|\tilde{u}(t)|^\alpha + K_2)|\tilde{u}(t)|\,\mathrm{d}t + M_3\|\tilde{u}\|_\infty \\
&\quad \leqslant C_0 M_1 h(|\bar{u}|)\|\tilde{u}\|_{L^2} + C_0 M_2 K_1 \|\tilde{u}\|_\infty^{1+\alpha} + C_0 M_2 K_2 \|\tilde{u}\|_\infty + M_3\|\tilde{u}\|_\infty
\end{aligned}$$

$$\leqslant \frac{1}{2a_1}\|\tilde u\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2}h^2(|\bar u|) + C_0M_2K_1\|\tilde u\|_\infty^{1+\alpha} + C_0M_2K_2\|\tilde u\|_\infty + M_3\|\tilde u\|_\infty$$
$$\leqslant \frac{T^2}{8\pi^2a_1}\|\dot u\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2}h^2(|\bar u|) + C_0M_2K_1\Bigl(\frac{T}{12}\Bigr)^{(1+\alpha)/2}\|\dot u\|_{L^2}^{1+\alpha} + C_0M_2K_2\Bigl(\frac{T}{12}\Bigr)^{1/2}\|\dot u\|_{L^2} + M_3\Bigl(\frac{T}{12}\Bigr)^{1/2}\|\dot u\|_{L^2}.$$

Hence, we have

$$\begin{aligned}\varphi(u) &= \frac12\int_0^T|\dot u(t)|^2\,\mathrm dt + \int_0^T F(t,u(t))\,\mathrm dt\\
&= \frac12\int_0^T|\dot u(t)|^2\,\mathrm dt + \int_0^T[F(t,u(t))-F(t,\bar u)]\,\mathrm dt + \int_0^T F(t,\bar u)\,\mathrm dt\\
&\geqslant \frac12\|\dot u\|_{L^2}^2 - \frac{T^2}{8\pi^2a_1}\|\dot u\|_{L^2}^2 - \frac{a_1(C_0M_1)^2}{2}h^2(|\bar u|)\\
&\quad - C_0M_2K_1\Bigl(\frac{T}{12}\Bigr)^{(1+\alpha)/2}\|\dot u\|_{L^2}^{1+\alpha}\\
&\quad - C_0M_2K_2\Bigl(\frac{T}{12}\Bigr)^{1/2}\|\dot u\|_{L^2} - M_3\Bigl(\frac{T}{12}\Bigr)^{1/2}\|\dot u\|_{L^2} + \int_0^T F(t,\bar u)\,\mathrm dt\\
&= \Bigl(\frac12 - \frac{T^2}{8\pi^2a_1}\Bigr)\|\dot u\|_{L^2}^2 + h^2(|\bar u|)\Bigl(h^{-2}(|\bar u|)\int_0^T F(t,\bar u)\,\mathrm dt - \frac{a_1C_0^2M_1^2}{2}\Bigr)\\
&\quad - \Bigl(\frac{T}{12}\Bigr)^{1/2}(C_0M_2K_2+M_3)\|\dot u\|_{L^2} - C_0M_2K_1\Bigl(\frac{T}{12}\Bigr)^{(1+\alpha)/2}\|\dot u\|_{L^2}^{1+\alpha}.\end{aligned}$$

As $\|u\|\to\infty$ if and only if $(|\bar u|^2+\|\dot u\|_{L^2}^2)^{1/2}\to\infty$, the above inequality and (2.1) imply that $\varphi(u)\to\infty$. Hence, by the least action principle, problem (1.1) has at least one solution which minimizes the functional $\varphi(u)$ in H_T^1. □

Proof of Theorem 1.2. First we prove that φ satisfies the (PS) condition. Assume that u_n is a (PS) sequence of φ, that is, $\varphi'(u_n)\to 0$ as $n\to\infty$ and $\{\varphi(u_n)\}$ is bounded. By (H3)′, we can choose $a_2 > T^2/4\pi^2$ such that

$$\limsup_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm dt < -\Bigl(\frac{a_2}{2}+\frac{\sqrt{a_2}\,T}{2\pi}\Bigr)C_0^2\int_0^T f^2(t)\,\mathrm dt. \tag{2.2}$$

In a way similar to the proof of Theorem 1.1, we have

$$\Bigl|\int_0^T(\nabla F(t,u_n(t)),\tilde u_n(t))\,\mathrm dt\Bigr|$$
$$\leqslant \frac{T^2}{8\pi^2a_2}\|\dot u_n\|_{L^2} + \frac{a_2(C_0M_1)^2}{2}h^2(|\bar u_n|) + C_0M_2K_1\Bigl(\frac{T}{12}\Bigr)^{(1+\alpha)/2}\|\dot u_n\|_{L^2}^{1+\alpha} + \Bigl(\frac{T}{12}\Bigr)^{1/2}(C_0M_2K_2+M_3)\|\dot u_n\|_{L^2}.$$

Hence, we get

$$\begin{aligned}\|\tilde{u}_n\| &\geqslant \langle \varphi'(u_n), \tilde{u}_n\rangle \\ &= \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t))\,\mathrm{d}t \\ &\geqslant \Big(1 - \frac{T^2}{8\pi^2 a_2}\Big)\|\dot{u}_n\|_{L^2}^2 - \frac{a_2(C_0M_1)^2}{2}\,h^2(|\bar{u}_n|) \\ &\quad - C_0M_2K_1\Big(\frac{T}{12}\Big)^{(1+\alpha)/2}\|\dot{u}_n\|_{L^2}^{1+\alpha} \\ &\quad - \Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2}.\end{aligned} \tag{2.3}$$

On the other hand, we find that

$$\|\tilde{u}_n\| \leqslant \frac{(T^2+4\pi^2)^{1/2}}{2\pi}\,\|\dot{u}_n\|_{L^2}. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\begin{aligned}\frac{a_2C_0^2M_1^2}{2}\,h^2(|\bar{u}_n|) &\geqslant \Big(1 - \frac{T^2}{8\pi^2 a_2}\Big)\|\dot{u}_n\|_{L^2}^2 - C_0M_2K_1\Big(\frac{T}{12}\Big)^{(1+\alpha)/2}\|\dot{u}_n\|_{L^2}^{1+\alpha} \\ &\quad - \Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2} - \|\tilde{u}_n\| \\ &\geqslant \Big(1 - \frac{T^2}{8\pi^2 a_2}\Big)\|\dot{u}_n\|_{L^2}^2 - C_0M_2K_1\Big(\frac{T}{12}\Big)^{(1+\alpha)/2}\|\dot{u}_n\|_{L^2}^{1+\alpha} \\ &\quad - \Big[\frac{(T^2+4\pi^2)^{1/2}}{2\pi} + C_0M_2K_2\Big(\frac{T}{12}\Big)^{1/2} + M_3\Big(\frac{T}{12}\Big)^{1/2}\Big]\|\dot{u}_n\|_{L^2}^2 \\ &\geqslant \frac{1}{2}\,\|\dot{u}_n\|_{L^2}^2 + C_1,\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}C_1 = \min_{s\in[0,\infty)}\Big\{&\frac{4\pi^2a_2 - T^2}{8\pi^2a_2}\,s^2 - \Big(\frac{T}{12}\Big)^{(1+\alpha)/2}C_0M_2K_1s^{1+\alpha} \\ &- \Big[\frac{(T^2+4\pi^2)^{1/2}}{2\pi} + C_0M_2K_2\Big(\frac{T}{12}\Big)^{1/2} + M_3\Big(\frac{T}{12}\Big)^{1/2}\Big]s\Big\}.\end{aligned}$$

Notice that $a_2 > T^2/4\pi^2$ implies $-\infty < C_1 < 0$. Hence, it follows from (2.5) that

$$\|\dot{u}_n\|_{L^2}^2 \leqslant a_2C_0^2M_1^2h^2(|\bar{u}_n|) - 2C_1, \tag{2.6}$$

and

$$\|\dot u_n\|_{L^2} \leqslant \sqrt{a_2}C_0M_1h(|\bar u_n|) + C_2, \tag{2.7}$$

where $C_2 > 0$. By the proof of Theorem 1.1, we have

$$\begin{aligned}
&\left|\int_0^T [F(t,u_n(t)) - F(t,\bar u_n)]\,\mathrm{d}t\right| \\
&\quad\leqslant C_0M_1h(|\bar u_n|)\|\tilde u_n\|_{L^2} + C_0M_2K_1\|\tilde u_n\|_\infty^{1+\alpha} \\
&\qquad + (C_0M_2K_2 + M_3)\|\tilde u_n\|_\infty \\
&\quad\leqslant \frac{\pi}{\sqrt{a_2}\,T}\,\|\tilde u_n\|_{L^2}^2 + \frac{\sqrt{a_2}\,TC_0^2M_1^2}{4\pi}\,h^2(|\bar u_n|) \\
&\qquad + C_0M_2K_1\|\tilde u_n\|_\infty^{1+\alpha} + (C_0M_2K_2 + M_3)\|\tilde u_n\|_\infty \\
&\quad\leqslant \frac{T}{4\pi\sqrt{a_2}}\|\dot u_n\|_{L^2}^2 + \frac{\sqrt{a_2}\,TC_0^2M_1^2}{4\pi}\,h^2(|\bar u_n|) \\
&\qquad + \Big(\frac{T}{12}\Big)^{(1+\alpha)/2}C_0M_2K_1\|\dot u_n\|_\infty^{1+\alpha} \\
&\qquad + \Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2 + M_3)\|\dot u_n\|_{L^2}.
\end{aligned} \tag{2.8}$$

It follows from the boundedness of $\varphi(u_n)$ and from (2.6)–(2.8) that

$$\begin{aligned}
C_3 &\leqslant \varphi(u_n) \\
&= \frac{1}{2}\,\|\dot u_n\|_{L^2}^2 + \int_0^T [F(t,u_n(t)) - F(t,\bar u_n)]\,\mathrm{d}t + \int_0^T F(t,\bar u_n)\,\mathrm{d}t \\
&\leqslant \Big(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\Big)\|\dot u_n\|_{L^2}^2 + \frac{\sqrt{a_2}\,TC_0^2M_1^2}{4\pi}\,h^2(|\bar u_n|) \\
&\quad + \Big(\frac{T}{12}\Big)^{(1+\alpha)/2}C_0M_2K_1\|\dot u_n\|_{L^2}^{1+\alpha} \\
&\quad + \Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2 + M_3)\|\dot u_n\|_{L^2} + \int_0^T F(t,\bar u_n)\,\mathrm{d}t \\
&\leqslant \Big(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\Big)(a_2C_0^2M_1^2h^2(|\bar u_n|) - 2C_1) + \frac{\sqrt{a_2}\,TC_0^2M_1^2}{4\pi}\,h^2(|\bar u_n|) \\
&\quad + \Big(\frac{T}{12}\Big)^{(1+\alpha)/2}C_0M_2K_1(\sqrt{a_2}C_0M_1h(|\bar u_n|) + C_2)^{1+\alpha} \\
&\quad + \Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2 + M_3)(\sqrt{a_2}C_0M_1\,h(|\bar u_n|) + C_2) + \int_0^T F(t,\bar u_n)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{a_2}{2}+\frac{\sqrt{a_2}\,T}{2\pi}\Big)C_0^2M_1^2h^2(|\bar u_n|)-\Big(1+\frac{T}{2\pi\sqrt{a_2}}\Big)C_1\\
&\quad+\int_0^T F(t,\bar u_n)\,\mathrm{d}t+\Big(\frac{T}{12}\Big)^{(1+\alpha)/2}C_0M_2K_1(\sqrt{a_2}C_0M_1h(|\bar u_n|)+C_2)^{1+\alpha}\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2+M_3)(\sqrt{a_2}C_0M_1h(|\bar u_n|)+C_2)\\
&\leqslant \Big(\frac{a_2}{2}+\frac{\sqrt{a_2}\,T}{2\pi}\Big)C_0^2M_1^2h^2(|\bar u_n|)-\Big(1+\frac{T}{2\pi\sqrt{a_2}}\Big)C_1+\int_0^T F(t,\bar u_n)\,\mathrm{d}t\\
&\quad+\Big(\frac{T}{12}\Big)^{(1+\alpha)/2}C_0M_2K_12^{\alpha}[(\sqrt{a_2}C_0M_1h(|\bar u_n|))^{(1+\alpha)}+C_2^{1+\alpha}]\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2+M_3)(\sqrt{a_2}C_0M_1h(|\bar u_n|)+C_2)\\
&= h^2(|\bar u_n|)\Big[h^{-2}(|\bar u_n|)\int_0^T F(t,\bar u_n)\,\mathrm{d}t+\Big(\frac{a_2}{2}+\frac{\sqrt{a_2}\,T}{2\pi}\Big)C_0^2M_1^2\\
&\quad+2^{\alpha}\Big(\frac{a_2T}{12}\Big)^{(1+\alpha)/2}C_0^{2+\alpha}M_2M_1^{1+\alpha}K_1h^{\alpha-1}(|\bar u_n|)\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2+M_3)\sqrt{a_2}C_0M_1\,h^{-1}(|\bar u_n|)\Big]\\
&\quad+2^{\alpha}\Big(\frac{T}{12}\Big)^{(1+\alpha)/2}C_0M_2K_1C_2^{1+\alpha}+\Big(\frac{T}{12}\Big)^{1/2}(C_0M_2K_2+M_3)C_2\\
&\quad-\Big(1+\frac{T}{2\pi\sqrt{a_2}}\Big)C_1.
\end{aligned}
$$

The above inequality and (2.2) imply that $\{\bar u_n\}$ is bounded. Hence, $\{u_n\}$ is bounded by (2.6) and (H1). Arguing as in Proposition 4.1 in [3], we conclude that φ satisfies the (PS) condition.

In order to use the saddle point theorem ([6, Theorem 4.6]), we only need to verify the following conditions:

(1) $\varphi(u)\to-\infty$ as $|u|\to\infty$ in $\mathbb{R}^n$.

(2) $\varphi(u)\to\infty$ as $\|u\|\to\infty$ in $\tilde H_T^1$, where $\tilde H_T^1=\{u\in H_T^1\colon\ \bar u=0\}$.

In fact, from the property (iv) of (H1) and (H3)$'$, we have

$$\int_0^T F(t,u)\,\mathrm{d}t\to-\infty\quad\text{as } |u|\to\infty\ \text{ in } \mathbb{R}^n,$$

which together with (1.2) implies that

$$\varphi(u)=\int_0^T F(t,u)\,\mathrm{d}t\to-\infty\quad\text{as } |u|\to\infty\ \text{ in } \mathbb{R}^n.$$

Hence, (1) holds.

Next, for all $u \in \tilde{H}_T^1$, by (H1) and Sobolev's inequality, we have

$$
\begin{aligned}
&\left|\int_0^T [F(t,u(t)) - F(t,0)]\,\mathrm{d}t\right| \\
&\quad = \left|\int_0^T \int_0^1 (\nabla F(t, su(t)), u(t))\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\quad \leqslant \int_0^T f(t)h(|u(t)|)|u(t)|\,\mathrm{d}t + \int_0^T g(t)|u(t)|\,\mathrm{d}t \\
&\quad \leqslant \int_0^T f(t)(K_1|u(t)|^\alpha + K_2)|u(t)|\,\mathrm{d}t + M_3\|u\|_\infty \\
&\quad \leqslant M_2\|u\|_\infty(K_1\|u\|_\infty^\alpha + K_2) + M_3\|u\|_\infty \\
&\quad = M_2K_1\|u\|_\infty^{1+\alpha} + M_2K_2\|u\|_\infty + M_3\|u\|_\infty \\
&\quad \leqslant \left(\frac{T}{12}\right)^{(1+\alpha)/2} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} + \left(\frac{T}{12}\right)^{1/2}(M_2K_2 + M_3)\|\dot{u}\|_{L^2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T [F(t,u(t)) - F(t,0)]\,\mathrm{d}t + \int_0^T F(t,0)\,\mathrm{d}t \\
&\geqslant \frac{1}{2}\|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} \\
&\quad - \left(\frac{T}{12}\right)^{1/2}(M_2K_2 + M_3)\|\dot{u}\|_{L^2} + \int_0^T F(t,0)\,\mathrm{d}t
\end{aligned} \tag{2.9}
$$

for all $u \in \tilde{H}_T^1$. By Wirtinger's inequality, $\|u\| \to \infty$ in $\tilde{H}_T^1$ if and only if $\|\dot{u}\|_{L^2} \to \infty$, so from (2.9) we obtain $\varphi(u) \to \infty$ as $\|u\| \to \infty$ in $\tilde{H}_T^1$, i.e. (2) is verified. Hence, the proof of Theorem 1.2 is complete. □

P r o o f of Theorem 1.3. By virtue of (1.8) and (1.9), we can choose a constant $a_3 \in \mathbb{R}$ such that

$$
a_3 > \frac{3T^2}{\pi^2(12 - K_1TM_2C_0)} > 0, \tag{2.10}
$$

and

$$
\liminf_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t > \frac{a_3}{2}M_1^2C_0^2. \tag{2.11}
$$

It follows from (H1) with $\alpha = 1$, Sobolev's inequality and Wirtinger's inequality that

$$
\begin{aligned}
&\left|\int_0^T [F(t,u(t)) - F(t,\bar{u})]\,\mathrm{d}t\right| \\
&\quad = \left|\int_0^T \int_0^1 (\nabla F(t,\bar{u} + s\tilde{u}(t)), \tilde{u}(t))\,\mathrm{d}t\right| \\
&\quad \leqslant \int_0^T \int_0^1 f(t)h(|\bar{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T g(t)|\tilde{u}(t)|\,\mathrm{d}t \\
&\quad \leqslant \int_0^T \int_0^1 C_0 f(t)(h(|\bar{u}|) + sh(|\tilde{u}(t)|))|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T g(t)|\tilde{u}(t)|\,\mathrm{d}t \\
&\quad = \int_0^T C_0 f(t)\Big(h(|\bar{u}|) + \frac{1}{2}\,h(|\tilde{u}(t)|)\Big)|\tilde{u}(t)|\,\mathrm{d}t + M_3\|\tilde{u}\|_\infty \\
&\quad \leqslant C_0 h(|\bar{u}|)\bigg(\int_0^T f^2(t)\bigg)^{1/2}\bigg(\int_0^T |\tilde{u}(t)|\,\mathrm{d}t\bigg)^{1/2} \\
&\qquad + \frac{1}{2}\,M_2 C_0 h(\|\tilde{u}\|_\infty)\|\tilde{u}\|_\infty + M_3\|\tilde{u}\|_\infty \\
&\quad = C_0 M_1 h(|\bar{u}|)\|\tilde{u}\|_{L^2} + \frac{1}{2}\,M_2 C_0 h(\|\tilde{u}\|_\infty)\|\tilde{u}\|_\infty + M_3\|\tilde{u}\|_\infty \\
&\quad \leqslant \frac{1}{2a_3}\,\|\tilde{u}\|_{L^2}^2 + \frac{a_3 M_1^2 C_0^2}{2}\,h^2(|\bar{u}|) \\
&\qquad + \frac{1}{2}\,M_2 C_0 (K_1\|\tilde{u}\|_\infty + K_2)\|\tilde{u}\|_\infty + M_3\|\tilde{u}\|_\infty \\
&\quad \leqslant \Big(\frac{T^2}{8\pi^2 a_3} + \frac{T M_2 K_1 C_0}{24}\Big)\|\dot{u}\|_{L^2}^2 + \Big(\frac{T}{12}\Big)^{1/2}\Big(M_3 + \frac{M_2 K_2 C_0}{2}\Big)\|\dot{u}\|_{L^2} \\
&\qquad + \frac{a_3 M_1^2 C_0^2}{2}\,h^2(|\bar{u}|),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,\mathrm{d}t + \int_0^T [F(t,u(t)) - F(t,\bar{u})]\,\mathrm{d}t + \int_0^T F(t,\bar{u})\,\mathrm{d}t \\
&\geqslant \Big(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{T M_2 K_1 C_0}{24}\Big)\|\dot{u}\|_{L^2}^2 - \Big(\frac{T}{12}\Big)^{1/2}\Big(M_3 + \frac{M_2 K_2 C_0}{2}\Big)\|\dot{u}\|_{L^2} \\
&\quad - \frac{a_3 M_1^2 C_0^2}{2}\,h^2(|\bar{u}|) + \int_0^T F(t,\bar{u})\,\mathrm{d}t \\
&= \Big(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{T M_2 K_1 C_0}{24}\Big)\|\dot{u}\|_{L^2}^2 - \Big(\frac{T}{12}\Big)^{1/2}\Big(M_3 + \frac{M_2 K_2 C_0}{2}\Big)\|\dot{u}\|_{L^2} \\
&\quad + h^2(|\bar{u}|)\bigg(h^{-2}(|\bar{u}|)\int_0^T F(t,\bar{u})\,\mathrm{d}t - \frac{a_3 M_1^2 C_0^2}{2}\bigg).
\end{aligned}
$$

As $\|u\| \to \infty$ if and only if $(|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{1/2} \to \infty$, the above inequality and (2.10) and (2.11) imply that $\varphi(u) \to \infty$. Hence, by the least action principle, problem (1.1) has at least one solution which minimizes the functional $\varphi(u)$ in H_T^1. $\square$

P r o o f of Theorem 1.4. First we prove that φ satisfies the (PS) condition. Assume that u_n is a (PS) sequence of φ, that is, $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. By (H4), we can choose $a_4 \in \mathbb{R}$ such that

$$a_4 > \frac{6T^2}{\pi^2(24 - TM_2K_1C_0)} > 0 \tag{2.12}$$

and

$$\limsup_{|x|\to\infty} h^{-2}(|x|) \int_0^T F(t,x)\,\mathrm{d}t < -\Big[\frac{(12 + TM_2K_1C_0)a_4}{24 - TM_2K_1C_0} + \frac{T}{\pi}\sqrt{\frac{6a_4}{24 - TM_2K_1C_0}}\Big] M_1^2C_0^2. \tag{2.13}$$

In a way similar to the proof of Theorem 1.3, we obtain

$$\begin{aligned}\Big|\int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))\,\mathrm{d}t\Big| &\leqslant \Big(\frac{T^2}{8\pi^2a_4} + \frac{TM_2K_1C_0}{24}\Big)\|\dot{u}_n\|_{L^2}^2 + \Big(\frac{T}{12}\Big)^{1/2}\Big(M_3 + \frac{M_2K_2C_0}{2}\Big)\|\dot{u}_n\|_{L^2} \\ &\quad + \frac{a_4M_1^2C_0^2}{2}\,h^2(|\bar{u}_n|).\end{aligned}$$

Hence, we have

$$\begin{aligned}\|\tilde{u}_n\| &\geqslant \langle\varphi'(u_n), \tilde{u}_n\rangle \\ &= \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))\,\mathrm{d}t \\ &\geqslant \Big(1 - \frac{T^2}{8\pi^2a_4} - \frac{TM_2K_1C_0}{24}\Big)\|\dot{u}_n\|_{L^2}^2 - \frac{a_4(C_0M_1)^2}{2}\,h^2(|\bar{u}_n|) \\ &\quad - \Big(\frac{T}{12}\Big)^{1/2}\Big(\frac{C_0M_2K_2}{2} + M_3\Big)\|\dot{u}_n\|_{L^2},\end{aligned}$$

which together with (2.4) implies that

$$\begin{aligned}\frac{a_4(C_0M_1)^2}{2}\,h^2(|\bar{u}_n|) &\geqslant \Big(1 - \frac{T^2}{8\pi^2a_4} - \frac{TM_2K_1C_0}{24}\Big)\|\dot{u}_n\|_{L^2}^2 \\ &\quad - \Big[\frac{(T^2+4\pi^2)^{1/2}}{2\pi} + \Big(\frac{T}{12}\Big)^{1/2}\Big(\frac{C_0M_2K_2}{2} + M_3\Big)\Big]\|\dot{u}_n\|_{L^2} \\ &\geqslant \frac{1}{2}\Big(1 - \frac{TM_2K_1C_0}{24}\Big)\|\dot{u}_n\|_{L^2}^2 + C_4,\end{aligned}$$

where

$$C_4 = \min_{s\in[0,\infty)} \Big\{ \Big(\frac{1}{2} - \frac{T^2}{8\pi^2 a_4} - \frac{TM_2K_1C_0}{48}\Big) s^2 - \Big[\frac{(T^2+4\pi^2)^{1/2}}{2\pi} + \Big(\frac{T}{12}\Big)^{1/2}\Big(\frac{1}{2}C_0M_2K_2 + M_3\Big)\Big] s \Big\}.$$

It follows from (2.12) that $-\infty < C_4 < 0$, so we obtain

$$\|\dot u_n\|_{L^2}^2 \leqslant \frac{24\, a_4 M_1^2 C_0^2}{24 - TM_2K_1C_0}\, h^2(|\bar u_n|) - \frac{48 C_4}{24 - TM_2K_1C_0} \tag{2.14}$$

and

$$\|\dot u_n\|_{L^2} \leqslant \frac{\sqrt{24\, a_4} M_1 C_0}{\sqrt{24 - TM_2K_1C_0}}\, h(|\bar u_n|) + C_5. \tag{2.15}$$

By the proof of Theorem 1.3, we have

$$\begin{aligned}
&\Big|\int_0^T (F(t,u_n(t)) - F(t,\bar u_n))\,\mathrm{d}t\Big| \\
&\quad\leqslant C_0M_1h(|\bar u_n|)\|\tilde u_n\|_{L^2} + \frac{M_2C_0}{2}\, h(\|\tilde u_n\|_\infty)\|\tilde u_n\|_\infty + M_3\|\tilde u_n\|_\infty \\
&\quad\leqslant \frac{\pi}{2T}\sqrt{\frac{24 - TM_2K_1C_0}{6a_4}}\,\|\tilde u_n\|_{L^2}^2 + \frac{T}{\pi}\sqrt{\frac{3a_4}{2(24 - TM_2K_1C_0)}}\, M_1^2C_0^2h^2(|\bar u_n|) \\
&\qquad + \frac{M_2C_0}{2}\,\|\tilde u_n\|_\infty(K_1\|\tilde u_n\|_\infty + K_2) + M_3\|\tilde u_n\|_\infty \\
&\quad\leqslant \Big(\frac{T}{8\pi}\sqrt{\frac{24 - TM_2K_1C_0}{6a_4}} + \frac{TM_2K_1C_0}{24}\Big)\|\dot u_n\|_{L^2}^2 \\
&\qquad + \frac{T}{\pi}\sqrt{\frac{3a_4}{2(24 - TM_2K_1C_0)}}\, M_1^2C_0^2h^2(|\bar u_n|) \\
&\qquad + \Big(\frac{T}{12}\Big)^{1/2}\Big(M_3 + \frac{K_2M_2C_0}{2}\Big)\|\dot u_n\|_{L^2}.
\end{aligned}$$

It follows from the boundedness of $\varphi(u_n)$, (2.14), (2.15), and the above inequality that

$$\begin{aligned}
C_6 &\leqslant \varphi(u_n) \\
&= \frac{1}{2}\|\dot u_n\|_{L^2}^2 + \int_0^T [F(t,u_n(t)) - F(t,\bar u_n)]\,\mathrm{d}t + \int_0^T F(t,\bar u_n)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left[\frac{1}{2}+\frac{T}{8\pi}\sqrt{\frac{24-TM_2K_1C_0}{6a_4}}+\frac{TM_2K_1C_0}{24}\right]\|\dot u_n\|_{L^2}^2+\int_0^T F(t,\bar u_n)\,\mathrm{d}t\\
&\quad+\frac{T}{\pi}\sqrt{\frac{3a_4}{2(24-TM_2K_1C_0)}}\,M_1^2C_0^2h^2(|\bar u_n|)\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}\Big(M_3+\frac{K_2M_2C_0}{2}\Big)\|\dot u_n\|_{L^2}\\
&\leqslant \left[\frac{1}{2}+\frac{T}{8\pi}\sqrt{\frac{24-TM_2K_1C_0}{6a_4}}+\frac{TM_2K_1C_0}{24}\right]\\
&\quad\times\Big(\frac{24a_4M_1^2C_0^2}{24-TM_2K_1C_0}\,h^2(|\bar u_n|)-\frac{48C_4}{24-TM_2K_1C_0}\Big)\\
&\quad+\frac{T}{\pi}\sqrt{\frac{3a_4}{2(24-TM_2K_1C_0)}}\,M_1^2C_0^2h^2(|\bar u_n|)+\int_0^T F(t,\bar u_n)\,\mathrm{d}t\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}\Big(M_3+\frac{K_2M_2C_0}{2}\Big)\Big(\frac{\sqrt{24a_4}M_1C_0}{\sqrt{24-TM_2K_1C_0}}\,h(|\bar u_n|)+C_5\Big)\\
&=\left[\frac{(12+TM_2K_1C_0)a_4}{24-TM_2K_1C_0}+\frac{T}{\pi}\sqrt{\frac{6a_4}{24-TM_2K_1C_0}}\right]M_1^2C_0^2h^2(|\bar u_n|)\\
&\quad+\int_0^T F(t,\bar u_n)\,\mathrm{d}t+\Big(\frac{T}{12}\Big)^{1/2}\Big(M_3+\frac{K_2M_2C_0}{2}\Big)\frac{\sqrt{24a_4}M_1C_0}{\sqrt{24-TM_2K_1C_0}}\,h(|\bar u_n|)\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}\Big(M_3+\frac{K_2M_2C_0}{2}\Big)C_5\\
&\quad-\frac{48C_4}{24-TM_2K_1C_0}\left[\frac{1}{2}+\frac{T}{8\pi}\sqrt{\frac{24-TM_2K_1C_0}{6a_4}}+\frac{TM_2K_1C_0}{24}\right]\\
&=h^2(|\bar u_n|)\Big\{h^{-2}(|\bar u_n|)\int_0^T F(t,\bar u_n)\,\mathrm{d}t\\
&\quad+\left[\frac{(12+TM_2K_1C_0)a_4}{24-TM_2K_1C_0}+\frac{T}{\pi}\sqrt{\frac{6a_4}{24-TM_2K_1C_0}}\right]M_1^2C_0^2\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}\Big(M_3+\frac{K_2M_2C_0}{2}\Big)\frac{\sqrt{24a_4}M_1C_0}{\sqrt{24-TM_2K_1C_0}}\,h^{-1}(|\bar u_n|)\Big\}\\
&\quad+\Big(\frac{T}{12}\Big)^{1/2}\Big(M_3+\frac{K_2M_2C_0}{2}\Big)C_5\\
&\quad-\frac{48C_4}{24-TM_2K_1C_0}\left[\frac{1}{2}+\frac{T}{8\pi}\sqrt{\frac{24-TM_2K_1C_0}{6a_4}}+\frac{TM_2K_1C_0}{24}\right].
\end{aligned}
$$

The above inequality and (2.13) imply that $\{\bar u_n\}$ is bounded. Hence, $\{u_n\}$ is bounded by (H1) and (2.14).

Similarly to the proof of Theorem 1.2, we only need to verify (1) and (2). It is easy to check (1) by (H4). Now, we verify that (2) holds. For $u\in\tilde H_T^1$, by (H1) and

Sobolev's inequality, we have

$$\begin{aligned}
&\Big|\int_0^T (F(t,u(t)) - F(t,0))\,\mathrm{d}t\Big| \\
&\quad = \Big|\int_0^T\int_0^1 (\nabla F(t,su(t)),u(t))\,\mathrm{d}s\,\mathrm{d}t\Big| \\
&\quad \leqslant \int_0^T\int_0^1 f(t)h(s|u(t)|)|u(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T g(t)|u(t)|\,\mathrm{d}t \\
&\quad \leqslant \int_0^T\int_0^1 C_0 f(t)(h(s|u(t)|)+h(0))|u(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T g(t)|u(t)|\,\mathrm{d}t \\
&\quad \leqslant \int_0^T\int_0^1 C_0 f(t)(K_1 s|u(t)| + K_2)|u(t)|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\quad + M_2C_0h(0)\|u\|_\infty + M_3\|u\|_\infty \\
&\quad = \frac{K_1M_2C_0}{2}\,\|u\|_\infty^2 + (M_2C_0h(0) + K_2M_2C_0 + M_3)\|u\|_\infty \\
&\quad \leqslant \frac{TK_1M_2C_0}{24}\,\|\dot u\|_{L^2}^2 + \Big(\frac{T}{12}\Big)^{1/2}(M_2C_0h(0) + K_2M_2C_0 + M_3)\|\dot u\|_{L^2}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|\dot u\|_{L^2}^2 + \int_0^T (F(t,u(t)) - F(t,0))\,\mathrm{d}t + \int_0^T F(t,0)\,\mathrm{d}t \\
&\geqslant \frac{12 - TM_2K_1C_0}{24}\,\|\dot u\|_{L^2}^2 - \Big(\frac{T}{12}\Big)^{1/2}(M_2C_0h(0) + K_2M_2C_0 + M_3)\|\dot u\|_{L^2} \\
&\quad + \int_0^T F(t,0)\,\mathrm{d}t
\end{aligned}$$

for all $u \in \tilde H_T^1$. By Wirtinger's inequality, $\|u\| \to \infty$ in $\tilde H_T^1$ if and only if $\|\dot u\|_{L^2} \to \infty$. So from the above inequality we have $\varphi(u) \to \infty$ as $\|u\| \to \infty$, i.e. (2) is verified. Hence, the proof of Theorem 1.4 is complete. □

3. Examples

In this section, we give four examples to illustrate our results.

Example 3.1. Let

$$F(t,x) = \Big(\frac{1}{2}T - t\Big)\ln^{3/2}(1+|x|^2) + \Big(\frac{2}{3}T - t\Big)\ln(1+|x|^2).$$

It is easy to see that

$$|\nabla F(t,x)| \leqslant \frac{3}{2}\Big|\frac{1}{2}T - t\Big|\ln^{1/2}(1+|x|^2) + \Big|\frac{2}{3}T - t\Big|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$. We can see that (1.5) holds with $f(t) = \frac{3}{2}\left|\frac{1}{2}T - t\right|$, $g(t) = \left|\frac{2}{3}T - t\right|$ and $h(|x|) = \ln^{1/2}(1+|x|^2)$. Let $C_0 = 2$. It is easy to check that (H1) holds. However, $F(t,x)$ does not satisfy (1.6). In fact,

$$\frac{T^2C_0^2}{8\pi^2}\int_0^T f^2(t)\,\mathrm{d}t = \frac{9T^5}{96\pi^2}.$$

Let $T^3 < \frac{16}{9}\pi^2$, then we have

$$\liminf_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t = \frac{T^2}{6} > \frac{9T^5}{96\pi^2} = \frac{T^2C_0^2}{8\pi^2}\int_0^T f^2(t)\,\mathrm{d}t.$$

Hence, according to Theorem 1.1, problem (1.1) has at least one solution which minimizes the functional φ in H_T^1.

Example 3.2. Let

$$F(t,x) = \frac{\sin(2\pi t/T)}{2}\ln^{3/2}(1+|x|^2) + \Big(\frac{2}{5}T - t\Big)\ln(1+|x|^2).$$

It is easy to see that

$$|\nabla F(t,x)| \leqslant \frac{3}{4}\Big|\sin\frac{2\pi t}{T}\Big|\ln^{1/2}(1+|x|^2) + \Big|\frac{2}{5}T - t\Big|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$. We can see that (1.5) holds with $f(t) = \frac{3}{4}|\sin(2\pi t/T)|$, $g(t) = \left|\frac{2}{5}T - t\right|$ and $h(|x|) = \ln^{1/2}(1+|x|^2)$. Let $C_0 = 2$. It is easy to check that (H1) holds. However, $F(t,x)$ does not satisfy (H3). In fact,

$$\frac{3T^2C_0^2}{8\pi^2}\int_0^T f^2(t)\,\mathrm{d}t = \frac{27T^3}{64\pi^2}.$$

Let $T < \frac{32}{135}\pi^2$, then we have

$$\limsup_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t = -\frac{T^2}{10} < -\frac{27T^3}{64\pi^2} = -\frac{3T^2C_0^2}{8\pi^2}\int_0^T f^2(t)\,\mathrm{d}t.$$

Hence, by Theorem 1.2, problem (1.1) has at least one solution in H_T^1.

Example 3.3. Let

$$F(t,x) = \Big(\frac{5}{6}T - t\Big)\ln^2(1+|x|^2) + c(t)\ln(1+|x|^2),$$

where $c(t) \in L^1([0,T],\mathbb{R}^+)$. It is easy to see that

$$|\nabla F(t,x)| \leqslant 2\Big|\frac{5}{6}T - t\Big| \ln(1+|x|^2) + |c(t)|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$. Let $f(t) = 2\big|\frac{5}{6}T - t\big|$, $g(t) = |c(t)|$, $h(|x|) = \ln(1+|x|^2)$, $C_0 = 2$, $K_1 = 1$, $K_2 = 10$. Then we have

(i) $h(s) \leqslant h(t) \quad \forall\, s \leqslant t,\ s,t \in [0,\infty)$,

(ii) $h(s+t) \leqslant 2(h(s)+h(t)) \quad \forall\, s,t \in [0,\infty)$,

(iii) $0 \leqslant h(s) \leqslant s + 10 \quad \forall\, s,t \in [0,\infty)$,

(iv) $h(s) \to \infty \quad$ as $s \to \infty$.

By a direct computation, we get

$$\int_0^T f(t)\,\mathrm{d}t = 2\int_0^T \Big|\frac{5}{6}T - t\Big|\,\mathrm{d}t = \frac{13T^2}{18}$$

and

$$\int_0^T f^2(t)\,\mathrm{d}t = 4\int_0^T \Big|\frac{5}{6}T - t\Big|^2\,\mathrm{d}t = \frac{7T^3}{9}.$$

Let $T^3 < 108\pi^2/(126 + 13\pi^2)$. Then we have

$$\int_0^T f(t)\,\mathrm{d}t = \frac{13T^2}{18} < \frac{6}{T} = \frac{12}{TK_1C_0}$$

and

$$\liminf_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t = \frac{T^2}{3} > \frac{3T^2 \times 4 \times 7T^3/9}{2\pi^2(12 - 2T \times 13T^2/18)}$$
$$= \frac{3T^2C_0^2\int_0^T f^2(t)\,\mathrm{d}t}{2\pi^2(12 - K_1C_0T\int_0^T f(t)\,\mathrm{d}t)}.$$

Hence, by Theorem 1.1, problem (1.1) has at least one solution which minimizes the functional φ in H_T^1.

Example 3.4. Let

$$F(t,x) = \Big(\frac{2}{5}T - t\Big)\ln^2(1+|x|^2) + c(t)\ln(1+|x|^2),$$

where $c(t) \in L^1([0,T],\mathbb{R}^+)$. It is easy to see that

$$|\nabla F(t,x)| \leqslant 2\Big|\frac{2}{5}T - t\Big| \ln(1+|x|^2) + |c(t)|$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$. Let $f(t) = 2\left|\frac{2}{5}T - t\right|$, $g(t) = |c(t)|$, $h(|x|) = \ln(1+|x|^2)$, $C_0 = 2$, $K_1 = 1$, $K_2 = 10$. As shown in Example 3.3, we know that $h(s)$ satisfies (i)–(iv). By a direct computation, we have

$$\int_0^T f(t)\,\mathrm{d}t = 2\int_0^T \left|\frac{2}{5}T - t\right|\mathrm{d}t = \frac{13T^2}{25}$$

and

$$\int_0^T f^2(t)\,\mathrm{d}t = 4\int_0^T \left|\frac{5}{6}T - t\right|^2\mathrm{d}t = \frac{28T^3}{75}.$$

Let $T^3 < \frac{3}{2}$. Then we have

$$\int_0^T f(t)\,\mathrm{d}t = \frac{13T^2}{18} < \frac{6}{T} = \frac{12}{TK_1C_0}$$

and

$$\begin{aligned}\limsup_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,\mathrm{d}t \\ &= -\frac{T^2}{10} < -\frac{8064T^5}{25\pi^2(24-1.04T^3)^2} \\ &= -\frac{216T^2C_0^2\int_0^T f^2(t)\,\mathrm{d}t}{\pi^2\left(24-K_1C_0T\int_0^T f(t)\,\mathrm{d}t\right)^2}.\end{aligned}$$

Hence, by Theorem 1.1, problem (1.1) has at least one solution in H_T^1.

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