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VARIANCE COMPONENTS AND AN ADDITIONAL EXPERIMENT*

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Abstract. Estimators of parameters of an investigated object can be considered after some time as insufficiently precise. Therefore, an additional measurement must be realized. A model of a measurement, taking into account both the original results and the new ones, has a litle more complicated covariance matrix, since the variance components occur in it. How to deal with them is the aim of the paper.

Keywords: additional experiment, variance components, insensitivity region

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1. Introduction

The necessity to realize an additional experiment can be illustrated by many examples. Instead of many others let us consider the following problem. Let the task be to build a bridge consisting of prefabricated parts over the river. We have to know whether the sizes of prefabricated parts are in agreement with the total length of the bridge. This can be determined from the coordinates of some points of the state geodetical network. Such points of the network are on both sides of the river, however the precision of their coordinates need not be sufficient for the purpose mentioned.

Therefore, some additional measurement must be realized by a device (distance-meter, theodolit, GPS), which is usually not the same as the measurement device used in the measurement of the state network. Thus the characteristcs of accuracy of both the measurements are included in the compound model of measurement. If they are not known in advance, they must be estimated from the data obtained from both the experiments.

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The problem is whether these estimators can be used for an estimation of the new coordinates of the state network points used for building the bridge. This leads to sensitivity analysis. Problems of sensitivity analysis in linear models are studied in [2], [4], [5], [6].

The formulation of the problem of an additional experiment leads to a little bit more complicated linear statistical model. Different structures of linear models are investigated in many books and articles, see e.g. [4], [1], [12], [14], [13], [7], [8].

In practice the problem of an additional experiment is more complicated, however, the essence of the problem has been shown.

To contribute to a solution of the problem is the aim of the paper.

2. Notation

The notation $\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$ means that the observation vector $\mathbf{Y}$ is an n-dimensional random vector with the mean value $E(\mathbf{Y})$ equal to $\mathbf{X}\boldsymbol{\beta}$, where $\mathbf{X}$ is an $n \times k$ known matrix and $\boldsymbol{\beta} \in \mathbb{R}^k$ (k-dimensional linear vector space) is an unknown vector parameter. The covariance matrix $\operatorname{Var}(\mathbf{Y})$ of the vector $\mathbf{Y}$ is $\operatorname{Var}(\mathbf{Y}) = \boldsymbol{\Sigma}$.

The compound model of the original and the additional experiment is

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n_1+n_2} \left[\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_1, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2 \end{pmatrix} \right],$$

where $\mathbf{Y}_1$ is the observation vector of the original model and $\mathbf{Y}_2$ is the observation vector of the additional experiment.

In the following text it is assumed that the rank $r(\mathbf{X}_1)$ is $k < n_1$ and the covariance matrices are of the form $\boldsymbol{\Sigma}_1 = \vartheta_1 \mathbf{V}_1$, $\boldsymbol{\Sigma}_2 = \vartheta_2 \mathbf{V}_2$. The known matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ are positive definite. The values $\vartheta_1, \vartheta_2 \in (0, \infty)$ are unknown. It is not assumed that $r(\mathbf{X}_2) = k < n_2$, since n_2 can be equal even to 1 (one additional measurement). The symbol $\mathbf{M}_X$ means $\mathbf{M}_X = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, where $\mathbf{I}$ is the identity matrix.

Let $\mathbf{C}_1 = \mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{X}_1$ and $\mathbf{C}_2 = \mathbf{X}_2'\mathbf{V}_2^{-1}\mathbf{X}_2$. In view of the assumptions the matrix $\mathbf{C}_1$ is regular, however, the matrix $\mathbf{C}_2$ need not be regular. The symbol $(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+$ means the Moore-Penrose [9] generalized inverse of the matrix $\mathbf{M}_X\mathbf{V}\mathbf{M}_X$ and under assumptions we have

$$(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+ = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}.$$

3. Preliminaries

Lemma 3.1. (i) *The BLUE (best linear unbiased estimator) of* $\boldsymbol{\beta}$ *in the original model is*

$$\hat{\boldsymbol{\beta}}^{(1)} = \mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 \sim_k (\boldsymbol{\beta}, \vartheta_1\mathbf{C}_1^{-1}).$$

(ii) *The* $\boldsymbol{\vartheta}$*-LBLUE (locally best linear unbiased estimator) of* $\boldsymbol{\beta}$ *in the compound model is*

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}) = \Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{C}_2\Big)^{-1}\Big(\mathbf{X}_1'\frac{1}{\vartheta_1}\mathbf{V}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\frac{1}{\vartheta_2}\mathbf{V}_2^{-1}\mathbf{Y}_2\Big),$$

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}) \sim_k \Big[\boldsymbol{\beta}, \Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{C}_2\Big)^{-1}\Big].$$

P r o o f. Proof is well known and therefore it is omitted (e.g., cf. [10]). □

Corollary 3.2. *The estimator* $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})$ *can be expressed as*

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}) = \hat{\boldsymbol{\beta}}^{(1)} + \Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{C}_2\Big)^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}(\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}^{(1)}),$$

where $\hat{\boldsymbol{\beta}}^{(1)} = \mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1$ *(estimator in the original model).*

P r o o f. The two following equalities must be utilized:

$$\Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{X}_2'\mathbf{V}_2^{-1}\mathbf{X}_2\Big)^{-1}$$
$$= \vartheta_1\mathbf{C}_1^{-1} - \vartheta_1\mathbf{C}_1^{-1}\mathbf{X}_2'(\vartheta_2\mathbf{V}_2 + \mathbf{X}_2\vartheta_1\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}\vartheta_1$$

and

$$\vartheta_1\mathbf{C}_1^{-1}\mathbf{X}_2'(\vartheta_2\mathbf{V}_2 + \mathbf{X}_2\vartheta_1\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1} = \Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{C}_2\Big)^{-1}\mathbf{X}_2'\frac{1}{\vartheta_2}\mathbf{V}_2^{-1}.$$

Thus

$$\begin{aligned}
\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}) &= \Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{X}_2'\mathbf{V}_2^{-1}\mathbf{X}_2\Big)^{-1}\Big(\mathbf{X}_1'\frac{1}{\vartheta_1}\mathbf{V}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\frac{1}{\vartheta_2}\mathbf{V}_2^{-1}\mathbf{Y}_2\Big)\\
&= \vartheta_1\mathbf{C}^{-1}\mathbf{X}_1'\frac{1}{\vartheta_1}\mathbf{V}_1^{-1}\mathbf{Y}_1 - \vartheta_1\mathbf{C}_1^{-1}\mathbf{X}_2'(\vartheta_2\mathbf{V}_2 + \mathbf{X}_2\vartheta_1\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\\
&\quad \times \mathbf{X}_2\mathbf{C}_1^{-1}\vartheta_1\mathbf{X}_1'\frac{1}{\vartheta_1}\mathbf{V}_1^{-1}\mathbf{Y}_1 + \Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{C}_2\Big)^{-1}\mathbf{X}_2'\frac{1}{\vartheta_2}\mathbf{V}_2^{-1}\mathbf{Y}_2\\
&= \hat{\boldsymbol{\beta}}^{(1)} + \Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{X}'\mathbf{V}_2^{-1}\mathbf{X}_2\Big)^{-1}\mathbf{X}_2'\frac{1}{\vartheta_2}\mathbf{V}_2^{-1}(\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}^{(1)})
\end{aligned}$$

(cf. e.g., [1], p. 79). □

The expression for $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})$ given in Corollary 3.2 is important from the viewpoint of practice. The value $\hat{\boldsymbol{\beta}}^{(1)}$ can be registered in the state documentation institute and cannot be changed without a serious reason. Thus the correction

$$\Big(\frac{1}{\vartheta_1}\mathbf{C}_1 + \frac{1}{\vartheta_2}\mathbf{C}_2\Big)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}(\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}^{(1)})$$

has been used during the time of the building the bridge only.

Lemma 3.3. *The random vectors*

$$\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_i}\Big|_{\vartheta=\vartheta_0}, \quad i = 1, 2,$$

are uncorrelated with the vector $\hat{\boldsymbol{\beta}}(\vartheta_{1,0},\vartheta_{2,0})$, *i.e.*

$$\mathrm{cov}_{\vartheta_0}\Big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0), \frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_i}\Big|_{\vartheta=\vartheta_0}\Big] = \mathbf{0}, \quad i = 1, 2.$$

Here $\boldsymbol{\vartheta}_0$ is an arbitrary however fixed vector of variance components.

P r o o f. In the following text the notation

$$\mathbf{D}_0^{-1} = \Big(\frac{\mathbf{C}_1}{\vartheta_{1,0}} + \frac{\mathbf{C}_2}{\vartheta_{2,0}}\Big)^{-1}, \quad \mathbf{D}^{-1} = \Big(\frac{\mathbf{C}_1}{\vartheta_1} + \frac{\mathbf{C}_2}{\vartheta_2}\Big)^{-1}$$

will be used.

Since

$$\begin{aligned}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}) &= \mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 + \mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}(\mathbf{Y}_2 - \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1)\\ &= \Big(\mathbf{I} - \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\Big)\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 + \mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\mathbf{Y}_2,\end{aligned}$$

we have

$$\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})}{\partial\vartheta_1} = -\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 + \mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\mathbf{Y}_2$$

and

$$\begin{aligned}
&\operatorname{cov}_\vartheta\Big[\hat{\boldsymbol\beta}(\boldsymbol\vartheta), \frac{\partial\hat{\boldsymbol\beta}(\vartheta_1,\vartheta_2)}{\partial\vartheta_1}\Big] \\
&\quad = -\Big(\mathbf{I}-\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\Big)\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\vartheta_1\mathbf{V}_1\mathbf{V}_1^{-1}\mathbf{X}_1\mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1} \\
&\qquad + \mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\vartheta_2\mathbf{V}_2\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\mathbf{X}_2\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1} \\
&\quad = -\mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1} + \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1} \\
&\qquad + \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1} \\
&\quad = -\mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1} + \mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1} - \mathbf{D}^{-1}\frac{1}{\vartheta_1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1} \\
&\qquad + \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1} = \mathbf{0}.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\frac{\partial\hat{\boldsymbol\beta}(\boldsymbol\vartheta)}{\partial\vartheta_2} &= -\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 \\
&\quad + \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 + \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\mathbf{Y}_2 \\
&\quad - \mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2^2}\mathbf{Y}_2 \\
&= \Big(\mathbf{I}-\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\Big)\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 \\
&\quad - \Big(\mathbf{I}-\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\Big)\mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2^2}\mathbf{Y}_2
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{cov}_\vartheta\Big[\hat{\boldsymbol\beta}(\boldsymbol\vartheta), \frac{\partial\hat{\boldsymbol\beta}(\vartheta_1,\vartheta_2)}{\partial\vartheta_2}\Big] \\
&\quad = \Big(\mathbf{I}-\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\Big)\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\vartheta_1\mathbf{V}_1\mathbf{V}_1^{-1}\mathbf{X}_1\mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\Big(\mathbf{I}-\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\Big) \\
&\qquad - \mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\vartheta_2\mathbf{V}_2\frac{\mathbf{V}_2^{-1}}{\vartheta_2^2}\mathbf{X}_2\mathbf{D}^{-1}\Big(\mathbf{I}-\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\Big) \\
&\quad = \mathbf{C}_1^{-1}\vartheta_1\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1} - \mathbf{D}^{-1}\Big(\mathbf{D}-\frac{\mathbf{C}_1}{\vartheta_1}\Big)\mathbf{C}_1^{-1}\vartheta_1\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1} - \mathbf{C}_1^{-1}\vartheta_1\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1} \\
&\qquad + \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{C}_1^{-1}\vartheta_1\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1} - \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1} + \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}
\end{aligned}$$

$$= -\mathbf{C}_1^{-1}\vartheta_1\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1} + \mathbf{D}^{-1}\Big(\mathbf{D} - \frac{\mathbf{C}_1}{\vartheta_1}\Big)\mathbf{C}_1^{-1}\vartheta_1\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}$$
$$+ \mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1} = \mathbf{0}.$$

□

4. Insensitivity region

The parameters ϑ_1, ϑ_2 are usually unknown and they must be estimated from the measured data. With respect to the assumptions the parameter ϑ_1 can be estimated in the original experiment, i.e.

$$\hat{\vartheta}_1 = \frac{\mathbf{Y}_1'(\mathbf{M}_{X_1}\mathbf{V}_1\mathbf{M}_{X_1})^+\mathbf{Y}_1}{n_1 - k}.$$

The estimator is unbiased and in the case of normally distributed observation vector $\mathbf{Y}_1$ it has the smallest dispersion among all unbiased estimators in the original model (cf. [6, pp. 81–85]).

The observation vector $\mathbf{Y}_2$ of the additional experiment cannot be used for an estimation of ϑ_2, since it can happen that $n_2 < k$.

Thus an estimator based on $\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}$ is considered.

Lemma 4.1. *The* $\boldsymbol{\vartheta}_0$-MINQUE *(minimum norm quadratic unbiased estimator) (in more detail cf.* [11]*) of* $\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix}$ *in the model*

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n_1+n_2} \left[\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}, \vartheta_1\begin{pmatrix}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix} + \vartheta_2\begin{pmatrix}\mathbf{0}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{V}_2\end{pmatrix}\right]$$

is $\hat{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}}$, *where*

$$\begin{pmatrix}\hat{\vartheta}_1\\ \hat{\vartheta}_2\end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix}[\mathbf{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]'\dfrac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}}[\mathbf{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]\\ [\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]'\dfrac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}[\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]\end{pmatrix},$$
$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0) = \mathbf{D}_0^{-1}\Big(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}}\mathbf{Y}_1 + \mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\mathbf{Y}_2\Big),$$

$$\{\mathbf{S}\}_{1,1} = n_1 - 2\mathrm{Tr}\Big(\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\Big) + \mathrm{Tr}\Big(\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\Big),$$

$$\{\mathbf{S}\}_{1,2} = \{\mathbf{S}\}_{2,1} = \mathrm{Tr}\Big(\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\Big),$$

$$\{\mathbf{S}\}_{2,2} = n_2 - 2\mathrm{Tr}\Big(\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\Big) + \mathrm{Tr}\Big(\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\Big).$$

(It is assumed that the matrix $\mathbf{S}$ is positive definite.)

In the case of a normally distributed vector $\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}$ *we have*

$$\mathrm{Var}_{\vartheta_0}\begin{pmatrix}\hat{\vartheta}_1\\ \hat{\vartheta}_2\end{pmatrix} = 2\mathbf{S}^{-1}.$$

P r o o f. It is sufficient to take into account the relationships (in more detail cf. [11])

$$\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}\begin{pmatrix}\mathbf{Y}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_1(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{Y}\\ \vdots\\ \mathbf{Y}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_p(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{Y}\end{pmatrix},$$

$$\{\mathbf{S}\}_{i,j} = \mathrm{Tr}[\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_j(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}], \quad i,j = 1,\dots,p,$$

for the $\boldsymbol{\vartheta}_0$-MINQUE of $\boldsymbol{\vartheta} = (\vartheta_1,\dots,\vartheta_p)'$ in the model

$$\mathbf{Y} \sim_n \Big(\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\Big),$$

where $r(\mathbf{X}_{n,k}) = k < n$, $\mathbf{V}_1,\dots,\mathbf{V}_p$, are symmetric and positive semidefinite, $\vartheta_i > 0$, $i = 1,\dots,p$, and $\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p}\vartheta_{i,0}\mathbf{V}_i$. The expression

$$\mathbf{Y}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{Y}$$

can be rewritten as

$$[\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}[\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)].$$

If

$$\mathbf{X} = \begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}, \quad \boldsymbol{\Sigma}_0 = \begin{pmatrix}\vartheta_{1,0}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \vartheta_{2,0}\mathbf{V}_2\end{pmatrix},$$

we obtain

$$
\begin{aligned}
&\{\mathbf{S}\}_{1,1} = \\
&\quad \mathrm{Tr}\Bigg\{\begin{pmatrix}\vartheta_{1,0}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\Bigg[\begin{pmatrix}\mathbf{V}_1^{-1}/\vartheta_{1,0}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{V}_2^{-1}/\vartheta_{2,0}\end{pmatrix}\\
&\quad -\begin{pmatrix}(\mathbf{V}_1^{-1}/\vartheta_{1,0})\mathbf{X}_1\\ (\mathbf{V}_1^{-2}/\vartheta_{2,0})\mathbf{X}_2\end{pmatrix}\mathbf{D}_0^{-1}\Big(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}},\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\Big)\Bigg]\begin{pmatrix}\vartheta_{1,0}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\\
&\quad \times\Bigg[\begin{pmatrix}\mathbf{V}_1^{-1}/\vartheta_{1,0}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{V}_2^{-1}/\vartheta_{2,0}\end{pmatrix}-\begin{pmatrix}(\mathbf{V}_1^{-1}/\vartheta_{1,0})\mathbf{X}_1\\ (\mathbf{V}_1^{-2}/\vartheta_{2,0})\mathbf{X}_2\end{pmatrix}\mathbf{D}_0^{-1}\Big(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}},\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\Big)\Bigg]\Bigg\}\\
&= \mathrm{Tr}\Bigg\{\Bigg[\begin{pmatrix}\mathbf{I}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}-\begin{pmatrix}\mathbf{X}_1\\ \mathbf{0}\end{pmatrix}\mathbf{D}_0^{-1}\Big(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}},\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\Big)\Bigg]\\
&\quad \times\Bigg[\begin{pmatrix}\mathbf{I}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}-\begin{pmatrix}\mathbf{X}_1\\ \mathbf{0}\end{pmatrix}\mathbf{D}_0^{-1}\Big(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}},\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\Big)\Bigg]\Bigg\}\\
&= \mathrm{Tr}\Bigg\{\begin{pmatrix}\mathbf{I}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}-2\begin{pmatrix}\mathbf{X}_1\\ \mathbf{0}\end{pmatrix}\mathbf{D}_0^{-1}\Big(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}},\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\Big)\\
&\quad +\begin{pmatrix}\mathbf{X}_1\\ \mathbf{0}\end{pmatrix}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}\Big(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}},\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\Big)\Bigg\}\\
&= n_1 - 2\mathrm{Tr}\Big(\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\Big)+\mathrm{Tr}\Big(\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\Big).
\end{aligned}
$$

Analogously the expressions for $\{\mathbf{S}\}_{1,2}$ and $\{\mathbf{S}\}_{2,2}$ can be obtained. □

Now the "plug-in" estimator of $\boldsymbol{\beta}$ can be calculated, i.e.

$$\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}}) = \mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1 + \Big(\frac{\mathbf{C}_1}{\hat{\vartheta}_1}+\frac{\mathbf{C}_2}{\hat{\vartheta}_2}\Big)^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\hat{\vartheta}_2}(\mathbf{Y}_2-\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1).$$

This estimator is of practical use in such a case only when it is known with sufficiently high probability that the actual values of ϑ_1 and ϑ_2 are in the insensitivity region. It is defined as follows.

Definition 4.2. Let $h(\boldsymbol{\beta}) = \mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in\mathbb{R}^k$ be a linear function of the parameter $\boldsymbol{\beta}$. The set

$$\mathcal{N}_h = \{\boldsymbol{\vartheta}\colon\ (\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0)'\mathbf{N}_h(\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_0) < a_h^2\}$$

with the property

$$\boldsymbol{\vartheta}\in\mathcal{N}_h \Rightarrow \mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta})] \leqslant (1+\varepsilon)^2\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]$$

is the insensitivity region for the function $h(\cdot)$ at the point $\boldsymbol{\vartheta}_0$. Here $\varepsilon > 0$ is a sufficiently small real number (for more detail cf. [5], [6]).

The matrix $\mathbf{N}_h$ and the number a_h^2 are given in Theorem 4.4.

Lemma 4.3. *We have*

$$\mathrm{Var}_{\vartheta_0}\left(\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_1}\Big|_{\vartheta=\vartheta_0}\right)=\frac{1}{\vartheta_{1,0}^2}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}-\frac{1}{\vartheta_{1,0}^2}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1},$$

$$\mathrm{cov}_{\vartheta_0}\left(\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_1}\Big|_{\vartheta=\vartheta_0},\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_2}\Big|_{\vartheta=\vartheta_0}\right)=-\frac{1}{\vartheta_{1,0}\vartheta_{2,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{2,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{1,0}}\mathbf{D}_0^{-1},$$

$$\mathrm{Var}_{\vartheta_0}\left(\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_2}\Big|_{\vartheta=\vartheta_0}\right)=\frac{1}{\vartheta_{2,0}^2}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}-\frac{1}{\vartheta_{2,0}^2}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}.$$

Proof. Since

$$\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)=\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1+\mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}(\mathbf{Y}_2-\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1),$$

we have (cf. also the proof of Lemma 3.3)

$$\begin{aligned}
\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_1} &= -\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{Y}_1+\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\mathbf{Y}_2\\
\Rightarrow\ &\mathrm{Var}_{\vartheta}\left(\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_1}\right)\\
&=\frac{1}{\vartheta_1}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{C}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\vartheta_1\mathbf{V}_1\mathbf{V}_1^{-1}\mathbf{X}_1\mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\frac{1}{\vartheta_1}\\
&\quad+\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1^2}\mathbf{D}^{-1}\\
&=\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\Big(\mathbf{D}-\frac{\mathbf{C}_1}{\vartheta_1}\Big)\vartheta_1\mathbf{C}_1^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\\
&\quad+\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\\
&=\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}-\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\\
&\quad+\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\frac{\mathbf{C}_2}{\vartheta_2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\\
&=\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\Big(\mathbf{D}-\frac{\mathbf{C}_1}{\vartheta_1}\Big)\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\\
&=\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}-\frac{1}{\vartheta_1^2}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}\frac{\mathbf{C}_1}{\vartheta_1}\mathbf{D}^{-1}.
\end{aligned}$$

The expressions for

$$\mathrm{cov}_{\vartheta_0}\left(\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_1}\Big|_{\vartheta=\vartheta_0},\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_2}\Big|_{\vartheta=\vartheta_0}\right)\quad\text{and}\quad\mathrm{Var}_{\vartheta_0}\left(\frac{\partial\hat{\boldsymbol{\beta}}(\vartheta_1,\vartheta_2)}{\partial\vartheta_2}\Big|_{\vartheta=\vartheta_0}\right)$$

can be derived analogously. □

Theorem 4.4. *The matrix* $\mathbf{N}_h$ *from Definition* 4.2 *is*

$$\mathbf{N}_h = \begin{pmatrix} \mathbf{h}'\mathrm{Var}_{\vartheta_0}\Big(\dfrac{\partial\hat{\boldsymbol\beta}}{\partial\vartheta_1}\Big)\mathbf{h}, & \mathbf{h}'\mathrm{cov}_{\vartheta_0}\Big(\dfrac{\partial\hat{\boldsymbol\beta}}{\partial\vartheta_1}, \dfrac{\partial\hat{\boldsymbol\beta}}{\partial\vartheta_2}\Big)\mathbf{h} \\ \mathbf{h}'\mathrm{cov}_{\vartheta_0}\Big(\dfrac{\partial\hat{\boldsymbol\beta}}{\partial\vartheta_2}, \dfrac{\partial\hat{\boldsymbol\beta}}{\partial\vartheta_1}\Big)\mathbf{h}, & \mathbf{h}'\mathrm{Var}_{\vartheta_0}\Big(\dfrac{\partial\hat{\boldsymbol\beta}}{\partial\vartheta_2}\Big)\mathbf{h} \end{pmatrix}$$

and

$$a_h^2 = 2\varepsilon\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\vartheta_0)].$$

The derivatives in the expression for $\mathbf{N}_h$ *are related to* $\boldsymbol\vartheta_0$.

P r o o f. Let $\boldsymbol\vartheta = \boldsymbol\vartheta_0 + \delta\boldsymbol\vartheta$. Then

$$\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0 + \delta\boldsymbol\vartheta) \approx \mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0) + \mathbf{h}'\frac{\partial\hat{\boldsymbol\beta}(\boldsymbol\vartheta)}{\partial\vartheta_1}\Big|_{\vartheta=\vartheta_0}\delta\vartheta_1 + \mathbf{h}'\frac{\partial\hat{\boldsymbol\beta}(\boldsymbol\vartheta)}{\partial\vartheta_2}\Big|_{\vartheta=\vartheta_0}\delta\vartheta_2$$

and in view of Lemma 3.3

$$\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta)] \approx \mathbf{h}'\mathrm{Var}\vartheta_0\big[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)\big]\mathbf{h} + \delta\boldsymbol\vartheta'\mathbf{N}_h\delta\boldsymbol\vartheta.$$

Since

$$\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta)] \leqslant (1+\varepsilon)^2\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)],$$

we have

$$\begin{aligned} \mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta)] &\approx \mathbf{h}'\mathrm{Var}_{\vartheta_0}\big[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)\big]\mathbf{h} + \delta\boldsymbol\vartheta'\mathbf{N}_h\delta\boldsymbol\vartheta \\ &\leqslant (1+\varepsilon)^2\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)] \\ \Rightarrow \sqrt{\mathbf{h}'\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]\mathbf{h} + \delta\boldsymbol\vartheta'\mathbf{N}_h\delta\boldsymbol\vartheta} &\leqslant (1+\varepsilon)\sqrt{\mathbf{h}'\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]\mathbf{h}}, \\ \sqrt{\mathbf{h}'\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]\mathbf{h} + \delta\boldsymbol\vartheta'\mathbf{N}_h\delta\boldsymbol\vartheta} &\approx \sqrt{\mathbf{h}'\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]\mathbf{h}}\Big(1 + \frac{1}{2}\frac{\delta\boldsymbol\vartheta'\mathbf{N}_h\delta\boldsymbol\vartheta}{\mathbf{h}'\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]\mathbf{h}}\Big) \\ &\leqslant (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]}, \\ \frac{1}{2}\frac{\delta\boldsymbol\vartheta'\mathbf{N}_h\delta\boldsymbol\vartheta}{\mathbf{h}'\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)\mathbf{h}} &\leqslant \varepsilon \Rightarrow \delta\boldsymbol\vartheta'\mathbf{N}_h\delta\boldsymbol\vartheta \leqslant 2\varepsilon\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta})\boldsymbol\vartheta_0)] \\ &\Rightarrow a_h^2 = 2\varepsilon\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]. \end{aligned}$$

$\square$

The expressions for the entries of the matrix $\mathbf{N}_h$ are given in Lemma 4.3.

The utilization of the above results can be described as follows.

(1) The value $\boldsymbol\vartheta_0$ of the vector $\boldsymbol\vartheta$ is chosen as near as possible to the actual value of $\boldsymbol\vartheta$.

(2) An iteration procedure

$$\hat{\boldsymbol{\vartheta}}^{(i+1)} = \mathbf{S}^{-1}(\hat{\boldsymbol{\vartheta}}^{(i)}) \begin{pmatrix} [\mathbf{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}}^{(i)})]' \frac{1}{\hat{\vartheta}_1^{(i)}} \mathbf{V}_1^{-1}[\mathbf{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}}^{(i)})] \\ [\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}}^{(i)})]' \frac{1}{\hat{\vartheta}_2^{(i)}} \mathbf{V}_2^{-1}[\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}}^{(i)})] \end{pmatrix}$$

is proceeding until a stable value $\hat{\boldsymbol{\vartheta}}$ is obtained. Then

$$\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}}) = \Big(\frac{\mathbf{C}_1}{\hat{\vartheta}_1} + \frac{\mathbf{C}_2}{\hat{\vartheta}_2}\Big)^{-1} \Big(\mathbf{X}_1' \frac{\mathbf{V}_1^{-1}}{\hat{\vartheta}_1}\mathbf{Y}_1 + \mathbf{X}_2' \frac{\mathbf{V}_2^{-1}}{\hat{\vartheta}_2}\mathbf{Y}_2\Big).$$

(3) The value $\hat{\boldsymbol{\vartheta}}$ is considered to be $\boldsymbol{\vartheta}_0$ and by help of it the matrix $\mathbf{N}_h$ and a_h^2 are calculated.

(4) When $\mathcal{N}_h$ is determined, it is necessary to check whether the actual value $\boldsymbol{\vartheta}$ is an element of $\mathcal{N}_h$. The covariance matrix $\mathrm{Var}_{\hat{\vartheta}}(\hat{\vartheta}) = 2\mathbf{S}^{-1}(\hat{\boldsymbol{\vartheta}})$ enables us to construct $(1-\alpha)$-confidence region $\mathcal{E}$ with sufficiently high level $1-\alpha$ of confidence. If $\mathcal{E} \subset \mathcal{N}_h$, then the plug-in estimator $\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}})$ of $\boldsymbol{\beta}$ is acceptable.

Remark 4.5. It is important to observe that the shift $\delta\boldsymbol{\vartheta} = \boldsymbol{\vartheta} - \boldsymbol{\vartheta}_0$ in the direction of $\boldsymbol{\vartheta}_0$ implies

$$\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] \approx \mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)].$$

In more detail (cf. Lemma 3.3)

$$\begin{aligned}
&\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \eta\boldsymbol{\vartheta}_0)] \\
&\quad\approx \mathrm{Var}_{\vartheta_0}\Big\{[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)] + \frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}'}\eta\boldsymbol{\vartheta}_0\Big\} \\
&\quad= \mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)] + \eta^2(\vartheta_{1,0})^2\mathrm{Var}_{\vartheta_0}\Big(\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\vartheta_1}\Big) \\
&\qquad+ \eta^2\vartheta_{1,0}\vartheta_{2,0}\mathrm{cov}_{\vartheta_0}\Big(\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\vartheta_1}, \frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\vartheta_2}\Big) \\
&\qquad+ \eta^2\vartheta_{2,0}\vartheta_{1,0}\mathrm{cov}_{\vartheta_0}\Big(\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\vartheta_2}, \frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\vartheta_1}\Big) + \eta^2(\vartheta_{2,0})^2\mathrm{Var}\Big(\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\vartheta_2}\Big) \\
&\quad= \mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)] + \eta^2\Big\{\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1} - \mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1} \\
&\qquad- \mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1} - \mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1} + \mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1} \\
&\qquad- \mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}\Big\} \\
&\quad= \mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)] + \eta^2\Big\{\mathbf{D}_0^{-1} - \mathbf{D}_0^{-1}\frac{\mathbf{C}_1}{\vartheta_{1,0}}\mathbf{D}_0^{-1} - \mathbf{D}_0^{-1}\frac{\mathbf{C}_2}{\vartheta_{2,0}}\mathbf{D}_0^{-1}\Big\} = \mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)].
\end{aligned}$$

Thus mainly such shifts $\delta\boldsymbol{\vartheta}$ cause the enlargement of the dispersion of the estimator for which $\delta\boldsymbol{\vartheta}'\boldsymbol{\vartheta}_0 = 0$.

It is of some interest to know also the sensitiveness of the variance

$$\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})]$$

to the shift $\delta\boldsymbol{\vartheta}$, i.e. to know

$$\mathrm{Var}_{\vartheta_0+\delta\vartheta}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})].$$

The corresponding insensitivity region $\mathcal{N}_{V,h}$ is defined as follows.

Definition 4.6. The set $\mathcal{N}_{V,h}$ is

$$\mathcal{N}_{V,h} = \{\delta\boldsymbol{\vartheta}\colon\ \mathrm{Var}_{\vartheta_0+\delta\vartheta}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] \leqslant (1+\varepsilon)^2 \mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]\}.$$

Lemma 4.7. *The set $\mathcal{N}_{V,h}$ at the point $\boldsymbol{\vartheta}_0$ can be expressed as*

$$\begin{aligned}
\mathcal{N}_{V,h} &= \{\delta\boldsymbol{\vartheta}\colon\ |\mathbf{q}_h'\delta\boldsymbol{\vartheta}| \leqslant 2\varepsilon\mathbf{h}'\mathbf{D}_0^{-1}\mathbf{h}\},\\
\mathbf{q}_h &= \left(\frac{\mathbf{h}'\mathbf{D}_0^{-1}\mathbf{C}_1\mathbf{D}_0^{-1}\mathbf{h}}{\vartheta_{1,0}^2}, \frac{\mathbf{h}'\mathbf{D}_0^{-1}\mathbf{C}_2\mathbf{D}_0^{-1}\mathbf{h}}{\vartheta_{2,0}^2}\right)'.
\end{aligned}$$

P r o o f. We have

$$\begin{aligned}
\mathrm{Var}_{\vartheta_0+\delta\vartheta}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] &= \left(\frac{\mathbf{C}_1}{\vartheta_{1,0}+\delta\vartheta_1} + \frac{\mathbf{C}_2}{\vartheta_{2,0}+\delta\vartheta_2}\right)^{-1}\\
&\approx \mathbf{D}_0^{-1} + \frac{\mathbf{D}_0^{-1}\mathbf{C}_1\mathbf{D}_0^{-1}}{\vartheta_{1,0}^2}\delta\vartheta_1 + \frac{\mathbf{D}_0^{-1}\mathbf{C}_2\mathbf{D}_0^{-1}}{\vartheta_{2,0}^2}\delta\vartheta_2\\
\Rightarrow \mathrm{Var}_{\vartheta_0+\delta\vartheta}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] &\approx \mathbf{h}'\mathbf{D}_0^{-1}\mathbf{h} + \mathbf{q}_h'\delta\boldsymbol{\vartheta}.
\end{aligned}$$

Due to Definition 4.6,

$$\begin{aligned}
\sqrt{\mathrm{Var}_{\vartheta_0+\delta\vartheta}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})]} &\approx \sqrt{\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)] + \mathbf{q}_h'\delta\boldsymbol{\vartheta}}\\
&\leqslant (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta_0}[\mathbf{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]}\\
\Rightarrow \sqrt{1 + \frac{\mathbf{q}_h'\delta\boldsymbol{\vartheta}}{\mathbf{h}'\mathbf{D}_0^{-1}\mathbf{h}}} \leqslant 1 + \varepsilon &\Rightarrow |\mathbf{q}_h'\delta\boldsymbol{\vartheta}| \leqslant 2\varepsilon\mathbf{h}'\mathbf{D}_0^{-1}\mathbf{h}.
\end{aligned}$$

$\square$

5. Numerical example

Let the regression function be $y = \beta_1 + \beta_2 x$, $x \in \mathbb{R}^1$, let the measurement in the original experiment be realized at the points $x = -2; -1; 0; 1; 2$, and let the covariance matrix of the vector $\mathbf{Y}_1$ be $\mathrm{Var}(\mathbf{Y}_1) = 1\mathbf{I}$. Since in this experiment the parameter β_2 is estimated with relatively large dispersion ($\mathrm{Var}(\hat{\beta}_2^{(1)}) = 0.1$), the additional experiment is realized, i.e. $Y_6 \sim_1 (\beta_1 + \beta_2 10, 0.1)$.

In this case we have

$$\mathbf{C}_1 = \begin{pmatrix} 5, & 0 \\ 0, & 10 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} 1, & 10 \\ 10, & 100 \end{pmatrix}, \quad \vartheta_{1,0} = 1, \quad \vartheta_{2,0} = 0.1,$$

$$\mathbf{D}_0^{-1} = \left(\frac{1}{\vartheta_{1,0}} \mathbf{C}_1 + \frac{1}{\vartheta_{2,0}} \mathbf{C}_2 \right)^{-1} = \begin{pmatrix} 0.196116, & -0.019417 \\ -0.019417, & 0.002913 \end{pmatrix},$$

i.e. $\mathrm{Var}_{\vartheta_0}[\hat{\beta}_2(\boldsymbol{\vartheta}_0)] = 0.002913 \ll 0.1 = \mathrm{Var}(\hat{\beta}_2^{(1)})$.

Let us investigate the insensitivity region from Definition 4.2, which is in this case interesting. Since (Lemma 4.3)

$$\mathrm{Var}_{\vartheta_0}\left(\frac{\partial \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial \vartheta_1} \right) = \frac{1}{\vartheta_{1,0}^2} \mathbf{D}_0^{-1} \frac{\mathbf{C}_1}{\vartheta_{1,0}} \mathbf{D}_0^{-1} - \frac{1}{\vartheta_{1,0}^2} \mathbf{D}_0^{-1} \frac{\mathbf{C}_1}{\vartheta_{1,0}} \mathbf{D}_0^{-1} \frac{\mathbf{C}_1}{\vartheta_{1,0}} \mathbf{D}_0^{-1} = \mathbf{0},$$

$$\mathrm{cov}_{\vartheta_0}\left(\frac{\partial \boldsymbol{\beta}(\hat{\boldsymbol{\vartheta}}_0)}{\partial \vartheta_1}, \frac{\partial \boldsymbol{\beta}(\hat{\boldsymbol{\vartheta}}_0)}{\partial \vartheta_2} \right) = -\frac{1}{\vartheta_{1,0}\vartheta_{2,0}} \mathbf{D}_0^{-1} \frac{\mathbf{C}_1}{\vartheta_{2,0}} \mathbf{D}_0^{-1} \frac{\mathbf{C}_2}{\vartheta_{1,0}} \mathbf{D}_0^{-1} = \mathbf{0},$$

$$\mathrm{cov}_{\vartheta_0}\left(\frac{\partial \boldsymbol{\beta}(\hat{\boldsymbol{\vartheta}}_0)}{\partial \vartheta_2}, \frac{\partial \boldsymbol{\beta}(\hat{\boldsymbol{\vartheta}}_0)}{\partial \vartheta_1} \right) = \left[\mathrm{cov}_{\vartheta_0}\left(\frac{\partial \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial \vartheta_1}, \frac{\partial \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial \vartheta_2} \right) \right]',$$

$$\mathrm{Var}_{\vartheta_0}\left(\frac{\partial \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial \vartheta_2} \right) = \frac{1}{\vartheta_{2,0}^2} \mathbf{D}_0^{-1} \frac{\mathbf{C}_2}{\vartheta_{2,0}} \mathbf{D}_0^{-1} - \frac{1}{\vartheta_{2,0}^2} \mathbf{D}_0^{-1} \frac{\mathbf{C}_2}{\vartheta_{2,0}} \mathbf{D}_0^{-1} \frac{\mathbf{C}_2}{\vartheta_{2,0}} \mathbf{D}_0^{-1} = \mathbf{0},$$

the matrix $\mathbf{N}_h$ from Definition 4.2, at the point $\boldsymbol{\vartheta}_0$, is $\mathbf{0}$ for any function $h(\boldsymbol{\beta}) = \mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^2$. The insensitivity region $\mathcal{N}_h$ seems to be the whole parametric space $(0, \infty) \times (0, \infty)$ of the parameter $\boldsymbol{\vartheta}$. However, it is necessary to keep in mind that the determination of $\mathcal{N}_h$ is based on the infinitesimal consideration.

In order to get an idea of the behaviour of $\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})]$ in the neighbourhood of the point $\boldsymbol{\vartheta}_0$ let us compare the matrices

$$\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)] = \left(\frac{\mathbf{C}_1}{\vartheta_{1,0}} + \frac{\mathbf{C}_2}{\vartheta_{2,0}} \right)^{-1} = \mathbf{D}_0^{-1} = \mathbf{D}^{-1}(\boldsymbol{\vartheta}_0), \quad \vartheta_{1,0} = 1, \quad \vartheta_{2,0} = 0.1$$

and

$$\begin{aligned} &\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] \\ &= \mathbf{D}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}) \left(\frac{\vartheta_{1,0}}{(\vartheta_{1,0} + \delta\vartheta_1)^2} \mathbf{C}_1 + \frac{\vartheta_{2,0}}{(\vartheta_{2,0} + \delta\vartheta_2)^2} \mathbf{C}_2 \right) \mathbf{D}^{-1}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})^{-1} \end{aligned}$$

for different $\delta\boldsymbol{\vartheta}$.

In view of Remark 4.5 any shift $\delta\boldsymbol{\vartheta} = k\begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$, $k \in \mathbb{R}^1$, implies the approximate equality

$$\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + k\boldsymbol{\vartheta}_0))] \approx \mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)].$$

Let $\delta\boldsymbol{\vartheta} = \begin{pmatrix} 0 \\ 0.05 \end{pmatrix}$, $\cos\varphi = \boldsymbol{\vartheta}_0'\delta\boldsymbol{\vartheta}/\sqrt{\boldsymbol{\vartheta}_0'\boldsymbol{\vartheta}_0\delta\boldsymbol{\vartheta}'\delta\boldsymbol{\vartheta}} = 0.0995$, $\varphi = 84.3^\circ$. (This shift is relatively large, i.e. 50% change of the value $\vartheta_{2,0} = 0.1$ and the direction of it is dangerous.) Nevertheless,

$$\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] = \begin{pmatrix} 0.196116, & -0.019417 \\ -0,019417, & 0.002915 \end{pmatrix},$$

which is practically the same as $\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]$.

Let $\delta\boldsymbol{\vartheta} = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}$. Then

$$\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta})] = \begin{pmatrix} 0.196118, & -0.019410 \\ -0,019410, & 0.002948 \end{pmatrix}.$$

Also in this case the agreement with the covariance matrix at the point $\boldsymbol{\vartheta}_0$ of the estimator $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)$ is very good.

The insensitivity region $\mathcal{N}_{V,h}$ (cf. Definition 4.6) is in the case $\mathbf{h} = (1,0)'$, $\varepsilon = 0.1$

$$\begin{aligned} \mathcal{N}_{V,(1,0)'} &= \{\delta\boldsymbol{\vartheta}\colon\ |\mathbf{q}_h'\delta\boldsymbol{\vartheta}| \leqslant 2\varepsilon\mathbf{h}'\mathbf{D}_0^{-1}\mathbf{h}\} \\ &= \{\delta\boldsymbol{\vartheta}\colon\ |0.196078\,\delta\vartheta_1 + 0.000379\,\delta\vartheta_2| \leqslant 0.039223\} \end{aligned}$$

and in the case $\mathbf{h} = (0,1)'$, $\varepsilon = 0.1$

$$\mathcal{N}_{V,(0,1)'} = \{\boldsymbol{\vartheta}\colon\ |0.001970\,\delta\vartheta_1 + 0.009434\,\delta\vartheta_2| \leqslant 0.000583\}.$$

The set $\mathcal{N}_{V,(1,0)'}$ is a strip orthogonal to $\mathbf{q}_{(1,0)'} = (0.196078, 0.000379)'$ of the width $\langle -0.2, 0.2\rangle$. The set $\mathcal{N}_{V,(0,1)'}$ is a strip orthogonal to $\mathbf{q}_{(0,1)'} = (0.001970, 0.009434)'$ of the width $\langle -0.060, 0.060\rangle$.

If $\delta\boldsymbol{\vartheta} \in \mathcal{N}_{V,(1,0)'}$ and $\delta\vartheta_2 = 0$, then $|\delta\vartheta_{1,\max}| \leqslant 0.2$.

If $\delta\boldsymbol{\vartheta} \in \mathcal{N}_{V,(1,0)'}$ and $\delta\vartheta_1 = 0$, then $|\delta\vartheta_{2,\max}| \leqslant 103.5$.

The variance of the estimator $\hat{\beta}_1$ is practically independent of the value ϑ_2.

In the case of $\mathcal{N}_{V,(0,1)'}$ the analogous values are

$\delta\vartheta_2 = 0 \Rightarrow |\delta\vartheta_{1,\max}| \leqslant 0.296$,

$\delta\vartheta_1 = 0 \Rightarrow |\delta\vartheta_{2,\max}| \leqslant 0.062$.

The variance of the estimator $\hat{\beta}_2$ is much more sensitive to ϑ_2 than the variance of the estimator $\hat{\beta}_1$.

In practice it is more sutitable to use $\sigma_i = \sqrt{\vartheta_i}$ instead of ϑ_i and therefore to express the admissible shift $\delta\boldsymbol{\vartheta}$ in terms of the quantities $\delta\sigma_1$, $\delta\sigma_2$.

Since $\delta\vartheta_i = 2\sigma_i\delta\sigma_i + (\delta\sigma_i)^2 \approx 2\sigma_i\delta\sigma_i$, $i = 1, 2$, we conclude that

$$\delta\sigma_i = \frac{\delta\vartheta_i}{2\sigma_i} = \frac{\delta\vartheta_i}{2\sqrt{\vartheta_i}}.$$

Thus

$$\begin{aligned}
|\delta\vartheta_{1,\max}| \leqslant 0.2 &\Rightarrow |\delta\sigma_{1,\max}| \leqslant 0.1,\\
|\delta\vartheta_{2,\max}| \leqslant 103.5 &\Rightarrow |\delta\sigma_{2,\max}| \leqslant 163.6,\\
|\delta\vartheta_{1,\max}| \leqslant 0.296 &\Rightarrow |\delta\sigma_{1,\max}| \leqslant 0.148,\\
|\delta\vartheta_{2,\max}| \leqslant 0.062 &\Rightarrow |\delta\sigma_{2,\max}| \leqslant 0.098.
\end{aligned}$$

References

[1] *E. Fišerová, L. Kubáček, P. Kunderová*: Linear Statistical Models; Regularity and Singularities. Academia, Praha, 2007.

[2] *S. Chatterjee, A. S. Hadi*: Sensitivity Analysis in Linear Regression. J. Wiley, New York, 1988.

[3] *D. E. A. Giles, V. K. Srivastava*: An unbiased eastimator of the covariance matrix of the mixed regression estimator. J. Am. Stat. Assoc. *86* (1991), 441–444.

[4] *L. Kubáček, L. Kubáčková, J. Volaufová*: Statistical Models with Linear Structures. Veda, Bratislava, 1995.

[5] *L. Kubáček, L. Kubáčková*: Nonsensitiveness regions in universal models. Math. Slovaca *50* (2000), 219–240.

[6] *L. Kubáček, L. Kubáčková*: Statistics and Metrology. Publishing House of Palacký University, Olomouc, 2000. (In Czech.)

[7] *J. Neter, W. Wassermann, M. H. Kutner*: Applied Linear Statistical Models, 3rd edition. Irwin, Boston, 1990.

[8] *C. R. Rao*: Unified theory of least squares. Commun. Stat., Part A—Theory and Methods *1* (1973), 1–8.

[9] *C. R. Rao, S. K. Mitra*: Generalized Inverse of Matrices and its Applications. John Wiley & Sons, New York-London-Sydney-Toronto, 1971.

[10] *C. R. Rao*: Linear Statistical Inference and Its Applications. John Wiley & Sons, New York-London-Sydney, 1973; Czech translation: Academia, Praha, 1978.

[11] *C. R. Rao, J. Kleffe*: Estimation of Variance Components and Applications. North-Holland, Amsterdam-New York-Oxford-Tokyo, 1988.

[12] *C. R. Rao, H. Toutenburg*: Linear Models. Least Squares and Alternatives, 2nd ed. Springer, New York, 1999.

[13] *G. A. F. Seber, A. J. Lee*: Linear Regression Analysis, 2nd ed. Wiley, New York, 2003.

[14] *S. Weisberg*: Applied Linear Regression, 3rd revised ed. John Wiley & Sons, New York, 1980.

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