

Anthony Uyi Afuwape; Mathew Omonigho Omeike  
Ultimate boundedness of some third order ordinary differential equations

*Mathematica Bohemica*, Vol. 137 (2012), No. 3, 355–364

Persistent URL: <http://dml.cz/dmlcz/142900>

## Terms of use:

© Institute of Mathematics AS CR, 2012

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ULTIMATE BOUNDEDNESS OF SOME THIRD ORDER  
ORDINARY DIFFERENTIAL EQUATIONS

ANTHONY UYI AFUWAPE, Medellín, M. O. OMEIKE, Abeokuta

(Received February 15, 2011)

*Abstract.* We prove the ultimate boundedness of solutions of some third order nonlinear ordinary differential equations using the Lyapunov method. The results obtained generalize earlier results of Ezeilo, Tejumola, Reissig, Tunç and others. The Lyapunov function used does not involve the use of signum functions as used by others.

*Keywords:* ultimate boundedness, complete Lyapunov function, differential equation of third-order

*MSC 2010:* 34K20

## 1. INTRODUCTION

Motivation for this paper comes from the generalization of the works by Reissig [18] and Tejumola [21] by Ezeilo in [13], the recent works of Afuwape and Omeike [6] and Ademola et al [1].

In exciting work, Ezeilo [13] investigated the equation of the form

$$\ddot{x} + \{\varphi_1(\dot{x}) + \varphi_2(x, \dot{x})\}\dot{x} + g_1(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

for ultimate boundedness generalizing the works of Reissig [18] on

$$\ddot{x} + \varphi_2(x, \dot{x})\dot{x} + g_1(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

and that of Tejumola [21] on

$$\ddot{x} + \varphi_1(\dot{x})\dot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}).$$

---

This research was supported by University of Antioquia Research Grant through SUI No. IN10132CE.

We shall consider here the equation

$$(1.1) \quad \ddot{x} + \{f_1(\ddot{x}) + f_2(\dot{x}, \ddot{x})\} + g(x, \dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

where  $f_1, f_2, g, h, p$  depend on the arguments displayed. Our assumptions on  $f_1, f_2, g, h,$  and  $p$  shall allow us to generalize the results of Ademola et al [1], Afuwape [2], [3] and a particular case of Afuwape and Omeike [4] concerning

$$\ddot{x} + f_1(\ddot{x}) + g_1(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x});$$

and Qian [17] and a particular case of Tunç [22] concerning

$$\ddot{x} + f_2(x, \dot{x})\ddot{x} + g(x, \dot{x}) = p(t).$$

The assumptions will also give us an opportunity to discuss the ultimate boundedness results which generalize the earlier ones. A good record of ultimate boundedness results of these types is recorded in the book [19], and the papers Hara [16], Afuwape and Omeike [6] and references therein. Also, the recent excellent book of Haddad [15] includes a good summary of the theoretical works on the subject.

Consider the third order nonlinear ordinary differential equation of the form (1.1), or its equivalent system form

$$(1.2) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\{f_1(z) + f_2(y, z)\} - g(x, y) - h(x) + p(t, x, y, z), \end{aligned}$$

where  $f_1, f_2, g, h$  and  $p$  are continuous in their respective arguments, and the dots denote differentiation with respect to  $t$ .

The object of this paper is to discuss the ultimate boundedness of solutions of Eq. (1.1). It is well known that the ultimate boundedness is a very important problem in the theory and applications of differential equations. An effective method for studying the ultimate boundedness of nonlinear differential equations is still Lyapunov's direct method (see [1]–[11]). In [13], incomplete Lyapunov functions augmented with signum functions and with certain restrictive conditions on the nonlinear functions were used. Our aim in this paper is to study a more general Eq. (1.1) for ultimate boundedness of solutions, using a complete Lyapunov function with less restrictive conditions on the nonlinear functions  $f_1, f_2, g, h$  and  $p$ . In the process, we shall be able to generalize earlier results of [1], [2], [3] and a particular case of [4], [17], [22].

## 2. MAIN RESULTS

Our main result is the following theorem.

**Theorem 2.1.** *In addition to the basic assumptions on the functions  $f_1$ ,  $f_2$ ,  $g$ ,  $h$  and  $p$ , assume that the following conditions are satisfied ( $a$ ,  $b$ ,  $c$ ,  $\nu$  and  $A$  being some positive constants):*

- (i)  $(f_1(z) + f_2(y, z))/z \geq a$  for all  $y, z \neq 0$ ,
- (ii)  $g(x, y)/y \geq b$  for all  $x, y \neq 0$ ,
- (iii)  $h(x)/x \geq \nu$  for all  $x \neq 0$ ,
- (iv)  $h'(x) \leq c$ ,
- (v)  $ab > c$ ,
- (vi)  $|p(t, x, y, z)| \leq A < \infty$  for all  $t \geq 0$  and for all  $x, y, z$ .

*Then every solution  $x(t)$  of (1.1) ultimately satisfies*

$$(2.1) \quad |x(t)| \leq D, \quad |\dot{x}(t)| \leq D, \quad |\ddot{x}(t)| \leq D$$

where  $D$  is a constant depending only on  $a$ ,  $b$ ,  $c$ ,  $\nu$  and  $A$ .

**Remark 2.1.** Theorem 2.1 generalizes the results of Ademola et al [1], if we set  $f_2(\dot{x}, \ddot{x}) = 0$ .

**Remark 2.2.** In using Lyapunov's theory, Theorem 2.1 gives a different method of discussing the works of Afuwape [2], [3] who used the frequency domain methods, with  $f_2(\dot{x}, \ddot{x}) \equiv 0$ .

**Remark 2.3.** Theorem 2.1 generalizes the results of Qian [17] and Tunç [22] if  $f_1(\ddot{x}) \equiv 0$ . This becomes obvious if we carry out some differentiations of  $f_2(\dot{x}, \ddot{x})$  and  $g(x, \dot{x})$  with respect to their variables  $x, \dot{x}, \ddot{x}$  to obtain the equivalent equation to that of [17] and [22]. However, the  $p(t)$  will be replaced by  $p(t, x, y, z)$ , with the appropriate conditions on it.

## 3. PRELIMINARIES

It is convenient here to consider, in place of Eq. (1.1), the system (1.2). In order to prove Theorem 2.1, we need to show that every solution  $(x(t), y(t), z(t))$  of (1.2) satisfies

$$(3.1) \quad |x(t)| \leq D, \quad |y(t)| \leq D, \quad |z(t)| \leq D$$

for all sufficiently large  $t$ , where  $D$  is a suitable constant. Set  $(x, y, z) \equiv (x(t), y(t), z(t))$ .

Our proof of (3.1) rests entirely on two properties (stated in the lemma below) of the function  $V(t) \equiv V(x, y, z)$  defined by

$$(3.2) \quad 2V(x, y, z) = \beta(1 - \beta)b^2x^2 + b(\beta + \alpha a^{-1})y^2 + \alpha a^{-1}z^2 + [z + ay + (1 - \beta)bx]^2$$

with  $0 < \beta < 1$ , and  $\alpha > 0$ .

**Lemma 3.1.** *Subject to the conditions of Theorem 2.1,  $V(0, 0, 0) = 0$  and there is a positive constant  $D_1$  depending only on  $a, b, c, \alpha$  and  $\delta$  such that*

$$(3.3) \quad V(t) \equiv V(x, y, z) \geq D_1(x^2 + y^2 + z^2)$$

for all  $x, y, z$ .

Let us set  $V(t) \equiv V(x(t), y(t), z(t))$ .

Furthermore, there are finite constants  $D_2, D_3$  dependent only on  $a, b, c, A, \nu, \delta$  and  $\alpha$  such that for any solution  $(x(t), y(t), z(t))$  of (1.2),

$$(3.4) \quad \frac{d}{dt}V \equiv \frac{d}{dt}V(x(t), y(t), z(t)) \leq -D_2$$

provided that  $x^2 + y^2 + z^2 \geq D_3$ .

**Proof of Lemma 3.1.** Clearly,  $V(0, 0, 0) = 0$ . Also, by rearranging (3.2) and choosing

$$D_1 \geq \min\{\beta(1 - \beta)b^2; b(\beta + \alpha a^{-1}); \alpha a^{-1}\}$$

we have (3.3).

To prove (3.4), we find that the derivative of  $V$  with respect to  $t$  along the solution path of (1.2), (after simplifications) gives

$$(3.5) \quad \begin{aligned} \frac{d}{dt}V(t)|_{(1.2)} = & -b(1 - \beta)xh(x) - \{ab\beta y^2 + a[ yg(x, y) - by^2]\} \\ & - \{(\alpha a^{-1} + 1)z(f_1 + f_2) - az^2\} \\ & + \{b^2(1 - \beta)xy - ayh(x) - b(1 - \beta)xg(x, y)\} \\ & + \{[b(\alpha a^{-1} + 1) + a^2]yz - (\alpha a^{-1} + 1)zg(x, y) - ay(f_1 + f_2)\} \\ & + \{ab(1 - \beta)xz - (\alpha a^{-1} + 1)zh(x) - b(1 - \beta)x(f_1 + f_2)\} \\ & + \{[b(1 - \beta)x + ay + (\alpha a^{-1} + 1)z]p(t, x, y, z)\}. \end{aligned}$$

A rearrangement of this shows that for  $x \neq 0; y \neq 0; z \neq 0$  we have

$$(3.6) \quad \frac{d}{dt}V(t)|_{(1.2)} = -W_1 - W_2 - W_3 - W_4 - W_5 - W_6 - W_7 + W_p$$

where

$$\begin{aligned}
 W_1 &= \gamma_1 b(1 - \beta)x^2 \left[ \frac{h(x)}{x} \right] + \left\{ \delta_1 a \left[ \frac{g(x, y)}{y} - b \right] y^2 + \eta_1 ab\beta y^2 \right\} \\
 &\quad + \left\{ \xi_1 (\alpha a^{-1} + 1) \left[ \frac{f_1 + f_2}{z} - a \right] z^2 + \mu_1 \alpha z^2 \right\}, \\
 W_2 &= \left\{ \gamma_2 b(1 - \beta)x^2 \left[ \frac{h(x)}{x} \right] + axy \left[ \frac{h(x)}{x} \right] + \eta_2 ab\beta y^2 \right\}, \\
 W_3 &= \left\{ \gamma_3 b(1 - \beta)x^2 \left[ \frac{h(x)}{x} \right] + b(1 - \beta) \left[ \frac{g(x, y)}{y} - b \right] xy + \delta_2 a \left[ \frac{g(x, y)}{y} - b \right] y^2 \right\}, \\
 W_4 &= \left\{ \delta_3 a \left[ \frac{g(x, y)}{y} - b \right] y^2 + (\alpha a^{-1} + 1) \left[ \frac{g(x, y)}{y} - b \right] yz + \mu_2 \alpha z^2 \right\}, \\
 W_5 &= \left\{ \eta_3 ab\beta y^2 + a \left[ \frac{f_1 + f_2}{z} - a \right] yz + \xi_2 (\alpha a^{-1} + 1) \left[ \frac{f_1 + f_2}{z} - a \right] z^2 \right\}, \\
 W_6 &= \left\{ \gamma_4 b(1 - \beta)x^2 \left[ \frac{h(x)}{x} \right] + b(1 - \beta) \left[ \frac{f_1 + f_2}{z} - a \right] xz \right. \\
 &\quad \left. + \xi_2 (\alpha a^{-1} + 1) \left[ \frac{f_1 + f_2}{z} - a \right] z^2 \right\}, \\
 W_7 &= \left\{ \gamma_5 b(1 - \beta)x^2 \left[ \frac{h(x)}{x} \right] + (\alpha a^{-1} + 1)xz \left[ \frac{h(x)}{x} \right] + \mu_3 \alpha z^2 \right\}, \\
 W_p &= \{ [b(1 - \beta)x + ay + (\alpha a^{-1} + 1)z]p(t, x, y, z) \}
 \end{aligned}$$

with

$$\begin{aligned}
 \sum_{i=1}^5 \gamma_i &= 1; \quad \sum_{i=1}^3 \delta_i = 1; \quad \sum_{i=1}^3 \eta_i = 1; \quad \sum_{i=1}^3 \xi_i = 1; \\
 \gamma_i &> 0, \quad \delta_j > 0, \quad \eta_j > 0, \quad \xi_j > 0, \quad i = 1, 2, 3, 4, 5; \quad j = 1, 2, 3.
 \end{aligned}$$

We note that for any two real numbers,  $u, v$ , and for  $s \in \{-1, +1\}$  there exists a constant  $k > 0$  such that

$$(su)v = \left( ksu + \frac{1}{2k}v \right)^2 - \left( k^2u^2 + \frac{1}{4k^2}v^2 \right) > - \left( k^2u^2 + \frac{1}{4k^2}v^2 \right).$$

Moreover, this inequality is retained when multiplied by any positive term.

By virtue of the conditions of the theorem on the nonlinear functions, we have a constant  $k_2 > 0$  such that

$$\begin{aligned}
 W_2 &= \left\{ \gamma_2 b(1 - \beta)x^2 \left[ \frac{h(x)}{x} \right] + axy \left[ \frac{h(x)}{x} \right] + \eta_2 ab\beta y^2 \right\} \\
 &\geq \left\{ \gamma_2 b(1 - \beta)\nu x^2 + a\nu xy + \eta_2 ab\beta y^2 \right\} \\
 &\geq \left\{ \nu [\gamma_2 b(1 - \beta) - k_2^2 a] x^2 + a \left[ \eta_2 b\beta - \frac{\nu}{4k_2^2} \right] y^2 \right\} \\
 &\geq 0, \quad \forall x, y, z \text{ whenever } \frac{\nu}{4\eta_2 b\beta} \leq k_2^2 < \frac{\gamma_2 b(1 - \beta)}{a}.
 \end{aligned}$$

In a similar manner, we have constants  $k_i > 0$ ,  $i = 3, 4, \dots, 7$ , such that

$$\begin{aligned}
 W_3 &\geq \left\{ b(1-\beta) \left[ \gamma_3 \nu - k_3^2 \left[ \frac{g(x,y)}{y} - b \right] \right] x^2 + \left[ \frac{g(x,y)}{y} - b \right] \left[ \delta_2 a - \frac{b(1-\beta)}{4k_3^2} \right] y^2 \right\} \\
 &\geq 0, \quad \forall x, y, z, \text{ whenever } \frac{b(1-\beta)}{4\delta_2 a} \leq k_3^2 < \frac{\gamma_3 \nu}{[g(x,y)/y - b]}; \\
 W_4 &\geq \left\{ \left[ \frac{g(x,y)}{y} - b \right] [\delta_3 a - k_4^2(\alpha^{-1} + 1)] y^2 + \left[ \mu_2 \alpha - \frac{(\alpha^{-1} + 1)[g(x,y)/y - b]}{4k_4^2} \right] z^2 \right\} \\
 &\geq 0, \quad \forall x, y, z, \text{ whenever } \frac{(\alpha^{-1} + 1)[g(x,y)/y - b]}{4\mu_2 \alpha} \leq k_4^2 < \frac{\delta_3 a}{(\alpha^{-1} + 1)}; \\
 W_5 &\geq \left\{ a \left[ \eta_3 b \beta - k_5^2 \left[ \frac{f_1 + f_2}{z} - a \right] \right] y^2 + \left[ \frac{f_1 + f_2}{z} - a \right] \left[ \xi_2(\alpha^{-1} + 1) - \frac{a}{4k_5^2} \right] z^2 \right\} \\
 &\geq 0, \quad \forall x, y, z, \text{ whenever } \frac{a}{4\xi_2(\alpha^{-1} + 1)} \leq k_5^2 < \frac{\eta_3 b \beta}{[(f_1 + f_2)/z - a]}; \\
 W_6 &\geq \left\{ b(1-\beta) \left[ \gamma_4 \nu - k_6^2 \left[ \frac{f_1 + f_2}{z} - a \right] \right] x^2 \right. \\
 &\quad \left. + \left[ \frac{f_1 + f_2}{z} - a \right] \left[ \xi_3(\alpha^{-1} + 1) - \frac{b(1-\beta)}{4k_6^2} \right] z^2 \right\} \\
 &\geq 0, \quad \forall x, y, z, \text{ whenever } \frac{b(1-\beta)}{4\xi_3(\alpha^{-1} + 1)} \leq k_6^2 < \frac{\gamma_4 \nu}{[(f_1 + f_2)/z - a]}.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 W_7 &\geq \left\{ \nu [\gamma_5 b(1-\beta) - k_7^2(\alpha^{-1} + 1)] x^2 + \left[ \mu_3 \alpha - \frac{(\alpha^{-1} + 1)\nu}{4k_7^2} \right] z^2 \right\} \\
 &\geq 0, \quad \forall x, y, z, \text{ whenever } \frac{(\alpha^{-1} + 1)\nu}{4\mu_3 \alpha} \leq k_7^2 < \frac{\gamma_5 b(1-\beta)}{(\alpha^{-1} + 1)}.
 \end{aligned}$$

Also, we have that

$$W_1 \geq \gamma_1 b(1-\beta)\nu x^2 + \delta_1 ab\beta y^2 + \mu_1 \alpha z^2 \geq D_4(x^2 + y^2 + z^2)$$

where  $0 < D_4 \leq \min\{\gamma_1 b(1-\beta)\nu; \delta_1 ab\beta; \mu_1 \alpha\}$ .

Moreover,

$$W_p \leq \{[b(1-\beta)|x| + a|y| + (\alpha^{-1} + 1)|z|]|p(t, x, y, z)|\} \leq D_5(|x| + |y| + |z|)$$

where  $D_5 = A \max\{b(1-\beta); a; (\alpha^{-1} + 1)\}$ .

Hence, using (3.6) we have

$$(3.7) \quad \dot{V} \leq -D_4(x^2 + y^2 + z^2) + D_6(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

where  $D_6 = 3^{\frac{1}{2}} D_5$ .

If we choose  $(x^2 + y^2 + z^2)^{\frac{1}{2}} \geq D_7 = 2D_6D_4^{-1}$ , inequality (3.7) implies that

$$\dot{V} \leq -\frac{1}{2}D_4(x^2 + y^2 + z^2).$$

We see at once that

$$\dot{V} \leq -D_8,$$

provided that  $x^2 + y^2 + z^2 \geq 2D_8D_4^{-1}$ ; and this completes the verification of (3.4), (with  $D_2 \equiv D_8$ ).

**Remark 3.1.** We note that in the work of Tunç [22], using the Lyapunov method only ended up with

$$\dot{V} \leq -\overline{D}(y^2 + z^2) + \overline{\overline{D}}(|y| + |z|)|p(t)|$$

which gave an incomplete nature of the function.

#### 4. PROOF OF THEOREM 2.1

Let  $(x(t), y(t), z(t))$  be any solution of (1.2). Then there is evidently a  $t_0 \geq 0$  such that

$$x^2(t_0) + y^2(t_0) + z^2(t_0) < D_3,$$

where  $D_3$  is the constant in the lemma; for otherwise, that is if

$$x^2(t) + y^2(t) + z^2(t) \geq D_3, \quad t \geq 0,$$

then, by (3.4),

$$\dot{V}(t) \leq -D_2 < 0, \quad t \geq 0,$$

and this in turn implies that  $V(t) \rightarrow -\infty$  as  $t \rightarrow \infty$ , which contradicts (3.3). Hence to prove (3.4) it will suffice to show that if

$$(4.1) \quad x^2(t) + y^2(t) + z^2(t) < D_9 \quad \text{for } t = T,$$

where  $D_9 \geq D_3$  is a finite constant, then there is a constant  $D_{10} > 0$ , depending on  $a, b, c, \delta, \alpha, \xi$  and  $D_9$ , such that

$$(4.2) \quad x^2(t) + y^2(t) + z^2(t) \leq D_{10} \quad \text{for } t \geq T.$$

Our proof of (4.2) is based essentially on an extension of an argument in the proof of [8; Lemma 1]. For any given constant  $d > 0$ , let  $S(d)$  denote the surface



$x^2 + y^2 + z^2 = d$ . Because  $V$  is continuous in  $x, y, z$  and tends to  $+\infty$  as  $x^2 + y^2 + z^2 \rightarrow \infty$ , there is evidently a constant  $D_{11} > 0$ , depending on  $D_9$  as well as on  $a, b, c, \delta, \xi$  and  $\alpha$ , such that

$$(4.3) \quad \min_{(x,y,z) \in S(D_{11})} V(x, y, z) > \max_{(x,y,z) \in S(D_9)} V(x, y, z).$$

It is easy to see from (4.1) and (4.3) that

$$(4.4) \quad x^2(t) + y^2(t) + z^2(t) < D_{11} \quad \text{for } t \geq T.$$

For suppose on the contrary that there is a  $t > T$  such that

$$x^2(t) + y^2(t) + z^2(t) \geq D_{11}.$$

Then, by (4.1) and by the continuity of the quantities  $x(t), y(t), z(t)$  in the argument displayed, there exist  $t_1, t_2, T < t_1 < t_2$  such that

$$(4.5) \quad x^2(t_1) + y^2(t_1) + z^2(t_1) = D_9,$$

$$(4.6) \quad x^2(t_2) + y^2(t_2) + z^2(t_2) = D_{11}$$

and such that

$$(4.7) \quad D_9 \leq x^2(t) + y^2(t) + z^2(t) \leq D_{11}, \quad t_1 \leq t \leq t_2.$$

But, writing  $V(t) \equiv V(x(t), y(t), z(t))$ , since  $D_9 \geq D_3$ , (4.7) obviously implies [in view of (3.4)] that

$$V(t_2) < V(t_1),$$

and this contradicts the conclusion [from (4.3) and (4.6)]

$$V(t_2) > V(t_1).$$

Hence (4.4) holds. This completes the proof of (3.4), and the theorem now follows.  $\square$

## References

- [1] Ademola, T. A., Ogundiran, M. O., Arawomo, P. O, Adesina, O. A: Boundedness results for a certain third order nonlinear differential equation. *Appl. Math. Comput.* *216* (2010), 3044–3049.
- [2] Afuwape, A. U.: Frequency-domain criteria for dissipativity of some third order differential equations. *An. Stiint. Univ. Al. I. Cuza Iasi, n. Ser., Sect. Ia* *24* (1978), 271–275.
- [3] Afuwape, A. U.: An application of the frequency-domain criteria for dissipativity of a certain third order non-linear differential equation. *Analysis* *1* (1981), 211–216.
- [4] Afuwape, A. U., Omeike, M. O.: Further ultimate boundedness of solutions of some system of third order nonlinear ordinary differential equations. *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math.* *43* (2004), 7–20.
- [5] Afuwape, A. U., Omeike, M. O.: Convergence of solutions of certain third order systems of nonlinear ordinary differential equations. *J. Nigerian Math. Soc.* *25* (2006), 1–12.
- [6] Afuwape, A. U., Omeike, M. O.: Convergence of solutions of certain non-homogeneous third order ordinary differential equations. *Kragujevac J. Math.* *31* (2008), 5–16.
- [7] Bereketoglu, H., Gyori, I.: On the boundedness of the solutions of a third-order nonlinear differential equation. *Dynam. Systems Appl.* *6* (1997), 263–270.
- [8] Chukwu, E. N.: On the boundedness of solutions of third order differential equations. *Ann. Mat. Pur. Appl.* *104* (1975), 123–149.
- [9] Ezeilo, J. O. C.: An elementary proof of a boundedness theorem for a certain third order differential equation. *J. Lond. Math. Soc.* *38* (1963), 11–16.
- [10] Ezeilo, J. O. C.: Some results for the solutions of a certain system of differential equations. *J. Math. Anal. Appl.* *6* (1963), 389–393.
- [11] Ezeilo, J. O. C.: A generalization of a boundedness theorem for a certain third order differential equation. *Proc. Cambridge Philos. Soc.* *63* (1967), 735–742.
- [12] Ezeilo, J. O. C.: New properties of the equation  $x''' + ax'' + bx' + h(x) = p(t, x, x', x'')$  for certain special values of the incrementary ratio  $y^{-1}\{h(x+y) - h(x)\}$  (P. Janssons, J. Mawhin, N. Rouche, eds.). *Equations differentielles et fonctionnelles non-lineares*, Hermann Publishing, Paris, 1973, pp. 447–462.
- [13] Ezeilo, J. O. C.: A generalization of some boundedness results by Reissig and Tejumola. *J. Math. Anal. Appl.* *41* (1973), 411–419.
- [14] Ezeilo, J. O. C.: A further result on the existence of periodic solutions of the equation  $\ddot{x} + \Psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + v(x, \dot{x}, \ddot{x}) = p(t)$  with a bound  $\nu$ . *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* *55* (1978), 51–57.
- [15] Haddad, W. A., Chellaboina, V. S.: *Nonlinear Dynamical Systems and Control—A Lyapunov-Based Approach*. Princeton University Press, Princeton, 2008.
- [16] Hara, T.: On a uniform ultimate boundedness of the solution of certain third order differential equations. *J. Math. Anal. Appl.* *80* (1981), 533–544.
- [17] Qian, C.: Asymptotic behavior of third-order nonlinear differential equations. *J. Math. Anal. Appl.* *284* (2003), 191–205.
- [18] Reissig, R.: Über die Existenz periodischer Lösungen bei einer nichtlinearen Differentialgleichung dritter Ordnung. *Math. Nachr.* *32* (1966), 83–88.
- [19] Reissig, R., Sansone, G., Conti, R.: *Nonlinear Differential Equations of Higher Order*. Noordhoff, Groningen, 1974.
- [20] Swick, K. E.: Boundedness and stability for a nonlinear third order differential equation. *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* *56* (1974), 859–865.
- [21] Tejumola, H. O.: On the boundedness and periodicity of solutions of certain third-order non-linear differential equations. *Ann. Mat. Pura Appl., IV Ser.* *83* (1969), 195–212.
- [22] Tunç, Cemil: Boundedness of solutions of a third order nonlinear differential equation. *J. Inequal. Pure Appl. Math.* *6* (2005), Article 3, 6 pp. (electronic).

- [23] *Yoshizawa, T.*: Stability Theory by Lyapunov's Second Method. Mathematical Society of Japan, Tokyo, 1966.

*Authors' addresses:* *Anthony Afuwape*, Instituto de Matemáticas, Universidad de Antioquia, Calle 67, No. 53-108, Medellín AA 1226, Colombia, e-mail: [aafuwape@yahoo.co.uk](mailto:aafuwape@yahoo.co.uk); *M. O. Omeike*, Department of Mathematics, University of Agriculture, Abeokuta, Nigeria, e-mail: [moomeike@yahoo.com](mailto:moomeike@yahoo.com).