Mathematica Bohemica

R. Roopkumar; C. Kalaivani

Some results on fuzzy proper functions and connectedness in smooth fuzzy topological spaces

Mathematica Bohemica, Vol. 137 (2012), No. 3, 311–332

Persistent URL: `http://dml.cz/dmlcz/142897`

SOME RESULTS ON FUZZY PROPER FUNCTIONS AND CONNECTEDNESS IN SMOOTH FUZZY TOPOLOGICAL SPACES

R. Roopkumar, C. Kalaivani, Karaikudi

(Received October 11, 2010)

Abstract. In this paper, we introduce the notion of the (α,β)-weakly smooth fuzzy continuous proper function and discuss its properties. We also study several notions of connectedness in smooth fuzzy topological spaces and establish that the product of connected sets (spaces) is not connected in any sense, as well as investigate continuous images of smooth connected sets (spaces) under (α,β)-weakly smooth fuzzy continuous functions.

Keywords: fuzzy proper function, smooth fuzzy topology, smooth fuzzy continuity

MSC 2010: 54A40

1. Introduction

The concept of fuzzy topology was introduced in 1968 by Chang [4], and Chang's fuzzy topology on a fuzzy set was studied by Chakraborty and Ahsanullah [3]. In 1980, Höhle [13] suggested that a topology can be viewed as an L-subset of a powerset. With this motivation, in 1985, Kubiak [17] and Šostak [26] independently extended the idea to the more general setting of L-subsets of L-powersets, where L is either a complete Heyting algebra with quasi-complementation [17], or its particular instance—the unit interval $[0,1]$ [26]. Höhle and Šostak [15], Kubiak and Šostak [18] developed the concept of an L-fuzzy topological space even further to situations where L is more general than $[0,1]$, in 1995 and 1997, respectively. The respective categories of L-fuzzy topological spaces and L-fuzzy continuous maps are studied in [14], [22], [23]. Ramadan [24] gave a similar definition of fuzzy topology on a fuzzy set in Šostak's sense under the name of "*smooth fuzzy topological spaces*". In the context of smooth fuzzy topological spaces, neighborhood structures [7], base and subbase [21], product topology [27], compactness [1], [6], [8], [9], separation axioms [27], gradation preserving functions [12], connectedness [2], [5], [20] were also studied.

A fuzzy proper function from a fuzzy set into a fuzzy set was introduced by Chakraborty and Ahsanullah [3] and discussed by various researchers [3], [5], [10], [24], [25]. Since fuzzy proper functions between fuzzy sets generalize functions between sets, whereas smooth fuzzy topologies on fuzzy sets extend classical topologies on sets, many standard results fail for fuzzy proper functions on smooth fuzzy topological spaces. In [25], it was pointed out that pointwise continuity of a fuzzy proper function at every fuzzy point belonging to a fuzzy set does not imply fuzzy continuity on the fuzzy set, with respect to different notions like smooth fuzzy continuity, weakly smooth fuzzy continuity, etc. To get these results, the concepts of the α-weakly smooth fuzzy continuity and the positive minimum smooth fuzzy topological space were introduced.

In this paper, we point out some situations where smooth fuzzy continuity of a fuzzy proper function F is not equivalent to any of the following statements.

▷ $F^{\rightarrow}(\mathrm{Cl}(C)) \leqslant \mathrm{Cl}(F^{\rightarrow}(C)),\ \forall C \leqslant \mu$.

▷ $F^{\leftarrow}(V^{\circ}) \leqslant (F^{\leftarrow}(V))^{\circ},\ \forall V \leqslant \nu$.

(For the above-mentioned closure and interior operators on fuzzy subsets in a smooth fuzzy topological space, we refer to Definition 2.4.) Further, we provide some sufficient conditions to get one of them by introducing (α,β)-weakly smooth fuzzy continuous functions. We also find the interrelations between α-weakly smooth fuzzy continuous functions and (α,β)-weakly smooth fuzzy continuous functions. With respect to connectedness in smooth fuzzy topological spaces, we note that for certain notions of connectedness, image of a connected space under an (α,β)-weakly fuzzy continuous function is connected with a weaker assumption than in Theorem 6.4 of [5] and for some other notions this result fails. We also show that the product of two connected smooth fuzzy topological spaces is not connected with respect to several definitions of connectedness for smooth fuzzy topological spaces.

2. Preliminaries

Throughout this paper X, S denote fixed non-empty sets, μ, ν denote fuzzy subsets of X, S, respectively, I denotes the unit interval $[0,1]$, I_0 denotes $(0,1]$ and I^X denotes the set of all fuzzy subsets of X. For $X=\{x_1,x_2,\ldots,x_n\}$ and $\lambda_i \in I$, for every $i \in \{1,2,\ldots,n\}$, by $\mu^{[\lambda_1,\lambda_2,\ldots,\lambda_n]}_{[x_1,x_2,\ldots,x_n]}$ we shall mean the fuzzy subset μ of X which maps x_i to λ_i for every $i=1,2,\ldots,n$. A fuzzy point [20] in X is defined by

$$P_x^{\lambda}(t)=\begin{cases}\lambda & \text{if } t=x,\\ 0 & \text{if } t\neq x,\end{cases}$$

where $0<\lambda\leqslant 1$. By $P_x^{\lambda}\in\mu$ we mean that $\lambda\leqslant\mu(x)$.

If $\{A_\alpha\colon\ \alpha \in J\} \subseteq I^X$, where J is an arbitrary index set, then the union and intersection of this collection of fuzzy subsets are defined respectively by $\Big(\bigvee_{\alpha\in J} A_\alpha\Big)(x) = \sup_{\alpha\in J} A_\alpha(x)$ and $\Big(\bigwedge_{\alpha\in J} A_\alpha\Big)(x) = \inf_{\alpha\in J} A_\alpha(x)$, for every $x \in X$. If $A, B \in I^X$ such that $A \geqslant B$, then we define the complement of B in A by $(A-B)(x) = A(x) - B(x)$ for every $x \in X$. We also use the notation $A \vee B$ and $A \wedge B$ to denote the union of A, B and the intersection of A, B respectively.

Lemma 2.1 [25]**.** *If $A \in I^X$, then $A = \bigvee \big\{P_x^\lambda\colon\ P_x^\lambda \in A\big\}$.*

Definition 2.2 [24]**.** A smooth fuzzy topology on a fuzzy set $\mu \in I^X$ is a map $\tau\colon\ \mathfrak{I}_\mu \to I,$ where $\mathfrak{I}_\mu = \big\{U \in I^X\colon\ U \leqslant \mu\big\}$, satisfying the following axioms:
(1) $\tau(\underline{0}) = \tau(\mu) = 1$,
(2) $\tau(A_1 \wedge A_2) \geqslant \tau(A_1) \wedge \tau(A_2), \forall A_1, A_2 \in \mathfrak{I}_\mu$,
(3) $\tau\Big(\bigvee_{i\in\Gamma} A_i\Big) \geqslant \bigwedge_{i\in\Gamma} \tau(A_i)$ for every family $(A_i)_{i\in\Gamma} \subseteq \mathfrak{I}_\mu$.

The pair (μ, τ) is called a smooth fuzzy topological space or simply sfts.

Definition 2.3 [2]**.** Let (μ, τ) be a smooth fuzzy topological space and $A \in \mathfrak{I}_\mu$. The mapping $\tau_A\colon\ \mathfrak{I}_A \to I$ defined by

$$\tau_A(U) = \bigvee \{\tau(K)\colon\ K \in \mathfrak{I}_\mu, K \wedge A = U\}$$

is a subspace smooth fuzzy topology induced over A by τ.

Definition 2.4. Let (μ, τ) be a smooth fuzzy topological space, $U \in \mathfrak{I}_\mu$ and $\alpha \in I_0$. Define

(1) $U^\circ = \bigvee\{V \in \mathfrak{I}_\mu\colon\ \tau(V) > 0, V \leqslant U\}$ [6],
(2) $(U, \alpha)^\circ = \bigvee\{V \in \mathfrak{I}_\mu\colon\ \tau(V) \geqslant \alpha, V \leqslant U\}$ [2],
(3) $\mathrm{Cl}(U) = \bigwedge\{K \in \mathfrak{I}_\mu\colon\ U \leqslant K, \tau(\mu - K) > 0\}$ [6],
(4) $\mathrm{Cl}(U, \alpha) = \bigwedge\{K \in \mathfrak{I}_\mu\colon\ U \leqslant K, \tau(\mu - K) \geqslant \alpha\}$ [2].

Definition 2.5 [27]**.** Let $\{(\mu_j, \tau_j)\colon\ j \in J\}$ be a family of smooth fuzzy topological spaces and $P_k\colon \prod X_j \to X_k$ the kth projection map. Let

$$\mathscr{S} = \{P_k^{\leftarrow}(U)\colon\ \tau_k(U) > 0,\ k \in J\}, \quad \text{where } P_k^{\leftarrow}(U)(\mathbf{x}) = U(P_k(\mathbf{x})),\ \forall \mathbf{x} \in \prod X_j$$

and let $\mathscr{B}_{\mathscr{S}}$ be the collection of all finite intersections of members of $\mathscr{S}$. Define $\prod_j \mu_j \in I^{\prod_j X_j}$ by $\Big(\prod_j \mu_j\Big)(\mathbf{x}) = \bigwedge_j \mu_j(x_j)$ for every $\mathbf{x} = (x_j) \in \prod_j X_j$ and define

$\tau\colon \mathfrak{I}_{\prod_j \mu_j} \to I$ by

$$\tau(U) = \begin{cases} \tau_k(V), & \text{if } U = P_k^{\leftarrow}(V), \\ \tau(E_1) \wedge \tau(E_2), & U = E_1 \wedge E_2 \text{ where } E_1, E_2 \in \mathscr{S}, \\ \bigvee_i \tau(W_i), & U = \bigvee_i W_i \text{ where each } W_i \in \mathscr{B}_{\mathscr{S}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then τ is called the product smooth fuzzy topology on $\prod_j \mu_j$.

Definition 2.6 [3]. Let $\mu \in I^X$ and $\nu \in I^S$. A fuzzy subset F of $X \times S$ is said to be a fuzzy proper function from μ to ν if

(1) $F(x,s) \leqslant \min\{\mu(x), \nu(s)\}$ for each $(x,s) \in X \times S$,

(2) for each $x \in X$ there exists a unique $s_0 \in S$ such that $F(x, s_0) = \mu(x)$ and $F(x,s) = 0$ if $s \neq s_0$.

Definition 2.7 [3]. Let F be a fuzzy proper function from μ to ν. If $U \leqslant \mu$ and $V \leqslant \nu$, then $F^{\leftarrow}(V)\colon X \to I$ and $F^{\rightarrow}(U)\colon S \to I$ are defined by

$$(F^{\rightarrow}(U))(s) = \bigvee_{x \in X} \{F(x,s) \wedge U(x)\}, \ \forall s \in S,$$
$$(F^{\leftarrow}(V))(x) = \bigvee_{s \in S} \{F(x,s) \wedge V(s)\}, \ \forall x \in X.$$

The inverse image of a fuzzy subset V under a fuzzy proper function F can be easily obtained as $(F^{\leftarrow}(V))(x) = \mu(x) \wedge V(s)$, where $s \in S$ is unique such that $F(x,s) = \mu(x)$.

Definition 2.8 [10]. A fuzzy proper function $F\colon \mu \to \nu$ is said to be

(1) injective (or one-to-one) if $F(x_1, s) > 0$ and $F(x_2, s) > 0$ for some $x_1, x_2 \in X$ and $s \in S$ imply $x_1 = x_2$;

(2) surjective (or onto) if for every $s \in S$ with $\nu(s) > 0$ there exists $x \in X$ such that $F(x,s) = \mu(x) > 0$.

For a fuzzy proper function $F\colon \mu \to \nu$, the following properties hold (see, e.g., [10]).

(1) $F^{\rightarrow}(F^{\leftarrow}(V)) \leqslant V$, $\forall V \leqslant \nu$.

(2) $F^{\leftarrow}(F^{\rightarrow}(U)) \geqslant U$, $\forall U \leqslant \mu$.

(3) $F^{\leftarrow}\Big(\bigvee_{j \in J} V_j\Big) = \bigvee_{j \in J} F^{\leftarrow}(V_j)$ where $V_j \leqslant \nu$, $\forall j \in J$.

(4) $F^{\leftarrow}\Big(\bigwedge_{j \in J} V_j\Big) = \bigwedge_{j \in J} F^{\leftarrow}(V_j)$ where $V_j \leqslant \nu$, $\forall j \in J$.

(5) If F is injective, then $F^{\leftarrow}(F^{\rightarrow}(U)) = U$, $\forall U \leqslant \mu$.

The following example shows that $F^{\rightarrow}(F^{\leftarrow}(V)) \neq V$ for some $V \leqslant \nu$, even if F is surjective. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu_{[x,y]}^{[0.7,0.4]} \in I^X$ and $\nu_{[s,t]}^{[0.8,0.7]} \in I^S$. If a fuzzy proper function $F\colon\ \mu \rightarrow \nu$ is defined by

$$F(x, s) = 0.7,\ F(x, t) = 0,\ F(y, s) = 0,\ F(y, t) = 0.4,$$

then for $V_{[s,t]}^{[0,0.5]} \in I^S$ we get $V \leqslant \nu$ and $F^{\rightarrow}(F^{\leftarrow}(V))_{[s,t]}^{[0,0.4]} \neq V$.

Lemma 2.9 [25]**.** *If $P_x^{\lambda} \in F^{\leftarrow}(V)$, then $F^{\rightarrow}(P_x^{\lambda}) \in V$.*

Definition 2.10 [24]**.** Let $F\colon\ (\mu, \tau) \rightarrow (\nu, \sigma)$ be a fuzzy proper function and let (μ, τ), (ν, σ) be smooth fuzzy topological spaces. Then F is said to be smooth fuzzy continuous on μ if $\tau(F^{\leftarrow}(V)) \geqslant \sigma(V)$, $\forall V \leqslant \nu$.

Definition 2.11 [24]**.** Let $F\colon\ (\mu, \tau) \rightarrow (\nu, \sigma)$ be a fuzzy proper function and let (μ, τ), (ν, σ) be smooth fuzzy topological spaces. Then F is said to be weakly smooth fuzzy continuous on μ if $\tau(F^{\leftarrow}(V)) > 0$ whenever $\sigma(V) > 0$, $\forall V \leqslant \nu$.

Definition 2.12 [25]**.** Let $F\colon\ (\mu, \tau) \rightarrow (\nu, \sigma)$ be a fuzzy proper function and let (μ, τ), (ν, σ) be smooth fuzzy topological spaces. Then F is said to be α-weakly smooth fuzzy continuous on μ if $\tau(F^{\leftarrow}(V)) \geqslant \alpha$ whenever $\sigma(V) \geqslant \alpha$, $\forall V \leqslant \nu$.

Definition 2.13 [25]**.** Let (μ, τ) be a smooth fuzzy topological space. Then τ is said to be a positive minimum smooth fuzzy topology if $\bigwedge\limits_{i \in \Gamma} \tau(U_i) > 0$ whenever $U_i \in \mathfrak{I}_{\mu}$ and $\tau(U_i) > 0$ for all $i \in \Gamma$.

Lemma 2.14. *Let $F\colon\ (\mu, \tau) \rightarrow (\nu, \sigma)$ be a smooth fuzzy continuous fuzzy proper function. If $G\colon\ (\mu, \tau) \rightarrow (F^{\rightarrow}(\mu), \sigma_{F^{\rightarrow}(\mu)})$ is defined by $G(x, s) = F(x, s), \forall (x, s) \in X \times S$, then G is smooth fuzzy continuous.*

P r o o f. Let F be smooth fuzzy continuous and let $V \leqslant F^{\rightarrow}(\mu)$. If $T \in \mathfrak{I}_{\nu}$ is such that $V = T \wedge F^{\rightarrow}(\mu)$, then $F^{\leftarrow}(V) = F^{\leftarrow}(T) \wedge F^{\leftarrow}(F^{\rightarrow}(\mu)) \geqslant F^{\leftarrow}(T) \wedge \mu = F^{\leftarrow}(T)$ and $F^{\leftarrow}(T) \geqslant F^{\leftarrow}(V)$. Therefore, $F^{\leftarrow}(T) = F^{\leftarrow}(V)$. Since $\tau(G^{\leftarrow}(V)) = \tau(F^{\leftarrow}(V)) = \tau(F^{\leftarrow}(T)) \geqslant \sigma(T)$, by smooth fuzzy continuity of $F\colon\ (\mu, \tau) \rightarrow (\nu, \sigma)$ and $\sigma_{F^{\rightarrow}(\mu)}(V) = \bigvee \{\sigma(T)\colon\ T \in \mathfrak{I}_{\nu}$ such that $T \wedge F^{\rightarrow}(\mu) = V\}$, the claimed result of the lemma follows. $\square$

Lemma 2.15. *Let $F\colon\ (\mu, \tau) \rightarrow (\nu, \sigma)$ be a weakly smooth fuzzy continuous fuzzy proper function. If G is defined as in the previous lemma, then $G\colon\ (\mu, \tau) \rightarrow (F^{\rightarrow}(\mu), \sigma_{F(\mu)})$ is weakly smooth fuzzy continuous.*

P r o o f. Let $V \leqslant F^{\rightarrow}(\mu)$ be such that $\sigma_{F^{\rightarrow}(\mu)}(V) > 0$. Then there exists $T \in \mathfrak{I}_{\nu}$ such that $T \wedge F^{\rightarrow}(\mu) = V$ and $\sigma(T) > 0$. Since F is weakly smooth fuzzy continuous, the equality $\tau(G^{\leftarrow}(V)) = \tau(F^{\leftarrow}(T))$ from the proof of the previous lemma provides $\tau(G^{\leftarrow}(V)) > 0$. Hence, G is weakly smooth fuzzy continuous. $\square$

3. (α,β)-weakly smooth fuzzy continuous functions

We first discuss two expected equivalent statements for continuous functions, in the context of (weakly) smooth fuzzy continuous functions.

Theorem 3.1. *Let $F\colon (\mu,\tau)\to(\nu,\sigma)$ be a fuzzy proper function. If F is weakly smooth fuzzy continuous, then $F^{\leftarrow}(V^{\circ})\leqslant (F^{\leftarrow}(V))^{\circ}$ for every $V\leqslant\nu$.*

Proof. Let $V\leqslant\nu$. Then

$$\begin{aligned}F^{\leftarrow}(V^{\circ}) &= F^{\leftarrow}\Big(\bigvee\{K\in\mathfrak{I}_{\nu}\colon\ \sigma(K)>0, K\leqslant V\}\Big)\\ &\leqslant \bigvee\{F^{\leftarrow}(K)\colon\ \sigma(K)>0, F^{\leftarrow}(K)\leqslant F^{\leftarrow}(V)\}\\ &\leqslant \bigvee\{F^{\leftarrow}(K)\colon\ \tau(F^{\leftarrow}(K))>0, F^{\leftarrow}(K)\leqslant F^{\leftarrow}(V)\}\\ &\leqslant \bigvee\{U\in\mathfrak{I}_{\mu}\colon\ \tau(U)>0, U\leqslant F^{\leftarrow}(V)\}\\ &= (F^{\leftarrow}(V))^{\circ}.\end{aligned}$$

Hence, $F^{\leftarrow}(V^{\circ})\leqslant (F^{\leftarrow}(V))^{\circ}$. □

The converse of the above theorem is not true.

Counterexample 3.2. Let $X=\{x,y\}$, $S=\{s,t\}$, $\mu^{[0.6,0.7]}_{[x,y]}\in I^X$, $\nu^{[0.8,0.9]}_{[s,t]}\in I^S$, $U^{[0.5-\frac{1}{n+1},0.6-\frac{1}{n+1}]}_{n[x,y]}$, $\forall n=1,2\ldots$, $\bigvee U_n=W^{[0.5,0.6]}_{[x,y]}$ and $V^{[0.5,0.6]}_{1[s,t]}$. If $\tau\colon\ \mathfrak{I}_{\mu}\to I$ is defined by

$$\tau(U)=\begin{cases}1, & U=\underline{0}\text{ or }\mu,\\ 1/n, & U=U_n,\\ 0, & \text{otherwise}\end{cases}$$

and $\sigma\colon\ \mathfrak{I}_{\nu}\to I$ is defined by

$$\sigma(V)=\begin{cases}1, & V=\underline{0}\text{ or }\nu,\\ 0.6, & V=V_1,\\ 0, & \text{otherwise,}\end{cases}$$

then (μ,τ) and (ν,σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F\colon\ (\mu,\tau)\to(\nu,\sigma)$ be defined by

$$F(x,s)=0.6,\ F(x,t)=0,\ F(y,s)=0,\ F(y,t)=0.7.$$

First, we claim that F is not weakly smooth fuzzy continuous. Since $F^{\leftarrow}(V_1)(x)=\mu(x)\wedge V_1(s)=0.6\wedge 0.5=0.5$ and $F^{\leftarrow}(V_1)(y)=\mu(y)\wedge V_1(t)=0.7\wedge 0.6=0.6$, we get that $F^{\leftarrow}(V_1)=W$ and hence $\sigma(V_1)=0.6>0=\tau(W)=\tau(F^{\leftarrow}(V_1))$. Therefore, F is not smooth fuzzy continuous. Next, we show that $F^{\leftarrow}(V^{\circ})\leqslant(F^{\leftarrow}(V))^{\circ}$ for every $V\in\mathfrak{I}_{\nu}$. Let $V\in\mathfrak{I}_{\nu}$ be arbitrary.

Case 1. $V_1 \leqslant V$.

Subcase (i). $V = V_1$. Then $F^{\leftarrow}(V^\circ) = F^{\leftarrow}(V_1^\circ) = F^{\leftarrow}(V_1) = W = W^\circ = (F^{\leftarrow}(V_1))^\circ = (F^{\leftarrow}(V))^\circ$.

Subcase (ii). $V = \nu$. Hence, $F^{\leftarrow}(V^\circ) = F^{\leftarrow}(\nu) = \mu = \mu^\circ = (F^{\leftarrow}(\nu))^\circ = (F^{\leftarrow}(V))^\circ$.

Subcase (iii). $V_1 \nleqslant V \nleqslant \nu$. Since $V^\circ = V_1$, it follows that $F^{\leftarrow}(V^\circ) = F^{\leftarrow}(V_1) = W = W^\circ = (F^{\leftarrow}(V_1))^\circ \leqslant (F^{\leftarrow}(V))^\circ$.

Case 2. $V_1 \nleqslant V$. In this case $V^\circ = \underline{0}$. Thus, $F^{\leftarrow}(V^\circ) = \underline{0} \leqslant (F^{\leftarrow}(V))^\circ$.

Theorem 3.3. *Let $F\colon (\mu,\tau) \to (\nu,\sigma)$ be a fuzzy proper function and (μ,τ) a positive minimum smooth fuzzy topological space. Then F is weakly smooth fuzzy continuous if and only if $F^{\leftarrow}(V^\circ) \leqslant (F^{\leftarrow}(V))^\circ$ for every $V \leqslant \nu$.*

Proof. As one part of the proof of this theorem is similar to the proof of Theorem 3.1, we show the other part of this proof only. Let $V \leqslant \nu$ be such that $\sigma(V) > 0$. Then we have $V = V^\circ$. Now, $F^{\leftarrow}(V) = F^{\leftarrow}(V^\circ) \leqslant (F^{\leftarrow}(V))^\circ$. Hence, $F^{\leftarrow}(V) = (F^{\leftarrow}(V))^\circ$. Since (μ,τ) is a positive minimum smooth fuzzy topological space, we have $\tau(U^\circ) > 0$, $\forall U \in \mathfrak{I}_\mu$. Therefore, $\tau(F^{\leftarrow}(V)) > 0$. Thus, F is weakly smooth fuzzy continuous. $\square$

Next, we show that neither smooth fuzzy continuity nor weakly smooth fuzzy continuity of F implies or is implied by $F^{\rightarrow}(\mathrm{Cl}(C)) \leqslant \mathrm{Cl}(F^{\rightarrow}(C))$, $\forall C \in \mathfrak{I}_\mu$. Since smooth fuzzy continuity implies weakly smooth fuzzy continuity, the following two counterexamples justify our statement.

Counterexample 3.4. Let $X = \{x,y\}$, $S = \{s,t\}$ and let $\mu^{[0.7,0.6]}_{[x,y]}$, $\nu^{[0.8,0.7]}_{[s,t]}$ be fuzzy subsets of X and S, respectively. Define fuzzy subsets $U_1 \leqslant \mu$, $V_1 \leqslant \nu$ by ${U_1}^{[0.6,0.5]}_{[x,y]}$, ${V_1}^{[0.6,0.5]}_{[s,t]}$.

If $\tau\colon \mathfrak{I}_\mu \to I$ is defined by

$$\tau(U) = \begin{cases} 1, & U = \underline{0} \text{ or } \mu, \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma\colon \mathfrak{I}_\nu \to I$ is defined by

$$\sigma(V) = \begin{cases} 1, & V = \underline{0} \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then obviously, (μ,τ), (ν,σ) are smooth fuzzy topological spaces.

Let the fuzzy proper function $F\colon (\mu,\tau)\to(\nu,\sigma)$ be defined by

$$F(x,s)=0.7,\ F(x,t)=0,\ F(y,s)=0,\ F(y,t)=0.6.$$

Since $F^{\leftarrow}(\underline{0})=\underline{0}$, $F^{\leftarrow}(\nu)=\mu$, $F^{\leftarrow}(V_1){}^{[0.6,0.5]}_{[x,y]}=U_1$, and $\tau(U_1)=0.6>0.5=\sigma(V_1)$, it follows that F is smooth fuzzy continuous on μ. For $C=F^{\leftarrow}(\nu-V_1){}^{[0.2,0.2]}_{[x,y]}$ we get that $\mathrm{Cl}(C)=\mathrm{Cl}(F^{\leftarrow}(\nu-V_1))=\mu$. Therefore, $F^{\rightarrow}(\mathrm{Cl}(C)){}^{[0.7,0.6]}_{[s,t]}$. But $F^{\rightarrow}(C)=F^{\rightarrow}(F^{\leftarrow}(\nu-V_1))\leqslant\nu-V_1$ implies that $\mathrm{Cl}(F^{\rightarrow}(C))\leqslant\mathrm{Cl}(\nu-V_1)=(\nu-V_1){}^{[0.2,0.2]}_{[s,t]}$. Hence, $F^{\rightarrow}(\mathrm{Cl}(C))>\mathrm{Cl}(F^{\rightarrow}(C))$.

Counterexample 3.5. Let $X=\{x,y\}$, $S=\{s,t\}$, $\mu^{[0.8,0.7]}_{[x,y]}$, $\nu^{[0.9,0.7]}_{[s,t]}$. Define fuzzy subsets $U_1\leqslant\mu$ and $V_1\leqslant\nu$ by $U_1{}^{[0.1,0.1]}_{[x,y]}$, $V_1{}^{[0.2,0.1]}_{[s,t]}$.

If $\tau\colon \mathfrak{I}_\mu\to I$ is defined by

$$\tau(U)=\begin{cases}1, & U=\underline{0}\text{ or }\mu,\\ 0.6, & U=U_1,\\ 0, & \text{otherwise}\end{cases}$$

and $\sigma\colon \mathfrak{I}_\nu\to I$ is defined by

$$\sigma(V)=\begin{cases}1, & V=\underline{0}\text{ or }\nu,\\ 0.5, & V=V_1,\\ 0, & \text{otherwise,}\end{cases}$$

then obviously, (μ,τ), (ν,σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F\colon (\mu,\tau)\to(\nu,\sigma)$ be defined by

$$F(x,s)=0.8, F(x,t)=0, F(y,s)=0, F(y,t)=0.7.$$

Since $\tau(F^{\leftarrow}(V_1){}^{[0.2,0.1]}_{[x,y]})=0$ and $\sigma(V_1)>0$, we have that F is not weakly smooth fuzzy continuous on μ. Next, we claim that $F^{\rightarrow}(\mathrm{Cl}(C))\leqslant\mathrm{Cl}(F^{\rightarrow}(C))$, $\forall C\in\mathfrak{I}_\mu$. Let $C\in\mathfrak{I}_\mu$.

Case 1. $F^{\rightarrow}(C)\leqslant\nu-V_1$.

Subcase (i). $F^{\rightarrow}(C)=\underline{0}$. Then $C=\underline{0}$. Hence, $F^{\rightarrow}(\mathrm{Cl}(C))=\underline{0}=\mathrm{Cl}(F^{\rightarrow}(C))$.

Subcase (ii). $\underline{0}\nleqslant C\leqslant\nu-V_1$. It follows that $\underline{0}\nleqslant C\leqslant F^{\leftarrow}(F^{\rightarrow}(C))\leqslant F^{\leftarrow}(\nu-V_1)=\mu-U_1$. Therefore, $\mathrm{Cl}(C)=\mu-U_1$ and hence we get $F^{\rightarrow}(\mathrm{Cl}(C))=F^{\rightarrow}(\mu-U_1)=\nu-V_1=\mathrm{Cl}(F^{\rightarrow}(C))$.

Case 2. $F^{\rightarrow}(C)\nleqslant\nu-V_1$. Then, we get $F^{\rightarrow}(\mathrm{Cl}(C))\leqslant\nu=\mathrm{Cl}(F^{\rightarrow}(C))$.

Definition 3.6. Let $F\colon (\mu,\tau)\to(\nu,\sigma)$ be a fuzzy proper function and let (μ,τ), (ν,σ) be smooth fuzzy topological spaces. Then F is said to be (α,β)-weakly smooth fuzzy continuous on μ if $\tau(F^{\leftarrow}(V))\geqslant\alpha$ whenever $V\in\mathfrak{I}_\nu$ and $\sigma(V)\geqslant\beta$.

Definition 3.7. Let $F\colon (\mu,\tau)\to(\nu,\sigma)$ be a fuzzy proper function and let (μ,τ), (ν,σ) be smooth fuzzy topological spaces and $P_x^\lambda\in\mu$. Then F is said to be (α,β)-weakly smooth fuzzy continuous at P_x^λ if $F^{\to}(P_x^\lambda)\in V\in\mathfrak{I}_\nu$ and $\sigma(V)\geqslant\beta$ imply the existence of $U\in\mathfrak{I}_\mu$ such that $\tau(U)\geqslant\alpha$, $P_x^\lambda\in U$ and $F^{\to}(U)\leqslant V$.

Theorem 3.8. *Let $F\colon (\mu,\tau)\to(\nu,\sigma)$ be a fuzzy proper function and let (μ,τ), (ν,σ) be smooth fuzzy topological spaces. Then F is (α,β)-weakly smooth fuzzy continuous at P_x^λ, $\forall P_x^\lambda\in\mu$ if and only if it is (α,β)-weakly smooth fuzzy continuous on μ.*

Though the proof of this theorem is similar to the proof of Theorem 3.15 in [25], for the sake of completeness we present the proof here.

Proof. Assume that F is (α,β)-weakly smooth fuzzy continuous on μ. Let $P_x^\lambda\in\mu$ and let $V\leqslant\nu$ be given such that $\sigma(V)\geqslant\beta$ and $F^{\to}(P_x^\lambda)\in V$. If we take $U=F^{\leftarrow}(V)$, then $F^{\to}(U)=F^{\to}(F^{\leftarrow}(V))\leqslant V$. We note that if $s\in S$ is unique such that $F(x,s)=\mu(x)$, then it follows immediately that $F^{\to}(P_x^\lambda)=P_s^\lambda$. Since $P_x^\lambda\in\mu$ and $P_s^\lambda\in V$, we get $U(x)=F^{\leftarrow}(V)(x)=\mu(x)\wedge V(s)\geqslant\lambda$ and hence, $P_x^\lambda\in U$. Next, by using the assumption on F, we also get that $\tau(U)=\tau(F^{\leftarrow}(V))\geqslant\alpha$. Thus, F is (α,β)-weakly smooth fuzzy continuous at P_x^λ. Conversely, assume that F is (α,β)-weakly smooth fuzzy continuous at every fuzzy point $P_x^\lambda\in\mu$. Let $V\leqslant\nu$ and $\sigma(V)\geqslant\beta$. For every $P_x^\lambda\in F^{\leftarrow}(V)$, by Lemma 2.9, we have $F^{\to}(P_x^\lambda)\in V$. Therefore, there exists $U_{x_\lambda}\leqslant\mu$ such that $P_x^\lambda\in U_{x_\lambda}$, $\tau(U_{x_\lambda})\geqslant\alpha$ and $F^{\to}(U_{x_\lambda})\leqslant V$. Then, by Lemma 2.1, we get $F^{\leftarrow}(V)=\bigvee U_{x_\lambda}$ and $\tau(F^{\leftarrow}(V))=\tau(\bigvee U_{x_\lambda})\geqslant\bigwedge\tau(U_{x_\lambda})\geqslant\alpha$. Thus, F is (α,β)-weakly smooth fuzzy continuous on μ. □

Proposition 3.9. *If $F\colon (\mu,\tau)\to(\nu,\sigma)$ is (α,β)-weakly smooth fuzzy continuous and $\alpha\geqslant\beta$, then F is α-weakly smooth fuzzy continuous as well as β-weakly smooth fuzzy continuous.*

Proof. Let $V\leqslant\nu$ with $\sigma(V)\geqslant\alpha$. Clearly, $\sigma(V)\geqslant\beta$. Then, by hypothesis, $\tau(F^{\leftarrow}(V))\geqslant\alpha$. Hence, F is α-weakly smooth fuzzy continuous.

Let $V\leqslant\nu$ with $\sigma(V)\geqslant\beta$. Then, by hypothesis, $\tau(F^{\leftarrow}(V))\geqslant\alpha\geqslant\beta$. Hence, F is β-weakly smooth fuzzy continuous. □

Proposition 3.10. *Let $\alpha\leqslant\beta$. If $F\colon (\mu,\tau)\to(\nu,\sigma)$ is α-weakly smooth fuzzy continuous or β-weakly smooth fuzzy continuous, then F is (α,β)-weakly smooth fuzzy continuous.*

Proof. Let $V\leqslant\nu$ with $\sigma(V)\geqslant\beta$. Clearly, $\sigma(V)\geqslant\alpha$. Then, by our hypothesis, $\tau(F^{\leftarrow}(V))\geqslant\alpha$. Hence, F is (α,β)-weakly smooth fuzzy continuous. Similarly, we can prove the other part of this theorem. □

Obviously, every α-weakly smooth fuzzy continuous function is an (α,α)-weakly smooth fuzzy continuous function. In fact, we justify by the following counterexample that the collection of all α-weakly smooth fuzzy continuous functions is properly contained in the collection of all (α,β)-weakly smooth fuzzy continuous functions.

Counterexample 3.11. Let $X=\{x,y\}$, $S=\{s,t\}$ and let $\mu_{[x,y]}^{[0.7,0.4]}$, $\nu_{[s,t]}^{[0.9,1]}$ be fuzzy subsets of X and S, respectively. Define fuzzy subsets $U_1\leqslant\mu$, $V_1,V_2\leqslant\nu$ by $U_{1[x,y]}^{[0.5,0.4]}$ and $V_{1[s,t]}^{[0.5,0]}$, $V_{2[s,t]}^{[0.5,0.6]}$. If $\tau\colon\ \mathfrak{I}_\mu\to I$ is defined by

$$\tau(U)=\begin{cases}1, & U=\underline{0}\text{ or }\mu,\\ 0.6, & U=U_1,\\ 0, & \text{otherwise}\end{cases}$$

and $\sigma\colon\ \mathfrak{I}_\nu\to I$ is defined by

$$\sigma(V)=\begin{cases}1, & V=\underline{0}\text{ or }\nu,\\ 0.7, & V=V_1,\\ 1, & V=V_2,\\ 0, & \text{otherwise,}\end{cases}$$

then (μ,τ) and (ν,σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F\colon\ (\mu,\tau)\to(\nu,\sigma)$ be defined by

$$F(x,s)=0.7,\ F(x,t)=0,\ F(y,s)=0,\ F(y,t)=0.4.$$

Fix $\alpha=0.6$ and $\beta=0.75$. First, we claim that F is (α,β)-weakly smooth fuzzy continuous. Here $\sigma(\underline{0})\geqslant\beta$, $\sigma(\nu)\geqslant\beta$, $\sigma(V_2)\geqslant\beta$. Clearly, $\tau(F^{\leftarrow}(\underline{0}))=\tau(\underline{0})\geqslant\alpha$, $\tau(F^{\leftarrow}(\nu))=\tau(\mu)\geqslant\alpha$ and $\tau(F^{\leftarrow}(V_2))=\tau(U_1)=\alpha$. Hence, F is (α,β)-weakly smooth fuzzy continuous. Next, we claim that F is not γ-weakly smooth fuzzy continuous for every $\gamma\in I_0$.

Case 1. $0<\gamma\leqslant0.7$. We note that $\sigma(V_1)=0.7\geqslant\gamma$ but $\tau(F^{\leftarrow}(V_1))=0<\gamma$. Hence, F is not γ-weakly smooth fuzzy continuous.

Case 2. $0.7<\gamma\leqslant1$. Here $\sigma(V_2)=1\geqslant\gamma$ and $\tau(F^{\leftarrow}(V_2))=\tau(U_1)=0.6<\gamma$. Hence, F is not γ-weakly smooth fuzzy continuous.

Theorem 3.12. *Let $F\colon\ (\mu,\tau)\to(\nu,\sigma)$ be a fuzzy proper function. Then F is (α,β)-weakly smooth fuzzy continuous if and only if $F^{\leftarrow}((V,\alpha)^\circ)\leqslant(F^{\leftarrow}(V),\beta)^\circ$ for every $V\leqslant\nu$.*

P r o o f. The proof of this theorem is similar to the proof of Theorem 3.3. $\square$

However, (α,β)-weakly smooth fuzzy continuity of F does not imply (and is not implied by) $F^{\rightarrow}(\mathrm{Cl}(C,\alpha)) \leqslant \mathrm{Cl}(F^{\rightarrow}(C),\beta)$, $\forall C \in \mathfrak{I}_{\mu}$. Counterexamples 3.4, 3.5 justify this statement for the case $\alpha=\beta=0.5$.

4. Connectedness

In this section we discuss smooth connectedness, smooth α-connectedness, smooth Q-connectedness and α-fuzzy μ-connectedness of smooth fuzzy topological spaces.

4.1. Smooth connectedness.

Definition 4.1 [5]. Let μ be a fuzzy subset of X. Then $E \leqslant \mu$ is said to be maximal if for every $x \in X$, $E(x) \neq 0$ implies $E(x) = \mu(x)$.

Definition 4.2 [5]. Let (μ,τ) be a smooth fuzzy topological space. Then $U \leqslant \mu$ is said to be clopen if $\tau(U) > 0$ and $\tau(\mu - U) > 0$.

Definition 4.3 (Cf. [5]). A smooth topological space (μ,τ) is said to be smooth connected if it has no proper non-zero maximal fuzzy clopen set.

Definition 4.4 (Cf. [5]). $E \leqslant \mu$ is said to be smooth connected if (E,τ_E) is smooth connected.

Theorem 6.4 of [5] states that a continuous image of a connected Chang's fuzzy topological space is connected if the fuzzy proper function is one-to-one and onto. We claim that the assumptions one-to-one and onto on the fuzzy proper function are redundant. Actually, we prove our claim in a more general setup, in the form of the following theorem.

Theorem 4.5. *Let $F\colon (\mu,\tau) \to (\nu,\sigma)$ be a fuzzy proper function. If F is weakly smooth fuzzy continuous and μ is smooth connected, then $F^{\rightarrow}(\mu)$ is also smooth connected.*

Proof. Assume that $F^{\rightarrow}(\mu)$ is not smooth connected. Then $F^{\rightarrow}(\mu)$ has a non-zero proper maximal fuzzy clopen set V. Since V is non-zero and maximal, there exists $s \in S$ with $0 \neq V(s) = (F^{\rightarrow}(\mu))(s)$. If $F(x,s) = 0$ for every $x \in X$, then $(F^{\rightarrow}(\mu))(s) = \bigvee_{x\in X} \{F(x,s) \wedge \mu(x)\} = 0$, which leads to a contradiction. Therefore, we conclude that there exists $x_0 \in X$ such that $F(x_0,s) \neq 0$ and hence, $\mu(x_0) = F(x_0,s) \neq 0$. Therefore, $(F^{\leftarrow}(V))(x_0) = \mu(x_0) \wedge V(s) \neq 0$. To prove that $F^{\leftarrow}(V)$ is a maximal fuzzy set in μ, let $(F^{\leftarrow}(V))(x_1) \neq 0$ for some $x_1 \in X$. If $s_1 \in S$ is unique such that $F(x_1,s_1) \neq 0$, from $0 \neq (F^{\leftarrow}(V))(x_1) = \mu(x_1) \wedge V(s_1)$ we conclude that $V(s_1) \neq 0$ and $\mu(x_1) \neq 0$. Since V is a maximal subset of $F^{\rightarrow}(\mu)$, we have $V(s_1) = (F^{\rightarrow}(\mu))(s_1)$. From

$$\begin{aligned}(F^{\leftarrow}(V))(x_1) &= \mu(x_1) \wedge (F^{\rightarrow}(\mu))(s_1) \\ &= \mu(x_1) \wedge \bigvee_{x\in X} \{F(x, s_1) \wedge \mu(x)\} \\ &\geqslant \mu(x_1) \wedge F(x_1, s_1) \wedge \mu(x_1) \\ &= \mu(x_1) \wedge \mu(x_1) \wedge \mu(x_1) = \mu(x_1)\end{aligned}$$

and $(F^{\leftarrow}(V))(x_1) = \mu(x_1) \wedge (F^{\rightarrow}(\mu))(s_1) \leqslant \mu(x_1)$, it follows that $(F^{\leftarrow}(V))(x_1) = \mu(x_1) \neq 0$. Next, we show that $F^{\leftarrow}(V)$ is a fuzzy clopen set in μ. Since V is a fuzzy clopen set in $F^{\rightarrow}(\mu)$, we have $\sigma_{F^{\rightarrow}(\mu)}(V) > 0$ and $\sigma_{F^{\rightarrow}(\mu)}(F^{\rightarrow}(\mu) - V) > 0$. Since $F\colon (\mu, \tau) \rightarrow (\nu, \sigma)$ is weakly smooth fuzzy continuous, by Lemma 2.15, $F\colon (\mu, \tau) \rightarrow (F^{\rightarrow}(\mu), \tau_{F^{\rightarrow}(\mu)})$ is also weakly smooth fuzzy continuous. Therefore, immediately we obtain $\tau(F^{\leftarrow}(V)) > 0$ and $\tau(F^{\leftarrow}(F^{\rightarrow}(\mu) - V)) > 0$. We claim that $F^{\leftarrow}(F^{\rightarrow}(\mu) - V) = \mu - F^{\leftarrow}(V)$. We note that if $s \in S$ is unique such that $F(x, s) \neq 0$, then $(F^{\rightarrow}(\mu))(s) = \bigvee_{y\in X} \{F(y, s) \wedge \mu(y)\} \geqslant F(x, s) \wedge \mu(x) = \mu(x)$ and if $V(s) \neq 0$, then $V(s) = F^{\rightarrow}(\mu)(s)$. Therefore, we get

$$\begin{aligned}(F^{\leftarrow}(F^{\rightarrow}(\mu) - V))(x) &= \mu(x) \wedge ((F^{\rightarrow}(\mu))(s) - V(s)) \\ &= \begin{cases} \mu(x) & \text{if } V(s) = 0, \\ (\mu(x) \wedge ((F^{\rightarrow}(\mu))(s) - (F^{\rightarrow}(\mu))(s)) = 0 & \text{if } V(s) \neq 0 \end{cases}\end{aligned}$$

and

$$\begin{aligned}(\mu - F^{\leftarrow}(V))(x) &= \mu(x) - (\mu(x) \wedge V(s)) \\ &= \begin{cases} \mu(x) & \text{if } V(s) = 0, \\ \mu(x) - (\mu(x) \wedge F^{\rightarrow}(\mu)(s)) = 0 & \text{if } V(s) \neq 0. \end{cases}\end{aligned}$$

Therefore, $\tau(\mu - F^{\leftarrow}(V)) > 0$ and hence, $F^{\leftarrow}(V)$ is a non-zero proper maximal fuzzy clopen set of μ, which contradicts the assumption that μ is connected. Thus, $F^{\rightarrow}(\mu)$ is connected. □

Corollary 4.6. *Let $F\colon (\mu, \tau) \rightarrow (\nu, \sigma)$ be a fuzzy proper function. If F is smooth fuzzy continuous and μ is smooth connected, then $F^{\rightarrow}(\mu)$ is also smooth connected.*

P r o o f. It is easy to prove that every smooth fuzzy continuous function is weakly smooth fuzzy continuous. Hence, the corollary follows from Theorem 4.5. □

Definition 4.7. Let (μ, τ) be a smooth fuzzy topological space and let $\alpha \in I_0$. Then $U \leqslant \mu$ is said to be α-clopen if $\tau(U) \geqslant \alpha$ and $\tau(\mu - U) \geqslant \alpha$.

Definition 4.8. A smooth fuzzy topological space (μ, τ) is said to be smooth α-connected if it has no proper non-zero maximal fuzzy α-clopen set.

Definition 4.9. Let (μ, τ) be a smooth fuzzy topological space and $E \leqslant \mu$. E is said to be smooth α-connected if (E, τ_E) is smooth α-connected, where τ_E is the subspace smooth fuzzy topology on E induced by μ.

Theorem 4.10. *Let $F\colon (\mu, \tau) \to (\nu, \sigma)$ be a fuzzy proper function. If F is (α, β)-weakly smooth fuzzy continuous and μ is smooth α-connected, then $F^{\rightarrow}(\mu)$ is smooth β-connected in ν.*

Proof of this theorem is similar to that of Theorem 4.5.

Theorem 4.11. *Let (μ, τ) be a smooth fuzzy topological space. (μ, τ) is smooth connected iff (μ, τ) is smooth α-connected for every $\alpha \in I_0$.*

Proof. Assume that (μ, τ) is connected. Suppose (μ, τ) is not smooth α-connected. Then it has a non-zero proper maximal fuzzy α-clopen set V. By using $\tau(V) \geqslant \alpha > 0$ and $\tau(\mu - V) \geqslant \alpha > 0$, we get that V is a non-zero proper maximal fuzzy clopen set V of (μ, τ), which is a contradiction. Conversely, assume that (μ, τ) is smooth α-connected for every $\alpha \in I_0$. If (μ, τ) is not smooth connected, then it has a non-zero proper maximal fuzzy clopen set V. Hence, it follows that $\tau(V) > 0$ and $\tau(\mu - V) > 0$. If $\beta = \min\{\tau(V), \tau(\mu - V)\}$, then $\beta \in I_0$, $\tau(V) \geqslant \beta$ and $\tau(\mu - V) \geqslant \beta$. Therefore, V is a non-zero proper maximal fuzzy β-clopen set in (μ, τ). Thus, (μ, τ) is not smooth β-connected for some $\beta \in I_0$, which contradicts the assumption. Hence, (μ, τ) is smooth connected. $\square$

Next, we show by a counterexample that there exists a smooth fuzzy topological space which is α-smooth connected for some $\alpha \in I_0$ but not smooth connected.

Counterexample 4.12. Let $X = \{x, y\}$. Define $\mu \in I^X$ by $\mu_{[x,y]}^{[0.7,0.8]}$ and $U_1, U_2 \leqslant \mu$ by $U_{1[x,y]}^{[0.7,0]}$, $U_{2[x,y]}^{[0,0.8]}$. If $\tau\colon \mathfrak{I}_\mu \to I$ is defined by

$$\tau(U) = \begin{cases} 1, & U = \underline{0} \text{ or } \mu, \\ 0.4, & U = U_1, \\ 0.3, & U = U_2, \\ 0, & \text{otherwise}, \end{cases}$$

then (μ, τ) is a smooth fuzzy topological space. Here, (μ, τ) has no non-zero proper maximal fuzzy 0.5-clopen set but it has a non-zero proper maximal fuzzy clopen set U_1. Hence, (μ, τ) is smooth 0.5-connected but not smooth connected.

Next, we show that the product of smooth connected spaces need not be smooth connected.

Counterexample 4.13. Let $X=\{x,y\}$, $S=\{s,t\}$, $\mu_{[x,y]}^{[0.7,0.6]}$, $\nu_{[s,t]}^{[0.8,0.9]}$, $V_{1[s,t]}^{[0.7,0]}$, $V_{2[s,t]}^{[0,0.7]}$ and $V_{3[s,t]}^{[0.7,0.7]}$. We define $\tau\colon \mathfrak{I}_\mu\to I$ by

$$\tau(U)=\begin{cases}1, & U=\underline{0}\text{ or }\mu,\\ 0, & \text{otherwise}\end{cases}$$

and $\sigma\colon \mathfrak{I}_\nu\to I$ by

$$\sigma(V)=\begin{cases}1, & V=\underline{0}\text{ or }\nu,\\ 0.5, & V=V_1,\\ 0.3, & V=V_2,\\ 0.6, & V=V_3,\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, (μ,τ) and (ν,σ) are smooth connected. We claim that $\mu\times\nu$ is not smooth connected, where $(\mu\times\nu)(x,s)=\mu(x)\wedge\nu(s)$ for every $(x,s)\in X\times S$. The product topology (see Definition 2.5) on $\mu\times\nu$ is given by

$$\varrho(U)=\begin{cases}1, & U=\underline{0}\text{ or }\mu\times\nu,\\ 0.5, & U=\mu\times V_1,\\ 0.3, & U=\mu\times V_2,\\ 0, & \text{otherwise.}\end{cases}$$

One can verify that

$$\mu\times\nu_{[(x,s),(x,t),(y,s),(y,t)]}^{[0.7,0.7,0.6,0.6]},\ (\mu\times V_1)_{[(x,s),(x,t),(y,s),(y,t)]}^{[0.7,0,0.6,0]},\ (\mu\times V_2)_{[(x,s),(x,t),(y,s),(y,t)]}^{[0,0.7,0,0.6]}$$

and $\mu\times V_3=\mu\times\nu$. Hence, $\mu\times V_1$ is a non-zero proper maximal fuzzy clopen subset of $\mu\times\nu$. Thus, $\mu\times\nu$ is not smooth connected.

4.2. Smooth Q-connectedness.

Definition 4.14 (Cf. [20]). Let (μ,τ) be a smooth fuzzy topological space. Two fuzzy sets U_1 and U_2 in (μ,τ) are said to be smooth Q-separated if there exists G_i with $\tau(\mu-G_i)>0\ (i=1,2)$ such that $G_i\geqslant U_i\ (i=1,2)$ and $G_1\wedge U_2=\underline{0}=G_2\wedge U_1$.

It is obvious that U_1 and U_2 are smooth Q-separated if and only if $\mathrm{Cl}(U_1)\wedge U_2=\underline{0}=\mathrm{Cl}(U_2)\wedge U_1$.

Definition 4.15 (cf. [20]). A fuzzy set $U \in \Im_\mu$ is called smooth Q-disconnected if it can be written as a union of two non-empty smooth Q-separated sets C and D.

(μ, τ) is said to be smooth Q-connected if it is not smooth Q-disconnected. The following example shows that the image of a smooth Q-connected space need not be smooth Q-connected under a smooth continuous injective function.

Counterexample 4.16. Let $X = \{x, y\}$, $S = \{s, t\}$. If $\mu^{[0.5,0.6]}_{[x,y]}$, $\nu^{[1,1]}_{[s,t]}$, $U^{[0.5,0.5]}_{1[x,y]}$, $V^{[0.5,1]}_{1[s,t]}$, $V^{[1,0.5]}_{2[s,t]}$, $V^{[0.5,0.5]}_{3[s,t]}$, then $U_1 \leqslant \mu \in I^X$, $V_1, V_2, V_3 \leqslant \nu \in I^S$. We define smooth fuzzy topologies τ on μ and σ on ν by

$$\tau(U) = \begin{cases} 1, & U = \underline{0} \text{ or } \mu, \\ 0.5, & U = U_1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma(V) = \begin{cases} 1, & V = \underline{0} \text{ or } \nu, \\ 0.4, & V = V_1, \\ 0.3, & V = V_2, \\ 0.5, & V = V_3, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F\colon (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x, s) = 0.5,\ F(x, t) = 0,\ F(y, s) = 0,\ F(y, t) = 0.6.$$

Obviously, F is injective. Since $F^{\leftarrow}(\underline{0}) = \underline{0}$, $F^{\leftarrow}(\nu) = \mu$, $F^{\leftarrow}(V_1) = \mu$, $F^{\leftarrow}(V_2) = U_1$ and $F^{\leftarrow}(V_3) = U_1$, we get that F is smooth fuzzy continuous. Next, we claim that U_1 is smooth Q-connected. Suppose $U_1 = C_1 \vee C_2$, where $C_1, C_2 \in \Im_\mu \setminus \{\underline{0}\}$.

Case 1. $C_1(x) = 0.5$, $C_1(y) = 0.5$, $0 \leqslant C_2(x) \leqslant 0.5$, $0 \leqslant C_2(y) \leqslant 0.5$.

Since C_1, $\mathrm{Cl}(C_1)$ are non-zero at both points of X and C_2 is non-zero on X, it obviously follows that $\mathrm{Cl}(C_1) \wedge C_2 \neq \underline{0}$ and $C_1 \wedge \mathrm{Cl}(C_2) \neq \underline{0}$.

Case 2. $C_1(x) = 0.5$, $C_2(y) = 0.5$, $0 \leqslant C_1(y) \leqslant 0.5$, $0 \leqslant C_2(x) \leqslant 0.5$.

Here, $\mathrm{Cl}(C_1) = \mu$ and $\mathrm{Cl}(C_2) = \mu$. Hence, $C_1 \wedge \mathrm{Cl}(C_2) \neq \underline{0}$, $C_2 \wedge \mathrm{Cl}(C_1) \neq \underline{0}$.

One can verify the remaining cases by interchanging C_1 and C_2 in both Case 1 and Case 2. Hence, U_1 is smooth Q-connected. But $(F^{\rightarrow}(U_1))^{[0.5,0.5]}_{[s,t]}$ is not smooth Q-connected, because $F^{\rightarrow}(U_1) = A^{[0.5,0]}_{[s,t]} \vee B^{[0,0.5]}_{[s,t]}$ and $A \wedge \mathrm{Cl}(B) = A \wedge (\nu - V_2) = \underline{0}$ and $\mathrm{Cl}(A) \wedge B = (\nu - V_1) \wedge B = \underline{0}$.

The following example shows that the product of smooth Q-connected spaces need not be smooth Q-connected

Counterexample 4.17. Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu^{[0.7,0.6]}_{[x,y]} \in I^X$, $\nu^{[0.8,0.9]}_{[s,t]} \in I^S$, $V^{[0.7,0]}_{1[s,t]}$, $V^{[0,0.7]}_{2[s,t]}$ and $V^{[0.7,0.7]}_{3[s,t]}$. If $\tau\colon \Im_\mu \to I$ is defined by

$$\tau(U) = \begin{cases} 1, & U = \underline{0} \text{ or } \mu, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma\colon \Im_\nu \to I$ is defined by

$$\sigma(V) = \begin{cases} 1, & V = \underline{0} \text{ or } \nu, \\ 0.6, & V = V_1, \\ 0.5, & V = V_2, \\ 0.5, & V = V_3, \\ 0, & \text{otherwise,} \end{cases}$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. We claim that μ is smooth Q-connected. Suppose $\mu = A \vee B$, where $A, B \in \Im_\mu \setminus \{\underline{0}\}$. Then $\mathrm{Cl}(A) = \mathrm{Cl}(B) = \mu$ and hence, $A \wedge \mathrm{Cl}(B) \neq \underline{0}$ and $B \wedge \mathrm{Cl}(A) \neq \underline{0}$. Therefore, we conclude that μ is smooth Q-connected. Suppose $\nu = W_1 \vee W_2$, where $W_1, W_2 \in \Im_\nu \setminus \{\underline{0}\}$.

Since the possible $K \in \Im_\nu$ with $\sigma(\nu - K) > 0$ are

$$K = \underline{0}, \nu, (\nu - V_1)^{[0.1,0.9]}_{[s,t]}, (\nu - V_2)^{[0.8,0.2]}_{[s,t]} \text{ and } (\nu - V_3)^{[0.1,0.2]}_{[s,t]},$$

the closure of any non-zero fuzzy subset of ν is non-zero at both points of S and hence, $W_1 \wedge \mathrm{Cl}(W_2) \neq \underline{0}$, $W_2 \wedge \mathrm{Cl}(W_1) \neq \underline{0}$. Hence, ν is smooth Q-connected. Next, we show that $\mu \times \nu$ is not smooth Q-connected. The product topology (cf. Counterexample 4.13) on $\mu \times \nu$ is given by

$$\varrho(U) = \begin{cases} 1, & U = \underline{0} \text{ or } \mu \times \nu, \\ 0.6, & U = \mu \times V_1, \\ 0.5, & U = \mu \times V_2, \\ 0, & \text{otherwise.} \end{cases}$$

We can write $\mu \times \nu = (\mu \times V_1) \vee (\mu \times V_2)$. Since $\mathrm{Cl}(\mu \times V_1) = \mu \times V_1$ and $\mathrm{Cl}(\mu \times V_2) = \mu \times V_2$, we get that $(\mu \times V_1) \wedge \mathrm{Cl}(\mu \times V_2) = \underline{0}$ and $(\mu \times V_2) \wedge \mathrm{Cl}(\mu \times V_1) = \underline{0}$. Hence, $\mu \times \nu$ is not smooth Q-connected.

4.3. α-fuzzy μ-connectedness.

Definition 4.18 [2]. Let $\mu \in I^X$ and let $U, V \in \mathfrak{I}_\mu$. Then U and V are said to be quasi-coincident with respect to μ (written as $U\, q\, V[\mu]$) if there exists $x \in X$ such that $U(x) + V(x) > \mu(x)$. If U and V are not quasi-coincident with respect to μ, we write $U \not{q}\, V[\mu]$.

Definition 4.19 [2]. Let (μ, τ) be a smooth fuzzy topological space, $\alpha \in I_0$ and $U_1, U_2 \in \mathfrak{I}_\mu$. Then U_1, U_2 are said to be α-fuzzy μ-separated if $U_1 \not{q} \operatorname{Cl}(U_2, \alpha)[\mu]$ and $U_2 \not{q} \operatorname{Cl}(U_1, \alpha)[\mu]$.

Definition 4.20 [2]. $U \in \mathfrak{I}_\mu$ is said to be α-fuzzy μ-connected if it cannot be expressed as the union of two α-fuzzy μ-separated sets.

Theorem 4.21 [2]. *Let (μ, τ) be a smooth fuzzy topological space. Then $U_1, U_2 \in \mathfrak{I}_\mu$ are α-fuzzy μ-separated if and only if there are $V_1, V_2 \in \mathfrak{I}_\mu$ with $\tau(V_1) \geqslant \alpha$ and $\sigma(V_2) \geqslant \alpha$ such that $U_1 \leqslant V_1, U_2 \leqslant V_2$, $U_1 \not{q}\, V_2[\mu]$ and $U_2 \not{q}\, V_1[\mu]$.*

Theorem 4.22. *Let (μ, τ), (ν, σ) be smooth fuzzy topological spaces and let $F\colon (\mu, \tau) \to (\nu, \sigma)$ be an (α, β)-weakly smooth fuzzy continuous and injective mapping. If $U \in \mathfrak{I}_\mu$ is α-fuzzy μ-connected, then $F^{\to}(U)$ is β-fuzzy $F^{\to}(\mu)$-connected.*

Proof. Suppose $F^{\to}(U)$ is not β-fuzzy $F^{\to}(\mu)$-connected. Then $F^{\to}(U) = U_1 \vee U_2$, where U_1, U_2 are β-fuzzy μ-separated. Then by Theorem 4.21, there are $V_1, V_2 \in \mathfrak{I}_\mu$ with $\sigma(V_1) \geqslant \beta$ and $\sigma(V_2) \geqslant \beta$ such that $U_1 \leqslant V_1$, $U_2 \leqslant V_2$, $U_1 \not{q}\, V_2[F^{\to}(\mu)]$ and $U_2 \not{q}\, V_1[F^{\to}(\mu)]$. Since F is (α, β)-weakly smooth fuzzy continuous, we have $\tau(F^{\leftarrow}(V_1)) \geqslant \alpha$, $\tau(F^{\leftarrow}(V_2)) \geqslant \alpha$. Since $V_1 \leqslant F^{\to}(\mu)$ and F is injective, we also have $F^{\leftarrow}(V_1) \leqslant F^{\leftarrow}(F^{\to}(\mu)) = \mu$ and hence, $F^{\leftarrow}(U_1) \leqslant \mu$. Now,

$$\begin{aligned}(F^{\leftarrow}(U_1))(x) + (F^{\leftarrow}(V_2))(x) &= \mu(x) \wedge U_1(s) + \mu(x) \wedge V_2(s) \\ &\leqslant U_1(s) + V_2(s) \leqslant (F^{\to}(\mu))(s) = \mu(x).\end{aligned}$$

Hence, $F^{\leftarrow}(U_1) \not{q}\, F^{\leftarrow}(V_2)[\mu]$. Similarly, we obtain that $F^{\leftarrow}(U_2) \not{q}\, F^{\leftarrow}(V_1)[\mu]$. Employing Theorem 4.21 once more, we conclude that $F^{\leftarrow}(U_1)$ and $F^{\leftarrow}(U_2)$ are α-fuzzy μ-separated sets. Since F is injective, $U = F^{\leftarrow}(F^{\to}(U)) = F^{\leftarrow}(U_1 \vee U_2) = F^{\leftarrow}(U_1) \vee F^{\leftarrow}(U_2)$, which is a contradiction. $\square$

The following example shows that the image of an α-fuzzy μ-connected space need not be β-fuzzy ν-connected under a non-injective (α, β)-weakly smooth fuzzy continuous function.

Counterexample 4.23. Let $X = \{x, y, z\}$, $S = \{s, t\}$ and let $\mu^{[0.5,0.5,0.1]}_{[x,y,z]}$, $\nu^{[1,1]}_{[s,t]}$ be fuzzy subsets of X and S, respectively. Define the fuzzy subset $V_1 \leqslant \nu$ by $V^{[0.5,0.5]}_{1[s,t]}$. If $\tau\colon \mathfrak{I}_\mu \to I$ is defined by

$$\tau(U) = \begin{cases} 1, & U = \underline{0} \text{ or } \mu, \\ 0, & \text{otherwise} \end{cases}$$

and $\sigma\colon \mathfrak{I}_\nu \to I$ is defined by

$$\sigma(V) = \begin{cases} 1, & V = \underline{0} \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise}, \end{cases}$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F\colon (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$\begin{aligned} F(x,s) &= 0.5, & F(x,t) &= 0, \\ F(y,s) &= 0, & F(y,t) &= 0.5, \\ F(z,s) &= 0, & F(z,t) &= 0.1. \end{aligned}$$

It is clear that F is not injective. Let $\alpha = \beta = 0.5$. We observe that $F^{\leftarrow}(\underline{0}) = \underline{0}$ and $F^{\leftarrow}(\nu) = \mu$, $F^{\leftarrow}(V_1) = \mu$. Hence, F is (α, β)-weakly smooth fuzzy continuous. Next, we claim that μ is α-fuzzy μ-connected. Let $\mu = C_1 \vee C_2$, where $C_1, C_2 \in \mathfrak{I}_\mu \setminus \{\underline{0}\}$.

Case 1. $C_i = \mu$ for some $i \in \{1, 2\}$ and $C_j \neq \underline{0}$ for $j \in \{1, 2\} \setminus \{i\}$.

Since $\mathrm{Cl}(C_i, \alpha) = \mu$ and $\mathrm{Cl}(C_j, \alpha) \neq \underline{0}$ we get $C_1 \; q \; \mathrm{Cl}(C_2, \alpha)[\mu]$ and $C_2 \; q \; \mathrm{Cl}(C_1, \alpha)[\mu]$.

Case 2. $\underline{0} \nleqslant C_i \nleqslant \mu$ for each $i \in \{1, 2\}$.

In this case, $\mu = C_1 \vee C_2$ implies that there exist two points $a, b \in X$ such that $C_1(a) = \mu(a)$ and $C_2(b) = \mu(b)$. Since $C_i \neq \underline{0}$, we have $\mathrm{Cl}(C_i, \alpha) = \mu$, $i = 1, 2$. Therefore, $C_1 \; q \; \mathrm{Cl}(C_2, \alpha)[\mu]$ and $C_2 \; q \; \mathrm{Cl}(C_1, \alpha)[\mu]$. Hence, μ is α-fuzzy μ-connected.

Next, we claim that $(F^{\rightarrow}(\mu))^{[0.5,0.5]}_{[s,t]}$ is not β-fuzzy ν-connected. We write $F^{\rightarrow}(\mu) = A \vee B$, where $A^{[0.5,0]}_{[s,t]}$ and $B^{[0,0.5]}_{[s,t]}$. Since $\mathrm{Cl}(A, \beta) = \mathrm{Cl}(B, \beta) = (\nu - V_1)^{[0.5,0.5]}_{[s,t]}$, we have $A \not{q} \; \mathrm{Cl}(B, \beta)[\nu]$ and $B \not{q} \; \mathrm{Cl}(A, \beta)[\nu]$. Hence, $F^{\rightarrow}(\mu)$ is not β-fuzzy ν-connected.

The following example shows that the product of α-fuzzy connected spaces need not be α-fuzzy connected.

Counterexample 4.24. Let $X = \{x, y\}$ and $S = \{s, t\}$. Define $\mu^{[0.7,0.6]}_{[x,y]} \in I^X$, $\nu^{[0.8,0.9]}_{[s,t]} \in I^S$, $V_1, V_2, V_3 \leqslant \nu$ by $V^{[0.7,0]}_{1[s,t]}$, $V^{[0,0.7]}_{2[s,t]}$, $V^{[0.7,0.7]}_{3[s,t]}$. Define τ on μ and σ on ν by

$$\tau(U) = \begin{cases} 1, & U = \underline{0} \text{ or } \mu, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V)=\begin{cases}1, & V=\underline{0} \text{ or } \nu,\\ 0.6, & V=V_1,\\ 0.5, & V=V_2,\\ 0.5, & V=V_3,\\ 0, & \text{otherwise.}\end{cases}$$

Assume that $\mu = A \vee B$, where $A, B \in \mathfrak{I}_\mu \setminus \{\underline{0}\}$. Fix $\alpha = 0.5$. Since $\mathrm{Cl}(A, 0.5) = \mathrm{Cl}(B, 0.5) = \mu$, we get $A\ q\ \mathrm{Cl}(B, 0.5)[\mu]$ and $B\ q\ \mathrm{Cl}(A, 0.5)[\mu]$. Thus, μ is α-fuzzy μ-connected. To prove that ν is α-fuzzy ν-connected, let $\nu = W_1 \vee W_2$, where $W_1, W_2 \in \mathfrak{I}_\nu \setminus \{\underline{0}\}$. First, we observe that if $K \in \mathfrak{I}_\nu$ is such that $\sigma(\nu - K) > 0$, then K is non-zero at both points of S. Hence, for every non-zero fuzzy subset W of ν, $\mathrm{Cl}(W, 0.5)$ is non-zero at both points of S.

Case 1. $W_1(s) = 0.8$, $W_1(t) = 0.9$, $0 \leqslant W_2(s) \leqslant 0.8$ and $0 \leqslant W_2(t) \leqslant 0.9$.

Since $W_1 = \nu = \mathrm{Cl}(W_1, 0.5)$ and $\underline{0} \neq W_2 \leqslant \mathrm{Cl}(W_2, 0.5)$, we get $W_2\ q\ \mathrm{Cl}(W_1, 0.5)[\nu]$ and $\mathrm{Cl}(W_2, 0.5)\ q\ W_1[\nu]$.

Case 2. $W_1(s) = 0.8$, $W_2(t) = 0.9$, $0 \leqslant W_1(t) \leqslant 0.9$ and $0 \leqslant W_2(s) \leqslant 0.8$.

$W_1 \neq \underline{0}$ implies that $\mathrm{Cl}(W_1, 0.5) \neq \underline{0}$. Therefore, $W_2(t) + \mathrm{Cl}(W_1, 0.5)(t) > \nu(t)$. Similarly, since $\mathrm{Cl}(W_2, 0.5) \neq \underline{0}$, we get $W_1(s)+\mathrm{Cl}(W_2, 0.5)(s) > \nu(s)$. The remaining cases can be verified by interchanging W_1 and W_2 in both Case 1 and Case 2. Hence, ν is α-fuzzy connected.

Recall that the product topology on $\mu \times \nu$ is given by

$$\varrho(U)=\begin{cases}1, & U=\underline{0} \text{ or } \mu\times\nu,\\ 0.6, & U=\mu\times V_1,\\ 0.5, & U=\mu\times V_2,\\ 0, & \text{otherwise.}\end{cases}$$

Now, we write $\mu \times \nu = (\mu \times V_1) \vee (\mu \times V_2)$. Here $\mathrm{Cl}(\mu \times V_1, 0.5) = \mu \times V_1$, $\mathrm{Cl}(\mu \times V_2, 0.5) = \mu \times V_2$ and hence, $\mu \times V_1 + \mathrm{Cl}(\mu \times V_2, 0.5) = \mu \times \nu$, $\mu \times V_2 + \mathrm{Cl}(\mu \times V_1, 0.5) = \mu \times \nu$. Hence, $(\mu \times V_1) \not{q}\ \mathrm{Cl}(\mu \times V_2, 0.5)[\mu \times \nu]$ and $(\mu \times V_2) \not{q}\ \mathrm{Cl}(\mu \times V_1, 0.5)[\mu \times \nu]$. Hence, $\mu \times \nu$ is not α-fuzzy connected.

5. Conclusion

In this article we have properly generalized the concept of α-weak smooth fuzzy continuity [25] of a fuzzy proper function to (α,β)-weakly smooth fuzzy continuity, in the framework of smooth fuzzy topological spaces. We proved that weak smooth fuzzy continuity of a fuzzy proper function $F\colon(\mu,\tau)\to(\nu,\sigma)$ implies $F^{\leftarrow}(V^{\circ})\leqslant(F^{\leftarrow}(V))^{\circ}$, $\forall V\leqslant\nu$, and the converse is not true. However, by imposing a condition, namely, positive minimum smooth topology on the domain of F, the converse follows. At the same time, (α,β)-weak smooth fuzzy continuity is equivalent to $F^{\leftarrow}((V,\beta)^{\circ})\leqslant(F^{\leftarrow}(V),\alpha)^{\circ}$, $\forall V\leqslant\nu$. But weak smooth fuzzy continuity ((α,β)-weak smooth fuzzy continuity) neither implies nor is implied by $F^{\rightarrow}(\mathrm{Cl}(A))\leqslant\mathrm{Cl}(F^{\rightarrow}(A))$, $\forall A\leqslant\mu$ ($F^{\rightarrow}(\mathrm{Cl}(A,\alpha))\leqslant\mathrm{Cl}(F^{\rightarrow}(A),\beta)$, $\forall A\leqslant\mu$). As one of the main results, we proved that a fuzzy proper function F is (α,β)-weakly smooth fuzzy continuous on a fuzzy set μ if and only if it is (α,β)-weakly smooth fuzzy continuous at every fuzzy point $P_x^{\lambda}\in\mu$.

Regarding connectedness in smooth fuzzy topological spaces, we observed that the product of two connected spaces need not be connected, with respect to any of the three notions of connectedness introduced in [2], [5], [20]. As a better version of "*The image of a smooth connected space under a smooth fuzzy continuous, injective fuzzy proper function is connected*" (Theorem 6.4 of [5]), we proved that the image of a smooth fuzzy connected space is connected under a weakly smooth fuzzy continuous function. We also proved that the image of a smooth Q-connected space need not be smooth Q-connected under a smooth continuous injective fuzzy proper function, whereas the image of an α-fuzzy μ-connected fuzzy set under an (α,β)-weakly smooth fuzzy continuous injective fuzzy proper function is β-fuzzy $F^{\rightarrow}(\mu)$-connected.

Therefore, many classical results in crisp topological spaces do not hold in the context of smooth fuzzy topological spaces. Hence, there is a good scope for further research in this area. To be specific, one can further extend the work presented in this paper to the more general setting of L-fuzzy topological spaces [16], $(L;M)$-fuzzy topological spaces [19], or M-fuzzy topological L-fuzzy spaces [11] (which incorporate smooth fuzzy topological spaces of this paper, but not their respective smooth fuzzy continuous functions), where L, M are lattices with convenient properties. The next step would be then the variable-basis approach in the sense of S. E. Rodabaugh [22], where both L and M can vary through a category of certain lattice-theoretic structures.

Acknowledgement. The authors express their deep gratitude to the referee for his/her valuable comments towards the improvement of the paper.

References

[1] *H. Aygün, S. E. Abbas*: Some good extensions of Šostak's L-fuzzy topology. Hacet. J. Math. Stat. *36* (2007), 115–125.

[2] *S. E. Abbas*: On smooth fuzzy subspaces. Int. J. Math. Math. Sci. *66* (2004), 3587–3602.

[3] *M. K. Chakraborty, T. M. G. Ahsanullah*: Fuzzy topology on fuzzy sets and tolerance topology. Fuzzy Sets Syst. *45* (1992), 103–108.

[4] *C. L. Chang*: Fuzzy topological spaces. J. Math. Anal. Appl. *24* (1968), 182–189.

[5] *A. K. Chaudhuri, P. Das*: Some results on fuzzy topology on fuzzy sets. Fuzzy Sets Syst. *56* (1993), 331–336.

[6] *M. Demirici*: On several types of compactness in smooth topological spaces. Fuzzy Sets Syst. *90* (1997), 83–88.

[7] *M. Demirici*: Neighborhood structures of smooth topological spaces. Fuzzy Sets Syst. *92* (1997), 123–128.

[8] *M. Demirici*: Three topological structures of smooth topological spaces. Fuzzy Sets Syst. *101* (1999), 185–190.

[9] *M. K. El Gayyar, E. E. Kerre*: Almost compactness and near compactness in smooth topological spaces. Fuzzy Sets Syst. *62* (1994), 193–202.

[10] *M. A. Fath Alla, F. S. Mahmoud*: Fuzzy topology on fuzzy sets, functions with fuzzy closed graphs, strong fuzzy closed graphs. J. Fuzzy Math. *9* (2001), 525–533.

[11] *C. Guido*: Powerset operators based approach to fuzzy topologies on fuzzy sets, topological and algebraic structures in fuzzy sets. A handbook of recent developments in the Mathematics of fuzzy sets (S. E. Rodabaugh and E. P. Klement, eds.). Trends Log. Stud. Log. Libr., vol. 20, Kluwer Academic Publishers, Dordrecht, 2003, pp. 401–413.

[12] *R. N. Hazra, S. K. Samanta, K. C. Chattopadhyay*: Fuzzy topology redefined. Fuzzy Sets Syst. *45* (1992), 79–82.

[13] *U. Höhle*: Upper semicontinuous fuzzy sets and applications. J. Math. Anal. Appl. *78* (1980), 659–673.

[14] *U. Höhle, S. E. Rodabaugh (eds.)*: Mathematics of fuzzy Sets: Logic, Topology, and Measure Theory. Kluwer Academic Publishers, Dordrecht, 1999.

[15] *U. Höhle, A. P. Šostak*: A general theory of fuzzy topological spaces. Fuzzy Sets Syst. *73* (1995), 131–149.

[16] *U. Höhle, A. P. Šostak*: Axiomatic foundations of fixed-basis fuzzy topology. Mathematics of fuzzy Sets: Logic, Topology, and Measure Theory (U. Höhle, S. E. Rodabaugh, eds.). Kluwer Academic Publishers, Dordrecht, 1999, pp. 123–272.

[17] *T. Kubiak*: On fuzzy topologies. Ph.D. thesis, Adam Mickiewicz University. Poznań, Poland, 1985.

[18] *T. Kubiak, A. P. Šostak*: Lower set-valued fuzzy topologies. Quaestiones Math. *20* (1997), 423–429.

[19] *T. Kubiak, A. P. Šostak*: Foundations of the theory of (L, M)-fuzzy topological spaces. Abstracts of the 30th Linz Seminar on Fuzzy Set Theory (U. Bodenhofer, B. De Baets, E. P. Klement, S. Saminger-Platz, eds.). Johannes Kepler Universität, Linz, 2009, pp. 70–73.

[20] *P. P. Ming, L. Y. Ming*: Fuzzy topology, I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. J. Math. Anal. Appl. *76* (1980), 571–599.

[21] *W. Peeters*: Subspaces of smooth fuzzy topologies and initial smooth fuzzy structures. Fuzzy Sets Syst. *104* (1999), 423–433.

[22] *S. E. Rodabaugh*: Categorical foundations of variable-basis fuzzy topology. Mathematics of fuzzy Sets: Logic, Topology, and Measure Theory (U. Höhle, S. E. Rodabaugh, eds.). Kluwer Academic Publishers, Dordrecht, 1999, pp. 273–388.

[23] *S. E. Rodabaugh*: Powerset operator foundations for Poslat fuzzy set theories and topologies. Mathematics of fuzzy Sets: Logic, Topology, and Measure Theory (U. Höhle, S. E. Rodabaugh, eds.). Kluwer Academic Publishers, Dordrecht, 1999, pp. 91–116.
[24] *A. A. Ramadan, M. A. Fath Alla, S. E. Abbas*: Smooth fuzzy topology on fuzzy sets. J. Fuzzy Math. *10* (2002), 59–68.
[25] *R. Roopkumar, C. Kalaivani*: Continuity of fuzzy proper function on Šostak's I-fuzzy topological spaces. Commun. Korean Math. Soc. *26* (2011), 305–320.
[26] *A. P. Šostak*: On a fuzzy topological structure. Rend. Circ. Matem. Palermo Ser. II. *11* (1985), 89–103.
[27] *R. Srivastava*: On separation axioms in a newly defined fuzzy topology. Fuzzy Sets Syst. *62* (1994), 341–346.

Authors' addresses: *R. Roopkumar, C. Kalaivani*, Alagappa University, Department of Mathematics, Karaikudi-630003, India, e-mail: roopkumarr@rediffmail.com, kalai_posa@yahoo.co.in.