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ON THE LOCALIZATION OF THE SPECTRUM FOR QUASI-SELFADJOINT EXTENSIONS OF A CARLEMAN OPERATOR

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Abstract. In the present work, using a formula describing all scalar spectral functions of a Carleman operator A of defect indices $(1,1)$ in the Hilbert space $L^2(X,\mu)$ that we obtained in a previous paper, we derive certain results concerning the localization of the spectrum of quasi-selfadjoint extensions of the operator A.

Keywords: defect indices, integral operator, quasi-selfadjoint extension, spectral theory

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1. Preliminaries

Let A be a closed symmetric operator with a dense domain $D(A)$ in a separable Hilbert space H endowed with an inner product $(\cdot,\cdot)$.

Let $\mathfrak{M}_\lambda$ denote the range of the operator $(A-\lambda I)$, then its orthogonal complement in H

$$\mathfrak{N}_{\overline{\lambda}} = H \ominus \mathfrak{M}_\lambda$$

coincides with the eigenspace corresponding to the eigenvalue $\overline{\lambda}$ of the operator A^*. The sets $D(A)$, $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$ $(\operatorname{Im}\lambda \neq 0)$ are linearly independent, hence according to von Neumann (see [1], [11]), the domain of the adjoint operator A^* admits the representation

$$D(A^*) = D(A) \oplus \mathfrak{N}_\lambda \oplus \mathfrak{N}_{\overline{\lambda}}, \tag{1.1}$$

and

$$A^* f = Af_0 + \lambda\varphi_\lambda + \overline{\lambda}\varphi_{\overline{\lambda}} \tag{1.2}$$

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with $f_0 \in D(A)$, $\varphi_\lambda \in \mathfrak{N}_\lambda$ and $\varphi_{\overline{\lambda}} \in \mathfrak{N}_{\overline{\lambda}}$. The numbers $m = \dim \mathfrak{N}_\lambda$ and $n = \dim \mathfrak{N}_{\overline{\lambda}}$ do not change when λ belongs to the half-plane $\operatorname{Im} \lambda > 0$. Then A is said to be of defect indices (m, n). The formulas (1.1) and (1.2) show that A is selfadjoint iff it is of defect indices $(0, 0)$.

Further, let M and $\widetilde{M}$ be two subspaces of H such that $M \subset \widetilde{M}$. The number n is called the dimension of $\widetilde{M}$ modulo M (denoted $\dim_M \widetilde{M}$, i.e. $\dim \widetilde{M} = n \pmod{M}$) if there is n, and no more than n vectors $f_1, f_2, \ldots, f_n$ in $\widetilde{M}$ such that

$$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n \in M$$

implies that

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

A quasi-selfadjoint extension of A of defect indices (m, m) $(m < \infty)$ is an arbitrary linear operator B which satisfies the conditions

$$A \subset B \subset A^*,$$
$$\dim D(B) = m \pmod{D(A)}$$

but is not a selfadjoint extension of the operator A.

For simplicity we restrict ourselves to the case of operators of defect indices $(1, 1)$. We shall assume that the operator A is simple (i.e. there exists no subspace invariant under A such that the restriction of A to this subspace is selfadjoint).

We recall that a number λ is called a regular point of the operator A if the operator $(A - \lambda I)^{-1}$ (I denotes the identity operator in H) exists, is bounded, and is defined in the whole space. The spectrum of the operator A is defined as the complement of the set of its regular points. In ([1], Appendix I, Section 5), it is proved that the spectrum of a quasi-selfadjoint extension B of a simple symmetric operator A of defect indices $(1, 1)$ consists of the spectral kernel (i.e., the complement of the set of all points of regular type) of A and the eigenvalues, and the set of the eigenvalues lies wholly either in the upper or in the lower half-plane.

2. Carleman operators of second class

One can find necessary information about Carleman operators, for example, in [7], [13], [18], [19], [20]. Let X be an arbitrary set, μ a σ-finite measure on X (μ is defined on a σ-algebra of subsets of X, we do not indicate this σ-algebra), $L^2(X, \mu)$ the Hilbert space of square integrable functions with respect to μ. For short, instead of writing 'μ-measurable', 'μ-almost everywhere' and '$\mathrm{d}\mu(x)$' we write 'measurable', 'a.e.' and '$\mathrm{d}x$'.

A linear operator $A\colon\ D(A) \longrightarrow L^2(X,\mu)$, where the domain $D(A)$ is a dense linear manifold in $L^2(X,\mu)$, is said to be integral if there exists a measurable function K on $X \times X$, a kernel, such that, for every $f \in D(A)$,

$$Af(x) = \int_X K(x,y)f(y)\,\mathrm{d}y \quad \text{a.e.} \tag{2.1}$$

A kernel K on $X \times X$ is a Carleman kernel if $K(x,y) \in L^2(X,\mu)$ for almost every fixed x, that is to say

$$\int_X |K(x,y)|^2\,\mathrm{d}y < \infty \quad \text{a.e.}$$

The integral operator A defined by (2.1) is called a Carleman operator if K is a Carleman kernel. Since the closure of a Carleman operator always exists and is itself a Carleman operator [20], we can suppose also that A is closed.

Now we consider the Carleman integral operators (2.1) of second class that were introduced in [7], [3] generated by symmetric kernels of the form

$$K(x,y) = \sum_{p=0}^{\infty} a_p \psi_p(x)\overline{\psi_p(y)},$$

where the overbar denotes complex conjugation. Here $\{\psi_p(x)\}_{p=0}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ such that

$$\sum_{p=0}^{\infty} |\psi_p(x)|^2 < \infty \quad \text{a.e.},$$

and $\{a_p\}_{p=0}^{\infty}$ is a real number sequence verifying

$$\sum_{p=0}^{\infty} a_p^2|\psi_p(x)|^2 < \infty \quad \text{a.e.}$$

We call $\{\psi_p(x)\}_{p=0}^{\infty}$ a Carleman sequence (we refer for instance to [20], Section 6.2).

Moreover, we assume that there exists a number sequence $\{\gamma_p\}_{p=0}^{\infty} \neq 0$ such that

$$\sum_{p=0}^{\infty} \gamma_p \psi_p(x) = 0 \quad \text{a.e.} \tag{2.2}$$

and

$$\sum_{p=0}^{\infty} \left|\frac{\gamma_p}{a_p - \lambda}\right|^2 < \infty. \tag{2.3}$$

Under the conditions (2.2) and (2.3), the symmetric operator $A=(A^*)^*$ is of defect indices $(1,1)$ (see [3]) with

$$A^*f(x)=\sum_{p=0}^{\infty}a_p(f,\psi_p)\psi_p(x),$$
$$D(A^*)=\Big\{f\in L^2(X,\mu)\colon \sum_{p=0}^{\infty}a_p(f,\psi_p)\psi_p(x)\in L^2(X,\mu)\Big\}.$$

Moreover, in [4], we saw that

$$\begin{cases}\varphi_\lambda(x)=\displaystyle\sum_{p=0}^{\infty}\frac{\gamma_p}{a_p-\lambda}\psi_p(x)\in\mathfrak{N}_{\overline{\lambda}}, & \lambda\in\mathbb{C},\ \lambda\neq a_k,\ k=1,2,\dots,\\ \varphi_{a_k}(x)=\psi_k(x)\end{cases}$$

with $\mathfrak{N}_{\overline{\lambda}}$ the defect space of A.

We denote by $\mathfrak{L}_\psi$ the sub-space of $L^2(X,\mu)$ generated by the sequence $\{\psi_p(x)\}_{p=0}^{\infty}$. It is clear that the orthogonal complement $\mathfrak{L}_\psi^\perp=L^2(X,\mu)\ominus\mathfrak{L}_\psi$ is contained in $D(A)$ and cancels the operator A. As $\mathfrak{L}_\psi$ is reduced by A (see [1]), we consider A on $\mathfrak{L}_\psi$. Then we have (see [4]) for all $f\in\mathfrak{L}_\psi$ and for almost all $x\in X$:

$$f(x)=\int_{-\infty}^{+\infty}\frac{(f,\varphi_\sigma)\varphi_\sigma(x)}{(\sigma^2+1)|(\varphi_\sigma,\mathring{\varphi}_{\rm i})|}\,{\rm d}\varrho(\sigma),\tag{2.4}$$

$$\|f\|^2=\int_{-\infty}^{+\infty}\frac{|(f,\varphi_\sigma)|^2}{(\sigma^2+1)|(\varphi_\sigma,\mathring{\varphi}_{\rm i})|}\,{\rm d}\varrho(\sigma),\tag{2.5}$$

with

$$\mathring{\varphi}_{\rm i}=\frac{\varphi_{\rm i}}{\|\varphi_{\rm i}\|}(\varphi_{\rm i}\in\mathfrak{N}_{-{\rm i}}),$$

and

$$\varrho(\sigma)=\frac{1}{\pi}\lim_{\tau\to+0}\int_0^\sigma\Re\frac{1+\omega(t+{\rm i}\tau)C(t+{\rm i}\tau)}{1-\omega(t+{\rm i}\tau)C(t+{\rm i}\tau)}\,{\rm d}t,\tag{2.6}$$

were $\omega(\lambda)$ is an analytical function on the upper half-plane Π^+ with $|\omega(\lambda)|\leqslant 1$ $(\operatorname{Im}\lambda>0)$ ($\operatorname{Im}\lambda$ the imaginary part of λ) and $C(\lambda)$ is the function

$$C(\lambda)=\frac{[1-\omega(\lambda)](f,\varphi_{\overline{\lambda}})}{[\omega(\lambda)\chi(\lambda)-1](\lambda+{\rm i})(\varphi_\lambda,\varphi_{\rm i})}\quad(\operatorname{Im}\lambda>0)$$

with $\chi(\lambda)$ the characteristic function of A (see [4], [1]).

Let $\mathfrak{P}$ be the set of all functions $\varrho(\sigma)$ defined by (2.6) (see [5]). We call such a function $\varrho(\sigma)$ the scalar spectral function of the operator A. This function characterizes the spectrum of the quasi-selfadjoint extension A_ω of the operator A associated with the analytic function $\omega(\lambda)$. The spectrum of this extension is the set of points of growth of $\varrho(\sigma)$. We recall here (see [5]) that $\varrho(\sigma)$ is called orthogonal scalar spectral function if it corresponds to a constant function $\omega(\lambda)$ with $|\omega| \equiv 1$.

Now let us look more closely at the function $\varrho(\sigma)$ given by (2.6). It is clear that the homographic function $(1+z)/(1-z)$ transforms the circle $|z| = 1$ into the real line $\mathbb{R}$. So if

$$\omega(\lambda) = \varkappa \tag{2.7}$$

with $|\varkappa| = 1$, then

$$\Re \frac{1+\varkappa C(\sigma)}{1-\varkappa C(\sigma)} = 0$$

for all $\sigma \in \mathbb{R}$ except at points σ satisfying

$$1 - \varkappa C(\sigma) = 0.$$

We infer that the function $\varrho(\sigma)$ associated with $\varkappa$ has jumps at points of the spectrum of the selfadjoint extension $A_\varkappa$ associated with $\varkappa$. This spectrum is formed by the zeros of the equation $C(\sigma) = \overline{\varkappa}$.

We denote by $\mathfrak{G}_0$ the convex hull of these functions:

$$\mathfrak{G}_0 = \Big\{ \varrho(\sigma) = \sum_{k=1}^{n} \alpha_k \varrho_{\varkappa_k},\ \alpha_k > 0,\ \sum_{k=1}^{n} \alpha_k = 1 \Big\},$$

and $\mathfrak{G} = \overline{\mathfrak{G}_0}$, for the convergence at each point of continuity.

For any function $\varrho(\sigma) \in \mathfrak{G}_0$ we have:

$$\begin{aligned} \varrho(\sigma) &= \frac{1}{\pi} \lim_{\tau\to+0} \int_0^\sigma \Re\Big[\sum_{k=1}^{n} \alpha_k \frac{1+\varkappa_k C(\lambda)}{1-\varkappa_k C(\lambda)} \Big] \,\mathrm{d}t \\ &= \frac{1}{\pi} \lim_{\tau\to+0} \int_0^\sigma \Re\Big[\frac{1+\omega(\lambda) C(\lambda)}{1-\omega(\lambda) C(\lambda)} \Big] \,\mathrm{d}t \quad (\lambda = \sigma + \mathrm{i}\tau) \end{aligned} \tag{2.8}$$

where $\omega(\lambda)$ is the analytical function corresponding to $\varrho(\sigma) \in \mathfrak{P}$.

Let now $\mathfrak{M}$ be the set of all analytic functions $\varphi(z)$ on the unit disc $K = \{z \in C\colon\ |z| < 1\}$ satisfying, $|\varphi(z)| \leqslant 1$, $z \in K$ and admitting the representation

$$\varphi(z) = \frac{\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}(1 - z\mathrm{e}^{\mathrm{i}t})^{-1} \,\mathrm{d}S(t)}{\int_0^{2\pi} (1 - z\mathrm{e}^{\mathrm{i}t})^{-1} \,\mathrm{d}S(t)} \tag{2.9}$$

where $S(t)$ is a monotonic nondecreasing function with total variation equal to one, i.e. $\int_0^{2\pi} \mathrm{d}S(t) = 1$. We denote by $\mathfrak{M}_0$ the set of all functions $\varphi(z) \in \mathfrak{M}$ with $S(t)$ a step function with a finite number of jumps. Consequently, from (2.8) and (2.9), we find easily that

$$\begin{aligned} \omega(\lambda) &= \frac{\sum_{k=1}^{n} \alpha_k \varkappa_k (1 - \varkappa_k C(\lambda))^{-1}}{\sum_{k=1}^{n} (1 - \varkappa_k C(\lambda))^{-1}} \\ &= \frac{\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}t} (1 - C(\lambda)\mathrm{e}^{\mathrm{i}t})^{-1} \,\mathrm{d}S(t)}{\int_{-\infty}^{+\infty} (1 - C(\lambda)\mathrm{e}^{\mathrm{i}t})^{-1} \,\mathrm{d}S(t)} = \varphi(C(\lambda)), \end{aligned} \tag{2.10}$$

with $\varphi(z) \in \mathfrak{M}_0$.

3. Description of the spectrum of quasi-selfadjoint extensions of a Carleman operator

In this section we will study the spectrum of the quasi-selfadjoint extension A_ω of the Carleman operator A which equals the set of all points of growth of its spectral scalar function $\varrho(t) \in \mathfrak{G}$ (see [1], [19]).We recall ([5], Theorem 2.1) that for all $\varrho_\omega(t) = \varrho(t) \in \mathfrak{G}$ there corresponds an analytic function $\omega(\lambda) = \varphi(C(\lambda))$ with $\varphi(z) \in \mathfrak{M}$.

In the previous section we have observed that the spectrum of a selfadjoint extension $A_\varkappa$ of the Carleman operator A associated with $\varkappa$ ($|\varkappa| = 1$) coincides with the set of all solutions of the equation

$$C(\sigma) = \overline{\varkappa}. \tag{3.1}$$

Let $\Delta_p = [a_p, a_{i(p)}]$ ($p = 1, 2, \ldots$) be the interval of the real line $\mathbb{R}$ such that a_p and $a_{i(p)}$ be consecutive (i.e., exist no other a_k between a_p and $a_{i(p)}$). The characteristic function $C(\lambda)$ applies to each interval, namely, for every p, k and $\zeta \in \Delta_p$ there exists a unique $\eta \in \Delta_k$ such that

$$C(\zeta) = C(\eta).$$

We denote by Γ the spectrum of the quasi selfadjoint extension A_ω of the Carleman operator A whose scalar spectral function is

$$\varrho_\omega(t) = \varrho(t).$$

Theorem 1. (1) *If $\varrho(t) \in \mathfrak{G}_0$, then for all p ($p = 1, 2, \ldots$) Γ contains only a finite number n of points in each interval Δ_p, i.e.*

$$\Gamma \cap \Delta_p = \{\sigma_p^1, \sigma_p^2, \ldots, \sigma_p^n\}.$$

(2) *If $\varrho(t) \in \mathfrak{G}$, we have for all p ($p = 1, 2, \ldots$)*

$$\{z\colon\ z = \mathrm{e}^{\mathrm{i}t},\ t \in \Gamma \cap \Delta_p\} = \{z\colon\ z = \mathrm{e}^{\mathrm{i}t},\ t \in \Gamma\}. \tag{3.2}$$

If $\varrho(t) \in \mathfrak{G}$, we have for all p ($p = 1, 2, \ldots$)

$$\{z\colon\ z = \mathrm{e}^{\mathrm{i}t},\ t \in \Gamma \cap \Delta_p\} = \{z\colon\ z = \mathrm{e}^{\mathrm{i}t},\ t \in \Gamma\}. \tag{3.3}$$

Proof. Let $\varrho(t) \in \mathfrak{G}_0$. Then $\omega(t)$ associated with $\varrho(t)$ is the rational function (2.10). Therefore the equation

$$\omega(\lambda)C(\lambda) = 1$$

admits only n solutions in each interval $\Delta_p = [a_p, a_{i(p)}]$, ($p = 1, 2, \ldots$). Indeed, as noted earlier in this section, for each p, q and $\sigma_q \in \Delta_q$, there is a single point $\sigma_p^q \in \Delta_p$ such that

$$C(\sigma_p^q) = C(\sigma_q).$$

By applying the function φ to this equality we obtain, using (2.10), that

$$\omega(\sigma_p^q) = \omega(\sigma_q).$$

By (2.7), we have also

$$\omega(\sigma_q) = \varkappa_q.$$

Hence

$$\omega(\sigma_p^q) = \varkappa_q.$$

Now by the equality (3.1) it follows that

$$C(\sigma_p^q) = \overline{\varkappa_q}.$$

Then

$$\omega(\sigma_p^q)C(\sigma_p^q) = |\varkappa_q|^2 = 1 \quad (q = 1, 2, \ldots, n;\ p = 1, 2, \ldots),$$

and so

$$\Gamma \cap \Delta_p = \{\sigma_p^1, \sigma_p^2, \ldots, \sigma_p^n\}.$$

This proves the first assumption. To see the second point, we argue as follows.

- ▷ First, if $\varrho(t) \in \mathfrak{G}_0$, then equality (3.3) follows from the bijection established by the characteristic function $C(t)$ between Δ_p's.
- ▷ Now if $\varrho(t) \in \mathfrak{G}$ and $\varrho(t) \notin \mathfrak{G}_0$, then there is a sequence of scalar spectral functions $\varrho_n(t) \in \mathfrak{G}_0$ wich converges to $\varrho(t)$. Since equality (3.3) is true for $\varrho_n(t)$ for any n, it is also true for $\varrho(t)$. □

Theorem 2. *Let E be a closed set contained in the interval $\Delta_p = [a_p, a_{i(p)}]$. Then there is $\varrho(t) \in \mathfrak{G}$ such that the spectrum Γ of the quasi-selfadjoint extension A_ω of the Carleman operator A having $\varrho(t)$ as the scalar spectral function satisfies the equality*

$$\Gamma \cap \Delta_p = E.$$

P r o o f. We choose a countable set

$$\Omega = \{\sigma_p^1, \sigma_p^2, \ldots\} \subset \Delta_p,$$

dense in E. It is clear that if we denote

$$\Omega_n = \{\sigma_p^1, \sigma_p^2, \ldots, \sigma_p^n\},$$

then

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

Let

$$C(\sigma_p^k) = \overline{\varkappa_k} \quad (k = 1, 2, \ldots, n),$$

and for all n $(n = 1, 2, \ldots)$, let us form the spectral function by setting

$$\varrho_n(t) = \sum_{k=1}^{n-1} \frac{1}{2^k} \varrho_{\varkappa_k}(t) + \frac{1}{2^{n-1}} \varrho_{\varkappa_n}(t),$$

where $\varrho_{\varkappa_k}(t)$ denotes the orthogonal spectral function associated with $\varkappa_k$ $(k = 1, 2, \ldots, n)$.

Clearly, $\varrho_n(t) \in \mathfrak{G}_0$. We will show that $\varrho_n(t)$ converges pointwise as n tends to ∞. We start by introducing the function

$$S_n(t) = \int_{-\infty}^{t} \frac{\mathrm{d}\varrho_n(\sigma)}{\sigma^2 + 1}.$$

According to the formula (2.5) $S_n(t)$ is a distribution function, i.e.,

$$S_n(+\infty) = \lim_{t \to +\infty} S_n(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\varrho_n(\sigma)}{\sigma^2 + 1} = \|\mathring{\varphi}_{\mathrm{i}}\|^2 = 1$$

and

$$S_n(-\infty) = \lim_{t \to -\infty} S_n(t) = 0.$$

Since

$$S_n(t) = \sum_{k=1}^{n-1} \frac{1}{2^k} S_{\varkappa_k}(t) + \frac{1}{2^{n-1}} S_{\varkappa_n}(t),$$

we have

$$\begin{aligned} |S_{n+n_0}(t) - S_n(t)| &= \left| \sum_{k=n}^{n+n_0-1} \frac{1}{2^k} S_{\varkappa_k}(t) + \frac{1}{2^{n+n_0-1}} S_{\varkappa_{n+n_0}}(t) - \frac{1}{2^{n-1}} S_{\varkappa_n}(t) \right| \\ &\leqslant \sum_{k=n}^{n+n_0-1} \frac{1}{2^k} + \frac{1}{2^{n+n_0-1}} - \frac{1}{2^{n-1}}. \end{aligned}$$

It is clear that this quantity tends to 0 as n tends to ∞. Therefore, at each point t, $S_n(t)$ converges to a limit, denoted by $S(t)$.

Thus $\varrho_n(t)$ converges to $\varrho(t)$ as n tends to ∞ and

$$S(t) = \int_{-\infty}^{t} \frac{\mathrm{d}\varrho(\sigma)}{\sigma^2 + 1}.$$

The spectrum of $\varrho_n(t)$ is $\{\sigma_\varrho^1, \sigma_\varrho^2, \ldots, \sigma_\varrho^n\} = \Omega_n$, consequently the spectrum of $\varrho(t)$ is $\overline{\Omega} = E$. □

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