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On Decomposable Almost Pseudo Conharmonically Symmetric Manifolds

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Abstract

The object of the present paper is to study decomposable almost pseudo conharmonically symmetric manifolds.

Key words: almost pseudo conharmonically symmetric manifold, decomposable manifold, scalar curvature, torse-forming vector field

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1 Introduction

Let (M^n, g) be a Riemannian manifold of dimension $n, (n \geq 3)$. It is well know that the conformal transformations of (M^n, g) do not change the angle between two vectors at a point. But a harmonic function, which is defined by a vanishing Laplacian, is not transformed into a harmonic function by the conformal transformation in general. The condition under which a harmonic function remains invariant was studied by Ishii [11] who introduced the conharmonic transformation as a subgroup of conformal transformation satisfying a special differential equation and defined the conharmonic curvature tensor a geometrical invariant under conharmonic transformation.

The conharmonic curvature K of type $(0, 4)$ of a Riemannian manifold (M^n, g) $(n > 3)$ is given by [11].

$$\begin{aligned} K(Y,Z,U,V) = {} & R(Y,Z,U,V) - \frac{1}{n-2}\,[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) \\ & + S(Y,V)g(Z,U) - S(Z,V)g(Y,U)] \end{aligned} \tag{1}$$

where S is the Ricci tensor of the manifold of type $(0, 2)$. In [17] Shaikh and Hui found out that the conharmonic curvature K satisfies all the symmetries properties of the Riemannian curvature tensor R. The conharmonic tensor K has many applications in physical field. Abdussattar [1] showed its phsical significance in general relativity. This tensor has also been studied by Siddiqui and Ahsan [18], Ghosh, De and Taleshian [10] and many others.

A non-flat Riemannian manifold (M^n, g) $(n \geq 2)$ is called *an almost pseudo symmetric manifold* whose curvature tensor R of type $(0,4)$ satisfies the condition [6]

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & [A(X)+B(X)]\, R(Y,Z,U,V) \\ & + A(Y)R(X,Z,U,V) + A(Z)R(Y,X,U,V) \\ & + A(U)R(Y,Z,X,V) + A(V)R(Y,Z,U,X) \end{aligned} \tag{2}$$

where A and B are nowhere vanishing 1-forms such that

$$A(X) = g(X,\rho) \quad \text{and} \quad B(X) = g(X,Q) \tag{3}$$

for all X and ρ and Q are the vector fields associated with the 1-forms A and B, respectively. An n-dimensional almost pseudo symmetric manifold has been denoted by $A(PS)_n$. If $A = B$ in (2), then the manifold reduces to a pseudo symmetric manifold $(PS)_n$ introduced by Chaki [3]. It is pointed out that the notion of pseudo symmetry in the sense of Chaki [3] is different from that of Deszcz [8]. Pseudo symmetric spaces, generalized symmetric spaces, were also studied by Mikeš [13, 14]. It is to be noted that the almost pseudo symmetric manifold is not a praticular case of a weakly symmetric manifold $(WS)_n$ introduced by Tamássy and Binh [20].

The notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta [4], recurrent manifolds introduced by Walker [21], conformally recurrent manifolds by Adati and Miyazawa [2], projective symmetric manifolds by Soos [19], projective-symmetric and projectively recurrent affinely connected spaces by Mikeš [12], pseudo conformally symmetric spaces by De and Biswas [5], almost pseudo conformally symmetric manifolds by De and Gazi [7], weakly conharmonically symmetric manifolds by Shaikh and Hui [17], etc.

The present paper is concerned with a non-conharmonic flat Riemannian manifold (M^n, g) $(n > 3)$ whose conharmonic curvature tensor K satisfies the condition

$$\begin{aligned}(\nabla_X K)(Y,Z,U,V) = {} & [A(X)+B(X)]\, K(Y,Z,U,V) \\ & + A(Y)K(X,Z,U,V) + A(Z)K(Y,X,U,V) \\ & + A(U)K(Y,Z,X,V) + A(V)K(Y,Z,U,X) \end{aligned} \tag{4}$$

where A and B have the meaning already stated. Such a manifold will be called *an almost pseudo conharmonic symmetric manifold* and denoted by $A(PCHS)_n$. Since the conformal curvature tensor vanishes identically for $n = 3$, we assume the condition $n > 3$ throughout the paper.

The paper is organized as follows: In Section 2, it deals with an $A(PCHS)_n$. In Section 3, it is concerned with a decomposable $A(PCHS)_n$ and exactly defined each decomposition of an $A(PCHS)_n$. In Section 4, it is shown that the integral curves of the unit torse-forming vector field ρ in an Einstein $A(PCHS)_n$ are geodesics. Hence it is also found that the vector field Q is the torse-forming vector field and its the integral curves are geodesics. Finally in Section 5, it is given non-trivial two examples of a decomposable $A(PCHS)_n$.

2 $A(PCHS)_n$

L denotes the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S of type $(0,2)$, that is

$$g(LX,Y)=S(X,Y) \tag{5}$$

Let $\{e_i\}$, $(1\leq i\leq n)$ be an orthonormal basis of the tangent space at any point of the manifold. From (1), we have

$$\begin{aligned} F(Z,U)=\sum_{i=1}^{n}K(Z,e_i,e_i,U)=\sum_{i=1}^{n}K(e_i,Z,U,e_i)=-\frac{r}{n-2}g(Z,U)\\ \sum_{i=1}^{n}K(e_i,e_i,Y,Z)=\sum_{i=1}^{n}K(Y,Z,e_i,e_i)=0 \end{aligned} \tag{6}$$

where r is the scalar curvature of the manifold.

From (4) and (6), it follows that

$$\begin{aligned} (\nabla_X F)(Z,U) &= -\frac{r}{n-2}\{[A(X)+B(X)]\,g(Z,U)\\ &\quad + A(Z)g(X,U)+A(U)g(Z,X)\}\\ &\quad + A\left(R(X,Z)U\right)-\frac{1}{n-2}\{S(Z,U)A(X)-S(X,U)A(Z)\\ &\quad + A(LX)g(Z,U)-A(LZ)g(X,U)\}\\ &\quad + A\left(R(X,U)Z\right)-\frac{1}{n-2}\{S(U,Z)A(X)-S(X,Z)A(U)\\ &\quad + A(LX)g(U,Z)-A(LU)g(X,Z)\} \end{aligned} \tag{7}$$

putting $Z=U=e_i$ in (7) and taking summation over i, $(1\leq i\leq n)$, it follows from (6) that

$$\nabla_X r=r\left\{\left(\frac{n+4}{n}\right)A(X)+B(X)\right\} \tag{8}$$

Hence we can state the following:

Theorem 1 *The scalar curvature r of an $A(PCHS)_n$ is satisfied the following relation:*

$$\nabla_X r=r\left\{\left(\frac{n+4}{n}\right)A(X)+B(X)\right\}\quad \text{for all } X \tag{9}$$

We now suppose that an $A(PCHS)_n$ of non-zero constant scalar curvature. Then, from (9) and $r \neq 0$, we get

$$\left(\frac{n+4}{n}\right) A(X) + B(X) = 0 \quad \text{for all } X \tag{10}$$

Thus we can state the following:

Theorem 2 *The two associated 1-forms in an $A(PCHS)_n$ of non-zero constant scalar curvature are linearly dependent.*

3 The torse-forming vector field ρ

We consider an $A(PCHS)_n$ defined by (4) which is an Einstein manifold. Then its Ricci tensor S satisfies

$$S(Z,U) = \frac{r}{n} g(Z,U) \tag{11}$$

It follows that

$$dr(Z) = 0 \quad \text{and} \quad (\nabla_X S)(Z,U) = 0 \tag{12}$$

We suppose that ρ is a unit torse-forming vector [22] defined by

$$\nabla_X \rho = \lambda X + w(X)\rho \tag{13}$$

where λ is a non-zero scalar and w is a non-zero 1-form, called *the scalar* and *1-form of the vector field* ρ, respectively. Some properties of torse forming vector fields in Riemannian spaces have been studied by Rachunek and Mikeš [15] and various mathematicians.

Now, due to (12), we get

$$(\nabla_X S)(Z,\rho) = 0 \tag{14}$$

Remembering that $(\nabla_X S)(Z,\rho) = \nabla_X S(Z,\rho) - S(\nabla_X Z,\rho) - S(Z,\nabla_X \rho)$ and using (3) and (11), we have

$$\frac{r}{n}(\nabla_X A)(Z) - S(Z,\nabla_X \rho) = 0 \tag{15}$$

Substituting (13) in (15) and using (11), we obtain

$$\frac{r}{n}(\nabla_X A)(Z) - \lambda S(Z,X) - \frac{r}{n} w(X) A(Z) = 0 \tag{16}$$

Putting $Z = \rho$ in (16) and remembering that ρ is a unit vector, thus the equation (16) takes the form

$$(\nabla_X A)(\rho) = \lambda A(X) + w(X) \tag{17}$$

Since ρ is a unit vector, we get

$$(\nabla_X A)(\rho) = -A(\nabla_X \rho) \tag{18}$$

From (13) and (18), it follows that

$$(\nabla_X A)(\rho) = -\lambda A(X) - w(X) \tag{19}$$

From (17) and (19), we get $w(X) = -\lambda A(X)$. It means that

$$\lambda = -w(\rho) \tag{20}$$

Substituting (20) in (13), we obtain

$$\nabla_X \rho = -w(\rho)X + w(X)\rho \tag{21}$$

Hence it follows that $\nabla_\rho \rho = 0$. Therefore we can state the following:

Theorem 3 *If in an Einstein $A(PCHS)_n$ the vector field ρ is a unit torse-forming vector field, then the integral curves of the vector ρ are geodesics.*

Remembering that an Einstein manifold is a constant scalar curvature, so from Theorem 2, two associated 1-forms in an Einstein $A(PCHS)_n$ whose scalar curvature is non-zero are linearly dependent. Also, it follows from (3) and (10) that

$$\rho = -\left(\frac{n}{n+4}\right) Q \tag{22}$$

In virtue of (21) and (22), the vector field Q is also a torse-forming vector field. Thus, from Theorem 3, the integral curves of the vector field Q are also geodesics.

4 Decomposable $A(PCHS)_n$

A Riemannian manifold (M^n, g) is called *decomposable* if it can be expressed as the product $M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, namely, if coordinates can be found so that its metric takes the form [16]

$$ds^2 = g_{ij}dx^i dx^j = \overline{g}_{ab}dx^a dx^b + g^\star_{\alpha\beta}dx^\alpha dx^\beta \tag{23}$$

where $\overline{g}_{ab}$ are functions of $x^1, x^2, \ldots, x^p$ and $g^\star_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \ldots,$ x^n only: $a, b, c, \ldots$ are taken to have range from 1 to p and $\alpha, \beta, \gamma, \ldots$ are taken to have range from $p+1$ to n. The two parts of (23) are the metrics of M_1^p $(p \geq 2)$ and M_2^{n-p} $(n-p \geq 2)$ which are called *the decomposable spaces of* M^n [9].

In virtue of (23), it follows that

$$g_{ab} = \overline{g}_{ab}, \quad g_{\alpha\beta} = g^\star_{\alpha\beta}, \quad g^{ab} = \overline{g}^{ab}, \quad g^{\alpha\beta} = g^{\star\alpha\beta}, \quad g_{a\beta} = g^{a\beta} = 0 \tag{24}$$

Let (M^n, g) now be a decomposable Riemannian manifold such that it is the form: $M^n = M_1^p \times M_2^{n-p}$ $(2 \leq p \leq n-2)$. Throughout the paper each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by a 'star' is assumed to be from M_2.

Let $\overline{X}, \overline{Y}, \overline{Z}, \overline{U}, \overline{V} \in \chi(M_1)$ and $X^\star, Y^\star, Z^\star, U^\star, V^\star \in \chi(M_2)$. In decomposable Riemannian manifold the following relations hold [23]:

$$\begin{aligned}
R(X^\star, \overline{Y}, \overline{Z}, \overline{U}) &= 0 = R(\overline{X}, Y^\star, \overline{Z}, U^\star) = R(\overline{X}, Y^\star, Z^\star, U^\star) \\
(\nabla_{X^\star} R)(\overline{Y}, \overline{Z}, \overline{U}, \overline{V}) &= 0 = (\nabla_{\overline{X}} R)(\overline{Y}, Z^\star, \overline{U}, V^\star) = (\nabla_{X^\star} R)(\overline{Y}, Z^\star, \overline{U}, V^\star) \\
R(\overline{X}, \overline{Y}, \overline{Z}, \overline{U}) &= \overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{U}); \\
R(X^\star, Y^\star, Z^\star, U^\star) &= R^\star(X^\star, Y^\star, Z^\star, U^\star) \\
S(\overline{X}, \overline{Y}) &= \overline{S}(\overline{X}, \overline{Y}); \\
S(X^\star, Y^\star) &= S^\star(X^\star, Y^\star) \\
(\nabla_{\overline{X}} S)(\overline{Y}, \overline{Z}) &= \left(\overline{\nabla}_{\overline{X}} S\right)(\overline{Y}, \overline{Z}); \\
(\nabla_{X^\star} S)(Y^\star, Z^\star) &= (\nabla^\star_{X^\star} S)(Y^\star, Z^\star) \\
r &= \overline{r} + r^\star
\end{aligned} \tag{25}$$

We consider a decomposable $A(PCHS)_n$, which is decomposable M_1^p and M_2^{n-p} $(2 \leq p \leq n-2)$. Then, using (25), it follows from (1) that

$$\begin{aligned}
K(X^\star, \overline{Y}, \overline{Z}, \overline{U}) &= K(\overline{X}, Y^\star, Z^\star, U^\star) = 0 \\
K(X^\star, \overline{Y}, \overline{Z}, U^\star) &= -\frac{1}{n-2}\left[S(\overline{Y}, \overline{Z}) g(X^\star, U^\star) + S(X^\star, U^\star) g(\overline{Y}, \overline{Z})\right]
\end{aligned} \tag{26}$$

from (4), on the manifold M_1 we have

$$\begin{aligned}
(\nabla_{\overline{X}} K)(\overline{Y}, \overline{Z}, \overline{U}, \overline{V}) &= \left[A(\overline{X}) + B(\overline{X})\right] K(\overline{Y}, \overline{Z}, \overline{U}, \overline{V}) \\
&+ A(\overline{Y}) K(\overline{X}, \overline{Z}, \overline{U}, \overline{V}) + A(\overline{Z}) K(\overline{Y}, \overline{X}, \overline{U}, \overline{V}) \\
&+ A(\overline{U}) K(\overline{Y}, \overline{Z}, \overline{X}, \overline{V}) + A(\overline{V}) K(\overline{Y}, \overline{Z}, \overline{U}, \overline{X})
\end{aligned} \tag{27}$$

replacing $\overline{X}$ and $X^\star$ in (27) and using (25) and (26), it follows that

$$\left[A(X^\star) + B(X^\star)\right] K(\overline{Y}, \overline{Z}, \overline{U}, \overline{V}) = 0 \tag{28}$$

Similarly, replasing $\overline{Y}$ and $Y^\star$, we have

$$A(Y^\star) K(\overline{X}, \overline{Z}, \overline{U}, \overline{V}) = 0 \tag{29}$$

putting $\overline{X} = X^\star$ and $\overline{U} = U^\star$ in (27), we get

$$A(\overline{Y}) K(X^\star, \overline{Z}, U^\star, \overline{V}) + A(\overline{Z}) K(\overline{Y}, X^\star, U^\star, \overline{V}) + A(\overline{V}) K(\overline{Y}, \overline{Z}, U^\star, X^\star) = 0 \tag{30}$$

Similarly, putting $\overline{Y} = Y^\star$ and $\overline{V} = V^\star$ in (27), we obtain

$$\begin{aligned}
\left[A(\overline{X}) + B(\overline{X})\right] K(Y^\star, \overline{Z}, \overline{U}, V^\star) &+ A(\overline{Z}) K(Y^\star, \overline{X}, \overline{U}, V^\star) \\
&+ A(\overline{U}) K(Y^\star, \overline{Z}, \overline{X}, V^\star) = 0
\end{aligned} \tag{31}$$

Setting $\overline{X} = X^\star, \overline{Y} = Y^\star$ and $\overline{V} = V^\star$ in (27), we get

$$\begin{aligned}(\nabla_{X^\star} K)(Y^\star, \overline{Z}, \overline{U}, V^\star) &= [A(X^\star) + B(X^\star)]\, K(Y^\star, \overline{Z}, \overline{U}, V^\star)\\ &+ A(Y^\star) K(X^\star, \overline{Z}, \overline{U}, V^\star)\\ &+ A(V^\star) K(Y^\star, \overline{Z}, \overline{U}, X^\star)\end{aligned} \tag{32}$$

In the similar way from (27), we have the following relations:

$$\begin{aligned}A(Z^\star)K(\overline{Y}, \overline{X}, U^\star, V^\star) &+ A(U^\star)K(\overline{Y}, Z^\star, \overline{X}, V^\star)\\ &+ A(V^\star)K(\overline{Y}, Z^\star, U^\star, \overline{X}) = 0\end{aligned} \tag{33}$$

$$\left[A(\overline{X}) + B(\overline{X})\right] K(Y^\star, Z^\star, U^\star, V^\star) = 0 \tag{34}$$

$$A(\overline{Y}) K(X^\star, Z^\star, U^\star, V^\star) = 0 \tag{35}$$

$$\begin{aligned}(\nabla_{X^\star} K)(Y^\star, Z^\star, U^\star, V^\star) &= [A(X^\star) + B(X^\star)]\, K(Y^\star, Z^\star, U^\star, V^\star)\\ &+ A(Y^\star) K(X^\star, Z^\star, U^\star, V^\star)\\ &+ A(Z^\star) K(Y^\star, X^\star, U^\star, V^\star)\\ &+ A(U^\star) K(Y^\star, Z^\star, X^\star, V^\star)\\ &+ A(V^\star) K(Y^\star, Z^\star, U^\star, X^\star)\end{aligned} \tag{36}$$

Thus from (28), (29), (34) and (35) we can state the following:

Theorem 4 *Let an $A(PCHS)_n$ be a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ $(2 \le p \le n-2)$. Then one of the decomposition is conharmonically flat and on the other is $A = B = 0$.*

Let us now deal with each decomposition individually. Let one of the decomposition be conharmonically flat. Then we get

$$K(\overline{Y}, \overline{Z}, \overline{U}, \overline{V}) = 0 \quad \text{for} \quad \overline{Y}, \overline{Z}, \overline{U}, \overline{V} \in \chi(M_1)$$

Using (1), we obtain

$$\begin{aligned}R(\overline{Y}, \overline{Z}, \overline{U}, \overline{V}) = \frac{1}{n-2}\big[&S(\overline{Z}, \overline{U}) g(\overline{Y}, \overline{V}) - S(\overline{Y}, \overline{U}) g(\overline{Z}, \overline{V})\\ &+ S(\overline{Y}, \overline{V}) g(\overline{Z}, \overline{U}) - S(\overline{Z}, \overline{V}) g(\overline{Y}, \overline{U})\big]\end{aligned}$$

contracting over $\overline{Y}$ and $\overline{V}$, we get

$$S(\overline{Z}, \overline{U}) = \frac{\overline{r}}{(n-p)} g(\overline{Z}, \overline{U}) \tag{37}$$

Hence we see that the manifold M_1 is an Einstein manifold. So it is of constant curvature.

Again, taking the contraction over $\overline{Z}$ and $\overline{U}$, we obtain

$$(n-2p)\overline{r} = 0 \tag{38}$$

It implies that either $\overline{r} = 0$ or $p = \frac{n}{2}$.

Let us consider the other decomposition, that is, $A = B = 0$ on M_2. Then from (32), we get

$$(\nabla_{X^\star} K)\,(Y^\star, \overline{Z}, \overline{U}, V^\star) = 0$$

which implies that

$$(\nabla_{X^\star} S)\,(Y^\star, V^\star) = 0 \tag{39}$$

Hence we see that the manifold M_2 is Ricci-symmetric. By virtue of (36), it follows that

$$(\nabla_{X^\star} K)\,(Y^\star, Z^\star, U^\star, V^\star) = 0 \tag{40}$$

From (1), we get

$$\begin{aligned}(\nabla_{X^\star} R)\,(Y^\star, Z^\star, U^\star, V^\star) \;=\; & \frac{1}{n-2}\{g(Y^\star, V^\star)\nabla_{X^\star} S(Z^\star, U^\star) \\ & - g(Z^\star, V^\star)\nabla_{X^\star} S(Y^\star, U^\star) \\ & + g(Z^\star, U^\star)\nabla_{X^\star} S(Y^\star, V^\star) \\ & - g(Y^\star, U^\star)\nabla_{X^\star} S(Z^\star, V^\star)\}\end{aligned} \tag{41}$$

Contracting over $Y^\star$ and $V^\star$, we obtain

$$(\nabla_{X^\star} S)\,(Z^\star, U^\star) = \frac{1}{p}\,(\nabla_{X^\star} r^\star)\, g(Z^\star, U^\star) \tag{42}$$

Substituting (42) in (41), we have

$$\begin{aligned}(\nabla_{X^\star} R)\,(Y^\star, Z^\star, U^\star, V^\star) \;=\; & \frac{2}{(n-2)p}\,(\nabla_{X^\star} r^\star)\,\{g(Y^\star, V^\star)g(Z^\star, U^\star) \\ & - g(Z^\star, V^\star)g(Y^\star, U^\star)\}\end{aligned} \tag{43}$$

Contracting in (42) over $Z^\star$ and $U^\star$, we get

$$(\nabla_{X^\star} r^\star) = 0 \tag{44}$$

From (43) and (44), it follows that

$$(\nabla_{X^\star} R)\,(Y^\star, Z^\star, U^\star, V^\star) = 0 \tag{45}$$

Hence M_2 is a locally symmetric manifold. Moreover, from (40), we can say that M_2 is a conharmonically symmetric manifold.

If it is repeated the above operations for (34) and (35), then it is obtained similar results. Therefore we can all state the following:

Theorem 5 *Let $A(PCHS)_n$ be a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p}$ $(2 \le p \le n-2)$. Then the following holds:*

(i) If one of the decomposition is conharmonically flat, then it is of constant curvature. Also, either its the scalar curvature vanishes or its dimension is equal to half of that of M.

(ii) If 1-forms A and B vanish on the other, then both this decomposition is Ricci-symmetric and locally symmetric, also it is conharmonically symmetric.

Now, contracting in (30) over $X^\star$ and $U^\star$, we get

$$A(\overline{Y})\left\{(n-p)S(\overline{Z},\overline{V})+r^\star g(\overline{Z},\overline{V})\right\}-A(\overline{Z})\left\{(n-p)S(\overline{Y},\overline{V})+r^\star g(\overline{Y},\overline{V})\right\}=0 \tag{46}$$

Again contracting over $\overline{Z}$ and $\overline{V}$, we obtain

$$A(L\overline{Y})=\left\{\frac{(p-1)}{(n-p)}r^\star+\overline{r}\right\}A(\overline{Y}) \tag{47}$$

Repeating similar operation for (31), we have

$$\begin{aligned} 0 = & \left[A(\overline{X})+B(\overline{X})\right]\left\{(n-p)S(\overline{Z},\overline{U})+r^\star g(\overline{Z},\overline{U})\right\} \\ & + A(\overline{Z})\left\{(n-p)S(\overline{X},\overline{U})+r^\star g(\overline{X},\overline{U})\right\} \\ & + A(\overline{U})\left\{(n-p)S(\overline{Z},\overline{X})+r^\star g(\overline{Z},\overline{X})\right\} \end{aligned} \tag{48}$$

and

$$A(\overline{X})\left\{(n-p)\overline{r}+(p+2)r^\star\right\}+B(\overline{X})\left\{(n-p)\overline{r}+pr^\star\right\}+2(n-p)A(L\overline{X})=0 \tag{49}$$

From (47), it follows that

$$3A(\overline{X})+B(\overline{X})=0 \tag{50}$$

Thus we can state the following.

Theorem 6 *Let (M,g) be a decomposable Riemannian manifold, $M_1^p\times M_2^{n-p}$, $(2\le p\le n-2)$. If M is an $A(PCHS)_n$, then the following relations are satisfied*

$$A(L\overline{X})=\left\{\frac{(p-1)}{(n-p)}r^\star+\overline{r}\right\}A(\overline{X}) \quad \textit{and} \quad 3A(\overline{X})+B(\overline{X})=0$$

on M_1.

Contracting in (33) over $\overline{Y},\overline{X}$ and $Z^\star, V^\star$, respectively, we obtain

$$A(LU^\star)=\left\{r^\star+\frac{(p-1)}{p}\overline{r}\right\}A(U^\star)$$

on M_2. Hence we can state the following.

Theorem 7 *Let (M,g) be a decomposable Riemannian manifold, $M_1^p\times M_2^{n-p}$, $(2\le p\le n-2)$. If M is an $A(PCHS)_n$, then the following relation is satisfied*

$$A(LU^\star)=\left\{r^\star+\frac{(p-1)}{p}\overline{r}\right\}A(U^\star)$$

on M_2.

5 Examples of a decomposable $A(PCHS)_n$

Example 1 Let $M^n = \left\{(x^1, x^2, x^3, \ldots, x^n) \in \mathbb{R}^n : 0 < x^4 < 1\right\}$ be a manifold endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = \left[\left(x^4\right)^{\frac{4}{3}} - 1\right]\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + \delta_{ab}dx^a dx^b \quad (51)$$

where δ_{ab} is the Kronecker delta and each index runs over $1, 2, \ldots, n$. Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\begin{aligned} \Gamma^1_{14} &= \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \\ \Gamma^4_{11} &= \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{1/3} \\ R_{1441} &= R_{2442} = R_{3443} = -\frac{2}{9}(x^4)^{-2/3} \end{aligned} \quad (52)$$

and the components obtained by the symmetry properties.

In the metric considered, the covariant and contravariant components of the metric are as follows

$$\begin{aligned} g_{11} = g_{22} = g_{33} = \left(x^4\right)^{\frac{4}{3}}, \quad g_{44} = g_{55} = \cdots = g_{nn} = 1 \\ g^{11} = g^{22} = g^{33} = \left(x^4\right)^{-\frac{4}{3}}, \quad g^{44} = g^{55} = \cdots = g^{nn} = 1 \end{aligned} \quad (53)$$

Due to (52) and (53), the non-vanishing components of the Ricci tensor are

$$S_{11} = S_{22} = S_{33} = -\frac{2}{9}(x^4)^{-2/3}, S_{44} = -\frac{2}{3}(x^4)^{-2} \quad (54)$$

From $r = g^{ij}S_{ij} = g^{11}S_{11} + g^{22}S_{22} + g^{33}S_{33} + \cdots + g^{nn}S_{nn}$, using (53) and (54), we can be easly seen that the scalar curvature of (M^n, g) is the following

$$r = -\frac{4}{3}(x^4)^{-2} \quad (55)$$

Therefore (M^n, g) is of non-zero and non-constant scalar curvature.

Now let us calculate the conharmonic curvature tensor K. In virtue of (1), we obtain that the only non-vanishing components of the conharmonic curvature tensor K of (M^n, g) are

$$\begin{aligned} K_{1221} &= K_{1331} = K_{2332} = \frac{4}{9(n-2)}\left(x^4\right)^{\frac{2}{3}} \\ K_{1441} &= K_{2442} = K_{3443} = -\frac{2(n-6)}{9(n-2)}\left(x^4\right)^{-\frac{2}{3}} \\ K_{pqqp} &= \frac{2}{9(n-2)}\left(x^4\right)^{-\frac{2}{3}}, \quad (1 \le p \le 4), \quad (5 \le q \le n) \end{aligned} \quad (56)$$

and the components obtained by the symmetry properties. Hence (M^n, g) is of non-conharmonic flat. From (56), it can be easily shown that the only non-zero terms of $\nabla_l K_{ikjm}$ are

$$\begin{aligned}
\nabla_4 K_{1221} &= \nabla_4 K_{1331} = \nabla_4 K_{2332} = -\frac{8}{9(n-2)}\left(x^4\right)^{-\frac{1}{3}} \\
\nabla_4 K_{1441} &= \nabla_4 K_{2442} = \nabla_4 K_{3443} = \frac{4(n-6)}{9(n-2)}\left(x^4\right)^{-\frac{5}{3}} \\
\nabla_4 K_{pqqp} &= -\frac{4}{9(n-2)}\left(x^4\right)^{-\frac{5}{3}}, \quad (1 \le p \le 4), \quad (5 \le q \le n)
\end{aligned} \tag{57}$$

All other components of $\nabla_l K_{ikjm}$ vanish identically. Thus our M^n with the considered metric g in (51) is a Riemannian manifold with non-zero scalar curvature which is neither conharmonically symmetric nor conharmonically flat.

If we consider the 1-forms

$$A_i = 0 \ \text{ for all } i \qquad \text{and} \qquad B_i = \begin{cases} -\dfrac{2}{(x^4)}, & \text{for } i = 4 \\ 0, & \text{otherwise} \end{cases} \tag{58}$$

then (4) reduces with these 1-forms to the following equations. It is sufficient to check this equations in order to verify (4) in M^n:

1. $\nabla_4 K_{1221} = [A_4 + B_4] K_{1221} + A_1 K_{4221} + A_2 K_{1421} + A_2 K_{1241} + A_1 K_{1224}$
2. $\nabla_4 K_{1331} = [A_4 + B_4] K_{1331} + A_1 K_{4331} + A_3 K_{1431} + A_3 K_{1341} + A_1 K_{1334}$
3. $\nabla_4 K_{1441} = [A_4 + B_4] K_{1441} + A_1 K_{4441} + A_4 K_{1441} + A_2 K_{1441} + A_1 K_{1444}$
4. $\nabla_4 K_{pqqp} = [A_4 + B_4] K_{pqqp} + A_p K_{4qqp} + A_q K_{p4qp} + A_q K_{pq4p} + A_p K_{pqq4}$

As for the case other than (1)–(4), the components of K_{ikjm} and $\nabla_l K_{ikjm}$ vanish identically and the equation (4) holds trivially. It can be easly seen that the relations (1)–(4) are satisfied. Thus (M^n, g) is an $A(PCHS)_n$ with non-zero scalar curvature and conharmonically recurrent. It can be stated the following:

Theorem 8 *Let* $M^n = \left\{(x^1, x^2, x^3, \ldots, x^n) \in \mathbb{R}^n : 0 < x^4 < 1\right\}$ *be an open subset of* $\mathbb{R}^n$ *equipped with the metric*

$$ds^2 = g_{ij}dx^i dx^j = \left[\left(x^4\right)^{\frac{4}{3}} - 1\right]\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + \delta_{ab}dx^a dx^b$$

where δ_{ab} *is the Kronecker delta and each index runs over* $1, 2, \ldots, n$. *Then* (M^n, g) *is a conharmonically recurrent* $A(PCHS)_n$ *with non-zero and non-constant scalar curvature which is neither conharmonically symmetric nor conharmonically flat.*

Example 2 Let M^4 be an open subset of $\mathbb{R}^4$ equipped with the metric

$$ds^2 = g_{ij}dx^i dx^j = \left(x^4\right)^{\frac{4}{3}} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (dx^4)^2 \qquad (59)$$

Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are in the form (52). In the metric considered, the covariant and contravariant components of the metric are as follows

$$\begin{aligned} g_{11} = g_{22} = g_{33} &= \left(x^4\right)^{\frac{4}{3}}, \quad g_{44} = 1 \\ g^{11} = g^{22} = g^{33} &= \left(x^4\right)^{-\frac{4}{3}}, \quad g^{44} = 1 \end{aligned} \qquad (60)$$

Due to (52) and (60), the non-vanishing components of the Ricci tensor are the same as in (54). Therefore, by performing the same as calculation, we can easily see that the scalar curvature of (M^4, g) is $r = -\frac{4}{3}(x^4)^{-2}$. That is, (M^4, g) is of non-zero and non-constant scalar curvature.

In virtue of (1) we obtain that the only non-vanishing components of the conharmonic curvature tensor K of (M^4, g) are

$$\begin{aligned} K_{1221} = K_{1331} = K_{2332} &= \frac{2}{9}\left(x^4\right)^{\frac{2}{3}} \\ K_{1441} = K_{2442} = K_{3443} &= \frac{2}{9}\left(x^4\right)^{-\frac{2}{3}} \end{aligned} \qquad (61)$$

and the components obtained by the symmetry properties. From (61), the only non-zero terms of $\nabla_l K_{ikjm}$ are

$$\begin{aligned} \nabla_4 K_{1221} = \nabla_4 K_{1331} = \nabla_4 K_{2332} &= -\frac{4}{9}\left(x^4\right)^{-\frac{1}{3}} \\ \nabla_4 K_{1441} = \nabla_4 K_{2442} = \nabla_4 K_{3443} &= -\frac{4}{9}\left(x^4\right)^{-\frac{5}{3}} \end{aligned} \qquad (62)$$

All other components of $\nabla_l K_{ikjm}$ vanish identically. Thus our M^4 with the considered metric g in (59) is a Riemannian manifold with non-vanishing scalar curvature which is neither conharmonically symmetric nor conharmonically flat.

if we consider the 1-forms

$$A_i = 0 \;\text{ for all } i \qquad \text{and} \qquad B_i = \begin{cases} -\dfrac{2}{(x^4)}, & \text{for } i = 4 \\ 0, & \text{otherwise} \end{cases}$$

then (4) reduces with these 1-forms to the following equations. It is sufficient to check this equations in order to verify (4) in M^4:

1. $\nabla_4 K_{1221} = [A_4 + B_4] K_{1221} + A_1 K_{4221} + A_2 K_{1421} + A_2 K_{1241} + A_1 K_{1224}$
2. $\nabla_4 K_{1331} = [A_4 + B_4] K_{1331} + A_1 K_{4331} + A_3 K_{1431} + A_3 K_{1341} + A_1 K_{1334}$
3. $\nabla_4 K_{1441} = [A_4 + B_4] K_{1441} + A_1 K_{4441} + A_4 K_{1441} + A_2 K_{1441} + A_1 K_{1444}$

As for the case other than (1)–(3), the components of each term K_{ikjm} and $\nabla_l K_{ikjm}$ vanish identically and the equation (4) holds trivially. It can be easly seen that the relations (1)–(3) are satisfied. Thus (M^4, g) is an $A(PCHS)_n$ with non-vanishing and non-constant scalar curvature and conharmonically recurrent. It can be stated the following:

Theorem 9 *Let M^4 be an open subset of $\mathbb{R}^4$ equipped with the metric*

$$ds^2 = g_{ij}dx^i dx^j = \left(x^4\right)^{\frac{4}{3}} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (dx^4)^2$$

Then (M^4, g) is a conharmonically recurrent $A(PCHS)_n$ with non-vanishing and non-constant scalar curvature which is neither conharmonically symmetric nor conharmonically flat.

Let us denote the metric (59) by $\overline{g}$ and $(M^4, \overline{g})$ be a Riemannian manifold in Example 2. Also, let $(\mathbb{R}^{n-4}, g^\star)$ be an $(n-4)$-dimensional Euclidean space whose the metric $g^\star$ is a standard metric. Then we can say that (M^n, g) in the Example 1 is a product manifold of $(M^4, \overline{g})$ and $(\mathbb{R}^{n-4}, g^\star)$. Hence we can state the following:

Theorem 10 *Let (M^n, g), $(n > 4)$ be a Riemannian manifold equipped with the metric given in (51). Then (M^n, g) is a decomposable almost pseudo conharmonic symmetric manifold $M^n = (M^4, \overline{g}) \times (\mathbb{R}^{n-4}, g^\star)$ whose the scalar curvature is non-vanishing and non-constant and neither conharmonically flat nor conharmonically symmetric but conharmonically recurrent.*

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