Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 51 (2012), No. 1, 89--100

Persistent URL: `http://dml.cz/dmlcz/142876`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **51**, 1 (2012) 89–100

A New Sequence Space Defined by a Sequence of Orlicz Functions over n-Normed Spaces

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(Received March 16, 2012)

Abstract

In this paper we introduce a new sequence space $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$ defined by a sequence of Orlicz functions $\mathcal{M} = (M_k)$ and study some topological properties of this sequence space.

Key words: paranorm space, invariant mean, orlicz function, Musielak–orlicz function, n-normed space, solid

2000 Mathematics Subject Classification: 40D05, 40A05

1 Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler[2] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[11]. Since then, many others have studied this concept and obtained various results, see Gunawan ([3], [4]), Gunawan and Mashadi [5] and many others. Let $n \in \mathbb{N}$ and X be a linear space of dimension d, where $d \geq n \geq 2$ over the field $\mathbb{K}$ ($\mathbb{K}$ is the field of real or complex numbers). A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions:

1. $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;
2. $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{K}$, and
4. $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called a n-norm on X and the pair $(X, \|\cdot,\ldots,\cdot\|)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm, $\|x_1, x_2, \ldots, x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$ and $\|.\|_E$ denotes the Euclidean norm. Let $(X, \|\cdot,\ldots,\cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot,\ldots,\cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \ldots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| \colon i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every } z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot,\ldots,\cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every } z_1, \ldots, z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. A complete n-normed space is said to be a n-Banach space.

Let X be a linear metric space. A function $p\colon X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$;
2. $p(-x) = p(x)$, for all $x \in X$;
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;
4. if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [20], Theorem 10.4.2, P-183). For more details about sequence spaces see ([6], [12], [16], [18]).

Let l_∞ and c denotes the Banach spaces of bounded and convergent sequences $x = (x_k)_{k=1}^\infty$ respectively. Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits and T be the operator defined on l_∞ by $T((x_n)_{n=1}^\infty) = (x_{\sigma(n)})_{n=1}^\infty$.

A positive linear functional φ, with $\|\varphi\| = 1$, is called a σ-mean or an invariant mean if $\varphi(x) = \varphi(Tx)$ for all $x \in l_\infty$.

A sequence $x = (x_k)$ is said to be σ-convergent, denoted by $x \in V_\sigma$, if $\varphi(x)$ takes the same value, called σ-lim x, for all σ-means φ. We have

$$V_\sigma = \left\{ x = (x_n) \colon \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in } n,\ L = \sigma - \lim x \right\},$$

for $m \geq 0$, $n > 0$, where, $t_{m,n}(x) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^m(n)}}{m+1}$, and $t_{-1,n} = 0$ (see schaefer [19]), where $\sigma^m(n)$ denotes the m^{th} iterate of σ at n. In particular, if σ is a translation, a σ-mean is often called a Banach limit and V_σ reduces to f, the set of almost convergent sequences (see Lorentz [8]). Subsequently invariant mean have been studied by Ahmad and Mursaleen [1] and many others.

A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.

A sequence space E is said to be monotone if it contains the canonical preimages of all its step spaces.

An orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w \colon \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 \colon \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \right\}.$$

It is shown in [7] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak–Orlicz function see ([10], [14]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) \colon u \geq 0\},\ k = 1, 2, \ldots$$

is called the complementary function of a Musielak–Orlicz function $\mathcal{M}$. For a given Musielak–Orlicz function $\mathcal{M}$, the Musielak–Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \{x \in w \colon I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$
$$h_{\mathcal{M}} = \{x \in w \colon I_{\mathcal{M}}(cx) < \infty \text{for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \Big\{ k > 0 \colon I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1 \Big\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) : k > 0 \right\}.$$

Mursaleen [13] defined the sequence space

$$BV_\sigma = \left\{ x \in l_\infty \colon \sum_m |\varphi_{m,n}(x)| < \infty, \text{ uniformly in } n \right\},$$

where $\varphi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x)$, assuming that $t_{m,n}(x) = 0$, for $m = -1$. Note that for any sequences $x = (x_k)$, $y = (y_k)$ and scalar λ we have

$$\varphi_{m,n}(x + y) = \varphi_{m,n}(x) + \varphi_{m,n}(y)$$

and

$$\varphi_{m,n}(\lambda x) = \lambda \varphi_{m,n}(x).$$

Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function, $p = (p_m)$ be any sequence of strictly positive real numbers and $r \geq 0$ the sequence space $BV_\sigma(\mathcal{M}, p, r)$ defined by Raj, Sharma and Sharma [15].

Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $w(X)$ denotes X-valued sequence spaces. Let $p = (p_m)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. In the present paper we define the sequence space:

$$BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|) =$$
$$\Big\{ x = (x_k) \in w(X) \colon \sum_{m=1}^{\infty} \frac{1}{m^r} \Big[\sup_{k \geq 0} u_k M_k \Big(\| \frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1} \| \Big) \Big]^{p_m} < \infty,$$
$$\text{uniformly in } n \text{ and for some } \rho > 0 \Big\}.$$

For $\mathcal{M}(x) = x$, we get

$$BV_\sigma(u, p, r, \|\cdot, \ldots, \cdot\|) =$$

$$\Big\{x = (x_k) \in w(X) : \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $p = p_m = 1$ for all m, we get

$$BV_\sigma(\mathcal{M}, u, r, \|\cdot, \ldots, \cdot\|) =$$

$$\Big\{x = (x_k) \in w(X)\colon \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big] < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $r = 0$, we get

$$BV_\sigma(\mathcal{M}, u, p, \|\cdot, \ldots, \cdot\|) =$$

$$\Big\{x = (x_k) \in w(X) : \sum_{m=1}^{\infty} \Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $\mathcal{M}(x) = x$ and $r = 0$, we get

$$BV_\sigma(u, p, \|\cdot, \ldots, \cdot\|) =$$

$$\Big\{x = (x_k) \in w(X)\colon \sum_{m=1}^{\infty} \Big[\sup_{k\geq 0} u_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $p = p_m = 1$ for all m and $r = 0$, we get

$$BV_\sigma(\mathcal{M}, u, \|\cdot, \ldots, \cdot\|) =$$

$$\Big\{x = (x_k) \in w(X) : \sum_{m=1}^{\infty} \Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big] < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

For $\mathcal{M}(x) = x$, $p = p_m = 1$ for all m and $r = 0$, we get,

$$BV_\sigma(u, \|\cdot, \ldots, \cdot\|) =$$

$$\Big\{x = (x_k) \in w(X) : \sum_{m=1}^{\infty} \Big[\sup_{k\geq 0} u_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big] < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

If we take $u = u_k = 1$ for all k we get,

$$BV_\sigma(\mathcal{M}, p, r, \|\cdot, \dots, \cdot\|) =$$

$$\Big\{x = (x_k) \in w(X)\colon \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \dots, z_{n-1}\|\Big)\Big]^{p_m} < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H,\ K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The aim of this paper is to examine some topological properties and inclusion relations between above defined sequence spaces.

2 Some properties of sequence space $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \dots, \cdot\|)$

Theorem 2.1 *The sequence space $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \dots, \cdot\|)$ is a linear space over the field of complex numbers $\mathbb{C}$.*

Proof Let $x, y \in BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \dots, \cdot\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho_1}, z_1, \dots, z_{n-1}\Big)\Big]^{p_m} < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho_1 > 0$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(y)}{\rho_2}, z_1, \dots, z_{n-1}\Big)\Big]^{p_m} < \infty,$$

$$\text{uniformly in } n \text{ and for some } \rho_2 > 0.$$

Define

$$\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2).$$

Since $\mathcal{M} = (M_k)$ is non-decreasing, convex and so by using inequality (1.1), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(\alpha x + \beta y)}{\rho_3}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} \\
&\quad \leq \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\alpha\varphi_{m,n}(x)}{\rho_3}, z_1, \ldots, z_{n-1}\| \\
&\qquad\qquad + \|\frac{\beta\varphi_{m,n}(y)}{\rho_3}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} \\
&\quad \leq K \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho_1}, z_1, \ldots, z_{n-1}\|\Big)\Big] \\
&\quad + K \sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(y)}{\rho_2}, z_1, \ldots, z_{n-1}\|\Big)\Big] \\
&\qquad\qquad < \infty, \text{ uniformly in } n.
\end{aligned}$$

This proves that $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$ is a linear space over the field $\mathbb{C}$ of complex numbers. □

Theorem 2.2 *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_m)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, the space $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$ is a paranormed space with the paranorm defined by*

$$g(x) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \\ \textit{uniformly in } n\Big\},$$

where $H = \max(1, \sup p_m)$.

Proof It is clear that $g(x) = g(-x)$. Since $M(0) = 0$, we get $g(0) = 0$. By using Theorem 2.1, for $\alpha = \beta = 1$, we get

$$g(x + y) \leq g(x) + g(y).$$

For the continuity of scalar multiplication, let $\lambda \neq 0$ be any complex numbers, then by definition, we have

$$g(\lambda x) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(\lambda x)}{\rho}, z_1, \ldots, z_{n-1}\Big)\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \\ \text{uniformly in } n\Big\}.$$

$$g(\lambda x) = \inf_{n\geq 1}\Big\{(|\lambda|s)^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(\lambda x)}{s|\lambda|}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \text{ uniformly in } n\Big\},$$

where $s = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_m} \leq \max(1, |\lambda|^q)$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^q)\inf_{n\geq 1}\Big\{s^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m_r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}}{s}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}\Big)^{\frac{1}{H}}\Big) \leq 1, \text{ uniformly in } n\Big\} = \max(1, |\lambda|^q)g(x),$$

and therefore $g(\lambda x)$ converges to zero in $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$. Now let x be fixed element in $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$, there exist $\rho > 0$ such that

$$g(x) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m_r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \text{ uniformly in } n\Big\}.$$

Now

$$g(\lambda x) = \inf_{n\geq 1}\Big\{\rho^{\frac{p_n}{H}} : \Big(\sum_{m=1}^{\infty}\frac{1}{m_r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(\lambda x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}\Big)^{\frac{1}{H}} \leq 1, \text{ uniformly in } n\Big\} \longrightarrow 0 \text{ as } \lambda \longrightarrow 0.$$

This completes the proof. □

Theorem 2.3 *Suppose that* $0 < p_m \leq q_m < \infty$, *for each* $m \in \mathbb{N}$ *and* $r \geq 0$. *Then*

(i) $BV_\sigma(\mathcal{M}, u, p, \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(\mathcal{M}, u, q, \|\cdot, \ldots, \cdot\|)$,

(ii) $BV_\sigma(\mathcal{M}, u, \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(\mathcal{M}, u, r, \|\cdot, \ldots, \cdot\|)$.

Proof (i) Suppose that $x \in BV_\sigma(\mathcal{M}, u, p, \|\cdot, \ldots, \cdot\|)$. This implies that

$$\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} \leq 1$$

for sufficiently large value of m, say $m \geq m_0$ for some fixed $m_0 \in \mathbb{N}$. Since $\mathcal{M} = (M_k)$ is non-decreasing, we have

$$\sum_{m=m_0}^{\infty} \Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{q_m}$$

$$\leq \sum_{m=m_0}^{\infty} \Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z-1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} < \infty.$$

Hence $x \in BV_\sigma(\mathcal{M}, u, q, \|\cdot, \ldots, \cdot\|)$.

(ii) Suppose that $x \in BV_\sigma(\mathcal{M}, u, \|\cdot, \ldots, \cdot\|)$. This implies that

$$\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big] \leq 1,$$

for sufficiently large value of m, say $m = m_0$ for fixed $m_0 \in \mathbb{N}$. Since $\mathcal{M} = (M_k)$ is non-decreasing, we have

$$\sum_{m=m_0}^{\infty} \frac{1}{m^r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]$$

$$\leq \sum_{m=m_0}^{\infty} \Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big] < \infty.$$

Hence $x \in BV_\sigma(\mathcal{M}, u, r, \|\cdot, \ldots, \cdot\|)$. □

Corollary 2.1 *(i) If* $0 < p_m \leq 1$ *for each m, then*

$$BV_\sigma(\mathcal{M}, u, p, \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(\mathcal{M}, u, \|\cdot, \ldots, \cdot\|).$$

(ii) If $p_m \geq 1$ *for all m, then*

$$BV_\sigma(\mathcal{M}, u, \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(\mathcal{M}, u, p, \|\cdot, \ldots, \cdot\|).$$

Proof It follows from the above Theorem. □

Theorem 2.4 *The sequence space* $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$ *is solid.*

Proof Let $x \in BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$. This implies that

$$\sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} < \infty.$$

Let (α_m) be the sequence of scalars such that $|\alpha_m| \leq 1$ for all $m \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\alpha_m\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}$$

$$\leq \sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\geq 0} u_k M_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} < \infty.$$

Hence $\alpha x \in BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$ for all sequences of scalars (α_m) with $|\alpha_m| \le 1$ for all $m \in \mathbb{N}$ whenever $x \in BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$. □

Corollary 2.2 *The sequence space* $BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$ *is monotone.*

Proof It follows from the above Theorem. □

Theorem 2.5 *Let* $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$, $\mathcal{M}'' = (M''_k)$ *are sequences of Orlicz functions satisfying* Δ_2*-condition and* $r, r_1, r_2 \ge 0$. *Then we have*

(i) If $r > 1$ *then* $BV_\sigma(\mathcal{M}', u, p, r, \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(\mathcal{M} \circ \mathcal{M}', u, p, r, \|\cdot, \ldots, \cdot\|)$,

(ii) $BV_\sigma(\mathcal{M}', u, p, r, \|\cdot, \ldots, \cdot\|) \bigcap BV_\sigma(\mathcal{M}'', u, p, r, \|\cdot, \ldots, \cdot\|)$

$$\subseteq BV_\sigma(\mathcal{M}' + \mathcal{M}'', u, p, r, \|\cdot, \ldots, \cdot\|),$$

(iii) If $r_1 \le r_2$ *then* $BV_\sigma(\mathcal{M}, u, p, r_1, \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(\mathcal{M}, u, p, r_2, \|\cdot, \ldots, \cdot\|)$.

Proof (i) Since $\mathcal{M}' = (M'_k)$ is continuous at origin from right for all k, for $\epsilon > 0$ there exists $0 < \delta < 1$ such that $0 \le C \le \delta$ implies $M'_k(C) < \epsilon$. If we define

$$I_1 = \Big\{m \in \mathbb{N}\colon \sup_{k\ge 0} u_k M'_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big) \le \delta, \text{ for some } \rho > 0\Big\},$$

$$I_2 = \Big\{m \in \mathbb{N}\colon \sup_{k\ge 0} u_k M'_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big) > \delta, \text{ for some } \rho > 0\Big\},$$

when $\sup_{k\ge 0} u_k M'_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big) > \delta$, we get

$$\sup_{k\ge 0} u_k M_k\Big(\sup_{k\ge 0} M'_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big)$$

$$\le \Big\{2 \sup_{k\ge 0} u_k M_k(1)/\delta\Big\} \sup_{k\ge 0} u_k M'_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big).$$

Hence for $x \in BV_\sigma(\mathcal{M}', u, p, r, \|\cdot, \ldots, \cdot\|)$ and $r > 1$, we have

$$\sum_{m=1}^{\infty} m^{-r}\Big[\sup_{k\ge 0} u_k (M_k \circ M'_k)\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}$$

$$= \sum_{m\in I_1} m^{-r}\Big[\sup_{k\ge 0} u_k (M_k \circ M'_k)\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}$$

$$+ \sum_{m\in I_2} m^{-r}\Big[\sup_{k\ge 0} u_k (M_k \circ M'_k)\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}$$

$$\le \sum_{m\in I_1} m^{-r}[\epsilon]^{p_m}$$

$$+ \sum_{m\in I_2} m^{-r}\Big[\Big\{\sup_{k\ge 0} 2u_k M_k(1)/\delta\Big\} \sup_{k\ge 0} u_k M'_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}$$

$$\le \max(\epsilon^h, \epsilon^H) \sum_{m=1}^{\infty} m^{-r} + \max\Big(\Big\{\frac{2M_k(1)}{\delta}\Big\}^h, \Big\{\frac{2M_k(1)}{\delta}\Big\}^H\Big),$$

where $0 < h = \inf p_m \leq p_m \leq H = \sup_n p_m < \infty$.

(ii) The proof follows from the following inequality

$$\begin{aligned} m^{-r}\Big[\sup_{k\geq 0} u_k(M'_k + M''_k)\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} \\ \leq Km^{-r}\Big[u_k M'_k\Big(\|\frac{|\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m} \\ + Km^{-r}\Big[u_k M''_k\Big(\|\frac{\varphi_{m,n}(x)}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_m}. \end{aligned}$$

(iii) The proof is straight forward. □

Corollary 2.3 *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions satisfying Δ_2-condition. Then we have*

(i) If $r > 1$, then $BV_\sigma(u, p, r, \|\cdot, \ldots, \cdot\|) \subset BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$,

(ii) $BV_\sigma(\mathcal{M}, u, p, \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(\mathcal{M}, u, p, r, \|\cdot, \ldots, \cdot\|)$,

(iii) $BV_\sigma(u, p, , \|\cdot, \ldots, \cdot\|) \subseteq BV_\sigma(u, p, r, , \|\cdot, \ldots, \cdot\|)$,

(iv) $BV_\sigma(\mathcal{M}) \subseteq BV_\sigma(\mathcal{M}, u, r, , \|\cdot, \ldots, \cdot\|)$.

Proof The proof follows from the above theorem. □

Acknowledgement The authors thank the referee for his valuable suggestions that improved the presentation of the paper.

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