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SECOND ORDER DIFFERENCE INCLUSIONS OF MONOTONE TYPE

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Abstract. The existence of anti-periodic solutions is studied for a second order difference inclusion associated with a maximal monotone operator in Hilbert spaces. It is the discrete analogue of a well-studied class of differential equations.

Keywords: anti-periodic solution, maximal monotone operator, Yosida approximation

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1. Introduction

We are concerned with the second order difference inclusion

$$\begin{cases} u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, \quad 1 \leqslant i \leqslant N \\ u_0 = -u_{N+1}, \ u_1 - u_0 = -a_N(u_{N+1} - u_N), \end{cases} \tag{1.1}$$

where A is a nonlinear (possibly multivalued) maximal monotone operator in a real Hilbert space H, θ_i, $c_i > 0$ and $f_i \in H$ $(1 \leqslant i \leqslant N)$ are given finite sequences, and $a_N = 1/\theta_1\theta_2\ldots\theta_N$. Denote by $D(A)$ the domain of A.

The inclusion from (1.1) is the discrete variant of the continuous differential inclusion $pu'' + ru' \in Au + f$ a.e. on $[0,T]$ that has been intensely studied. See for example the papers [9], [1] and the monograph [8]. Anti-periodic solutions for such a class of differential equations were investigated in [2], [4], while the discrete analogue for $p \equiv 1$, $r \equiv 0$ was treated in [5]. In this case $\theta_i \equiv 1$. In [10], the authors study the asymptotic behavior of the bounded solution for the second order on half-axis. Existence and asymptotic behavior results for equation (1.1) for $i \geqslant 1$ and various boundary conditions have been obtained in [7]. For finite sets of i $(1 \leqslant i \leqslant N)$, in [6] the authors analyzed the continuous dependence of the solution on the operator A, the sequence f_i and the boundary conditions $u_0 = a$, $u_{N+1} = b$.

The structure of the paper is the following. In the next section we find some auxiliary results related to the maximal monotonicity of the operator

$$\mathcal{B}u = \{(-u_{i+1} + (1+\theta_i)u_i - \theta_i u_{i-1})_{1\leqslant i\leqslant N}\} \tag{1.2}$$

with the domain

$$D(\mathcal{B}) = \{u = (u_i)_{1\leqslant i\leqslant N},\ u_0 = -u_{N+1},\ u_1 - u_0 = -a_N(u_{N+1} - u_N)\}. \tag{1.3}$$

Denoting by $\mathcal{A}$ the operator

$$\mathcal{A}u = \{(c_1 v_1, \ldots, c_N v_N),\ v_i \in Au_i,\ 1 \leqslant i \leqslant N\},\ D(\mathcal{A}) = D(A)^N, \tag{1.4}$$

problem (1.1) can be written as $-f \in (\mathcal{A} + \mathcal{B})(u),\ f = (f_1, \ldots, f_N)$.

Section 3 is devoted to the existence of the solution of the boundary value problem (1.1). The main result of the paper is established here and an application to PDE is presented.

Recall that if A is maximal monotone and if $J_\lambda = (I + \lambda A)^{-1}$, $A_\lambda = (I - J_\lambda)/\lambda$ are its resolvent and its Yosida approximation, respectively, then $x = J_\lambda x + \lambda A_\lambda x,\ A_\lambda x \in A(J_\lambda x)$. Properties of maximal monotone operators can be found in [8].

In [3], [11] the authors studied second-order boundary value problems for discrete inclusions and applied the fixed-point techniques and a priori bound methods to obtain the existence of solutions. However, in these papers the boundary conditions are of Dirichlet type and so do not apply directly to the problem herein.

2. Auxiliary results

Note that, if A is maximal monotone in H, then $\mathcal{A}$ from (1.4) is maximal monotone in $H^N = H \times \ldots \times H$ (N times). We study now the maximal monotonicity of $\mathcal{B}$ in the Hilbert space H^N endowed with the scalar product

$$\langle (u_i)_{1\leqslant i\leqslant N}, (v_i)_{1\leqslant i\leqslant N}\rangle = \sum_{i=1}^{N} a_i(u_i, v_i). \tag{2.1}$$

Here $(\cdot,\cdot)$ is the scalar product in H and a_i is given by

$$a_0 = 1,\ \ a_i = \frac{1}{\theta_1 \ldots \theta_i},\quad 1 \leqslant i \leqslant N. \tag{2.2}$$

Observe that

$$a_i\theta_i = a_{i-1}, \quad 1 \leqslant i \leqslant N+1. \tag{2.3}$$

This Hilbert space is equivalent to H^N endowed with the scalar product $\langle (u_i)_{1\leqslant i\leqslant N}, (v_i)_{1\leqslant i\leqslant N}\rangle = \sum\limits_{i=1}^{N}(u_i, v_i)$. The only difference between the two Hilbert spaces is that the operator $\mathcal{B}$ introduced in (1.2)–(1.3) is monotone only in H^N with the scalar product (2.1). To show this, we begin with the existence results for the auxiliary boundary value problems, with $c, d \in \mathbb{R}$ given:

$$\begin{gathered} l_{i+1} - (2+\theta_i)l_i + \theta_i l_{i-1} = 0, \quad 1 \leqslant i \leqslant N, \\ l_0 = c, \;\; l_{N+1} = -c, \end{gathered} \tag{2.4}$$

$$\begin{gathered} m_{i+1} - (2+\theta_i)m_i + \theta_i m_{i-1} = 0, \quad 1 \leqslant i \leqslant N, \\ m_1 - m_0 = a_N d, \;\; m_{N+1} - m_N = -d. \end{gathered} \tag{2.5}$$

Lemma 2.1. *If $c \in \mathbb{R}$ and c_i, $\theta_i > 0$, $1 \leqslant i \leqslant N$, problem* (2.4) *has a unique solution $l = (l_i)_{1\leqslant i\leqslant N} \in \mathbb{R}^N$. Moreover, we can choose c such that $l_1 - l_0 + a_N(l_{N+1} - l_N) \neq 0$.*

Proof. Problem (2.4) has the form (6.1.13) from [8], page 143. Applying Theorem 6.1.2 in [8], one deduces that (2.4) admits a unique solution $l = (l_i)_{1\leqslant i\leqslant N} \in \mathbb{R}^N$. Let $l_0 = c$ and $l_1 \in \mathbb{R}$ be fixed. Then we can compute $l_2, l_3, \ldots, l_{N+1}$ in terms of l_1. By the boundary condition $l_{N+1} = -c$, we find $l_1 = c[2\theta_1(2+\theta_2)-1]/(8+4\theta_1+2\theta_2+2\theta_1\theta_2)$. Then we can choose c such that the condition $l_1 - l_0 + a_N(l_{N+1} - l_N) \neq 0$ is satisfied.

Lemma 2.2. *Let $d < 0$ be given and let c_i, $\theta_i > 0$, $1 \leqslant i \leqslant N$. Then problem* (2.5) *admits a unique solution $m = (m_i)_{1\leqslant i\leqslant N} \in \mathbb{R}^N$. In addition, we can choose $d < 0$ such that $m_0 + m_{N+1} \neq 0$.*

Proof. Let $m_0 \in \mathbb{R}$ be arbitrary fixed. Then $m_1 = m_0 + a_N d$ and from (2.5) we infer that $m_i = \alpha_i m_0 + \beta_i$, $1 \leqslant i \leqslant N$, with $\alpha_i > 0, \beta_i > 0$, $\alpha_{i+1} - \alpha_i > 0$ and $\beta_i - \beta_{i+1} - d > 0$ (if $d < 0$) for $1 \leqslant i \leqslant N$. By the boundary condition $m_{N+1} - m_N = -d$, one obtains $m_0 = (\beta_N - \beta_{N+1} - d)/(\alpha_{N+1} - \alpha_N)$. This m_0 exists and is positive. In addition, we can easily find that

$$m_0 + m_{N+1} = \frac{(\beta_N - \beta_{N+1} - d) - d\alpha_{N+1} + \alpha_{N+1}\beta_N - \beta_{N+1}\alpha_N}{\alpha_{N+1} - \alpha_N} > 0,$$

because $\alpha_{N+1} - \alpha_N > 0$, $\beta_N - \beta_{N+1} - d > 0$ and $\alpha_{N+1}\beta_N - \beta_{N+1}\alpha_N = -\theta_1\theta_2\ldots\theta_N a_N d = -d > 0$. The lemma is proved.

Now we can prove the maximal monotonicity of the operator $\mathcal{B}$ from (1.2) – (1.3).

Proposition 2.3. *The operator $\mathcal{B}$ defined in* (1.2)–(1.3) *is maximal monotone in the weighted Hilbert space H^N with the scalar product* (2.1).

P r o o f. To prove that $\mathcal{B}$ is monotone with respect to the scalar product (2.1), let $u = (u_i)_{1\leqslant i\leqslant N}$, $v = (v_i)_{1\leqslant i\leqslant N}$ be two sequences in the domain $D(\mathcal{B})$ of $\mathcal{B}$ and let $\varphi_i = a_{i-1}(u_i - u_{i-1})$, $\psi_i = a_{i-1}(v_i - v_{i-1})$, $1 \leqslant i \leqslant N$. In view of (1.2), (1.3) and (2.3), we can write

$$\begin{aligned}
\langle \mathcal{B}(u) - \mathcal{B}(v), u - v\rangle &= -\sum_{i=1}^{N}(\varphi_{i+1} - \varphi_i - \psi_{i+1} + \psi_i, u_i - v_i)\\
&= \sum_{i=1}^{N} a_i\|u_{i+1} - u_i - v_{i+1} + v_i\|^2 - \sum_{i=1}^{N}(\varphi_i - \psi_i, u_{i+1} - u_i - v_{i+1} + v_i)\\
&\quad - \sum_{i=1}^{N}(\varphi_{i+1} - \varphi_i - \psi_{i+1} + \psi_i, u_{i+1} - v_{i+1})\\
&= \sum_{i=1}^{N} a_i\|u_{i+1} - u_i - v_{i+1} + v_i\|^2\\
&\quad + \sum_{i=1}^{N}[(\varphi_i - \psi_i, u_i - v_i) - (\varphi_{i+1} - \psi_{i+1}, u_{i+1} - v_{i+1})]\\
&= \sum_{i=1}^{N} a_i\|u_{i+1} - u_i - v_{i+1} + v_i\|^2 + (u_1 - u_0 - v_1 + v_0, u_1 - v_1)\\
&\quad - a_N(u_{N+1} - u_N - v_{N+1} + v_N, u_{N+1} - v_{N+1}).
\end{aligned}$$

Since $u, v \in D(\mathcal{B})$, one obtains

$$\langle \mathcal{B}(u) - \mathcal{B}(v), u - v\rangle = \sum_{i=1}^{N} a_i\|u_{i+1} - u_i - v_{i+1} + v_i\|^2 + \|u_1 - u_0 - v_1 + v_0\|^2 \geqslant 0.$$

Thus $\mathcal{B}$ is monotone in H^N with the scalar product (2.1).

We now prove that $\mathcal{B}$ is maximal monotone, i.e. $R(\mathcal{B} + I) = H$ (see Minty's Theorem 1.4.13, [8]). Therefore, for every sequence $(h_i)_{1\leqslant i\leqslant N} \in H^N$, we are looking for $u = (u_i)_{1\leqslant i\leqslant N} \in H^N$ such that

$$\begin{aligned}
&u_{i+1} - (2 + \theta_i)u_i + \theta_i u_{i-1} = h_i, 1 \leqslant i \leqslant N,\\
&u_0 = -u_{N+1},\ u_1 - u_0 = -a_N(u_{N+1} - u_N).
\end{aligned} \tag{2.6}$$

We search the solution of (2.6) in the form

$$u_i = v_i + l_i x + m_i y, \quad 1 \leqslant i \leqslant N, \tag{2.7}$$

where $x, y \in H$ and l_i, m_i, v_i are solutions of the boundary value problems (2.4), (2.5) and

$$\tag{2.8} \begin{gathered} v_{i+1} - (2+\theta_i)v_i + \theta_i v_{i-1} = h_i, \quad 1 \leqslant i \leqslant N \\ v_0 = 0, \quad v_1 = 0, \end{gathered}$$

respectively. The sequence u_i in (2.7) verifies the equation from (2.6) for all $x, y \in H$. The boundary conditions from (2.6) become

$$\begin{gathered} (l_0 + l_{N+1})x + (m_0 + m_{N+1})y = -v_{N+1}, \\ [l_1 - l_0 + a_N(l_{N+1} - l_N)]x + [m_1 - m_0 + a_N(m_{N+1} - m_N)]y = -a_N(v_{N+1} - v_N). \end{gathered}$$

Lemmas 2.1 and 2.2, together with the boundary conditions from (2.4), (2.5), guarantee the existence and uniqueness of $x, y \in H$:

$$x = \frac{-a_N(v_{N+1} - v_N)}{l_1 - l_0 + a_N(l_{N+1} - l_N)}, \quad y = \frac{-v_{N+1}}{m_0 + m_{N+1}}.$$

Hence $\mathcal{B}$ is maximal monotone with respect to the scalar product (2.1).

3. The main result

In this section we establish the existence of a solution to the boundary value problem (1.1). The main ingredient of the proof is Proposition 2.3.

Theorem 3.1. *Assume that $A\colon D(A) \subseteq H \to H$ is maximal monotone in H, $0 \in D(A)$, $\theta_i, c_i > 0$, $f_i \in H$, $1 \leqslant i \leqslant N$, $a_N = 1/\theta_1\theta_2\ldots\theta_N$. Then the boundary value problem* (1.1) *has a unique solution $u = (u_i)_{1\leqslant i\leqslant N} \in D(A)^N$.*

Proof. Denote by $A_\lambda = (I - (I+\lambda A)^{-1})/\lambda$ and $\mathcal{A}_\lambda = (I - (I+\lambda\mathcal{A})^{-1})/\lambda$ the Yosida approximations of the operators A and $\mathcal{A}$, respectively. Recall that $\mathcal{A}$ defined through (1.4) is maximal monotone in H^N. Since $\mathcal{A}_\lambda$ is also maximal monotone and everywhere defined and $\mathcal{B}$ is maximal monotone with respect to the scalar product (2.1), the sum $\mathcal{A}_\lambda + \mathcal{B}$ is maximal monotone. Consequently, the operator $\mathcal{A}_\lambda + \mathcal{B} + \omega I$ is surjective for every $\omega > 0$, i.e. for any sequence $f = (f_i)_{1\leqslant i\leqslant N} \in H^N$, the problem

$$\tag{3.1} \begin{gathered} u_{i+1}^{\lambda\omega} - (1+\theta_i)u_i^{\lambda\omega} + \theta_i u_{i-1}^{\lambda\omega} = c_i A_\lambda u_i^{\lambda\omega} + \omega u_i^{\lambda\omega} + f_i, \quad 1 \leqslant i \leqslant N, \\ u_0^{\lambda\omega} = -u_{N+1}^{\lambda\omega}, \ u_1^{\lambda\omega} - u_0^{\lambda\omega} = -a_N(u_{N+1}^{\lambda\omega} - u_N^{\lambda\omega}) \end{gathered}$$

has a unique solution $u^{\lambda\omega} = (u_i^{\lambda\omega})_{1\leqslant i\leqslant N} \in H^N$.

S t e p 1. We first prove the boundedness with respect to λ and ω of the sequence $u_i^{\lambda\omega}$. Without loss of generality, we suppose that $0 \in A0$. Otherwise, we replace A by $\tilde{A} = A - A^0 0$ and f_i by $\tilde{f}_i = f_i + c_i A^0 0$, where $A^0 x$ is the element of the minimum norm of the set Ax.

One multiplies (3.1) by $a_i u_i^{\lambda\omega}$ and sums up from $i = 1$ to $i = N$. Using (2.3) and the monotonicity of A_λ, we get

$$\omega \sum_{i=1}^{N} a_i \|u_i^{\lambda\omega}\|^2 \leqslant \sum_{i=1}^{N} [a_i(u_{i+1}^{\lambda\omega} - u_i^{\lambda\omega}, u_i^{\lambda\omega}) - a_{i-1}(u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}, u_{i-1}^{\lambda\omega})] - \sum_{i=1}^{N} a_{i-1}\|u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}\|^2 - \sum_{i=1}^{N} a_i(f_i, u_i^{\lambda\omega}).$$

This implies that

$$\omega \sum_{i=1}^{N} a_i \|u_i^{\lambda\omega}\|^2 + a_N\|u_{N+1}^{\lambda\omega} - u_N^{\lambda\omega}\|^2 + \sum_{i=1}^{N} a_{i-1}\|u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}\|^2 \leqslant \Big(\sum_{i=1}^{N} a_i\|f_i\|^2\Big)^{1/2} \Big(\sum_{i=1}^{N} a_i\|u_i^{\lambda\omega}\|^2\Big)^{1/2}.$$

Hence we have obtained that

$$\sum_{i=1}^{N} a_i\|u_i^{\lambda\omega}\|^2 \leqslant K_1,\ \sum_{i=1}^{N} a_{i-1}\|u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}\|^2 \leqslant K_2,\ \ \|u_{N+1}^{\lambda\omega} - u_N^{\lambda\omega}\| \leqslant K_3, \tag{3.2}$$

where K_1, K_2, K_3 and all K_j below are positive constants. By (3.1) we find also that

$$\|A_\lambda u_i^{\lambda\omega}\| \leqslant K_4, \quad 1 \leqslant i \leqslant N. \tag{3.3}$$

S t e p 2. We now show that $u^{\lambda\omega}$ is strongly convergent in H^N as $\lambda \searrow 0$, for every fixed ω. To do this, we subtract (3.1) for λ and for μ and multiply this difference by $a_i(u_i^{\lambda\omega} - u_i^{\mu\omega})$. Summing up from $i = 1$ to $i = N$ and employing (2.3), we derive that

$$\begin{aligned}\omega \sum_{i=1}^{N} a_i\|u_i^{\lambda\omega} - u_i^{\mu\omega}\|^2 &+ \sum_{i=1}^{N} a_{i-1}\|u_i^{\lambda\omega} - u_i^{\mu\omega} - u_{i-1}^{\lambda\omega} + u_{i-1}^{\mu\omega}\|^2 \\ &\leqslant a_N(u_{N+1}^{\lambda\omega} - u_{N+1}^{\mu\omega} - u_N^{\lambda\omega} + u_N^{\mu\omega}, u_N^{\lambda\omega} - u_N^{\mu\omega}) \\ &\quad - (u_1^{\lambda\omega} - u_1^{\mu\omega} - u_0^{\lambda\omega} + u_0^{\mu\omega}, u_0^{\lambda\omega} - u_0^{\mu\omega}) \\ &\quad + \sum_{i=1}^{N} a_i c_i(A_\lambda u_i^{\lambda\omega} - A_\mu u_i^{\mu\omega}, J_\lambda u_i^{\lambda\omega} - J_\mu u_i^{\mu\omega}) \\ &\quad + \sum_{i=1}^{N} a_i c_i(A_\lambda u_i^{\lambda\omega} - A_\mu u_i^{\mu\omega}, \lambda A_\lambda u_i^{\lambda\omega} - \mu A_\mu u_i^{\mu\omega}).\end{aligned}$$

Since A is monotone and $A_\lambda x \in A(J_\lambda x)$, we have via (3.3)

$$\omega \sum_{i=1}^{N} a_i \|u_i^{\lambda\omega} - u_i^{\mu\omega}\|^2 + \sum_{i=1}^{N} a_{i-1} \|u_i^{\lambda\omega} - u_i^{\mu\omega} - u_{i-1}^{\lambda\omega} + u_{i-1}^{\mu\omega}\|^2 \\ + a_N \|u_{N+1}^{\lambda\omega} - u_{N+1}^{\mu\omega} - u_N^{\lambda\omega} + u_N^{\mu\omega}\|^2 \leqslant K_5(\lambda + \mu).$$

This estimate shows the strong convergence as $\lambda \searrow 0$ of the sequences $u_i^{\lambda\omega}$ and $u_i^{\lambda\omega} - u_{i-1}^{\lambda\omega}$, $1 \leqslant i \leqslant N$. Let $u_i^{\lambda\omega} \to v_i^\omega$ as $\lambda \searrow 0$. Since $A_\lambda u_i^{\lambda\omega}$ is bounded with respect to λ and ω, it is weakly convergent on a subsequence, say $A_\lambda u_i^{\lambda\omega} \rightharpoonup w_i^\omega$ as $\lambda \searrow 0$ in H. Then $J_\lambda u_i^{\lambda\omega} = u_i^{\lambda\omega} - \lambda A_\lambda u_i^{\lambda\omega} \to v_i^\omega$ as $\lambda \searrow 0$. Passing to the limit as $\lambda \searrow 0$ in $A_\lambda u_i^{\lambda\omega} \in A(J_\lambda u_i^{\lambda\omega})$ and in (3.1), one finds that $v_i^\omega \in D(A)$, $w_i^\omega \in A v_i^\omega$ and

$$\begin{aligned} &v_{i+1}^\omega - (1+\theta_i) v_i^\omega + \theta_i v_{i-1}^\omega \in c_i A v_i^\omega + \omega v_i^\omega + f_i, \quad 1 \leqslant i \leqslant N, \\ &v_0^\omega = -v_{N+1}^\omega, \quad v_1^\omega - v_0^\omega = -a_N (v_{N+1}^\omega - v_N^\omega). \end{aligned} \tag{3.4}$$

The solution of this problem is bounded because of (3.2):

$$\|v_i^\omega\| \leqslant K_6, \ 1 \leqslant i \leqslant N. \tag{3.5}$$

S t e p 3. We prove that $v_i^\omega - v_{i-1}^\omega$ is strongly convergent as $\omega \to 0$, $1 \leqslant i \leqslant N+1$. To this end, by (3.4) for ω and γ and by the monotonicity of A we deduce that

$$a_N \|v_{N+1}^\omega - v_{N+1}^\gamma - v_N^\omega + v_N^\gamma\|^2 + \sum_{i=1}^{N} a_{i-1} \|v_i^\omega - v_i^\gamma - v_{i-1}^\omega + v_{i-1}^\gamma\|^2 \\ \leqslant (\omega + \gamma) \sum_{i=1}^{N} a_i c_i (v_i^\omega, v_i^\gamma) \leqslant K_7 (\omega + \gamma).$$

This shows the desired strong convergence. Writing (3.4) in the form

$$v_{i+1}^\omega - v_i^\omega - \theta_i (v_i^\omega - v_{i-1}^\omega) - \omega v_i^\omega - f_i \in c_i A v_i^\omega, \quad 1 \leqslant i \leqslant N$$

and employing the maximal monotonicity of A together with the weak convergence of v_i^ω (say $v_i^\omega \rightharpoonup u_i$), $1 \leqslant i \leqslant N$, it follows that $u_i \in D(A)$ and $u = (u_i)_{1 \leqslant i \leqslant N}$ verifies the problem (1.1). The uniqueness can be easily obtained. This completes the proof.

An e x a m p l e. Denote by $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$ a bounded domain with the boundary $\partial\Omega$ smooth enough. Let $\beta\colon D(\beta) \subseteq \mathbb{R} \to \mathbb{R}$ be a maximal monotone, densely defined operator in $\mathbb{R}$, and let A be the operator $Au = -\Delta u$ with the domain $D(A) = \{u \in H^2(\Omega), -\partial u/\partial\eta \in \beta(u) \text{ a.e. on } \partial\Omega\}$, where $\partial/\partial\eta$ is the outward normal derivative. It is known that this operator is maximal monotone in the Hilbert space $H = L^2(\Omega)$ (see for example [8]). As a consequence of Theorem 3.1, we can state the following existence result for the boundary value problem

$$u_{i+1}(x) - (1+\theta_i)u_i(x) + \theta_i u_{i-1}(x) = -c_i \Delta u_i(x) + f_i(x), \quad x \in \Omega, \quad 1 \leqslant i \leqslant N$$
$$-\partial u_i(x)/\partial \eta \in \beta(u_i(x)), \quad x \in \partial\Omega$$
$$u_0(x) = -u_{N+1}(x), \quad u_1(x) - u_0(x) = -a_N[u_{N+1}(x) - u_N(x)], \quad x \in \Omega.$$

Proposition 3.2. *Let $\beta\colon D(\beta) \subseteq \mathbb{R} \to \mathbb{R}$ be a maximal monotone densely defined operator on $\mathbb{R}$ such that $0 \in \beta(0)$, $f_i \in H = L^2(\Omega)$, $c_i, \theta_i > 0$, $1 \leqslant i \leqslant N$. Then the above boundary value problem has a unique solution $u = (u_i)_{1\leqslant i\leqslant N} \in D(A)^N$.*

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