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NONLINEAR MIXED VOLTERRA-FREDHOLM INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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Abstract. The aim of the present paper is to investigate the global existence of mild solutions of nonlinear mixed Volterra-Fredholm integrodifferential equations, with nonlocal condition. Our analysis is based on an application of the Leray-Schauder alternative and rely on a priori bounds of solutions.

Keywords: global existence, Volterra-Fredholm integrodifferential equation, Leray-Schauder alternative, nonlocal condition

MSC 2010: 45N05, 47G20, 34K30, 47D09

1. Introduction

Let X be a Banach space with the norm $\|\cdot\|$. Let $B = C([0,b],X)$ be the Banach space of all continuous functions from $[0,b]$ into X endowed with the supremum norm

$$\|x\|_B = \sup\{\|x(t)\|\colon\ t \in [0,b]\}.$$

This paper is concerned with the global existence of solutions for the initial value problem for the nonlinear mixed Volterra-Fredholm integrodifferential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - u(t,x(t))] = Ax(t) + f\Big(t, x(t), \int_0^t k(t,s,x(s))\,\mathrm{d}s, \int_0^b h(t,s,x(s))\,\mathrm{d}s\Big), \tag{1.1}$$

$$x(0) = x_0 + g(x), \tag{1.2}$$

where A is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X, $f\colon [0,b]\times X\times X\times X\to X$, $k,h\colon [0,b]\times[0,b]\times X\to X$, $u\colon [0,b]\times X\to X$ are functions and $g\colon C([0,b],X)\to X$ is a given function, and x_0 is a given element of X.

The nonlocal condition, which is a generalization of the classical initial condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski [3], [4]. In a few past years, several authors have been devoted to the study of existence, uniqueness, boundedness and other properties of solutions of the equations (1.1)–(1.2) or their special forms by using different techniques, see [1], [2], [3], [4], [8], [10], [12], [13] and the references cited therein. Our general formulation of (1.1)–(1.2) is an attempt to generalize the results in [5], [6], [10], [11], [12], [13]. We are motivated by the model of the mixed integrodifferential equation established by Dhakne and Kendre [6], and influenced by the work of Byszewski [3].

The aim of the present paper is to study the global existence of solutions of the equations (1.1)–(1.2). The main tool used in our analysis is based on an application of the Leray-Schauder alternative and relies on a priori bounds of solutions. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

The outline of the paper is as follows. In Section 2, we present the preliminaries and hypotheses. Section 3 is concerned with the main result. Finally, we give an example to illustrate the application of our theorem in Section 4.

2. Preliminaries

We list some preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. Let A be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X and $f\in L^1(0,b;X)$. The function $x\in C([0,b],X)$ given by

$$\begin{aligned}(2.1)\quad x(t) &= T(t)[(x_0+g(x))-u(0,x_0+g(x))]+T(t)u(t,x(t))\\ &\quad+\int_0^t T(t-s)f\left(s,x(s),\int_0^s k(s,\tau,x(\tau))\,\mathrm{d}\tau,\int_0^b h(s,\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s,\\ &\qquad t\in[0,b],\end{aligned}$$

is called the mild solution of the initial value problem (1.1)–(1.2).

For completeness we state here the following fixed point result by Granas in ([7], p. 61).

Lemma 2.2 (Leray-Schauder Alternative)**.** *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F\colon S \to S$ be a completely continuous operator, and let*

$$\varepsilon(F) = \{x \in S\colon\ x = \lambda F x \ \text{ for some } \ 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ is unbounded or F has a fixed point.

We consider the following hypotheses:

(H_1) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in X such that

$$\|T(t)\| \leqslant N_0$$

for some $N_0 \geqslant 1$.

(H_2) There exist nonnegative constants c_1 and c_2 such that

$$\|u(t, x(t))\| \leqslant c_1\|x\| + c_2$$

for every $t \in [0, b]$ and $x \in X$.

(H_3) There exists a constant G such that

$$\|g(x)\| \leqslant G$$

for $x \in X$.

(H_4) There exists a continuous function $p\colon\ [0, b] \to \mathbb{R}_+$ such that

$$\left\|\int_0^t k(t, s, x(s))\,\mathrm{d}s\right\| \leqslant p(t)\|x\|$$

for every $t \geqslant s \geqslant 0$ and $x \in X$.

(H_5) There exists a continuous function $q\colon\ [0, b] \to \mathbb{R}_+$ such that

$$\left\|\int_0^b h(t, s, x(s))\,\mathrm{d}s\right\| \leqslant q(t)\|x\|$$

for every $t, s \in [0, b]$ and $x \in X$.

(H_6) There exists a continuous function $l\colon\ [0, b] \to \mathbb{R}_+$ such that

$$\|f(t, x, y, z)\| \leqslant l(t)K(\|x\| + \|y\| + \|z\|)$$

for every $t \in [0,b]$ and $x, y, z \in X$, where $K\colon \mathbb{R}_+ \to (0,\infty)$ is a continuous nondecreasing function satisfying

$$K(\alpha(t)x) \leqslant \alpha(t)K(x),$$

and $\alpha(t)$ is defined as the function p.

(H_7) For each $t \in [0,b]$ the function $f(t,\cdot,\cdot,\cdot)\colon [0,b]\times X\times X\times X \to X$ is continuous and for each $x,y,z \in X$ the function $f(\cdot,x,y,z)\colon [0,b]\times X\times X\times X \to X$ is strongly measurable.

(H_8) For each $t,s \in [0,b]$ the functions $k(t,s,\cdot), h(t,s,\cdot)\colon [0,b]\times[0,b]\times X \to X$ are continuous and for each $x \in X$ the functions $k(\cdot,\cdot,x), h(\cdot,\cdot,x)\colon [0,b]\times[0,b]\times X \to X$ are strongly measurable.

(H_9) For every positive integer m there exists $\alpha_m \in L^1(0,b)$ such that

$$\sup_{\|x\|\leqslant m, \|y\|\leqslant m, \|z\|\leqslant m} \|f(t,x,y,z)\| \leqslant \alpha_m(t) \quad \text{for } t \in [0,b] \text{ a.e.}$$

3. Global existence

Theorem 3.1. *Suppose that the hypotheses* (H_1)–(H_9) *hold. Then the initial value problem* (1.1)–(1.2) *has a solution x on $[0,b]$ provided b satisfies*

$$\int_0^b M(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{K(s)}, \tag{3.1}$$

where

$$c = \frac{N_0[\|x_0\| + G + \sup_{\|y\|\leqslant G}\|u(0,x_0+y)\| + c_2]}{1-N_0c_1}, \quad N_0c_1 < 1,$$

and

$$M(t) = \frac{N_0 l(t)[1+p(t)+q(t)]}{1-N_0c_1} \quad \textit{for } t \in [0,b].$$

Proof. We define an operator $F\colon B \to B$ for each $t \in [0,b]$ by

$$\begin{aligned}(Fx)(t) &= T(t)[(x_0+g(x)) - u(0,x_0+g(x))] + T(t)u(t,x(t)) \\ &\quad + \int_0^t T(t-s) f\left(s, x(s), \int_0^s k(s,\tau,x(\tau))\,\mathrm{d}\tau, \int_0^b h(s,\tau,x(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.\end{aligned} \tag{3.2}$$

In order to use the Leray-Schauder Alternative, we obtain an a priori bound for the solution of the integral equation $x = \lambda F(x)$, $\lambda \in (0,1)$. If x^λ is a solution of $x = \lambda F(x)$, $\lambda \in (0,1)$, then by using (3.2) and the hypotheses (H_1)–(H_6) we have

$$\begin{aligned}
\|x^\lambda(t)\| &\leqslant N_0[\|x_0\| + G + \sup_{\|y\|\leqslant G} \|u(0, x_0+y)\|] + N_0[c_1\|x^\lambda(t)\| + c_2] \\
&\quad + \int_0^t N_0 l(s) K(\|x^\lambda(s)\| + p(s)\|x^\lambda(s)\| + q(s)\|x^\lambda(s)\|)\,\mathrm{d}s \\
&\leqslant N_0[\|x_0\| + G + \sup_{\|y\|\leqslant G} \|u(0, x_0+y)\|] + N_0[c_1\|x^\lambda(t)\| + c_2] \\
&\quad + \int_0^t N_0 l(s)(1+p(s)+q(s)) K(\|x^\lambda(s)\|)\,\mathrm{d}s \\
&\leqslant \frac{N_0[\|x_0\| + G + \sup_{\|y\|\leqslant G}\|u(0,x_0+y)\| + c_2]}{1 - N_0 c_1} \\
&\quad + \frac{N_0}{1-N_0c_1}\int_0^t l(s)(1+p(s)+q(s))K(\|x^\lambda(s)\|)\,\mathrm{d}s.
\end{aligned} \tag{3.3}$$

Denoting by $r_\lambda(t)$ the right-hand side of the inequality (3.3), we find that

$$\begin{aligned}
r'_\lambda(t) &\leqslant \frac{N_0}{1-N_0c_1}\,[l(t)(1+p(t)+q(t))K(r_\lambda(t))] \\
\frac{r'_\lambda(t)}{K(r_\lambda(t))} &\leqslant M(t).
\end{aligned} \tag{3.4}$$

Integrating (3.4) from 0 to t and using the change of variables $t \to s = r_\lambda(t)$ and the condition (3.1), we obtain

$$\int_c^{r_\lambda(t)} \frac{\mathrm{d}s}{K(s)} \leqslant \int_0^t M(s)\,\mathrm{d}s \leqslant \int_0^b M(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{K(s)}. \tag{3.5}$$

From this inequality and the mean value theorem we observe that there exists a constant γ independent of $\lambda \in (0,1)$ such that $r_\lambda(t) \leqslant \gamma$ for $t \in [0,b]$, which implies that the set $\{r_\lambda : \lambda \in (0,1)\}$ is bounded in B and hence that $\{x^\lambda : \lambda \in (0,1)\}$ is bounded in B.

Next we prove that F is completely continuous. Let $B_m = \{x \in B \colon \|x\|_B \leqslant m\}$ for some $m \geqslant 1$. We first show that F maps B_m into an equicontinuous family of functions with values in X. Let $x \in B_m$ and $t_1, t_2 \in [0,b]$. Then if $\varepsilon < t_1 < t_2 \leqslant b$, then

$$
\begin{aligned}
(3.6)\quad & \|(Fx)(t_1)-(Fx)(t_2)\| \\
&\leqslant \|T(t_1)-T(t_2)\|[\|x_0\|+G+\sup_{\|y\|\leqslant G}\|u(0,x_0+y)\|] \\
&\quad + N_0[c_1\|x(t_1)\|+c_2] \\
&\quad + N_0[c_1\|x(t_2)\|+c_2] + \int_0^{t_1-\varepsilon}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s \\
&\quad + \int_{t_1-\varepsilon}^{t_1}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2}\|T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s \\
&\leqslant \|T(t_1)-T(t_2)\|[\|x_0\|+G+\sup_{\|y\|\leqslant G}\|u(0,x_0+y)\|] \\
&\quad + N_0[2mc_1+2c_2] + \int_0^{t_1-\varepsilon}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s \\
&\quad + \int_{t_1-\varepsilon}^{t_1}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s + \int_{t_1}^{t_2}\|T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s \\
&\leqslant \|T(t_1)-T(t_2)\|[\|x_0\|+G+\sup_{\|y\|\leqslant G}\|u(0,x_0+y)\|] \\
&\quad + N_0[2mc_1+2c_2] + \int_0^{t_1-\varepsilon}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s \\
&\quad + \int_{t_1-\varepsilon}^{t_1}\|T(t_1-s)-T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s + \int_{t_1}^{t_2}\|T(t_2-s)\|\alpha_m(s)\,\mathrm{d}s.
\end{aligned}
$$

The right-hand side of inequality (3.6) is independent of $x \in B_m$ and tends to zero as $t_2 \to t_1$, since $T(t)$ is continuous for $t \in [0,b]$ and the compactness $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Now we show that FB_m is uniformly bounded. From the definition of the operator F, hypotheses (H_1)–(H_6) and the fact that $\|x\|_B \leqslant m$, we obtain

$$
\begin{aligned}
&\|(Fx)(t)\| \\
&\quad\leqslant \|T(t)\|[\|(x_0+g(x))-u(0,x_0+g(x))\|] + \|T(t)u(t,x(t))\| \\
&\qquad + \int_0^t \|T(t-s)\|\left\|f\left(s,x(s),\int_0^s k(s,\tau,x(\tau))\,\mathrm{d}\tau,\int_0^b h(s,\tau,x(\tau))\,\mathrm{d}\tau\right)\right\|\mathrm{d}s \\
&\quad\leqslant N_0[\|x_0\|+G+\sup_{\|y\|\leqslant G}\|u(0,x_0+y)\|] + N_0[c_1\|x(t)\|+c_2] + N_0\int_0^t \alpha_m(s)\,\mathrm{d}s \\
&\quad\leqslant N_0[\|x_0\|+G+\sup_{\|y\|\leqslant G}\|u(0,x_0+y)\|] + N_0[c_1 m+c_2] + N_0\int_0^t \alpha_m(s)\,\mathrm{d}s.
\end{aligned}
$$

This implies that the set $\{(Fx)(t)\colon\ \|y\|_B \leqslant m,\ 0 \leqslant t \leqslant b\}$ is uniformly bounded in X and hence FB_m is uniformly bounded.

We have already shown that FB_m is an equicontinuous and uniformly bounded collection. To prove that F maps B_m into a precompact set in B, it is sufficient, by Arzela-Ascoli's Theorem, to show that the set $\{(Fx)(t)\colon\ x \in B_m\}$ is precompact in X for each $t \in [0, b]$. Let $0 < t \leqslant b$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B_m$ we define

$$
\begin{aligned}
(3.7)\quad (F_\varepsilon x)(t) &= T(t)[(x_0 + g(x)) - u(0, x_0 + g(x))] + T(t)u(t, x(t)) \\
&\quad + \int_0^{t-\varepsilon} T(t-s) f\biggl(s, x(s), \int_0^s k(s,\tau,x(\tau))\,\mathrm{d}\tau, \int_0^b h(s,\tau,x(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s.
\end{aligned}
$$

Since $T(t)$ is a compact operator, the set $Y_\varepsilon(t) = \{(F_\varepsilon x)(t)\colon\ x \in B_m\}$ is precompact in X for every ε, $0 < \varepsilon < t$. Moreover, for every $x \in B_m$, we get

$$
\begin{aligned}
&\|(Fx)(t) - (F_\varepsilon x)(t)\| \\
&\quad\leqslant \int_{t-\varepsilon}^t \|T(t-s)\| \biggl\| f\biggl(s, x(s), \int_0^s k(s,\tau,x(\tau)\biggr)\,\mathrm{d}\tau, \int_0^b h(s,\tau,x(\tau))\,\mathrm{d}\tau\biggr\|\,\mathrm{d}s \\
&\quad\leqslant N_0 \int_{t-\varepsilon}^t \alpha_m(s)\,\mathrm{d}s.
\end{aligned}
$$

This shows that there exist precompact sets arbitrarily close to the set $\{(Fx)(t)\colon\ x \in B_m\}$. Hence, the set $\{(Fx)(t)\colon\ x \in B_m\}$ is precompact in X.

It remains to show that $F\colon\ B \to B$ is continuous. Let $\{v_n\}$ be a sequence of elements of B converging to v in B. Then there exists an integer r such that $\|v_n\| \leqslant r$ for all n and $t \in B$. By hypotheses (H_7)–(H_9), we have

$$
\begin{aligned}
&f\biggl(t, v_n(t), \int_0^t k(t,s,v_n(s))\,\mathrm{d}s, \int_0^b h(t,s,v_n(s))\,\mathrm{d}s\biggr) \\
&\qquad \to f\biggl(t, v(t), \int_0^t k(t,s,v(s))\,\mathrm{d}s, \int_0^b h(t,s,v(s))\,\mathrm{d}s\biggr)
\end{aligned}
$$

for each $t \in I$. Since

$$
\begin{aligned}
&\biggl\| f\biggl(t, v_n(t), \int_0^t k(t,s,v_n(s))\,\mathrm{d}s, \int_0^b h(t,s,v_n(s))\,\mathrm{d}s\biggr) \\
&\qquad - f\biggl(t, v(t), \int_0^t k(t,s,v(s))\,\mathrm{d}s, \int_0^b h(t,s,v(s)))\,\mathrm{d}s\biggr\| \leqslant 2\alpha_r(t),
\end{aligned}
$$

dominated convergence yields

$$
\begin{aligned}
&\|(Fv_n)(t)-(Fv)(t)\| \\
&\qquad \leqslant \|T(t)[(g(v_n)-g(v))-(u(0,x_0+g(v_n)) \\
&\qquad\quad -u(0,x_0+g(v)))+(u(t,v_n(t))-u(t,v(t)))]\| \\
&\qquad\quad +N_0\int_0^t \bigg\|\bigg[f\bigg(s,v_n(s),\int_0^s k(s,\tau,v_n(\tau))\,\mathrm{d}\tau,\int_0^b h(s,\tau,v_n(\tau))\,\mathrm{d}\tau\bigg) \\
&\qquad\quad -f\bigg(s,v(s),\int_0^s k(s,\tau,v(\tau))\,\mathrm{d}\tau,\int_0^b (s,\tau,v(\tau))\,\mathrm{d}\tau\bigg)\bigg]\bigg\|\,\mathrm{d}s \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Thus F is continuous. This completes the proof that F is a completely continuous operator.

Finally, the set

$$\varepsilon(F)=\{x\in B\colon\ x=\lambda Fx,\ \ \lambda\in(0,1)\}$$

is bounded in B as was proved in the first part. Consequently, by Lemma 2.2, the operator F has a fixed point in B. This means that the initial value problem (1.1)–(1.2) has a solution. This completes the proof of the theorem. □

R e m a r k 3.2. We note that the advantage of our approach here is that, it yields simultaneously the existence of solution of (1)–(2) and maximal interval of existence. In the special case, if we take $M(t)=1$ in (3.1) and the integral on the right-hand side in (3.1) is assumed to diverge, then the solution of (1)–(2) exists for every $b<\infty$; that is, on the entire interval. Our result in Theorem 3.1 yields the existence of solution of (1)–(2) on $[0,b]$, if the integral on the right-hand side in (3.1) is divergent, i.e. $\int_c^\infty \mathrm{d}s/K(s)=\infty$. Thus Theorem 3.1 can be considered as a further extension of the well known theorem on the global existence of solution of ordinary differential equation due to Winter given in [14].

4. Application

In this section, we give an example to illustrate the application of our main result. Consider the nonlinear mixed partial integrodifferential equation of the form

$$
\begin{aligned}
(4.1)\qquad &\frac{\partial}{\partial t}[w(u,t)-v(t,w(u,t))]+\frac{\partial^2}{\partial t^2}w(u,t) \\
&\qquad = P\bigg(t,w(u,t),\int_0^t k_1(t,s,w(u,s))\,\mathrm{d}s,\int_0^b h_1(t,s,w(u,s))\,\mathrm{d}s\bigg),
\end{aligned}
$$

$$w(0,t) = w(\pi,t) = 0, \quad 0 \leqslant t \leqslant b, \tag{4.2}$$

$$w(u,0) = w_0(u) + g(w(u,t)), \quad 0 \leqslant u \leqslant \pi, \tag{4.3}$$

where $P\colon [0,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k_1, h_1\colon [0,b] \times [0,b] \times \mathbb{R} \to \mathbb{R}$, and $v\colon [0,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. We assume that the functions P, k_1, h_1, and v in (4.1)–(4.3) satisfy the following conditions.

(1) There exists a nonnegative constant G_1 such that

$$\|g\| \leqslant G_1$$

for $x \in \mathbb{R}$.

(2) There exists a nonnegative function p_1 defined on $[0,b]$ such that

$$\left| \int_0^t k_1(t,s,x)\,\mathrm{d}s \right| \leqslant p_1(t)|x|$$

for $t, s \in [0,b]$ and $x \in \mathbb{R}$.

(3) There exists a nonnegative function q_1 defined on $[0,b]$ such that

$$\left| \int_0^b h_1(t,s,x)\,\mathrm{d}s \right| \leqslant q_1(t)|x|$$

for $t, s \in [0,b]$ and $x \in \mathbb{R}$.

(4) There exist a nonnegative real valued continuous function l_1 defined on $[0,b]$ and a positive continuous increasing function K_1 defined on $\mathbb{R}_+$ such that

$$|P(t,x,y,z)| \leqslant l_1(t) K_1(|x| + |y| + |z|)$$

for $t \in [0,b]$ and $x, y, z \in \mathbb{R}$.

(5) There exist nonnegative constants L_1 and L_2 such that

$$|v(t,x)| \leqslant L_1|x| + L_2.$$

(6) For every positive integer m_1 there exists $\alpha_{m_1} \in L^1[0,b]$ such that

$$\sup_{|x| \leqslant m_1, |y| \leqslant m_1, |z| \leqslant m_1} |P(t,x,y,z)| \leqslant \alpha_{m_1}(t)$$

for $0 \leqslant t \leqslant b$ a.e.

Let $X = L^2[0,\pi]$. We define an operator $A\colon X \to X$ by $Ax = -x''$ with the domain $D(A) = \{x \in X\colon x, x'$ are absolutely continuous, $x'' \in X$ and $x(0) = x(\pi) = 0\}$. Then the operator A can be expressed as

$$Ax = \sum_{n=1}^{\infty} n^2 (x, x_n) x_n, \quad x \in D(A),$$

where $x_n(u) = (\sqrt{2/\pi}) \sin nu$, $n = 1, 2, 3 \ldots$, is the orthogonal set of eigenvectors of A and A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geqslant 0$, and is given by

$$T(t)x = \sum_{n=1}^{\infty} \exp(-n^2 t)(x, x_n) x_n, \quad x \in X.$$

Now, the analytic semigroup $T(t)$ being compact, there exists a constant N_0 such that $|T(t)| \leqslant N_0$ for each $t \in [0, b]$.

Suppose that the condition

$$\frac{N_0}{1 - N_0 L_1} \int_0^b l_1(s)(1 + p_1(s) + q_1(s)) \, \mathrm{d}s < \int_c^{\infty} \frac{\mathrm{d}s}{K_1(s)}$$

is satisfied, where

$$c = \frac{N_0[\|x_0\| + G_1 + \sup_{\|y\| \leqslant G_1} \|u(0, x_0 + y)\| + L_2]}{1 - N_0 L_1}, \quad N_0 L_1 < 1.$$

Define functions $f\colon [0, b] \times X \times X \times X \to X$, $k, h\colon [0, b] \times [0, b] \times X \to X$, and $u\colon [0, b] \times X \to X$ as follows:

$$\begin{aligned} f(t, x, y, z)(u) &= P(t, x(u), y(u), z(u)), \\ k(t, s, x)(u) &= k_1(t, s, x(u)), \\ h(t, s, x)(u) &= h_1(t, s, x(u)), \end{aligned}$$

and

$$u(t, x)(u) = u(t, x(u))$$

for $t \in [0, b]$, $x, y, z \in X$ and $0 \leqslant u \leqslant \pi$. Due to the above choices of the functions and the generator A, the equations (4.1)–(4.3) can be formulated as an abstract nonlinear mixed integrodifferential equation in the Banach space X:

$$\text{(4.4)} \quad \frac{\mathrm{d}}{\mathrm{d}t}[x(t) - u(t, x(t))] = Ax(t) + f\Big(t, x(t), \int_0^t k(t, s, x(s)) \, \mathrm{d}s, \int_0^b h(t, s, x(s)) \, \mathrm{d}s\Big),$$

$$\text{(4.5)} \quad x(0) = x_0 + g(x).$$

Since all the hypotheses of Theorem 3.1 are satisfied, therefore Theorem 3.1 can be applied to guarantee the solution of the nonlinear mixed partial integrodifferential equation (4.1)–(4.3).

References

[1] *K. Balachandran, M. Chandrasekaran*: Existence of solutions of nonlinear integrodifferential equation with nonlocal condition. J. Appl. Math. Stochastic Anal. *10* (1997), 279–288.

[2] *K. Balachandran*: Existence and uniqueness of mild and strong solutions of nonlinear integrodifferential equations with nonlocal condition. Differ. Equ. Dyn. Syst. *6* (1998), 159–165.

[3] *L. Byszewski*: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. *162* (1991), 494–505.

[4] *L. Byszewski*: Existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. Zesz. Nauk. Politech. Rzesz. 121, Mat. Fiz. *18* (1993), 109–112.

[5] *M. B. Dhakne, S. D. Kendre*: On a nonlinear Volterra integrodifferential equation in Banach spaces. Math. Inequal. Appl. *9* (2006), 725–735.

[6] *M. B. Dhakne, S. D. Kendre*: On an abstract nonlinear integrodifferential equation. Proceedings of Second International Conference on Nonlinear Systems, December 2007. Bulletin of Marathwada Mathematical Society *8* (2007), 12–22.

[7] *J. Dugundji, A. Granas*: Fixed Point Theory. I: Monografie Matematyczne. PWN, Warszawa, 1982.

[8] *S. K. Ntouyas, P. C. Tsamatos*: Global existence for semilinear evolution equations with nonlocal conditions. J. Math. Anal. Appl. *210* (1997), 679–687.

[9] *S. K. Ntouyas, P. C. Tsamatos*: Global existence for second-order semilinear ordinary and delay integrodifferential equations with nonlocal conditions. Appl. Anal. *67* (1997), 245–257.

[10] *B. G. Pachpatte*: Applications of the Leray-Schauder alternative to some Volterra integral and integrodifferential equations. Indian J. Pure Appl. Math. *26* (1995), 1161–1168.

[11] *A. Pazy*: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York, 1983.

[12] *H. L. Tidke, M. B. Dhakne*: On global existence of solutions of abstract nonlinear mixed integrodifferential equation with nonlocal condition. Commun. Appl. Nonlinear Anal. *16* (2009), 49–59.

[13] *H. L. Tidke*: Existence of global solutions to nonlinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions. Electron. J. Differ. Equ., paper No. 55 *2009* (2009), 1–7.

[14] *A. Winter*: The nonlocal existence problem for ordinary differential equations. Am. J. Math. *67* (1945), 277–284.

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