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LOWER BOUND AND UPPER BOUND OF OPERATORS ON BLOCK WEIGHTED SEQUENCE SPACES

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Abstract. Let $A=(a_{n,k})_{n,k\geqslant 1}$ be a non-negative matrix. Denote by $L_{v,p,q,F}(A)$ the supremum of those L that satisfy the inequality

$$\|Ax\|_{v,q,F}\geqslant L\|x\|_{v,p,F},$$

where $x\geqslant 0$ and $x\in \ell_p(v,F)$ and also $v=(v_n)_{n=1}^{\infty}$ is an increasing, non-negative sequence of real numbers. If $p=q$, we use $L_{v,p,F}(A)$ instead of $L_{v,p,p,F}(A)$. In this paper we obtain a Hardy type formula for $L_{v,p,q,F}(H_\mu)$, where H_μ is a Hausdorff matrix and $0<q\leqslant p\leqslant 1$. Another purpose of this paper is to establish a lower bound for $\|A_W^{NM}\|_{v,p,F}$, where A_W^{NM} is the Nörlund matrix associated with the sequence $W=\{w_n\}_{n=1}^{\infty}$ and $1<p<\infty$. Our results generalize some works of Bennett, Jameson and present authors.

Keywords: lower bound, weighted sequence space, Hausdorff matrix, Euler matrix, Cesàro matrix, Hölder matrix, Gamma matrix

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1. Introduction

Let $v=(v_n)_{n=1}^{\infty}$ be an increasing, non-negative sequence of real numbers with $v_1=v_2=1$ and $\sum\limits_{n=1}^{\infty} v_n/n=\infty$. For $p\in\mathbb{R}\setminus\{0\}$, let $\ell_p(v)$ denote the space of all real sequences $x=\{x_k\}_{k=1}^{\infty}$, such that

$$\|x\|_{v,p}:=\biggl(\sum_{k=1}^{\infty} v_k x_k^p\biggr)^{1/p}<\infty.$$

Next, assume that F is a partition of positive integers. If $F=(F_n)$, where each (F_n) is a finite interval of positive integers and

$$\max F_n<\min F_{n+1}\quad (n=1,2,3,\ldots),$$

we denote by $\ell_p(v,F)$ the space of all real sequences $x = \{x_k\}_{k=1}^{\infty}$ such that

$$\|x\|_{v,p,F} := \left(\sum_{k=1}^{\infty} v_k |\langle x, F_k\rangle|^p\right)^{1/p} < \infty,$$

where $\langle x, F_k\rangle = \sum\limits_{j\in F_k} x_j$. This space is called the block weighted sequence space (see [1]).

For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$ is a partition of positive integers, $\ell_p(w,I) = \ell_p(w)$, and also $\|x\|_{w,p,F} = \|x\|_{w,p}$.

We write $x \geqslant 0$ if $x_k \geqslant 0$ for all k. We also write $x \uparrow$ for the case that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$. The symbol $x \downarrow$ is defined in a similar way. For $p, q \in \mathbb{R} \setminus \{0\}$, the lower bound involved here is the number $L_{w,p,q,F}(A)$ which is defined as the supremum of those L that obey the inequality

$$\|Ax\|_{v,q,F} \geqslant L\|x\|_{v,p,F},$$

where $x \geqslant 0$, $x \in \ell_p(v,F)$, and $A = (a_{n,k})_{n,k\geqslant 1}$ is a non-negative matrix operator from $\ell_p(v,F)$ into $\ell_q(v,F)$. Also, we consider the upper bounds U of the form

$$\|Ax\|_{v,p,F} \leqslant U\|x\|_{v,p,I}$$

for all non-negative sequences x in $\ell_p(v,I)$. We seek the smallest possible value of U, and denote the best upper bound by $\|A\|_{v,p,F}$ for a matrix operator A from $\ell_p(v,I)$ into $\ell_p(v,F)$. Obviously, we have

$$L_{v,p,F}(A) \leqslant \|A\|_{v,p,F}.$$

In Section 2 we generalize some techniques obtained by Chen and the present authors in [6], [12] and deduce a lower bound for the Hausdorff matrices. In Section 3, we also generalize Theorem 2.4 of [14] (also, Theorem 2.1 of [9]) to matrix operators from $\ell_p(v,I)$ into $\ell_p(v,F)$ and study the upper bound problem for some Nörlund matrices.

Throughout the paper, we denote the conjugate exponent of p by p^*, so that $p^* = p/(p-1)$. We also suppose that $F_1 = \{1\}$.

2. Hausdorff matrix operator

In this part, we are interested in the problem of finding the exact value of $L_{v,p,q,F}(A)$ for the case $A = H_\mu$, where $\mathrm{d}\mu$ is a Borel probability measure on [0,1] and $H_\mu = H_\mu(\theta) = (h_{n,k}(\theta))_{n,k\geqslant 1}$ is the Hausdorff matrix associated with $\mathrm{d}\mu$, defined by

$$h_{n,k}(\theta) = \begin{cases} \binom{n-1}{k-1} \int_0^1 \theta^{k-1}(1-\theta)^{n-k}\,\mathrm{d}\mu(\theta), & 1 \leqslant k \leqslant n, \\ 0, & k > n. \end{cases}$$

Clearly $h_{n,k} = \binom{n-1}{k-1} \Delta^{n-k}\mu_k$ for $n \geqslant k \geqslant 1$, where

$$\mu_k = \int_0^1 \theta^{k-1}\,\mathrm{d}\mu(\theta) \quad (k = 1, 2, \ldots)$$

and $\Delta^{n-k}\mu_k = \mu_k - \mu_{k+1}$.

The Hausdorff matrices contain some famous classes of matrices. These classes are as follows:

(i) Choice $\mathrm{d}\mu(\theta) = \alpha(1-\theta)^{\alpha-1}\,\mathrm{d}\theta$ gives the Cesàro matrix of order α;
(ii) choice $\mathrm{d}\mu(\theta) =$ point evaluation at $\theta = \alpha$ gives the Euler matrix of order α;
(iii) choice $\mathrm{d}\mu(\theta) = (|\log\theta|^{\alpha-1}/\Gamma(\alpha))\,\mathrm{d}\theta$ gives the Hölder matrix of order α;
(iv) choice $\mathrm{d}\mu(\theta) = \alpha\theta^{\alpha-1}\,\mathrm{d}\theta$ gives the Gamma matrix of order α.

The Cesàro, Hölder and Gamma matrices have non-negative entries whenever $\alpha > 0$, and also the Euler matrix is non-negative when $0 \leqslant \alpha \leqslant 1$.

In this section we exhibit a Hardy type formula for $L_{v,p,q,F}(H_\mu)$, where $0 < q \leqslant p \leqslant 1$. In particular, we apply our results to the Cesàro matrices, Hölder matrices and Gamma matrices which were recently considered in [2], [4], [5], [6], and [8] on the ℓ_p spaces and in [7], [10], [11], [12] on the usual weighted sequence spaces $\ell_p(v)$.

Proposition 2.1. *Let $0 < p < 1$ and let $A = (a_{n,k})$ be a lower triangular matrix with non-negative entries. If*

$$\sup_{n\geqslant 1} \sum_{k=1}^{n} a_{n,k} = R$$

and

$$\inf_{k\geqslant 1} \sum_{n=k}^{\infty} a_{n,k} = C > 0,$$

then $\|Ax\|_{v,p,F} \geqslant L\|x\|_{v,p,I}$ with

$$L \geqslant R^{1/p^*} C^{1/p}.$$

P r o o f. Applying Hölder's inequality, we have

$$\begin{aligned}\sum_{k=1}^{n} a_{n,k} v_k x_k^p &= \sum_{k=1}^{n} a_{n,k}^{1-p} (a_{n,k} v_k^{1/p} x_k)^p \\ &\leqslant \Big(\sum_{k=1}^{n} a_{n,k}\Big)^{1-p} \Big(\sum_{k=1}^{n} a_{n,k} v_k^{1/p} x_k\Big)^p \\ &\leqslant R^{1-p} \Big(\sum_{k=1}^{n} a_{n,k} v_k^{1/p}\Big)^p.\end{aligned}$$

Since v is increasing, we have

$$\begin{aligned}R^{1-p} \sum_{n=1}^{\infty} v_n \Big(\sum_{i\in F_n} \sum_{j=1}^{\infty} a_{i,j} x_j\Big)^p &= R^{1-p} \sum_{n=1}^{\infty} v_n \Big(\sum_{i\in F_n} \sum_{j=1}^{i} a_{i,j} x_j\Big)^p \\ &\geqslant R^{1-p} \sum_{n=1}^{\infty} \Big(\sum_{i\in F_n} \sum_{j=1}^{i} a_{i,j} v_j^{1/p} x_j\Big)^p \\ &\geqslant \sum_{n=1}^{\infty} \Big(\sum_{i\in F_n} \sum_{j=1}^{i} a_{i,j} v_j x_j^p\Big) \\ &= \sum_{j=1}^{\infty} v_j x_j^p \Big(\sum_{n=j}^{\infty} a_{n,j}\Big) \geqslant C \sum_{k=1}^{\infty} v_k \Big(\sum_{j\in I_k} x_j\Big)^p,\end{aligned}$$

and this leads to the desired inequality. □

For $\alpha \geqslant 0$, let $E(\alpha) = (e_{n,k}(\alpha))_{n,k\geqslant 1}$ denote the Euler matrix, defined by

$$e_{n,k}(\alpha) = \begin{cases} \binom{n-1}{k-1} \alpha^{k-1}(1-\alpha)^{n-k}, & n \geqslant k, \\ 0, & n < k \end{cases}$$

(cf. [2, p. 410]). For $\Omega \subset (0,1]$ we have

$$\int_{\Omega} e_{n,k}(\theta)\,\mathrm{d}\mu(\theta) = \mu(\Omega) \times \int_0^1 e_{n,k}(\theta)\,\mathrm{d}\lambda(\theta),$$

where $\mathrm{d}\lambda = (\chi_\Omega/\mu(\Omega))\,\mathrm{d}\mu$ is a Borel probability measure on [0,1] with $\lambda(\{0\}) = 0$. Hence the second part of ([3, Proposition 19.2]) can be generalized in the following way.

Proposition 2.2. *Suppose that $0 < p \leqslant 1$, $\Omega \subseteq [0,1]$ and $\mathrm{d}\mu$ is any Borel probability measure on $[0,1]$. If $\mu(\{0\}) = 0$ or $\Omega \subset (0,1]$, then the sequence $\big\|\big\{\int_\Omega e_{n,k}(\theta)\,\mathrm{d}\mu(\theta)\big\}_{n=k}^{\infty}\big\|_{v,p}$ increases with k.*

Proposition 2.3. *Let $0 < p \leqslant 1$. Then $L_{v,p,F}(E(\alpha)) \geqslant \alpha^{-1/p}$ for $0 < \alpha \leqslant 1$.*

Proof. We have $\sum\limits_{k=1}^{\infty} e_{n,k}(\alpha) = 1$ $(n \geqslant 1)$ and $\sum\limits_{n=1}^{\infty} e_{n,k}(\alpha) = \alpha^{-1}$ $(k \geqslant 1)$. Applying Proposition 2.1 to the case that $R = 1$ and $C = \alpha^{-1}$ we deduce that $L_{v,p,F}(E(\alpha)) \geqslant \alpha^{-1/p}$ for $0 < p < 1$. For $p = 1$, from the Fubini theorem and the monotonicity of (v_n) we deduce that

$$\begin{aligned}\|E(\alpha)x\|_{v,1,F} &= \sum_{n=1}^{\infty} v_n \langle E(\alpha)x, F_n\rangle \\ &= \sum_{n=1}^{\infty} v_n \Big(\sum_{i\in F_n}\sum_{k=1}^{\infty} e_{i,k}(\alpha)x_k\Big) \\ &\geqslant \sum_{i=1}^{\infty} v_i \Big(\sum_{n=1}^{\infty} e_{n,i}(\alpha)\Big)\Big(\sum_{j\in I_i} x_j\Big) \geqslant \alpha^{-1}\|x\|_{v,1,I},\end{aligned}$$

which gives the desired inequality. This completes the proof. $\square$

Now we are ready to introduce the basic theorem of this section.

Theorem 2.4. *We have*

$$L_{v,p,q,F}(H_\mu) \geqslant \int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta) \quad (0 < q \leqslant p \leqslant 1). \tag{2.1}$$

Moreover, the following statements are true:

(i) *For $p = q = 1$, (2.1) is an equality.*

(ii) *For $0 < q < p \leqslant 1$ and $F_n = I_n$, (2.1) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the right-hand side of (2.1) is infinity.*

Proof. Consider (2.1). Let $x \geqslant 0$ with $\|x\|_{v,p,F} = 1$. Then $\|x\|_{v,q,F} \geqslant \|x\|_{v,p,F} = 1$. Applying Minkowski's inequality and Proposition 2.3, we have

$$\begin{aligned}\|H_\mu x\|_{v,q,F} &= \Big(\sum_{n=1}^{\infty} v_n |\langle H_\mu x, F_n\rangle|^q\Big)^{1/q} \\ &= \Big(\sum_{n=1}^{\infty} v_n \Big(\sum_{j\in F_n}\sum_{k=1}^{\infty} h_{j,k}(\theta)x_k\Big)^q\Big)^{1/q} \\ &= \Big(\sum_{n=1}^{\infty} v_n \Big(\sum_{j\in F_n}\sum_{k=1}^{\infty} \binom{j-1}{k-1}\int_0^1 \theta^{k-1}(1-\theta)^{j-k}\,\mathrm{d}\mu(\theta)x_k\Big)^q\Big)^{1/q} \\ &= \Big(\sum_{n=1}^{\infty} v_n \Big(\int_0^1 \sum_{j\in F_n}\sum_{k=1}^{\infty} e_{j,k}(\theta)x_k\,\mathrm{d}\mu(\theta)\Big)^q\Big)^{1/q}\end{aligned}$$

$$
\begin{aligned}
&\geqslant \int_0^1 \biggl(\sum_{n=1}^\infty v_n \biggl(\sum_{j\in F_n}\sum_{k=1}^\infty e_{j,k}(\theta)x_k\biggr)^q\biggr)^{1/q} \,\mathrm{d}\mu(\theta)\\
&= \int_0^1 \|E(\theta)x\|_{v,q,F}\,\mathrm{d}\mu(\theta)\\
&\geqslant \biggl(\int_0^1 \theta^{-1/q}\,\mathrm{d}\mu(\theta)\biggr)\|x\|_{v,q,F} \geqslant \int_0^1 \theta^{-1/q}\,\mathrm{d}\mu(\theta).
\end{aligned}
$$

This leads to (2.1).

Now, consider (i). Let $e_2 = (0,1,0,\ldots)$. Then $e_2 \geqslant 0$ and $\|e_2\|_{v,1,F} = 1$. Since v is increasing and $v_1 = v_2 = 1$, we have

$$
\begin{aligned}
\|H_\mu e_2\|_{v,1,F} &= \sum_{n=1}^\infty v_n |\langle H_\mu e_2, F_n\rangle| = \sum_{n=2}^\infty v_n \biggl(\sum_{j\in F_n} h_{j,2}(\theta)\biggr)\\
&= \sum_{n=2}^\infty v_n \biggl(\sum_{j\in F_n}\int_0^1 \binom{j-1}{2-1}\theta(1-\theta)^{j-2}\,\mathrm{d}\mu(\theta)\biggr)\\
&= \int_0^1 \sum_{n=2}^\infty v_n \biggl(\sum_{j\in F_n} e_{j,2}(\theta)\biggr)\,\mathrm{d}\mu(\theta)\\
&\geqslant \int_0^1 \sum_{n=2}^\infty \sum_{j\in F_n} e_{j,2}(\theta)\,\mathrm{d}\mu(\theta)\\
&\geqslant \int_0^1 \sum_{n=2}^\infty e_{n,2}(\theta)\,\mathrm{d}\mu(\theta) = \int_{(0,1]} \theta^{-1}\,\mathrm{d}\mu(\theta).
\end{aligned}
$$

Hence

$$
L_{v,1,F}(H_\mu) \leqslant \int_{(0,1]} \frac{1}{\theta}\,\mathrm{d}\mu(\theta).
$$

Combining this with (2.1), we obtain (i).

Now, consider (ii). Obviously, (2.1) is an equality, if its right-hand side is infinity. For the case that $\mu(\{0\}) + \mu(\{1\}) = 1$, we have

$$
\begin{aligned}
\|H_\mu e_2\|_{v,q,F} &= \biggl(\sum_{n=1}^\infty v_n |\langle H_\mu e_2, F_n\rangle|^q\biggr)^{1/q} = \biggl(\sum_{n=2}^\infty v_n \biggl(\sum_{j\in F_n} h_{j,2}(\theta)\biggr)^q\biggr)^{1/q}\\
&\geqslant \biggl(\sum_{n=2}^\infty v_n \sum_{j\in F_n} h_{j,2}^q(\theta)\biggr)^{1/q} \geqslant \biggl(\sum_{n=2}^\infty v_n h_{n,2}^q(\theta)\biggr)^{1/q}\\
&= \biggl(\sum_{n=2}^\infty v_n \biggl(\binom{n-1}{1}\int_0^1 \theta(1-\theta)^{n-2}\,\mathrm{d}\mu(\theta)\biggr)^q\biggr)^{1/q}\\
&= \mu(\{1\}) = \int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta),
\end{aligned}
$$

where e_2 is defined as above. This implies that

$$L_{v,p,q,F}(H_\mu) \leqslant \int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta)$$

and consequently, (2.1) is an equality.

Conversely, let $0 < q < p \leqslant 1$, $F_n = I_n$, and assume that $\mu(\{0\}) + \mu(\{1\}) \neq 1$, and also

$$\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta) < \infty.$$

Then $\mu((0,1)) \neq 0$. Since $0 < q < 1$, we have

$$\sum_{n=0}^{\infty}(1-\theta)^n < \sum_{n=0}^{\infty}(1-\theta)^{nq}, \quad \theta \in (0,1) \tag{2.2}$$

Applying (2.2), Minkowski's inequality and the monotonicity of v, we have

$$\begin{aligned}\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta) &= \int_{(0,1]} \Big(\sum_{n=1}^{\infty}(1-\theta)^n\Big)^{1/q}\,\mathrm{d}\mu(\theta) \\ &< \int_{(0,1]} \Big(\sum_{n=1}^{\infty}(1-\theta)^{nq}\Big)^{1/q}\,\mathrm{d}\mu(\theta) \\ &\leqslant \Big\|\Big\{\int_{(0,1]}(1-\theta)^n\,\mathrm{d}\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_q \\ &\leqslant \Big\|\Big\{\int_{(0,1]}(1-\theta)^n\,\mathrm{d}\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q}.\end{aligned} \tag{2.3}$$

By virtue of (2.3) we can find $0 < \beta < 1$ such that

$$\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta) < \beta\Big\|\Big\{\int_{(0,1]}(1-\theta)^n\,\mathrm{d}\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q}. \tag{2.4}$$

We claim that

$$\begin{aligned}&L_{v,p,q,F}(H_\mu) \\ &\quad\geqslant \min\Big(\beta^{(q-p)/q}\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta), \beta\Big\|\Big\{\int_{(0,1]}(1-\theta)^n\,\mathrm{d}\mu(\theta)\Big\}_{n=1}^{\infty}\Big\|_{v,q}\Big).\end{aligned} \tag{2.5}$$

Let $x \geqslant 0$ with $\|x\|_{v,p,F} = 1$. We divide the proof into two cases: $x_{k_0} \geqslant \beta$ for some k_0 or $x_k < \beta$ for all k. For the first case, applying Proposition 2.2 it follows that

$$\begin{aligned}
\|H_\mu x\|_{v,q,F} &= \left(\sum_{n=1}^{\infty} v_n |\langle H_\mu x, F_n\rangle|^q\right)^{1/q} = \left(\sum_{n=1}^{\infty} v_n \left(\sum_{j\in F_n}\sum_{k=1}^{\infty} h_{j,k}x_k\right)^q\right)^{1/q} \\
&\geqslant \left(\sum_{n=1}^{\infty} v_n \sum_{j\in F_n}\left(\sum_{k=1}^{\infty} h_{j,k}x_k\right)^q\right)^{1/q} \geqslant \left(\sum_{n=1}^{\infty} v_n \left(\sum_{k=1}^{\infty} h_{n,k}x_k\right)^q\right)^{1/q} \\
&\geqslant x_{k_0}\left(\sum_{n=1}^{\infty} v_n h_{n,k_0}^q\right)^{1/q} \geqslant \beta\left\|\left\{\int_{(0,1]} e_{n,k_0}(\theta)\,\mathrm{d}\mu(\theta)\right\}_{n=k_0}^{\infty}\right\|_{v,q} \\
&\geqslant \beta\left\|\left\{\int_{(0,1]} e_{n,1}(\theta)\,\mathrm{d}\mu(\theta)\right\}_{n=1}^{\infty}\right\|_{v,q} = \beta\left\|\left\{\int_{(0,1]} (1-\theta)^n\,\mathrm{d}\mu(\theta)\right\}_{n=1}^{\infty}\right\|_{v,q}.
\end{aligned}$$

As for the second case, we have

$$x_k^q \geqslant \beta^{q-p}x_k^p \quad (\forall k \geqslant 1).$$

This implies

$$\|x\|_{v,q} = \left(\sum_{k=1}^{\infty} v_k x_k^q\right)^{1/q} \geqslant \beta^{(q-p)/q}\left(\sum_{k=1}^{\infty} v_k x_k^p\right)^{1/q} = \beta^{(q-p)/q}.$$

Applying (2.1), we deduce that

$$\begin{aligned}
\|H_\mu x\|_{v,q,F} &\geqslant \left(\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta)\right)\|x\|_{v,q,F} \\
&= \left(\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta)\right)\left(\sum_{n=1}^{\infty} v_n\left(\sum_{j\in F_n} x_j\right)^q\right)^{1/q} \\
&\geqslant \left(\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta)\right)\left(\sum_{n=1}^{\infty} v_n \sum_{j\in F_n} x_j^q\right)^{1/q} \\
&\geqslant \left(\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta)\right)\left(\sum_{n=1}^{\infty} v_n x_n^q\right)^{1/q} \\
&\geqslant \beta^{(q-p)/q}\left(\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta)\right).
\end{aligned}$$

Hence, no matter which case occurs, $\|H_\mu x\|_{v,q,F}$ is always greater than or equal to the minimum stated at the right-hand side of (2.5). This leads to (2.5). It is clear that $\beta^{(q-p)/q} > 1$. Putting (2.4) and (2.5) together, we get (ii). This completes the proof. $\square$

In the sequel, we present several special cases of Theorem 2.4.

Let $\mathrm{d}\mu(\theta) = \alpha(1-\theta)^{\alpha-1}\,\mathrm{d}\theta$, where $\alpha > 0$. Then H_μ reduces to the Cesàro matrix $C(\alpha)$ (see [2, p. 410]). For $0 < q \leqslant 1$, we have

$$\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta) = \alpha \int_{(0,1]} \theta^{-1/q}(1-\theta)^{\alpha-1}\,\mathrm{d}\theta = \infty.$$

Applying (2.1), we get the following result.

Corollary 2.5. *Let $\alpha > 0$. Then $L_{v,p,q,F}(C(\alpha)) = \infty$ for $0 < q \leqslant p \leqslant 1$.*

Next, consider the case $\mathrm{d}\mu(\theta) = (|\log\theta|^{\alpha-1}/\Gamma(\alpha))\,\mathrm{d}\theta$, where $\alpha > 0$. For this case, H_μ reduces to the Hölder matrix $H(\alpha)$ (see [2, p. 410]). We have

$$\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta) = \infty \quad (0 < q \leqslant 1).$$

Hence, the following corollary is a consequence of (2.1).

Corollary 2.6. *Let $\alpha > 0$. Then $L_{v,p,q,F}(H(\alpha)) = \infty$ for $0 < q \leqslant p \leqslant 1$.*

The third special case that we consider is $\mathrm{d}\mu(\theta) = \alpha\theta^{\alpha-1}\,\mathrm{d}\theta$, where $\alpha > 0$. Then H_μ becomes the Gamma matrix $\Gamma(\alpha)$ (see [2, p. 410]). We have

$$\int_{(0,1]} \theta^{-1/q}\,\mathrm{d}\mu(\theta) = \alpha \int_{(0,1]} \theta^{-1/q+\alpha-1}\,\mathrm{d}\theta = \begin{cases} \infty, & \alpha \leqslant 1/q, \\ \dfrac{\alpha}{\alpha - 1/q}, & \alpha > 1/q \end{cases}$$

Applying Theorem 2.1, we get the following corollary.

Corollary 2.7. *Let $\alpha > 0$ and $0 < q \leqslant p \leqslant 1$. Then $L_{v,p,q,F}(\Gamma(\alpha)) = \infty$ for $\alpha \leqslant 1q$. Also, we have $L_{v,p,q,F}(\Gamma(\alpha)) \geqslant \alpha/(\alpha - 1/q)$ for $\alpha > 1/q$.*

3. Nörlund matrix operator

Let $W = (w_n)_{n=1}^{\infty}$ be a sequence of non-negative numbers with $w_1 > 0$, set $W_n = \sum\limits_{k=1}^{n} w_k$, $n \geqslant 1$, and define the Nörlund matrix associated with $W = (w_n)$, $A_W^{NM} := A(w_n) = (a_{n,k})$, by

$$a_{n,k} = \begin{cases} \dfrac{w_{n-k+1}}{W_n}, & 1 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $A(w_n) = A(cw_n)$ for any $c > 0$, so we may as well assume that $w_1 = 1$. When all the w_n are 1, A_W^{NM} is the Cesàro matrix.

In this section we focus on the evaluation of the norm of A_W^{NM} as a matrix operator from $\ell_p(v, I)$ into $\ell_p(v, F)$. We indicate that the operator norm of A_W^{NM} is no less than $\max(1, \alpha p/(p-1))$, where $\alpha = \liminf\limits_{n\to\infty} nw_n/W_n$ and $1 < p < \infty$ (see Theorem 3.2). Our result generalizes [14, Theorem 2.4].

Proposition 3.1. *If $\sum a_n$ and $\sum b_n$ are series with positive terms, $\sum a_n$ is divergent and $b_n/a_n \to 1$ as $n \to \infty$, then $\sum\limits^N b_n / \sum\limits^N a_n \to 1$ as $N \to \infty$.*

Proof. See Lemma 2 of [9]. □

Theorem 3.2. *Suppose that $W = (w_n)_{n=1}^\infty$ is a non-negative, non-increasing sequence of real numbers with $w_1 = 1$. Then*

$$\|A_W^{NM}\|_{v,p,F} \geqslant \max\Bigl(1, \frac{\alpha p}{p-1}\Bigr)$$

where $\alpha = \liminf\limits_{n\to\infty} nw_n/W_n$ and $1 < p < \infty$.

Proof. Fix $\delta \in (0,1)$, and suppose $N \geqslant 1$ is sufficiently large so that $w_n/W_n \geqslant ((1-\delta)/n)\alpha$ for all $n \geqslant N$. Then $(w_{n-k+1})/W_n \geqslant ((1-\delta)/n)\alpha$ for all $n \geqslant N$ and $1 \leqslant k \leqslant n$, because the w_j are non-increasing.

Suppose $M > N$ and define $x = (x_k)$ by

$$x_k = \begin{cases} \dfrac{1}{k^{1/p}}, & N \leqslant k \leqslant M, \\ 0 & \text{otherwise.} \end{cases}$$

Using conventional notation, we have

$$\begin{aligned} \|(A_W^{NM})x\|_{v,p,F}^p &= \sum_{n=1}^\infty v_n |\langle A_W^{NM} x, F_n\rangle|^p \\ &= \sum_{n=1}^\infty v_n \Bigl(\sum_{j\in F_n} \sum_{k=1}^j \frac{w_{j-k+1}}{W_j} x_k\Bigr)^p \\ &\geqslant \sum_{n=1}^\infty v_n \sum_{j\in F_n} \Bigl(\frac{1}{W_j} \sum_{k=1}^j w_{j-k+1} x_k\Bigr)^p \\ &\geqslant \sum_{n=N}^M v_n \Bigl(\frac{1}{W_n} \sum_{k=N}^n w_{n-k+1} x_k\Bigr)^p \end{aligned}$$

$$
\begin{aligned}
&\geqslant \alpha^p(1-\delta)^p \sum_{n=N}^{M} v_n \left(\frac{1}{n}\sum_{k=N}^{n}\frac{1}{k^{1/p}}\right)^p \\
&\geqslant \alpha^p(1-\delta)^p \sum_{n=N}^{M} v_n \left(\frac{1}{n}\int_N^n \frac{1}{x^{1/p}}\,\mathrm{d}x\right)^p \\
&= \left(\frac{p}{p-1}\right)^p \alpha^p(1-\delta)^p \sum_{n=N}^{M} \frac{v_n}{n^p}\left(n^{1-1/p-N^{1-1/p}}\right)^p \\
&= \left(\frac{p}{p-1}\right)^p \alpha^p(1-\delta)^p \zeta_M \sum_{n=N}^{M} \frac{v_n}{n} \\
&= \left(\frac{p}{p-1}\right)^p \alpha^p(1-\delta)^p \zeta_M \sum_{n=N}^{M} v_n \left(\sum_{j\in I_n}\frac{1}{j}\right) \\
&= \left(\frac{p}{p-1}\right)^p \alpha^p(1-\delta)^p \zeta_M \|x\|_{v,p,I}^p,
\end{aligned}
$$

where $\zeta_M \to 1$ as $M \to \infty$, by Proposition 3.1.

It follows that the operator norm of A_W^{NM} is no less than $(1-\delta)\alpha p/(p-1)$, and since δ was arbitrary, the operator norm of A_W^{NM} is no less than $\alpha p/(p-1)$. Since $\|A_W^{NM} e_1\|_{v,p,F} \geqslant 1$ where $e_1 = (1,0,0,\ldots)$ (note that $A_W^{NM} e_1$ is the first column of A_W^{NM} and $v_1 = 1$, and also $F_1 = \{1\}$), it follows that the operator norm of A_W^{NM} is no less than 1, either. This completes the proof of the statement. □

Theorem 3.2 also generalizes ([7, Corollary 2.7.9]).

Corollary 3.3. *If the (w_n) of Theorem 3.2 tend to a positive limit, then*

$$\|A_W^{NM}\|_{v,p,F} \geqslant \frac{p}{p-1} \quad \forall p > 1.$$

P r o o f. It is easy to see that if (w_n) tends to a positive limit, then $nw_n/W_n \to 1$ as $n \to \infty$. □

Corollary 3.3 is an analogue of ([13, Corollary 3.3]) which is obtained in a different way.

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