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Nguyen Van Huan; Nguyen Van Quang

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THE DOOB INEQUALITY AND STRONG LAW OF LARGE NUMBERS FOR MULTIDIMENSIONAL ARRAYS IN GENERAL BANACH SPACES

Nguyen Van Huan and Nguyen Van Quang

Dedicated to Professor Nguyen Duy Tien on the occasion of his 70th birthday

We establish the Doob inequality for martingale difference arrays and provide a sufficient condition so that the strong law of large numbers would hold for an arbitrary array of random elements without imposing any geometric condition on the Banach space. Some corollaries are derived from the main results, they are more general than some well-known ones.

Keywords: the Doob inequality, strong law of large numbers, martingale difference array, Banach space

Classification: 60E15, 60F15, 60G42, 60B12

1. INTRODUCTION

Smythe [18] obtained the Kolmogorov strong law of large numbers (SLLN) for multidimensional arrays of random variables. The Brunk–Prokhorov SLLN for multidimensional arrays of random variables was established by Noszály and Tómács [11], Móricz et al. [9] and was extended to multidimensional arrays of random elements by Lagodowski [7]. The Rademacher–Menshov type SLLN for double arrays of quasi-orthogonal random variables was studied by Móricz [8]. Afterwards, Móricz et al. [10] and Rosalsky and Thanh [15] obtained some results on the SLLN problem for double arrays of p-orthogonal random elements.

The method of subsequences is a basic method to prove the SLLN and has been used by many authors. This method is to prove the desired result for a subsequence and then reduce the problem for the whole sequence to that for the subsequence. In so doing, a maximal inequality for cumulative sums is usually needed.

In this paper, we give an extension of the Doob inequality for martingale difference arrays and provide a sufficient condition so that the SLLN would hold for an arbitrary array of random elements taking values in a separable Banach space. These results are then used to obtain two maximal inequalities and some SLLNs for various classes of dependent arrays in which the Kolmogorov, Brunk–Prokhorov and Rademacher–Menshov type SLLNs will be generalized, the main technique used in our proofs is based on the method of subsequences. The rest of the paper is organized as follows. Some notations, definitions and a lemma needed in this paper will be presented in Section 2. Section 3 is devoted to the main results and their proofs.

2. PRELIMINARIES

Throughout this paper, the symbol C will denote a generic positive constant which is not necessarily the same one in each appearance. Let d be a positive integer, the set of all nonnegative integer d-dimensional lattice points will be denoted by $\mathbb{N}_0^d$, and the set of all positive integer d-dimensional lattice points will be denoted by $\mathbb{N}^d$. We shall write $\mathbf{1}$, $\mathbf{n}$, $\mathbf{n+1}$, $\mathbf{n-1}$, $\mathbf{2^n}$ for points $(1,1,\dots,1)$, $(n_1,n_2,\dots,n_d)$, $(n_1+1,n_2+1,\dots,n_d+1)$, $(n_1-1,n_2-1,\dots,n_d-1)$, $(2^{n_1},2^{n_2},\dots,2^{n_d})$, respectively. The notation $\mathbf{m}\preceq\mathbf{n}$ means that $m_i\leqslant n_i$ for all $i=1,2,\dots,d$, and the notation $\mathbf{m}\prec\mathbf{n}$ means that $m_i<n_i$ for all $i=1,2,\dots,d$. We define $|\mathbf{n}|=\prod_{i=1}^d n_i$, and the limit $\mathbf{n}\to\infty$ means that $|\mathbf{n}|\to\infty$.

Let $\{\omega_1(j),j\geqslant 1\}$, $\{\omega_2(j),j\geqslant 1\}$,...,$\{\omega_d(j),j\geqslant 1\}$ be strictly increasing sequences of positive integers with $w_i(1)=1$ $(1\leqslant i\leqslant d)$. For $\mathbf{m}\in\mathbb{N}_0^d$ and $\mathbf{n}\in\mathbb{N}^d$, we introduce the following notation:

$$\begin{aligned}
\omega_{\mathbf{n}} &= \big(\omega_1(n_1),\omega_2(n_2),\dots,\omega_d(n_d)\big),\\
\Delta_{\mathbf{n}} &= \{\mathbf{k}:\omega_{\mathbf{n}}\preceq\mathbf{k}\prec\omega_{\mathbf{n+1}}\},\quad \Delta^{(\mathbf{m})}=\{\mathbf{k}:\mathbf{2^m}\preceq\mathbf{k}\prec\mathbf{2^{m+1}}\},\\
\Delta_{\mathbf{n}}^{(\mathbf{m})} &= \Delta_{\mathbf{n}}\cap\Delta^{(\mathbf{m})},\quad \Lambda_{\mathbf{m}}=\{\mathbf{k}:\Delta_{\mathbf{k}}^{(\mathbf{m})}\neq\emptyset\},\\
r_{\mathbf{n}}^{(\mathbf{m})}(i) &= \min\{r:r\in[\omega_i(n_i),\omega_i(n_i+1))\cap[2^{m_i},2^{m_i+1})\}\quad (\mathbf{n}\in\Lambda_{\mathbf{m}},1\leqslant i\leqslant d),\\
\mathbf{r}_{\mathbf{n}}^{(\mathbf{m})} &= \big(r_{\mathbf{n}}^{(\mathbf{m})}(1),r_{\mathbf{n}}^{(\mathbf{m})}(2),\dots,r_{\mathbf{n}}^{(\mathbf{m})}(d)\big)\quad(\mathbf{n}\in\Lambda_{\mathbf{m}}),\\
\varphi(\mathbf{n}) &= \sum_{\mathbf{k}\in\mathbb{N}_0^d}\operatorname{card}(\Lambda_{\mathbf{k}})\,I_{\Delta^{(\mathbf{k})}}(\mathbf{n}),\quad \psi(\mathbf{n})=\max_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}}\varphi(\mathbf{k}),
\end{aligned}$$

where $\operatorname{card}(\Lambda_{\mathbf{k}})$ denotes the cardinality of the set $\Lambda_{\mathbf{k}}$ and $I_{\Delta^{(\mathbf{k})}}$ denotes the indicator function of the set $\Delta^{(\mathbf{k})}$. It is easy to verify that if $\omega(\mathbf{n})=\mathbf{2^{n-1}}$ $(\mathbf{n}\in\mathbb{N}^d)$, then $\varphi(\mathbf{n})=\psi(\mathbf{n})=1$ for all $\mathbf{n}\in\mathbb{N}^d$.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $\{\mathcal{F}_{\mathbf{n}},\mathbf{1}\preceq\mathbf{m}\preceq\mathbf{n}\preceq\mathbf{M}\}$ be a d-dimensional array of non-decreasing sub-σ-algebras of $\mathcal{F}$ related to the partial order $\preceq$ on $\mathbb{N}^d$. Let $\mathbf{E}$ be a real separable Banach space, let $\mathcal{B}(\mathbf{E})$ be the σ-algebra of all Borel sets in $\mathbf{E}$, and let $\{X_{\mathbf{n}},\mathbf{m}\preceq\mathbf{n}\preceq\mathbf{M}\}$ be a d-dimensional array of $\mathbf{E}$-valued random elements such that $X_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}/\mathcal{B}(\mathbf{E})$-measurable for all $\mathbf{n}\in\mathbb{N}^d$ $(\mathbf{m}\preceq\mathbf{n}\preceq\mathbf{M})$. Then $\{X_{\mathbf{n}},\mathcal{F}_{\mathbf{n}},\mathbf{m}\preceq\mathbf{n}\preceq\mathbf{M}\}$ is said to be an *adapted array*.

Let $\{X_{\mathbf{n}},\mathcal{F}_{\mathbf{n}},\mathbf{m}\preceq\mathbf{n}\preceq\mathbf{M}\}$ be an adapted array. For $\mathbf{n}\in\mathbb{N}_0^d$ $(\mathbf{m-1}\preceq\mathbf{n}\preceq\mathbf{M-1})$, we adopt the convention that $\mathcal{F}_{\mathbf{n}}=\{\emptyset,\Omega\}$ if there exists a positive integer i $(1\leqslant i\leqslant d)$ such that $n_i=m_i-1$ and set

$$\begin{aligned}
\mathcal{F}_{\mathbf{n}}^1 &= \bigvee_{m_i\leqslant l_i\leqslant M_i\ (2\leqslant i\leqslant d)}\mathcal{F}_{n_1l_2l_3\dots l_d}:=\bigvee_{l_2=m_2}^{M_2}\bigvee_{l_3=m_3}^{M_3}\cdots\bigvee_{l_d=m_d}^{M_d}\mathcal{F}_{n_1l_2l_3\dots l_d},\\
\mathcal{F}_{\mathbf{n}}^j &= \bigvee_{m_i\leqslant l_i\leqslant M_i\ (1\leqslant i\leqslant j-1)}\ \bigvee_{m_i\leqslant l_i\leqslant M_i\ (j+1\leqslant i\leqslant d)}\mathcal{F}_{l_1\dots l_{j-1}n_jl_{j+1}\dots l_d}\quad\text{if}\quad 1<j<d,\\
\mathcal{F}_{\mathbf{n}}^d &= \bigvee_{m_i\leqslant l_i\leqslant M_i\ (1\leqslant i\leqslant d-1)}\mathcal{F}_{l_1l_2\dots l_{d-1}n_d},
\end{aligned}$$

in the case $d=1$, set $\mathcal{F}_n^1=\mathcal{F}_n$.

An adapted array $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M}\}$ is said to be a *martingale difference array* if $\mathbb{E}(X_{\mathbf{n}} | \mathcal{F}^i_{\mathbf{n}-\mathbf{1}}) = 0$ for all $\mathbf{n} \in \mathbb{N}^d$ $(\mathbf{m} \preceq \mathbf{n} \preceq \mathbf{M})$ and for all $i = 1, 2, \dots, d$.

Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an array of $\mathbf{E}$-valued random elements, and let $\{\mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an array of sub-σ-algebras of $\mathcal{F}$. The array $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be a *blockwise martingale difference array* with respect to the blocks $\{\Delta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ if for each $\mathbf{k} \in \mathbb{N}^d$, $\{X_{\mathbf{l}}, \mathcal{F}_{\mathbf{l}}, \mathbf{l} \in \Delta_{\mathbf{k}}\}$ is a martingale difference array.

The notion of p-orthogonality was introduced by Howell and Taylor [4], and by Móricz et al. [10]. A d-dimensional array of $\mathbf{E}$-valued random elements $\{X_{\mathbf{k}}, \mathbf{1} \preceq \mathbf{m} \preceq \mathbf{k} \preceq \mathbf{M}\}$ is said to be *p-orthogonal* $(1 \leqslant p < \infty)$ if $\mathbb{E}\|X_{\mathbf{k}}\|^p < \infty$ for all $\mathbf{m} \preceq \mathbf{k} \preceq \mathbf{M}$ and

$$\mathbb{E}\Big\| \sum_{\mathbf{m} \preceq \mathbf{k} \preceq \mathbf{l}} a_{\pi_1(k_1)\pi_2(k_2)\dots\pi_d(k_d)} X_{\pi_1(k_1)\pi_2(k_2)\dots\pi_d(k_d)} \Big\|^p \leqslant \mathbb{E}\Big\| \sum_{\mathbf{m} \preceq \mathbf{k} \preceq \mathbf{n}} a_{\mathbf{k}} X_{\mathbf{k}} \Big\|^p$$

for all choices of $\mathbf{m} \preceq \mathbf{l} \preceq \mathbf{n} \preceq \mathbf{M}$, for all arrays $\{a_{\mathbf{k}}, \mathbf{m} \preceq \mathbf{k} \preceq \mathbf{n}\}$ of constants, and for all permutations $\pi_1, \pi_2, \dots, \pi_d$ of the integers $\{m_1, m_1 + 1, \dots, n_1\}$, $\{m_2, m_2 + 1, \dots, n_2\}$, $\dots$, $\{m_d, m_d + 1, \dots, n_d\}$, respectively.

An array of random elements $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be *blockwise p-orthogonal* (respectively, *blockwise independent*) with respect to the blocks $\{\Delta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ if for each $\mathbf{k} \in \mathbb{N}^d$, the array $\{X_{\mathbf{l}}, \mathbf{l} \in \Delta_{\mathbf{k}}\}$ is p-orthogonal (respectively, independent).

As in Quang and Huan [13], the set of all martingale difference arrays is larger than the set of all arrays of independent mean zero random elements. As a consequence of this remark, the set of all blockwise martingale difference arrays is also larger than the set of all arrays of blockwise independent mean zero random elements.

A Banach space $\mathbf{E}$ is said to be *p-smoothable* $(1 \leqslant p \leqslant 2)$ if (possibly after equivalent renorming)

$$\rho(\tau) = \sup\Big\{ \frac{\|x + y\| + \|x - y\|}{2} - 1, \ \forall\, x, y \in \mathbf{E}, \|x\| = 1, \ \|y\| = \tau \Big\} = \mathcal{O}(\tau^p).$$

Let $\{Y_j, j \geqslant 1\}$ be a symmetric Bernoulli sequence, let $\mathbf{E}^\infty = \mathbf{E} \times \mathbf{E} \times \mathbf{E} \times \dots$ and define $\mathscr{C}(\mathbf{E}) = \big\{(v_1, v_2, \dots.) \in \mathbf{E}^\infty : \sum_{j=1}^\infty Y_j v_j \quad \text{converges in probability}\big\}$. Then $\mathbf{E}$ is said to be of *Rademacher type p* $(1 \leqslant p \leqslant 2)$ if there exists a positive constant C such that

$$\mathbb{E}\Big\| \sum_{j=1}^\infty Y_j v_j \Big\|^p \leqslant C \sum_{j=1}^\infty \|v_j\|^p \quad \text{for all } (v_1, v_2, \dots.) \in \mathscr{C}(\mathbf{E}).$$

As in Assouad [1] and Woyczyński [20], a real separable Banach space $\mathbf{E}$ is p-smoothable (respectively, of Rademacher type p) $(1 \leqslant p \leqslant 2)$ if and only if for all $q \geqslant 1$, there exists a positive constant C such that for all $\mathbf{E}$-valued martingale difference sequences $\{X_j, \mathcal{F}_j, 1 \leqslant j \leqslant i\}$ (respectively, sequences of independent mean zero $\mathbf{E}$-valued random elements $\{X_j, 1 \leqslant j \leqslant i\}$),

$$\mathbb{E}\Big\| \sum_{j=1}^i X_j \Big\|^q \leqslant C\, \mathbb{E}\Big(\sum_{j=1}^i \|X_j\|^p \Big)^{\frac{q}{p}}. \tag{1}$$

We close this section by giving a lemma which is the multidimensional version of Lemma 2.2 of Huan et al. [5].

Lemma 2.1. Let $\Phi_1(.)$, $\Phi_2(.)$, …, $\Phi_d(.)$ be positive nondecreasing unbounded functions on $(0,\infty)$, and let $\{x_{\mathbf{n}}, \mathbf{n}\in\mathbb{N}_0^d\}$ be an array of real numbers such that

$$\lim_{\mathbf{n}\to\infty} x_{\mathbf{n}} = 0.$$

Then the condition

$$\sup_{\mathbf{n}\in\mathbb{N}_0^d}\Big(\prod_{i=1}^d \Phi_i(2^{n_i})\Big)^{-1}\sum_{\mathbf{0}\preceq\mathbf{k}\preceq\mathbf{n}}\prod_{i=1}^d \Phi_i(2^{k_i+1}) < \infty \tag{2}$$

implies

$$\lim_{\mathbf{n}\to\infty}\Big(\prod_{i=1}^d \Phi_i(2^{n_i})\Big)^{-1}\sum_{\mathbf{0}\preceq\mathbf{k}\preceq\mathbf{n}}\Big(\prod_{i=1}^d \Phi_i(2^{k_i+1})\Big)x_{\mathbf{k}} = 0.$$

3. MAIN RESULTS

The first pioneering work on two parameter martingales appeared in Cairoli and Walsh [2]. Afterwards, there were considerable amount of studies on two parameter martingales and their application in stochastic calculus. In one parameter situation, the ordering is unique following the natural distinction made between past and present. A similar obvious ordering does not seem to exist for two parameter case. As a consequence, there are several definitions for two parameter martingales and two parameter martingale differences.

Consider an array $\{\mathcal{F}_{ij}, i\geqslant 1, j\geqslant 1\}$ of sub-σ-algebras of $\mathcal{F}$, and set

$$\mathcal{F}_{ij}^1 = \bigvee_{l\geqslant 1}\mathcal{F}_{il},\quad \mathcal{F}_{ij}^2 = \bigvee_{k\geqslant 1}\mathcal{F}_{kj}\quad (i\geqslant 1, j\geqslant 1).$$

Cairoli and Walsh [2] introduced the $F4$ condition as follows: The array $\{\mathcal{F}_{ij}, i\geqslant 1, j\geqslant 1\}$ is said to satisfy the $F4$ condition if $\mathcal{F}_{ij}^1$ and $\mathcal{F}_{ij}^2$ are conditionally independent with respect to $\mathcal{F}_{ij}$ for all $i\geqslant 1$, $j\geqslant 1$, i. e.,

$$\mathbb{P}(AB|\mathcal{F}_{ij}) = \mathbb{P}(A|\mathcal{F}_{ij})\,\mathbb{P}(B|\mathcal{F}_{ij})$$

for all $i\geqslant 1$, $j\geqslant 1$ and for all $A\in\mathcal{F}_{ij}^1$, $B\in\mathcal{F}_{ij}^2$.

The two parameter martingale theory is established on the basis of the $F4$ condition. Many fundamental theorems and results in this theory do not hold if the $F4$ condition is not satisfied. In the following theorem, the Doob inequality for multiparameter martingale differences will be established without the $F4$ condition.

Theorem 3.1. Let q be a real number ($q>1$), let g be a nonnegative nondecreasing convex function, and let $\{X_{\mathbf{k}}, \mathcal{F}_{\mathbf{k}}, \mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}, \mathbf{n}\in\mathbb{N}^d\}$ be a martingale difference array. Then

$$\mathbb{E}\Big(\max_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} g(\|\sum_{\mathbf{1}\preceq\mathbf{l}\preceq\mathbf{k}} X_{\mathbf{l}}\|)\Big)^q \leqslant \Big(\frac{q}{q-1}\Big)^{qd}\,\mathbb{E}\Big(g(\|\sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}}\|)\Big)^q. \tag{3}$$

P r o o f. Without loss of generality, assume that $n_d \geqslant 2$. Clearly, for $d = 1$, the conclusion (3) follows from Doob's inequality (see, e. g., Gut [3], p. 505). Assume that (3) holds for $d = D - 1 \geqslant 1$, we wish to show that it holds for $d = D$. For $\mathbf{k} \in \mathbb{N}^D$ ($\mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}$), set

$$S_{\mathbf{k}} = \sum_{\mathbf{1}\preceq\mathbf{l}\preceq\mathbf{k}} X_{\mathbf{l}}, \quad Y_{k_D} = \max_{1\leqslant k_i\leqslant n_i\ (1\leqslant i\leqslant D-1)} g(\|S_{\mathbf{k}}\|).$$

Then

$$\begin{aligned} &\mathbb{E}(S_{k_1k_2\dots k_{D-1}k_D}|\mathcal{F}^D_{k_1k_2\dots k_{D-1}k_D-1}) \\ &= \mathbb{E}(S_{k_1k_2\dots k_{D-1}k_D-1}|\mathcal{F}^D_{k_1k_2\dots k_{D-1}k_D-1}) \\ &\quad + \mathbb{E}\Big(\sum_{1\leqslant l_i\leqslant k_i\ (1\leqslant i\leqslant D-1)} X_{l_1l_2\dots l_{D-1}k_D}|\mathcal{F}^D_{k_1k_2\dots k_{D-1}k_D-1}\Big) \\ &= S_{k_1k_2\dots k_{D-1}k_D-1}. \end{aligned}$$

It means that $\{S_{k_1k_2\dots k_{D-1}k_D}, \mathcal{F}^D_{k_1k_2\dots k_{D-1}k_D}, 1 \leqslant k_D \leqslant n_D\}$ is a martingale, and so $\{g(\|S_{k_1k_2\dots k_{D-1}k_D}\|), \mathcal{F}^D_{k_1k_2\dots k_{D-1}k_D}, 1 \leqslant k_D \leqslant n_D\}$ is a nonnegative submartingale. It is easy to show that $\{Y_{k_D}, \mathcal{F}^D_{k_1k_2\dots k_{D-1}k_D}, 1 \leqslant k_D \leqslant n_D\}$ is a nonnegative submartingale. Then by Doob's inequality,

$$\mathbb{E}\Big(\max_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} g(\|S_{\mathbf{k}}\|)\Big)^q = \mathbb{E}\Big(\max_{1\leqslant k_D\leqslant n_D} Y_{k_D}\Big)^q \leqslant \Big(\frac{q}{q-1}\Big)^q \mathbb{E}Y^q_{n_D}. \tag{4}$$

Set

$$X^{(D-1)}_{k_1k_2\dots k_{D-1}} = \sum_{k_D=1}^{n_D} X_{k_1k_2\dots k_{D-1}k_D}, \quad \mathcal{F}^{(D-1)}_{k_1k_2\dots k_{D-1}} = \bigvee_{1\leqslant k_D\leqslant n_D} \mathcal{F}_{k_1k_2\dots k_{D-1}k_D}.$$

Then $\{X^{(D-1)}_{k_1k_2\dots k_{D-1}}, \mathcal{F}^{(D-1)}_{k_1k_2\dots k_{D-1}}, (1,1,\dots,1) \preceq (k_1,k_2,\dots,k_{D-1}) \preceq (n_1,n_2,\dots,n_{D-1})\}$ is a martingale difference array. Therefore, by the inductive assumption,

$$\mathbb{E}Y^q_{n_D} \leqslant \Big(\frac{q}{q-1}\Big)^{q(D-1)} \mathbb{E}\big(g(\|S_{\mathbf{n}}\|)\big)^q. \tag{5}$$

Combining (4) and (5) yields that (3) holds for $d = D$. $\square$

In the next two corollaries, we use Theorem 3.1 to obtain two maximal inequalities for martingale difference arrays in p-smoothable Banach spaces and arrays of independent mean zero random elements in Rademacher type p Banach spaces, respectively. The first corollary generalizes the implication ((i)⇒(ii)) of Theorem 2.3 of Quang and Huan [14].

Corollary 3.2. Let $\mathbf{E}$ be a real separable p-smoothable Banach space ($1 \leqslant p \leqslant 2$). Then, for all $q \geqslant 1$, there exists a positive constant C such that for all martingale difference arrays $\{X_{\mathbf{k}}, \mathcal{F}_{\mathbf{k}}, \mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$,

$$\mathbb{E} \max_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} \Big\| \sum_{\mathbf{1}\preceq\mathbf{l}\preceq\mathbf{k}} X_{\mathbf{l}} \Big\|^q \leqslant C^d |\mathbf{n}|^{\max\{\frac{q}{p};1\}-1} \sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} \mathbb{E}\|X_{\mathbf{k}}\|^q. \tag{6}$$

P r o o f . It is well known that every real separable p-smoothable Banach space ($1 \leqslant p \leqslant 2$) is q-smoothable for all $q \in [1, p]$. So, without loss of generality, assume that $q \geqslant p$. On the other hand, since (6) holds in the case $p = q = 1$, we assume further that $q > 1$. By virtue of Theorem 3.1, it suffices to show that

$$\mathbb{E}\Big\| \sum_{\mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \Big\|^q \leqslant C^d |\mathbf{n}|^{\frac{q}{p}-1} \sum_{\mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}} \mathbb{E}\|X_{\mathbf{k}}\|^q. \tag{7}$$

Remark that for $d = 1$, (7) follows from (1) and Hölder's inequality. Assume that (7) holds for $d = D - 1 \geqslant 1$, we wish to show that it holds for $d = D$. For $\mathbf{k} \in \mathbb{N}^D$, set $S_{\mathbf{k}} = \sum_{\mathbf{1} \preceq \mathbf{l} \preceq \mathbf{k}} X_{\mathbf{l}}$. Then $\{S_{n_1 n_2 \ldots n_{D-1} k_D}, \mathcal{F}^D_{n_1 n_2 \ldots n_{D-1} k_D}, 1 \leqslant k_D \leqslant n_D\}$ is a martingale. By (1) and Hölder's inequality, we get

$$\begin{aligned}\mathbb{E}\|S_{\mathbf{n}}\|^q &\leqslant C\mathbb{E}\Big(\sum_{k_D=1}^{n_D} \Big\| \sum_{1 \leqslant k_i \leqslant n_i \ (1 \leqslant i \leqslant D-1)} X_{k_1 k_2 \ldots k_{D-1} k_D} \Big\|^p \Big)^{\frac{q}{p}} \\ &\leqslant C (n_D)^{\frac{q}{p}-1} \sum_{k_D=1}^{n_D} \mathbb{E}\Big\| \sum_{1 \leqslant k_i \leqslant n_i \ (1 \leqslant i \leqslant D-1)} X_{k_1 k_2 \ldots k_{D-1} k_D} \Big\|^q. \end{aligned} \tag{8}$$

For a fixed positive integer k_D ($1 \leqslant k_D \leqslant n_D$), set

$$Y^{(D-1)}_{k_1 k_2 \ldots k_{D-1}} = X_{k_1 k_2 \ldots k_{D-1} k_D}, \ \ \mathcal{F}^{(D-1)}_{k_1 k_2 \ldots k_{D-1}} = \bigvee_{1 \leqslant l_D \leqslant n_D} \mathcal{F}_{k_1 k_2 \ldots k_{D-1} l_D}. \tag{9}$$

It is easy to show that $\{Y^{(D-1)}_{k_1 k_2 \ldots k_{D-1}}, \mathcal{F}^{(D-1)}_{k_1 k_2 \ldots k_{D-1}}, (1, 1, \ldots, 1) \preceq (k_1, k_2, \ldots, k_{D-1}) \preceq (n_1, n_2, \ldots, n_{D-1})\}$ is also a martingale difference array. Therefore, by the inductive assumption,

$$\begin{aligned}&\mathbb{E}\Big\| \sum_{1 \leqslant k_i \leqslant n_i \ (1 \leqslant i \leqslant D-1)} Y^{(D-1)}_{k_1 k_2 \ldots k_{D-1}} \Big\|^q \\ &\leqslant C^{D-1} (n_1 n_2 \ldots n_{D-1})^{\frac{q}{p}-1} \sum_{1 \leqslant k_i \leqslant n_i \ (1 \leqslant i \leqslant D-1)} \mathbb{E}\|Y^{(D-1)}_{k_1 k_2 \ldots k_{D-1}}\|^q. \end{aligned} \tag{10}$$

Combining (8) – (10) yields that (7) holds for $d = D$. □

Corollary 3.3. Let $\mathbf{E}$ be a real separable Rademacher type p ($1 \leqslant p \leqslant 2$) Banach space. Then, for all $q \geqslant 1$, there exists positive constants C_1 and C_2 such that for all arrays of independent mean zero random elements $\{X_{\mathbf{k}}, \mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$ (with values in $\mathbf{E}$),

$$\mathbb{E} \max_{\mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}} \Big\| \sum_{\mathbf{1} \preceq \mathbf{l} \preceq \mathbf{k}} X_{\mathbf{l}} \Big\|^q \leqslant C_1 C_2^d |\mathbf{n}|^{\max\{\frac{q}{p};1\}-1} \sum_{\mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}} \mathbb{E}\|X_{\mathbf{k}}\|^q. \tag{11}$$

P r o o f . Without loss of generality, we only need to consider the case $q \geqslant p$ and $q > 1$. It is easy to show that $\big\{X_{\mathbf{k}}, \mathcal{F}_{\mathbf{k}} = \bigvee_{\mathbf{1} \preceq \mathbf{l} \preceq \mathbf{k}} \sigma(X_{\mathbf{l}}), \mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}\big\}$ is a martingale difference array. Then by using Theorem 3.1, Hölder's inequality and Proposition 2.1 of Woyczyński [20], we get (11). □

The second result provides a sufficient condition so that the SLLN would hold for an arbitrary array of random elements without imposing any geometric condition on the Banach space. For the convenience of the reader, we recall the definition of the function ψ: $\psi(\mathbf{n}) = \max_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} \varphi(\mathbf{k})$, $\mathbf{n} \in \mathbb{N}^d$.

Theorem 3.4. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an array of random elements in a real separable Banach space, and let $\Phi_1(.)$, $\Phi_2(.)$, $\ldots$,$\Phi_d(.)$ be positive nondecreasing unbounded functions on $(0,\infty)$ satisfying (2). Then the condition

$$\Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1})\Big)^{-1} \big(\psi(\mathbf{2^m})\big)^{\frac{1-q}{q}} \sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \Big\| \sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}} X_{\mathbf{t}} \Big\| \to 0 \tag{12}$$

a.s. as $\mathbf{m}\to\infty$ for some $q \geqslant 1$ implies

$$\Big(\prod_{i=1}^{d} \Phi_i(n_i)\Big)^{-1} \big(\psi(\mathbf{n})\big)^{\frac{1-q}{q}} \sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}} \to 0 \quad \text{a.s. as } \mathbf{n}\to\infty. \tag{13}$$

Proof. For $\mathbf{m} \in \mathbb{N}_0^d$, set

$$\gamma_{\mathbf{m}} = \Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1})\Big)^{-1} \big(\psi(\mathbf{2^m})\big)^{\frac{1-q}{q}} \sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \Big\| \sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}} X_{\mathbf{t}} \Big\|.$$

Then by (12) and Lemma 2.1,

$$\Big(\prod_{i=1}^{d} \Phi_i(2^{m_i})\Big)^{-1} \sum_{\mathbf{0}\preceq\mathbf{k}\preceq\mathbf{m}} \Big(\prod_{i=1}^{d} \Phi_i(2^{k_i+1})\Big)\gamma_{\mathbf{k}} \to 0 \quad \text{a.s. as } \mathbf{m}\to\infty. \tag{14}$$

Next, for $\mathbf{n} \in \mathbb{N}^d$, let $\mathbf{m} \in \mathbb{N}_0^d$ be such that $\mathbf{n} \in \Delta^{(\mathbf{m})}$. Then

$$\begin{aligned} 0 &\leqslant \Big(\prod_{i=1}^{d} \Phi_i(n_i)\Big)^{-1} \big(\psi(\mathbf{n})\big)^{\frac{1-q}{q}} \Big\| \sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}} X_{\mathbf{k}} \Big\| \\ &\leqslant \Big(\prod_{i=1}^{d} \Phi_i(n_i)\Big)^{-1} \big(\psi(\mathbf{n})\big)^{\frac{1-q}{q}} \sum_{\mathbf{0}\preceq\mathbf{k}\preceq\mathbf{m}} \Big(\prod_{i=1}^{d} \Phi_i(2^{k_i+1})\Big) \big(\psi(\mathbf{2^k})\big)^{\frac{q-1}{q}} \gamma_{\mathbf{k}} \\ &\leqslant \Big(\prod_{i=1}^{d} \Phi_i(2^{m_i})\Big)^{-1} \sum_{\mathbf{0}\preceq\mathbf{k}\preceq\mathbf{m}} \Big(\prod_{i=1}^{d} \Phi_i(2^{k_i+1})\Big)\gamma_{\mathbf{k}}. \end{aligned} \tag{15}$$

Combining (14) and (15) yields (13). □

The following corollary is a consequence of Theorem 3.4. In the special case where $\Phi_i(x) = x^{\alpha_i}$ ($\alpha_i > 0,\ 1 \leqslant i \leqslant d$), from Corollary 3.5 we get the main result of Thanh [19].

Corollary 3.5. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an array of random elements in a real separable Banach space, and let $\Phi_1(.)$, $\Phi_2(.)$, $\ldots$,$\Phi_d(.)$ be positive nondecreasing unbounded functions on $(0,\infty)$ such that

$$\sup_{j\geqslant 0} \frac{\Phi_i(2^{j+1})}{\Phi_i(2^j)} < \infty, \quad 1 \leqslant i \leqslant d, \tag{16}$$

$$\inf_{j\geqslant 0} \frac{\Phi_i(2^{j+1})}{\Phi_i(2^j)} > 1, \qquad 1 \leqslant i \leqslant d. \tag{17}$$

Then the condition

$$\Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1})\Big)^{-1} \max_{\mathbf{2^m} \preceq \mathbf{k} \prec \mathbf{2^{m+1}}} \Big\| \sum_{\mathbf{2^m} \preceq \mathbf{l} \preceq \mathbf{k}} X_{\mathbf{l}} \Big\| \to 0 \quad \text{a.s. as } \mathbf{m} \to \infty \tag{18}$$

implies

$$\Big(\prod_{i=1}^{d} \Phi_i(n_i)\Big)^{-1} \sum_{\mathbf{1} \preceq \mathbf{k} \preceq \mathbf{n}} X_{\mathbf{k}} \to 0 \quad \text{a.s. as } \mathbf{n} \to \infty. \tag{19}$$

P r o o f. We will show that the conditions (16) and (17) imply (2). Assume that (16) and (17) hold. Then by (17), there exists a positive constant $C < 1$ such that for all $i = 1, 2, \ldots, d$,

$$\frac{\Phi_i(2^j)}{\Phi_i(2^{j+1})} \leqslant C, \quad j \geqslant 0,$$

and so

$$\sup_{j\geqslant 0} \frac{1}{\Phi_i(2^j)} \sum_{s=0}^{j-1} \Phi_i(2^{s+1}) \leqslant \frac{1}{1-C} < \infty.$$

This and (16) imply that

$$\begin{aligned} &\sup_{j\geqslant 0} \frac{1}{\Phi_i(2^j)} \sum_{s=0}^{j} \Phi_i(2^{s+1}) \\ &= \sup_{j\geqslant 0} \Big(\frac{1}{\Phi_i(2^j)} \sum_{s=0}^{j-1} \Phi_i(2^{s+1}) + \frac{\Phi_i(2^{j+1})}{\Phi_i(2^j)} \Big) < \infty, \quad 1 \leqslant i \leqslant d. \end{aligned}$$

Therefore, (2) holds. The proof is completed by using Theorem 3.4 with $\omega(\mathbf{n}) = \mathbf{2^{n-1}}$ ($\mathbf{n} \in \mathbb{N}^d$). $\square$

In the proof of Corollary 3.5, we show that the conditions (16) and (17) imply (2). However, the reverse is not true. We now show, via an example, that (2) does not imply (17).

Example 3.6. For $x > 0$ and $1 \leqslant i \leqslant d$, let

$$\Phi_i(x) = \begin{cases} 2 & \text{if } 0 < x < 1, \\ 2^{2x^{(0)}+(-1)^{x^{(0)}}} & \text{if } x \geqslant 1, \end{cases}$$

where $x^{(0)}$ is a nonnegative integer such that $2^{x^{(0)}} \leqslant x < 2^{x^{(0)}+1}$. Then for each i $(1 \leqslant i \leqslant d)$, $\Phi_i(.)$ is a positive nondecreasing unbounded function on $(0, \infty)$ such that

$$\frac{\Phi_i(2^1) + \Phi_i(2^2) + \cdots + \Phi_i(2^{j+1})}{\Phi_i(2^j)} = \frac{1}{2^{2j+(-1)^j}} \sum_{s=0}^{j} 2^{2(s+1)+(-1)^{s+1}}$$
$$\leqslant \frac{16}{4^j} \sum_{s=0}^{j} 4^s < \infty \quad (j \geqslant 0),$$

and so (2) holds. However, (17) fails since

$$\frac{\Phi_i(2^{j+1})}{\Phi_i(2^j)} = \begin{cases} 16 & \text{if } j \text{ is odd}, \\ 1 & \text{if } j \text{ is even}, \end{cases}$$

for all $i = 1, 2, \ldots, d$.

The following corollary establishes a SLLN for martingale difference arrays in general Banach spaces.

Corollary 3.7. Let q be a real number $(q > 1)$, let $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a martingale difference array, and let $\Phi_1(.)$, $\Phi_2(.)$, $\ldots$,$\Phi_d(.)$ be positive nondecreasing unbounded functions on $(0, \infty)$ satisfying (16) and (17). Then the condition

$$\sum_{\mathbf{m} \in \mathbb{N}_0^d} \Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1}) \Big)^{-q} \mathbb{E} \Big\| \sum_{\mathbf{2^m} \preceq \mathbf{l} \prec \mathbf{2^{m+1}}} X_{\mathbf{l}} \Big\|^q < \infty \tag{20}$$

implies (19).

Proof. Since $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is a martingale difference array, by using the "tower property" of conditional expectation (i. e., if $X \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}, \mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$ $(\mathcal{H} \subset \mathcal{G})$, then $\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})|\mathcal{H}\big) = \mathbb{E}(X|\mathcal{H})$), we may show that $\{X_{\mathbf{k}}, \mathcal{F}_{\mathbf{k}}, \mathbf{k} \in \Delta^{(\mathbf{m})}\}$ is a martingale difference array for all $\mathbf{m} \in \mathbb{N}_0^d$. Then by Theorem 3.1 and (20),

$$\sum_{\mathbf{m} \in \mathbb{N}_0^d} \mathbb{E}\bigg(\Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1}) \Big)^{-1} \max_{\mathbf{2^m} \preceq \mathbf{k} \prec \mathbf{2^{m+1}}} \Big\| \sum_{\mathbf{2^m} \preceq \mathbf{l} \preceq \mathbf{k}} X_{\mathbf{l}} \Big\| \bigg)^q$$
$$\leqslant C \sum_{\mathbf{m} \in \mathbb{N}_0^d} \Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1}) \Big)^{-q} \mathbb{E} \Big\| \sum_{\mathbf{2^m} \preceq \mathbf{l} \prec \mathbf{2^{m+1}}} X_{\mathbf{l}} \Big\|^q < \infty.$$

Applying Markov's inequality, we get (18), and the conclusion (19) follows from Corollary 3.5. □

Next we apply Theorem 3.4 to various classes of dependent arrays. The next corollary extends the Brunk–Prokhorov SLLN to blockwise martingale difference arrays. For $p = q$, it boils down to a version which extends the Kolmogorov SLLN to blockwise martingale difference arrays. Corollary 3.8 also generalizes some results in [5, 12, 21].

Corollary 3.8. Let q be a real number ($q \geqslant 1$), let $\mathbf{E}$ be a real separable p-smoothable Banach space ($1 \leqslant p \leqslant 2$), let $\{X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a blockwise martingale difference array with respect to the blocks $\{\Delta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$, and let $\Phi_1(.)$, $\Phi_2(.)$, ..., $\Phi_d(.)$ be positive nondecreasing unbounded functions on $(0, \infty)$ satisfying (2).

(i) If

$$\sum_{\mathbf{n}\in\mathbb{N}^d} \Big(\prod_{i=1}^d \Phi_i(n_i)\Big)^{-q} |\mathbf{n}|^{\max\{\frac{q}{p};1\}-1} \mathbb{E}\|X_{\mathbf{n}}\|^q < \infty, \tag{21}$$

then (13) holds.

(ii) If

$$\sum_{\mathbf{n}\in\mathbb{N}^d} \Big(\prod_{i=1}^d \Phi_i(n_i)\Big)^{-q} \big(\varphi(\mathbf{n})\big)^{q-1} |\mathbf{n}|^{\max\{\frac{q}{p};1\}-1} \mathbb{E}\|X_{\mathbf{n}}\|^q < \infty,$$

then (19) holds.

Proof. (i) For $\mathbf{m} \in \mathbb{N}_0^d$, we define $\gamma_{\mathbf{m}}$ as in the proof of Theorem 3.4. Then by Hölder's inequality and Corollary 3.2,

$$\begin{aligned}
\mathbb{E}\gamma_{\mathbf{m}}^q &= \Big(\prod_{i=1}^d \Phi_i(2^{m_i+1})\Big)^{-q} \big(\psi(\mathbf{2^m})\big)^{1-q} \mathbb{E}\Big(\sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \Big\| \sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}} X_{\mathbf{t}}\Big\|\Big)^q \\
&\leqslant \Big(\prod_{i=1}^d \Phi_i(2^{m_i+1})\Big)^{-q} \frac{\big(\operatorname{card}(\Lambda_{\mathbf{m}})\big)^{q-1}}{\big(\psi(\mathbf{2^m})\big)^{q-1}} \sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \mathbb{E} \max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \Big\| \sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}} X_{\mathbf{t}}\Big\|^q \\
&\leqslant C\Big(\prod_{i=1}^d \Phi_i(2^{m_i+1})\Big)^{-q} |\mathbf{2^m}|^{\max\{\frac{q}{p};1\}-1} \sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \sum_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \mathbb{E}\|X_{\mathbf{l}}\|^q \\
&\leqslant C \sum_{\mathbf{2^m}\preceq\mathbf{k}\prec\mathbf{2^{m+1}}} \Big(\prod_{i=1}^d \Phi_i(k_i)\Big)^{-q} |\mathbf{k}|^{\max\{\frac{q}{p};1\}-1} \mathbb{E}\|X_{\mathbf{k}}\|^q.
\end{aligned}$$

It thus follows from (21) that $\sum_{\mathbf{m}\in\mathbb{N}_0^d} \mathbb{E}\gamma_{\mathbf{m}}^q < \infty$, and so the condition (12) is satisfied. Theorem 3.4 ensures that (13) holds.

(ii) For $\mathbf{m} \in \mathbb{N}_0^d$, set

$$\gamma_{\mathbf{m}} = \Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1})\Big)^{-1} \sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \Big\| \sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}} X_{\mathbf{t}} \Big\|.$$

Then

$$\begin{aligned}
\mathbb{E}\gamma_{\mathbf{m}}^q &= \Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1})\Big)^{-q} \mathbb{E}\Big(\sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \| \sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}} X_{\mathbf{t}} \|\Big)^q \\
&\leqslant \Big(\prod_{i=1}^{d} \Phi_i(2^{m_i+1})\Big)^{-q} \big(\mathrm{card}(\Lambda_{\mathbf{m}})\big)^{q-1} \sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}} \mathbb{E} \max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}} \Big\| \sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}} X_{\mathbf{t}} \Big\|^q \\
&\leqslant C \sum_{\mathbf{2^m}\preceq\mathbf{k}\prec\mathbf{2^{m+1}}} \Big(\prod_{i=1}^{d} \Phi_i(k_i)\Big)^{-q} \big(\varphi(\mathbf{k})\big)^{q-1} |\mathbf{k}|^{\max\{\frac{q}{p};1\}-1} \mathbb{E}\|X_{\mathbf{k}}\|^q.
\end{aligned}$$

Therefore, $\sum_{\mathbf{m}\in\mathbb{N}_0^d} \mathbb{E}\gamma_{\mathbf{m}}^q < \infty$, and so $\gamma_{\mathbf{m}} \to 0$ a.s. as $\mathbf{m} \to \infty$. Theorem 3.4 ensures that (19) holds. □

By using Corollary 3.3 and the same arguments as in the proof of Corollary 3.8, we get the next corollary which generalizes some results in [15, 16, 19, 20]. We omit its proof.

Corollary 3.9. Let q be a real number ($q \geqslant 1$), let $\mathbf{E}$ be a real separable Rademacher type p ($1 \leqslant p \leqslant 2$) Banach space, let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an array of blockwise independent random elements with respect to the blocks $\{\Delta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ and $\mathbb{E}X_{\mathbf{n}} = 0$ ($\mathbf{n} \in \mathbb{N}^d$), and let $\Phi_1(.)$, $\Phi_2(.)$, $\ldots$,$\Phi_d(.)$ be positive nondecreasing unbounded functions on $(0, \infty)$ satisfying (2). Then the assertions of Corollary 3.8 hold.

We next establish the Rademacher–Menshov type SLLN for arrays of blockwise p-orthogonal random elements in Rademacher type p ($1 \leqslant p \leqslant 2$) Banach spaces. The key tool for proving Corollary 3.10 is the d-dimensional version of Lemma 3.2 of Móricz et al. [10]. This corollary generalizes Theorem 3.1 of Móricz et al. [10].

Corollary 3.10. Let $\mathbf{E}$ be a real separable Rademacher type p ($1 \leqslant p \leqslant 2$) Banach space, let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a array of blockwise p-orthogonal random elements with respect to the blocks $\{\Delta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$, and let $\Phi_1(.)$, $\Phi_2(.)$, $\ldots$,$\Phi_d(.)$ be positive nondecreasing unbounded functions on $(0, \infty)$ satisfying (2).

(i) If

$$\sum_{\mathbf{n}\in\mathbb{N}^d} \Big(\prod_{i=1}^{d} \Phi_i(n_i)\Big)^{-p} \prod_{i=1}^{d} (\log n_i)^p \, \mathbb{E}\|X_{\mathbf{n}}\|^p < \infty, \tag{22}$$

then (13) holds.

(ii) If

$$\sum_{\mathbf{n}\in\mathbb{N}^d}\Big(\prod_{i=1}^{d}\Phi_i(n_i)\Big)^{-p}\big(\varphi(\mathbf{n})\big)^{p-1}\prod_{i=1}^{d}(\log n_i)^p\ \mathbb{E}\|X_{\mathbf{n}}\|^p<\infty,$$

then (19) holds.

Proof. The proof of both assertions is similar, and so we only prove the first. For $\mathbf{m}\in\mathbb{N}_0^d$, we define $\gamma_{\mathbf{m}}$ as in the proof of Theorem 3.4 (with $q=p$). Then

$$\begin{aligned}\mathbb{E}\gamma_{\mathbf{m}}^p&\leqslant\Big(\prod_{i=1}^{d}\Phi_i(2^{m_i+1})\Big)^{-p}\frac{\big(\operatorname{card}(\Lambda_{\mathbf{m}})\big)^{p-1}}{\big(\psi(\mathbf{2^m})\big)^{p-1}}\sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}}\mathbb{E}\max_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}}\Big\|\sum_{\mathbf{r}_{\mathbf{k}}^{(\mathbf{m})}\preceq\mathbf{t}\preceq\mathbf{l}}X_{\mathbf{t}}\Big\|^p\\&\leqslant C\Big(\prod_{i=1}^{d}\Phi_i(2^{m_i+1})\Big)^{-p}\sum_{\mathbf{k}\in\Lambda_{\mathbf{m}}}\prod_{i=1}^{d}(\log 2^{m_i})^p\sum_{\mathbf{l}\in\Delta_{\mathbf{k}}^{(\mathbf{m})}}\mathbb{E}\|X_{\mathbf{l}}\|^p\\&\leqslant C\sum_{\mathbf{2^m}\preceq\mathbf{k}\prec\mathbf{2^{m+1}}}\Big(\prod_{i=1}^{d}\Phi_i(k_i)\Big)^{-p}\prod_{i=1}^{d}(\log k_i)^p\mathbb{E}\|X_{\mathbf{k}}\|^p.\end{aligned}$$

It thus follows from (22) that $\sum_{\mathbf{m}\in\mathbb{N}_0^d}\mathbb{E}\gamma_{\mathbf{m}}^p<\infty$, and so $\gamma_{\mathbf{m}}\to 0$ a.s. as $\mathbf{m}\to\infty$. Theorem 3.4 ensures that (13) holds. □

Remark 3.11. It is interesting to observe that the maximal inequality is essential for applying Theorem 3.4. Móricz et al. [9] established a maximal inequality for arrays of $\mathcal{M}$-dependent random variables and obtain a sufficient condition for an array of blockwise $\mathcal{M}$-dependent random variables with respect to dyadic blocks to obey the SLLN

$$\Big(\prod_{i=1}^{d}\Phi_i(n_i)\Big)^{-1}\sum_{\mathbf{1}\preceq\mathbf{k}\preceq\mathbf{n}}X_{\mathbf{k}}\to 0\quad\text{a.s. as}\ \ \min\{n_1,n_2,\dots,n_d\}\to\infty.$$

In this article, we consider the limit $\mathbf{n}\to\infty$ which is equivalent to the limit $\max\{n_1,n_2,\dots,n_d\}\to\infty$. Clearly, if an array of real or complex numbers $\{S_{\mathbf{n}},\mathbf{n}\in\mathbb{N}^d\}$ is convergent to S as $\mathbf{n}\to\infty$, then it is convergent to S as $\min\{n_1,n_2,\dots,n_d\}\to\infty$ (Pringsheim convergence) and $\sup_{\mathbf{n}\in\mathbb{N}^d}|S_{\mathbf{n}}|<\infty$. However, the reverse is not true. By the same arguments as in the proof of Corollary 3.8, we can use Lemma 3 of Móricz et al. [9] to extend their main result to arrays of blockwise $\mathcal{M}$-dependent random variables with respect to arbitrary blocks. Moreover, the main result of Móricz et al. [9] can be extended further to arrays of blockwise $\mathcal{M}$-dependent random elements in Rademacher type p Banach spaces. In the one-dimensional case, two applications of Theorem 3.4 can be obtained by using Lemma 2.4 of Kim [6] and Theorem 2 of Shao [17], respectively.

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Nguyen Van Huan, Department of Mathematics and Applications, Saigon University, Ho Chi Minh City. Vietnam.
e-mail: vanhuandhdt@yahoo.com

Nguyen Van Quang, Department of Mathematics, Vinh University, Nghe An. Vietnam.
e-mail: nvquang@hotmail.com