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THE k-METRIC COLORINGS OF A GRAPH

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Abstract. For a nontrivial connected graph G of order n, the detour distance $D(u,v)$ between two vertices u and v in G is the length of a longest $u-v$ path in G. Detour distance is a metric on the vertex set of G. For each integer k with $1 \leqslant k \leqslant n-1$, a coloring $c\colon V(G) \to \mathbb{N}$ is a k-metric coloring of G if $|c(u)-c(v)| + D(u,v) \geqslant k+1$ for every two distinct vertices u and v of G. The value $\chi_m^k(c)$ of a k-metric coloring c is the maximum color assigned by c to a vertex of G and the k-metric chromatic number $\chi_m^k(G)$ of G is the minimum value of a k-metric coloring of G. For every nontrivial connected graph G of order n, $\chi_m^1(G) \leqslant \chi_m^2(G) \leqslant \ldots \leqslant \chi_m^{n-1}(G)$. Metric chromatic numbers provide a generalization of several well-studied coloring parameters in graphs. Upper and lower bounds have been established for $\chi_m^k(G)$ in terms of other graphical parameters of a graph G and exact values of k-metric chromatic numbers have been determined for complete multipartite graphs and cycles. For a nontrivial connected graph G, the anti-diameter $\operatorname{adiam}(G)$ is the minimum detour distance between two vertices of G. We show that the $\operatorname{adiam}(G)$-metric chromatic number of a graph G provides information on the Hamiltonian properties of the graph and investigate realization results and problems on this parameter.

Keywords: detour distance, metric coloring

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1. Introduction

By the Four Color Theorem, the regions of every map can be colored with four or fewer colors so that every two adjacent regions (regions sharing a common boundary) are assigned distinct colors. However, if a map M contains a large number of regions, then to make the map look more attractive, it may be advantageous to use several more colors to color the regions. One disadvantage of this approach, however, is that if many colors are used, then it is likely that some pairs of colors, even though different, are similar and may appear to be indistinguishable at a casual glance. One solution to this problem is to allow regions to be assigned the same or similar colors only when these regions are sufficiently far apart.

With each map M, there is associated a dual planar graph G whose vertices are the regions of M and where two vertices of G are adjacent if the corresponding regions of M are adjacent. If we use positive integers as the colors, we then assign colors i and j to distinct vertices u and v of G depending on the distance $d(u,v)$ between u and v (the length of a shortest $u-v$ path in G). In particular, we could agree to assign colors i and j to u and v only if $|i-j|+d(u,v)\geqslant k+1$ for some prescribed positive integer k. This gives rise to a coloring of the vertices of the graph G that is sometimes called a *k-radio coloring*, introduced in [1].

More formally, for a connected graph G having diameter $\operatorname{diam}(G)=d$ (the largest distance between two vertices of G) and an integer k with $1\leqslant k\leqslant d$, a *k-radio coloring* of G is an assignment c of colors (positive integers) to the vertices of G such that

$$|c(u)-c(v)|+d(u,v)\geqslant k+1 \tag{1}$$

for every two distinct vertices u and v of G. The *value* $\mathrm{rc}_k(c)$ of a k-radio coloring c of G is the maximum color assigned to a vertex of G by c and the *k-radio chromatic number* $\mathrm{rc}_k(G)$ of G is defined as $\mathrm{rc}_k(G)=\min\{\mathrm{rc}_k(c)\}$, where the minimum is taken over all k-radio colorings c of G. In fact, 1-radio colorings are ordinary proper colorings, 2-radio colorings are the much studied *$L(2,1)$-colorings*, d-radio colorings are *radio labelings* (see [1]), and $(d-1)$-radio colorings are *antipodal colorings* (see [2]). Thus k-radio colorings provide a generalization of these colorings in graphs.

The term "radio coloring" emanates from another interpretation of coloring the vertices of a graph, namely from the *Channel Assignment Problem*, which is the problem of obtaining an optimal assignment of channels to a specified set of radio transmitters according to some prescribed restrictions on the distances between transmitters as well as other factors, including their effective radiated power and antenna heights. The use of graph theory to study the Channel Assignment Problem and related problems dates back at least to 1970 (see Metzger [21]). In 1980, Hale [16] modeled the Channel Assignment Problem as both a frequency-distance constrained and frequency constrained optimization problem and discussed applications to important real world problems. Since then, a number of graph colorings have been inspired by the Channel Assignment Problem (see [3], [10]–[15], [17]–[19], [25]–[28] for example).

Radio colorings of graphs gave rise to other colorings of graphs defined in terms of another distance parameter. The *detour distance* $D(u,v)$ between two vertices u and v in a connected graph G is the length of a longest $u-v$ path in G. Thus if G is a connected graph of order n, then $d(u,v) \leqslant D(u,v) \leqslant n-1$ for every two vertices u and v in G and $D(u,v) = n-1$ if and only if G contains a Hamiltonian $u-v$ path. Furthermore, $d(u,v) = D(u,v)$ for every two vertices u and v in G if and only if G is a tree. As with standard distance, the detour distance is a metric on the vertex set of a connected graph. For a nontrivial connected graph G of order n and an integer k with $1 \leqslant k \leqslant n-1$, a coloring $c\colon V(G) \to \mathbb{N}$ is called a *k-metric coloring* of G if

$$|c(u) - c(v)| + D(u,v) \geqslant k+1 \tag{2}$$

for every two distinct vertices u and v of G. Therefore, a k-metric coloring of a tree T is, in fact, a k-radio coloring of T. The *value* $\chi_m^k(c)$ of a k-metric coloring c is the maximum color assigned by c to a vertex of G and the *k-metric chromatic number* $\chi_m^k(G)$ of G is the minimum value of a k-metric coloring of G, that is, $\chi_m^k(G) = \min\{\chi_m^k(c)\}$ where the minimum is taken over all k-metric colorings c of G. A k-metric coloring c of G whose value equals $\chi_m^k(G)$ is a *minimum k-metric coloring* of G. Since every k-radio coloring of a graph G is a k-metric coloring of G, it follows that $\chi_m^k(G) \leqslant \mathrm{rc}_k(G)$ and equality holds if G is a tree.

For a connected graph G of order $n \geqslant 3$ and an integer k with $1 \leqslant k \leqslant n-2$, every $(k+1)$-metric coloring is a k-metric coloring and so

$$\chi_m^1(G) \leqslant \chi_m^2(G) \leqslant \ldots \leqslant \chi_m^{n-1}(G). \tag{3}$$

The concept of k-metric coloring provides a generalization of two other colorings in literature. An $(n-2)$-metric coloring of G is a *Hamiltonian coloring* which was introduced in [7] and studied further in [5], [8], [22], [23] for example, while an $(n-1)$-metric coloring is a *Hamiltonian labeling*, which was introduced in [29] and studied further in [24], [30].

We now present some preliminary and useful information on k-metric colorings of graphs. For a nontrivial connected graph G, the diameter $\mathrm{diam}(G)$ of G is the maximum value of the minimum length of a $u-v$ path in G for all $u, v \in V(G)$. That is,

$$\mathrm{diam}(G) = \max_{u,v \in V(G)} \{\text{the minimum length of a } u-v \text{ path in } G\}.$$

So the diameter of a graph is a max-min concept. If we reverse this order and consider the corresponding min-max concept

$$\min_{u,v \in V(G)} \{\text{the maximum length of a } u-v \text{ path in } G\},$$

then we obtain the *anti-diameter* $\mathrm{adiam}(G)$ of G. Thus $\mathrm{adiam}(G)$ is the minimum detour distance between two vertices of G. If G is a nontrivial connected graph of order n, then $1 \leqslant \operatorname{adiam}(G) \leqslant n-1$. Furthermore, $\operatorname{adiam}(G) = 1$ if and only if G contains a bridge and $\operatorname{adiam}(G) = n-1$ if and only if G is Hamiltonian-connected (that is, every two vertices of G are connected by a Hamiltonian path). If H is a connected spanning subgraph of a nontrivial graph G, then $d_G(u,v) \leqslant d_H(u,v) \leqslant D_H(u,v) \leqslant D_G(u,v)$ for every two distinct vertices u and v and so $\operatorname{diam}(G) \leqslant \operatorname{diam}(H)$ while $\operatorname{adiam}(H) \leqslant \operatorname{adiam}(G)$. We are prepared to present two useful observations.

Observation 1.1. *If H is a connected spanning subgraph of a nontrivial graph G of order n, then $\chi_m^k(G) \leqslant \chi_m^k(H)$ for $1 \leqslant k \leqslant n-1$. In particular, if T is a spanning tree of G, then $\chi_m^k(G) \leqslant \chi_m^k(T) = \operatorname{rc}_k(T)$ for $1 \leqslant k \leqslant n-1$.*

Observation 1.2. *Let G be a nontrivial connected graph of order n and k an integer with $1 \leqslant k \leqslant n-1$. Then $\chi_m^k(G) = 1$ if and only if $\operatorname{adiam}(G) \geqslant k+1$.*

By Observation 1.2, $\chi_m^1(G) = 1$ if and only if $\operatorname{adiam}(G) \geqslant 2$, that is, $\chi_m^1(G) = 1$ if and only if G is a connected graph without a bridge (or G is 2-edge-connected). Furthermore, if G contains a bridge e, then e belongs to every spanning tree T of G. Since a proper 2-coloring of T is a 1-metric coloring of G, it follows that $\chi_m^1(G) = 2$. Therefore,

$$\chi_m^1(G) = \begin{cases} 1 & \text{if } G \text{ is bridgeless,} \\ 2 & \text{otherwise.} \end{cases} \tag{4}$$

We have seen in (3) that if G is a nontrivial connected graph of order n, then $\{\chi_m^k(G)\}_{k=1}^{n-1}$ is a nondecreasing sequence of positive integers. Furthermore, $\chi_m^k(G) \geqslant 2$ if and only if $k \geqslant \operatorname{adiam}(G)$. It was shown in [29] that if G is a nontrivial connected graph of order n, then $\chi_m^{n-1}(G) \geqslant n$ and $\chi_m^{n-1}(G) = n$ if G is Hamiltonian. Therefore, if G is a Hamiltonian graph of order n, then

$$\{\chi_m^k(G)\}_{k=1}^{n-1}\colon\ 1 = a_1,\ a_2,\ a_3,\ \ldots,\ a_{n-2},\ a_{n-1} = n,$$

where $1 \leqslant a_2 \leqslant a_3 \leqslant \ldots \leqslant a_{n-2} \leqslant n$ (if $n \geqslant 4$) and, in particular, G is Hamiltonian-connected if and only if

$$\{\chi_m^k(G)\}_{k=1}^{n-1}\colon\ 1,\ 1,\ \ldots,\ 1,\ n.$$

To illustrate these concepts, we consider the k-metric chromatic number of the Petersen graph P of order 10 for each integer k with $1 \leqslant k \leqslant 9$. Observe that for two distinct vertices u and v of P,

$$D(u,v) = \begin{cases} 8 & \text{if } uv \in E(P), \\ 9 & \text{if } uv \notin E(P). \end{cases}$$

Thus $\operatorname{adiam}(P) = 8$. Since it was shown in [24] that $\chi_m^8(P) = 3$ and $\chi_m^9(P) = 10$, we have

$$\{\chi_m^k(P)\}_{k=1}^9\colon\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 3,\ 10.$$

We refer to the book [4] for graph theory notation and terminology not described in this paper. We assume that all graphs under consideration are connected.

2. Bounds for k-metric chromatic numbers

It is convenient to introduce some notation. For a k-metric coloring c of a nontrivial connected graph G of order n, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G is called a *c-ordering* of G if

$$1 = c(v_1) \leqslant c(v_2) \leqslant \ldots \leqslant c(v_n) = \chi_m^k(c).$$

Observe that if $\chi_m^{k+l}(G) = 1$ for some positive integers k and l, then $\chi_m^k(G) = 1$ by (3). When $\chi_m^{k+l}(G) \geqslant 2$, we have the following.

Theorem 2.1. *Let G be a connected graph of order $n \geqslant 3$ and k an integer with $1 \leqslant k \leqslant n-2$. Then $\chi_m^{k+l}(G) \leqslant \chi_m^k(G) + l(n-1)$ for each positive integer $l \leqslant n-k-1$. Furthermore, if $\chi_m^{k+l}(G) \geqslant 2$, then $\chi_m^{k+l}(G) \geqslant \chi_m^k(G) + l$.*

Proof. In order to verify that $\chi_m^{k+l}(G) \leqslant \chi_m^k(G) + l(n-1)$ we show that $\chi_m^{k+1}(G) \leqslant \chi_m^k(G) + n - 1$. Let c be a minimum k-metric coloring of G. Then

$$|c(u) - c(v)| + D(u,v) \geqslant k+1 \tag{5}$$

for every two distinct vertices u and v of G. Suppose that $v_1, v_2, \ldots, v_n$ is a c-ordering of the vertices of G, where then $1 = c(v_1) \leqslant c(v_2) \leqslant \ldots \leqslant c(v_n) = \chi_m^k(G)$. Define a coloring c' of G by $c'(v_i) = c(v_i) + i - 1$ for $1 \leqslant i \leqslant n$. Hence

$$1 = c'(v_1) \leqslant c'(v_2) \leqslant \ldots \leqslant c'(v_n) = \chi_m^k(G) + n - 1. \tag{6}$$

For two distinct vertices v_i and v_j of G, where say $1 \leqslant i < j \leqslant n$, it then follows by (5) and (6) that

$$\begin{aligned} |c'(v_i) - c'(v_j)| + D(v_i,v_j) &= [(c(v_j) + j - 1] - [c(v_i) + i - 1] + D(v_i,v_j) \\ &= c(v_j) - c(v_i) + D(v_i,v_j) + (j-i) \\ &\geqslant k+2. \end{aligned}$$

Thus c' is a $(k+1)$-metric coloring of G. Therefore, $\chi_m^{k+1}(G) \leqslant \chi_m^k(c') = \chi_m^k(G) + n - 1$ and the result follows by induction.

Next, we show that $\chi_m^{k+l}(G) \geqslant \chi_m^k(G) + l$ if $\chi_m^{k+l}(G) \geqslant 2$. It suffices to show that $\chi_m^{k+1}(G) \geqslant \chi_m^k(G) + 1$ when $\chi_m^{k+1}(G) \geqslant 2$. Let c be a minimum $(k+1)$-metric coloring of G and let $v_1, v_2, \ldots, v_n$ be a c-ordering of the vertices of G, where then

$$1 = c(v_1) \leqslant c(v_2) \leqslant \ldots \leqslant c(v_n) = \chi_m^{k+1}(G).$$

Since $\chi_m^{k+1}(G) \geqslant 2$, there is a largest integer p with $1 \leqslant p \leqslant n-1$ such that $c(v_p) < c(v_n)$. Hence $c(v_{p+1}) = c(v_{p+2}) = \ldots = c(v_n)$. Define a coloring c' of G by

$$c'(v_i) = \begin{cases} c(v_i) & \text{if } 1 \leqslant i \leqslant p, \\ c(v_i) - 1 & \text{if } p+1 \leqslant i \leqslant n. \end{cases}$$

Observe that $v_1, v_2, \ldots, v_n$ is also a c'-ordering of the vertices of G. To see that c' is a k-metric coloring of G, let v_i and v_j be two distinct vertices of G, where say $1 \leqslant i < j \leqslant n$. If $j \leqslant p$ or $i \geqslant p+1$, then $|c'(v_i) - c'(v_j)| + D(v_i, v_j) = c(v_j) - c(v_i) + D(v_i, v_j) > k+1$; otherwise, $|c'(v_i) - c'(v_j)| + D(v_i, v_j) \geqslant [c(v_j) - 1] - c(v_i) + D(v_i, v_j) \geqslant k+1$. Thus c' is a k-metric coloring of G, as claimed. Therefore, $\chi_m^k(G) \leqslant c'(v_n) = c(v_n) - 1 = \chi_m^{k+1}(G) - 1$. □

By Theorem 2.1, for every connected graph G of order $n \geqslant 3$ and for each k with $\operatorname{adiam}(G) \leqslant k \leqslant n-1$, it follows that

$$1 \leqslant \chi_m^k(G) - \chi_m^{k-1}(G) \leqslant n - 1. \tag{7}$$

Since for trees the k-metric chromatic number equals the k-radio chromatic number,

$$\chi_m^k(K_{1,n-1}) = (k-1)(n-1) + 2$$

for $1 \leqslant k \leqslant n-1$ (see [9]). Hence, the upper bound in (7) is sharp for every possible value of k. On the other hand, the following result was established in [29].

Theorem 2.2 [29]**.** *For every connected graph G of order $n \geqslant 3$*

$$\chi_m^{n-2}(G) + 2 \leqslant \chi_m^{n-1}(G) \leqslant \chi_m^{n-2}(G) + (n-1)$$

and both bounds are sharp.

Thus, there is no connected graph G of order $n \geqslant 3$ such that $\chi_m^{n-1}(G) - \chi_m^{n-2}(G) = 1$. That is,

$$\chi_m^k(G) - \chi_m^{k-1}(G) \geqslant \begin{cases} 1 & \text{if } \operatorname{adiam}(G) \leqslant k \leqslant n-2, \\ 2 & \text{if } k = n-1 \end{cases} \tag{8}$$

and the bound is sharp, which we discuss next.

It was shown in [7], [29] that $\chi_m^{n-1}(C_n) = n$ and $\chi_m^{n-2}(C_n) = n-2$ for each $n \geqslant 3$. Hence, $\chi_m^{n-1}(C_n) - \chi_m^{n-2}(C_n) = 2$ for each $n \geqslant 3$. Also, there is a connected graph G of order $n \geqslant 4$ such that $\chi_m^k(G) - \chi_m^{k-1}(G) = 1$ for $\operatorname{adiam}(G) \leqslant k \leqslant n-2$. To see this, let $G = K_2 + \bigl(K_{\lceil n/2\rceil -1} \cup K_{\lfloor n/2\rfloor -1}\bigr)$ and let x and y be the two vertices having degree $n-1$. Then

$$D(u,v) = \begin{cases} \lceil n/2\rceil & \text{if } \{u,v\} = \{x,y\},\\ n-1 & \text{otherwise.}\end{cases}$$

Therefore, $\operatorname{adiam}(G) = \lceil n/2\rceil$ and $\chi_m^k(G) = 1$ for $1 \leqslant k \leqslant \lceil n/2\rceil - 1$. Furthermore, the coloring $c\colon V(G) \to \{1,2\}$ such that $c(v) = 2$ if and only if $v = x$ is an $\lceil n/2\rceil$-metric coloring of G. Hence, $\chi_m^{\lceil n/2\rceil}(G) = 2$. In fact, the coloring $c_k\colon V(G) \to \{1, k - \lceil n/2\rceil + 2\}$ such that $c(v) = k - \lceil n/2\rceil + 2$ if and only if $v = x$ is a k-metric coloring of G for $\lceil n/2\rceil \leqslant k \leqslant n-2$. It then follows by Theorem 2.1 that

$$\chi_m^{\lceil n/2\rceil}(G) = k - \lceil n/2\rceil + 2$$

for $\lceil n/2\rceil \leqslant k \leqslant n-2$. Finally, $\chi_m^{n-1}(G) = n$ since G is Hamiltonian. Thus,

$$\{\chi_m^k(G)\}_{k=1}^{n-1}\colon \begin{cases} 1,2,4 & \text{if } n = 4,\\ 1,1,2,5 & \text{if } n = 5,\\ 1,1,\ldots,1,2,3,\ldots,\lfloor n/2\rfloor, n & \text{if } n \geqslant 6.\end{cases}$$

Observe that C_4 and $K_4 - e$ attain the lower bounds in (8) for every k with $\operatorname{adiam}(G) \leqslant k \leqslant n-1$. Whether there is a connected graph G of order $n \geqslant 5$ that attain the lower bounds in (8) for *every* k with $\operatorname{adiam}(G) \leqslant k \leqslant n-1$ is not known.

Combining (4) and Theorems 2.1 and 2.2, we obtain the following corollary.

Corollary 2.3. *For every connected graph G of order $n \geqslant 3$ and $a = \operatorname{adiam}(G) \leqslant k \leqslant n-1$,*

$$\chi_m^k(G) \geqslant \begin{cases} k-a+2 & \text{if } k \leqslant n-2,\\ \max\{n-a+2, n\} & \text{if } k = n-1\end{cases}$$

and

$$\chi_m^k(G) \leqslant \begin{cases} (k-a+1)(n-1)+1 & \text{if } a \geqslant 2,\\ (k-1)(n-1)+2 & \text{if } a = 1.\end{cases}$$

In [7], [30], sharp upper bounds for the Hamiltonian chromatic number and Hamiltonian labeling number of a connected graph G were given in terms of the order n of G, namely,

$$1 \leqslant \chi_m^{n-2}(G) \leqslant (n-3)(n-1)+2,$$
$$n \leqslant \chi_m^{n-1}(G) \leqslant (n-2)(n-1)+2.$$

It was also shown that each of the upper bounds is attained if and only if G is a star while each of the lower bounds is attained if and only if G is Hamiltonian-connected. We next present an improved upper bound as well as a lower bound for $\chi_m^k(G)$ where $\operatorname{adiam}(G) \leqslant k \leqslant n-1$. In order to do this, we present an additional definition. For a nontrivial connected graph G of order n with $a = \operatorname{adiam}(G)$ and an integer k with $a \leqslant k \leqslant n-1$, let G_k be the graph such that $V(G_k) = V(G)$ and $uv \in E(G_k)$ if and only if $D_G(u,v) \leqslant k$. Let $\chi(G)$ denote the chromatic number of a graph G.

Theorem 2.4. *For a nontrivial connected graph G of order n with* $\operatorname{adiam}(G) = a$ *and an integer k with $a \leqslant k \leqslant n-1$,*

$$(k-a+1)\big[\chi(G_a)-1\big]+1 \leqslant \chi_m^k(G) \leqslant (k-a+1)\big[\chi(G_k)-1\big]+1.$$

Proof. We first verify the upper bound. Let $c_0\colon V(G_k) \to \{1,2,\ldots,\chi(G_k)\}$ be a proper $\chi(G_k)$-coloring of G_k and consider the coloring $c\colon V(G) \to \mathbb{N}$ of G defined by $c(v) = (k-a+1)\,[c_0(v)-1]+1$ for every $v \in V(G)$. We show that c is a k-metric coloring of G. Let $u, v \in V(G)$. We may also assume that $D_G(u,v) \leqslant k$. Observe that $uv \in E(G_k)$, implying that $c_0(u) \neq c_0(v)$. Hence,

$$\begin{aligned}|c(u)-c(v)| + D_G(u,v) &= (k-a+1)|c_0(u)-c_0(v)| + D_G(u,v)\\ &\geqslant (k-a+1)\cdot 1 + a = k+1.\end{aligned}$$

Therefore, $\chi_m^k(G) \leqslant \chi_m^k(c) = (k-a+1)\big[\chi(G_k)-1\big]+1$.

For the lower bound, suppose that c is an arbitrary k-metric coloring of G with $\chi = \chi_m^k(c)$. Then $V(G)$ can be partitioned into the sets $V_1, V_2, \ldots, V_{\lceil \chi/(k-a+1)\rceil}$ such that for each i

$$V_i = \{v \in V(G)\colon\ (i-1)(k-a+1)+1 \leqslant c(v) \leqslant i(k-a+1)\}.$$

Therefore, if two vertices u and v belong to V_i for some i, then $|c(u)-c(v)| < k-a+1$, implying that $D_G(u,v) \geqslant a+1$. Hence, $uv \notin E(G_a)$, that is, each V_i is an independent set in G_a. This in turn implies that $\chi(G_a) \leqslant \lceil \chi/(k-a+1)\rceil$. Therefore, $\chi \geqslant (k-a+1)\big[\chi(G_a)-1\big]+1$, which completes the proof. $\square$

The following is an immediate consequence of Theorem 2.4.

Corollary 2.5. *If G is a nontrivial connected graph with* $\operatorname{adiam}(G) = a$, *then* $\chi_m^a(G) = \chi(G_a)$.

3. Two well-known classes of graphs

In this section, we determine the k-metric chromatic numbers of two well-known classes of graph, namely complete multipartite graphs and cycles. We begin with complete bipartite graphs. For a star $K_{1,n-1}$ of order $n \geqslant 2$, we have seen that $\chi_m^k(K_{1,n-1}) = (k-1)(n-1)+2$ for $1 \leqslant k \leqslant n-1$. Hence, we consider complete bipartite graphs that are not stars here. Hamiltonian labeling numbers and Hamiltonian chromatic numbers of complete bipartite graphs in general were determined in [7], [29].

Theorem 3.1 [7], [29]. *For integers r and s with $2 \leqslant r \leqslant s$,*

$$\chi_m^{r+s-1}(K_{r,s}) = \begin{cases} 2r & \text{if } r = s, \\ (s-r)(r+s-1)+2r-1 & \text{if } r < s \end{cases}$$

and

$$\chi_m^{r+s-2}(K_{r,s}) = \begin{cases} r & \text{if } r = s, \\ (s-r-1)(r+s-1)+2r-1 & \text{if } r < s. \end{cases}$$

We now consider the k-metric chromatic number of $K_{r,s}$ $(2 \leqslant r \leqslant s)$ for all k with $1 \leqslant k \leqslant r+s-1$. Let V_1 and V_2 be the partite sets of $K_{r,s}$. Observe that if $u, v \in V(K_{r,s})$, then

$$D(u,v) = \begin{cases} 2r-2 & \text{if } u,v \in V_i \text{ and } |V_i| = r, \\ 2r-1 & \text{if } uv \in E(K_{r,s}), \\ 2r & \text{if } u,v \in V_i \text{ and } |V_i| = s > r. \end{cases} \tag{9}$$

Thus $\operatorname{adiam}(K_{r,s}) = 2r-2$. This gives us the following result for regular complete bipartite graphs.

Corollary 3.2. *For integers k and $r \geqslant 2$ with $1 \leqslant k \leqslant 2r-1$,*

$$\chi_m^k(K_{r,r}) = \begin{cases} 2r & \text{if } k = 2r-1, \\ r & \text{if } k = 2r-2, \\ 1 & \text{otherwise.} \end{cases}$$

It remains to consider the case with $2 \leqslant r < s$.

Theorem 3.3. *For integers r, s and k with $2 \leqslant r < s$ and $1 \leqslant k \leqslant r+s-1$*

$$\chi_m^k(K_{r,s}) = \begin{cases} (k-2r+1)(r+s-1)+2r-1 & \text{if } 2r-1 \leqslant k \leqslant r+s-1,\\ r & \text{if } k = 2r-2,\\ 1 & \text{otherwise.} \end{cases}$$

Proof. By Observation 1.2 and Theorem 3.1 together with (9) we may assume that $2r-2 \leqslant k \leqslant r+s-3$. By Corollary 2.5, we have $\chi_m^{2r-2}(K_{r,s}) = \chi(K_r \cup \overline{K}_s) = r$.

For $k = 2r-1$, let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = V(K_{r,s}) - V_1$ be the partite sets of $K_{r,s}$ and consider the coloring c of $K_{r,s}$ defined by $c(u_i) = 2i-1$ for $1 \leqslant i \leqslant r$ and $c(v) = 2$ for every $v \in V_2$. Then c is a $(2r-1)$-metric coloring of $K_{r,s}$ whose value is $2r-1$ and so $\chi_m^{2r-1}(K_{r,s}) \leqslant 2r-1$. On the other hand, by Theorems 2.1 and 3.1

$$\chi_m^{2r-1}(K_{r,s}) \geqslant \chi_m^{r+s-1}(K_{r,s}) - (s-r)(r+s-1) = 2r-1.$$

Hence, $\chi_m^{2r-1}(K_{r,s}) = 2r-1$. Furthermore,

$$\chi_m^k(K_{r,s}) = (k-2r+1)(r+s-1)+2r-1$$

for every k with $2r \leqslant k \leqslant r+s-3$ again by Theorems 2.1 and 3.1. □

With the aid of Corollary 3.2 and Theorem 3.3 we are able to determine the k-metric chromatic numbers of complete l-partite graphs for each $l \geqslant 2$.

Theorem 3.4. *Let G be a complete l-partite graph ($l \geqslant 2$) of order n such that the maximum of the cardinalities of its partite sets equals n_1.*

(a) *If $n_1 < n/2$, then $\{\chi_m^k(G)\}_{k=1}^{n-1}$: $1, 1, \ldots, 1, n$.*

(b) *If $n_1 = n/2$, then $\{\chi_m^k(G)\}_{k=1}^{n-1}$: $1, 1, \ldots, 1, n/2, n$.*

(c) *If $n_1 > n/2$, then*

$$\chi_m^k(G) = \begin{cases} [k-2(n-n_1)+1]\,(n-1)+2(n-n_1)-1 & \text{if } k \geqslant 2(n-n_1)-1,\\ n-n_1 & \text{if } k = 2(n-n_1)-2,\\ 1 & \text{otherwise.} \end{cases}$$

Proof. If $n_1 < n/2$, then $l \geqslant 3$ and for every pair u, v of nonadjacent vertices $\deg u + \deg v \geqslant 2(n-n_1) > n$. Then G is Hamiltonian-connected and the result is immediate.

If $n_1 \geqslant n/2$, then observe that G contains $H = K_{n_1, n-n_1}$ as a subgraph and so $\chi_m^k(G) \leqslant \chi_m^k(H)$. Furthermore, one can verify that $D_G(u,v) = D_H(u,v)$ for every two vertices u and v. Therefore, every k-metric coloring of G is a k-metric coloring of H, that is, $\chi_m^k(G) \geqslant \chi_m^k(H)$. The result now follows by Corollary 3.2 and Theorem 3.3. □

We next consider the k-metric chromatic numbers of cycles.

Theorem 3.5. *For positive integers k and n with $1 \leqslant k \leqslant n-1$,*

$$\chi_m^k(C_n) = \begin{cases} 2(k+1)-n & \text{if } \lceil n/2 \rceil \leqslant k \leqslant n-1, \\ 1 & \text{otherwise.} \end{cases}$$

P r o o f. Since $\operatorname{adiam}(C_n) = \lceil n/2 \rceil$, it follows by Observation 1.2 that $\chi_m^k(C_n) = 1$ if $1 \leqslant k \leqslant \lceil n/2 \rceil - 1$. Thus we may assume that $\lceil n/2 \rceil \leqslant k \leqslant n-1$. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. We define a coloring c of C_n as follows.

$$(10) \qquad c(v_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant n-k, \\ i-n+k+1 & \text{if } n-k+1 \leqslant i \leqslant \lfloor n/2 \rfloor, \\ c(v_{i-\lfloor n/2 \rfloor}) - \lceil n/2 \rceil + k + 1 & \text{if } \lfloor n/2 \rfloor + 1 \leqslant i \leqslant n. \end{cases}$$

Hence, $1 = c(v_1) \leqslant c(v_2) \leqslant \ldots \leqslant c(v_n) = 2(k+1)-n$. We show that c is a k-metric coloring of C_n, that is,

$$(11) \qquad |c(u) - c(v)| + D(u,v) \geqslant k+1$$

for every two distinct vertices u and v of C_n. Let $u = v_i$ and $v = v_j$ for some i, j with $1 \leqslant i < j \leqslant n$. Observe that $|c(u) - c(v)| \leqslant j - i$. If u and v are antipodal vertices of C_n, then $j \in \{i + \lfloor n/2 \rfloor, i + \lceil n/2 \rceil\}$ and so

$$|c(u) - c(v)| + D(u,v) \geqslant c(v_{i+\lfloor n/2 \rfloor}) - c(v_i) + \lceil n/2 \rceil = k+1.$$

Thus we may assume that u and v are not antipodal vertices of C_n. We consider two cases, according to whether $j \leqslant \lfloor n/2 \rfloor$ or $j \geqslant \lfloor n/2 \rfloor + 1$.

C a s e 1. $j \leqslant \lfloor n/2 \rfloor$. Then $D(u,v) = n - (j-i)$. If $j \leqslant n-k$, then

$$|c(u) - c(v)| + D(u,v) = 0 + D(u,v) = n - j + i \geqslant k + i \geqslant k+1.$$

If $i \geqslant n-k+1$, then $|c(u) - c(v)| = j - i$ and so

$$|c(u) - c(v)| + D(u,v) = n \geqslant k+1.$$

Otherwise, $c(u) = 1$ and $c(v) = j - n + k + 1$ and so

$$|c(u) - c(v)| + D(u,v) = k + i \geqslant k+1.$$

C a s e 2. $j \geqslant \lfloor n/2 \rfloor + 1$. Let S be the set of antipodal vertices of v such that $x \in S$ if and only if $c(x) \leqslant c(v)$. Observe that S is nonempty. Let $w \in S$ such that $c(w) = \max\{c(x)\colon\ x \in S\}$. Clearly $D(u,v) > D(v,w)$ since u is not an antipodal vertex of v. If $c(u) \leqslant c(w)$, then

$$\begin{aligned}|c(u) - c(v)| + D(u,v) &= [c(v) - c(u)] + D(u,v)\\ &> [c(v) - c(w)] + [c(w) - c(u)] + D(v,w)\\ &\geqslant |c(v) - c(w)| + D(v,w) \geqslant k+1.\end{aligned}$$

Finally, if $c(u) \geqslant c(w)$, then observe that $D(u,v) = d(u,w) + D(v,w) \geqslant [c(u) - c(w)] + D(v,w)$. Hence,

$$\begin{aligned}|c(u) - c(v)| + D(u,v) &= [c(v) - c(w)] + [c(w) + c(u)] + D(u,v)\\ &\geqslant |c(v) - c(w)| + D(v,w) \geqslant k+1.\end{aligned}$$

In each case, (11) is satisfied. Therefore, $\chi_m^k(C_n) \leqslant c(v_n) = 2(k+1) - n$.

It now only remains to be shown that $\chi_m^k(C_n) \geqslant 2(k+1) - n$ for $\lceil n/2 \rceil \leqslant k \leqslant n-1$. By Observation 1.2, $\chi_k^m(C_n) \geqslant 2$. We consider two cases, according to whether n is odd or n is even.

C a s e I. n is odd. We proceed by induction on k. For $k = \lceil n/2 \rceil$, observe by Corollary 2.5 that

$$\chi_m^{\lceil n/2 \rceil}(C_n) = \chi(C_n) = 3 = 2\,(\lceil n/2 \rceil + 1) - n.$$

Assume that $\chi_m^{k-1}(C_n) \geqslant 2k - n$ for some integer k with $k - 1 \geqslant \lceil n/2 \rceil$. It then follows by Theorem 2.1 that $\chi_m^k(C_n) \geqslant \chi_m^{k-1}(C_n) + 1 \geqslant 2k + 1 - n$. Assume, to the contrary, that $\chi_m^k(C_n) = 2k+1-n$ and let c' be a minimum k-metric coloring of C_n using the colors in $\mathbb{N}_{2k+1-n}$. Then there exist adjacent vertices u and v such that $c'(u) \leqslant k - \lfloor n/2 \rfloor$ and $c'(v) \geqslant k - \lfloor n/2 \rfloor + 1$. Let w be the antipodal vertex of both u and v. If $c'(w) \leqslant k - \lfloor n/2 \rfloor$, then

$$|c'(u) - c'(w)| + D(u,w) \leqslant [(k - \lfloor n/2 \rfloor) - 1] + \lceil n/2 \rceil = k,$$

while if $c'(w) \geqslant k - \lfloor n/2 \rfloor + 1$, then

$$|c'(v) - c'(w)| + D(v,w) \leqslant [(2k+1-n) - (k - \lfloor n/2 \rfloor + 1)] + \lceil n/2 \rceil = k.$$

In each case, we obtain a contradiction. Therefore, $\chi_m^k(C_n) \geqslant 2(k+1) - n$.

C a s e II. n is even. Again, we proceed by induction on k. For $k = n/2$, we have

$$\chi_m^{n/2}(C_n) = \chi\Big(\frac{n}{2}K_2\Big) = 2 = 2\,(n/2+1) - n.$$

Assume that $\chi_m^{k-1}(C_n) \geqslant 2k-n$ for some integer k with $k-1 \geqslant n/2$. Then $\chi_m^k(C_n) \geqslant \chi_m^{k-1}(C_n)+1 \geqslant 2k+1-n$. Assume, to the contrary, that $\chi_m^k(C_n) = 2k+1-n$ and let c' be a minimum k-metric coloring of C_n using the colors in $\mathbb{N}_{2k+1-n}$. Then there exist adjacent vertices u and v such that $c(u) \leqslant k-n/2+1$ and $c(v) \geqslant k-n/2+2$. Let w_u be the antipodal vertex of u. If $c(w_u) \leqslant k-n/2+1$, then

$$|c(u)-c(w_u)| + D(u,w_u) \leqslant [(k-n/2+1)-1] + n/2 = k,$$

which is impossible and so $c(w_u) \geqslant k-n/2+2$. However, this implies that

$$|c(v)-c(w_u)| + D(v,w_u) \leqslant [(2k+1-n)-(k-n/2+2)] + (n/2+1) = k,$$

which is another contradiction. Hence, $\chi_m^k(C_n) \geqslant 2(k+1)-n$. □

4. Realizable triples

Suppose that G is a nontrivial connected graph of order n. We have seen in the sequence $\{\chi_m^k(G)\}_{k=1}^{n-1}$ that the first term greater than 1 appears when $k = \mathrm{adiam}(G)$ and the sequence is strictly increasing thereafter by Theorem 2.1. For this reason, it is important to study the adiam(G)-metric chromatic number of a graph G. Furthermore, the adiam(G)-metric chromatic number is related to Hamiltonian properties of a graph G. For example, we have seen that if G is a Hamiltonian-connected graph of order n and $\mathrm{adiam}(G) = a$, then $\chi_m^a(G) = n$. In fact, more can be said. For a nontrivial connected graph G of order n with $a = \mathrm{adiam}(G)$ and an integer k with $a \leqslant k \leqslant n-1$, recall that the graph G_k has $V(G_k) = V(G)$ and $uv \in E(G_k)$ if and only if $D_G(u,v) \leqslant k$. By Corollary 2.5,

$$\chi_m^a(G) = \begin{cases} 2 & \text{if and only if } G_a \text{ is bipartite,} \\ n & \text{if and only if } G_a \text{ is complete.} \end{cases}$$

In particular, observe that $\chi_m^a(G) = n$ if and only if $D(u,v) = \mathrm{adiam}(G)$ for every two vertices u and v in G.

The *circumference* $\mathrm{cir}(G)$ of a graph G of order n containing a cycle is the length of a longest cycle in G. Thus $3 \leqslant \mathrm{cir}(G) \leqslant n$ and $\mathrm{cir}(G) = n$ if and only if G is Hamiltonian. Also, $\mathrm{adiam}(G) \leqslant \mathrm{cir}(G) - 1$.

Theorem 4.1. *Let G be a connected graph of order $n \geqslant 2$, $\operatorname{adiam}(G) = a$ and $\operatorname{cir}(G) = l$.*

(a) *$\chi_m^a(G) = n$ if and only if G is Hamiltonian-connected.*

(b) *If G is Hamiltonian but is not Hamiltonian-connected, then*

$$2 \leqslant \chi_m^a(G) \leqslant \lceil n/(n-a) \rceil \leqslant \lceil n/2 \rceil \leqslant a \leqslant n-2.$$

(c) *If G is 2-edge-connected but is not Hamiltonian, then*

$$\chi_m^a(G) \leqslant \begin{cases} n - l + \lceil l/(l-a) \rceil - 1 & \text{if } a+1 \leqslant l \leqslant 2a, \\ n - l + 1 & \text{if } l \geqslant 2a+1. \end{cases}$$

Proof. For (a), we have seen that $\chi_m^a(G) = n$ if G is Hamiltonian-connected. For the converse, we verify that if $\chi_m^a(G) = n$, then $a = n-1$. Assume, to the contrary, that $\chi_m^a(G) = n$ and $a \leqslant n-2$. Since $\chi_m^a(G) = n$, the detour distance between every two vertices in G equals a. Suppose that $uv \in E(G)$. Since $D(u,v) = a$, the edge uv belongs to an $(a+1)$-cycle C. Furthermore, since $n \geqslant a+2$ and G is connected, we may assume that there exists a vertex $w \in V(G) - V(C)$ that is adjacent to u. However then, $D(v,w) \geqslant a+1$ and this is a contradiction.

For (b), observe that $\chi_m^a(G) \geqslant 2$ and $a \leqslant n-2$. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G. Therefore, $a \geqslant \operatorname{adiam}(C) = \lceil n/2 \rceil$. To show that $\chi_m^a(G) \leqslant \lceil n/2 \rceil$, consider a coloring $c\colon V(G) \to \mathbb{N}$ defined by $c(v_i) = \lceil i/(n-a) \rceil$ for $1 \leqslant i \leqslant n$. Suppose that $1 \leqslant i < j \leqslant n$ and $c(v_i) = c(v_j)$. Then $1 \leqslant j - i \leqslant n-a-1$ and so $D_G(v_i, v_j) \geqslant n - (j-i) \geqslant a+1$, that is, c is an $\operatorname{adiam}(G)$-metric coloring of G. Therefore,

$$\chi_m^a(G) \leqslant c(v_n) = \lceil n/(n-a) \rceil \leqslant \lceil n/2 \rceil.$$

Finally, for (c), first observe that $3 \leqslant l \leqslant n-1$. Let $C = (v_1, v_2, \ldots, v_l, v_1)$ be a cycle of length l and $V(G) - V(C) = \{v_{l+1}, v_{l+2}, \ldots, v_n\}$. Since G is connected, we assume that $v_1 v_{l+1} \in E(G)$. If $a+1 \leqslant l \leqslant 2a$, then one can verify that the coloring $c_1\colon V(G) \to \mathbb{N}$ given by

$$c(v_i) = \begin{cases} \lceil i/(l-a) \rceil & \text{if } 1 \leqslant i \leqslant l, \\ \lceil l/(l-a) \rceil + i - l - 1 & \text{if } l+1 \leqslant i \leqslant n \end{cases}$$

is an $\operatorname{adiam}(G)$-coloring of G whose value is $n - l + \lceil l/(l-a) \rceil - 1$. Similarly, if $l \geqslant 2a+1$, then a coloring $c_2\colon V(G) \to \mathbb{N}$ defined by $c_2(v_i) = 1$ for $1 \leqslant i \leqslant l$ and $c_2(v_i) = i - l + 1$ for $l+1 \leqslant i \leqslant n$ has the desired property. $\square$

We have seen that if G is a connected graph of order $n \geqslant 3$ with $\operatorname{adiam}(G) = a$ and $\chi_m^a(G) = \chi$, then $1 \leqslant a \leqslant n-1$ and $2 \leqslant \chi \leqslant n$. Define a triple (a, χ, n) of integers with

$$1 \leqslant a \leqslant n-1, \;\; 2 \leqslant \chi \leqslant n \;\; \text{and } n \geqslant 3 \tag{12}$$

to be a *realizable triple* if there exists a connected graph G of order n with $\operatorname{adiam}(G) = a$ and $\chi_m^a(G) = \chi$. Next, we investigate the following question:

(13) Which triples (a, χ, n) of integers satisfying (12) are realizable?

For $3 \leqslant n \leqslant 5$, all realizable triples are shown in Table 1, where "○" indicates that the corresponding triple is realizable and "×" indicates that the corresponding triple is not realizable. In Table 1, the first table is for $n = 3$, the second table is for $n = 4$ and the third table is for $n = 5$. For example, the triple $(1, 2, 3)$ is realizable, while the triples $(2, 3, 4)$ and $(3, 4, 5)$ are not realizable.

a \ χ	2	3
1	○	×
2	×	○

a \ χ	2	3	4
1	○	×	×
2	○	×	×
3	×	×	○

a \ χ	2	3	4	5
1	○	×	×	×
2	○	○	×	×
3	○	○	×	×
4	×	×	×	○

Table 1. Realizable triples (a, χ, n) where $n \in \{3, 4, 5\}$

In general, the following triples of integers satisfying (12) are realizable.

Theorem 4.2. *Let a, χ and $n \geqslant 3$ be integers with $1 \leqslant a \leqslant n-1$ and $2 \leqslant \chi \leqslant n$. Then*

(a) $(1, \chi, n)$ *is realizable if and only if* $\chi = 2$,
(b) (a, n, n) *is realizable if and only if* $a = n-1$,
(c) $(n-1, \chi, n)$ *is realizable if and only if* $\chi = n$,
(d) $(a, n-1, n)$ *is realizable if and only if* $a = 1$ *and* $n = 3$,
(e) $(a, 2, n)$ *is realizable if and only if* $1 \leqslant a \leqslant n-2$,
(f) $(a, a+1, n)$ *is realizable for all* a *with* $1 \leqslant a \leqslant \lceil n/2 \rceil - 1$.

Proof. Observe that (a) is a consequence of (4) and the fact that a connected graph G has $\operatorname{adiam}(G) = 1$ if and only if G contains a bridge; while (b) and (c) follow Theorem 4.1(a) and the fact that a nontrivial connected graph G of order n has $\operatorname{adiam}(G) = n-1$ if and only if G is Hamiltonian-connected. Thus, it remains to verify (d)–(f).

For (d), we may assume that $2 \leqslant a \leqslant n-2$ by (a)–(c). Assume, to the contrary, that there exists a connected graph G of order $n \geqslant 4$ with $\operatorname{adiam}(G) = a$ and $\chi_m^a(G) = n-1$. Then G is 2-edge-connected and $\chi(G_a) = n-1$. Hence, $\omega(G_a) = n-1$, that is, there exists a vertex $v_0 \in V(G)$ such that $D_G(u,v) = a$ whenever $v_0 \notin \{u,v\}$. Since G is not a star, $E(G-v) \neq \emptyset$ for every $v \in V(G)$. We may therefore assume that $v_1v_2 \in E(G-v_0) \subseteq E(G)$ and so $D_G(v_1,v_2) = a$. Then G contains an $(a+1)$-cycle $C = (v_1, v_2, \ldots, v_{a+1}, v_1)$. Since $a+1 \leqslant n-1$, there exists a vertex $x \in V(G) - V(C)$ and we may further assume that $v_1x \in E(G)$. Then $D_G(v_i, x) \geqslant a+1$ for $i \in \{2, a+1\}$, which implies that $x = v_0$. Now since G is 2-edge-connected, $\deg x \geqslant 2$. If x is adjacent to another vertex belonging to C, say $v_lx \in E(G)$ for some l with $2 \leqslant l \leqslant \lceil a/2 \rceil + 1$, then $D_G(v_2, v_{l+1}) \geqslant a+1$, which is impossible. Therefore, $a \leqslant n-3$ and x is adjacent a vertex $y \in V(G) - [V(C) \cup \{x\}]$. However then, $D_G(v_2, y) \geqslant a+2$, which is also impossible. Thus (d) holds.

For (e), we may again assume that $2 \leqslant a \leqslant n-2$. Let $G = K_2 + \big(K_{a-1} \cup \overline{K}_{n-a-1}\big)$. Then it can be verified that $\operatorname{adiam}(G) = a$ and $D(u,v) = a$ if and only if $\deg u = \deg v = n-1$. Therefore, $G_a = K_2 \cup \overline{K}_{n-2}$ and so $\chi_m^a(G) = \chi(G_a) = 2$ by Corollary 2.5.

Finally for (f), consider the graph $G = K_1 + (K_a \cup K_{n-a-1})$. Observe that $\operatorname{adiam}(G) = a$ and $G_a = G$ if $n = 2a+1$ while $G_a = K_{a+1} \cup \overline{K}_{n-a-1}$ otherwise. Thus, $\chi_m^a(G) = \chi(G_a) = a+1$ by Corollary 2.5. $\square$

With the aid of Theorems 4.1 and 4.2, we are able to determine all realizable triples (a, χ, n) for $n \in \{6, 7\}$. These realizable triples are shown in Table 2, where the first table is for $n = 6$ and the second table is for $n = 7$ and where "$\circ$" indicates that the corresponding triple is realizable while "$\times$" indicates that the corresponding triple is not realizable.

a \ χ	2	3	4	5	6
1	$\circ$	$\times$	$\times$	$\times$	$\times$
2	$\circ$	$\circ$	$\times$	$\times$	$\times$
3	$\circ$	$\times$	$\times$	$\times$	$\times$
4	$\circ$	$\circ$	$\times$	$\times$	$\times$
5	$\times$	$\times$	$\times$	$\times$	$\circ$

a \ χ	2	3	4	5	6	7
1	$\circ$	$\times$	$\times$	$\times$	$\times$	$\times$
2	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$
3	$\circ$	$\times$	$\circ$	$\times$	$\times$	$\times$
4	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$
5	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$
6	$\times$	$\times$	$\times$	$\times$	$\times$	$\circ$

Table 2. Realizable triples (a, χ, n) for $n = 6, 7$

As an illustration, we show that all realizable triples $(a, \chi, 6)$ are exactly those given in the first table in Table 2. Recall that if H is a connected spanning subgraph of a graph G of order n, then $\operatorname{adiam}(H) \leqslant \operatorname{adiam}(G)$ and $\chi_m^k(G) \leqslant \chi_m^k(H)$ for $1 \leqslant k \leqslant n-1$ by Observation 1.1.

Theorem 4.3. *A triple* $(a,\chi,6)$ *of integers satisfying* (12) *is realizable if and only if it is one of the realizable triples shown in the first table of* Table 2, *that is,*

$$(a,\chi,6)\in\{(a,2,6)\colon\ 1\leqslant a\leqslant 4\}\cup\{(2,3,6),(4,3,6),(5,6,6)\}.$$

P r o o f. By Theorems 4.1 and 4.2, we only need to consider those triples $(a,\chi,6)$ with $2\leqslant a\leqslant 4$ and $\chi=3,4$. The triple $(4,3,6)$ is realizable since $K_{3,3}$ has the desired property and so is $(2,3,6)$ with the graph $K_1+(P_2\cup P_3)$. Thus, it remains to verify that none of the four triples $(3,3,6)$, $(2,4,6)$, $(3,4,6)$ and $(4,4,6)$ is realizable. Let G be an arbitrary connected graph of order 6. By Theorem 4.2(a)(c), we may assume that $2\leqslant \operatorname{adiam}(G)\leqslant 4$ and so every edge belongs to a cycle. Hence $3\leqslant \operatorname{cir}(G)\leqslant 6$. Note that if $\operatorname{cir}(G)=3$, then G must be disconnected or contain a bridge, so this is impossible. On the other hand, if $\operatorname{cir}(G)=6$, then $\chi_m^a(G)\in\{2,3\}$ if $\operatorname{adiam}(G)=4$ while $\chi_m^a(G)=2$ if $\operatorname{adiam}(G)=3$ by Theorem 4.1(b).

If $\operatorname{cir}(G)=5$, then G must contain the graph F in Figure 1 as a spanning subgraph since G is 2-edge-connected. Observe also that $\operatorname{adiam}(G)\geqslant\operatorname{adiam}(F)=3$ and $\chi_m^3(G)\leqslant\chi_m^3(F)=\chi(P_4\cup\overline{K}_2)=2$ (by Corollary 2.5). Furthermore, $\deg_G v_1=\deg_G v_2=2$ since otherwise $\operatorname{cir}(G)=6$. Therefore, $\operatorname{adiam}(G)\leqslant D_G(v_3,v_4)=3$. That is, $\operatorname{adiam}(G)=3$ and $\chi_m^a(G)=2$.

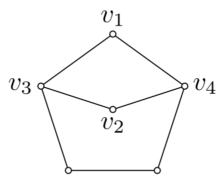

Figure 1. The graph F

Finally, suppose that $\operatorname{cir}(G)=4$. Then either (i) $G\in\{K_{2,4},K_{1,1,4}\}$ or (ii) $G\in\{K_1+(K_2\cup P_3),K_1+(K_2\cup K_3)\}$. If (i) occurs, then $\operatorname{adiam}(G)=2$ and $\chi_m^2(G)=2$; while if (ii) occurs, then $\operatorname{adiam}(G)=2$ and $\chi_m^2(G)=3$. Hence, $\operatorname{adiam}(G)=2$ and $\chi_m^a(G)\in\{2,3\}$ if $\operatorname{cir}(G)=4$. □

Based on the information obtained from the $\operatorname{adiam}(G)$-metric chromatic numbers of connected graphs G of order n with $3\leqslant n\leqslant 7$ and Theorems 4.1 and 4.2, we conclude with the following questions.

P r o b l e m 4.4. Let (a,χ,n) be a triple of integers satisfying (12) and $n\geqslant 8$.

(a) Is it true that no triple $(a,a+1,n)$ with $\lceil n/2\rceil\leqslant a\leqslant n-2$ is realizable?

(b) Is it true that no triple (a,χ,n) with $3\leqslant a+2\leqslant\chi\leqslant n$ is realizable?

(c) Is it true that a triple (a,χ,n) with $\lceil n/2\rceil+1\leqslant\chi\leqslant n$ is realizable if and only if $(a,\chi,n)=(n-1,n,n)$?

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