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Commutative Radical Rings II

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This paper, which is a continuation of [8], deals with further properties of commutative radical rings (i.e., rings equal to their Jacobson radical). In particular, radical rings whose additive and/or adjoint groups have finite torsionfree or Prüfer rank (or are minimax) are investigated.

0. Introduction

This paper is the second part of a comprehensive treatment concerning commutative radical rings, i.e., rings (generally without unit) which can arise as Jacobson radical of some (unitary) ring. As a tool, the adjoint (or circle) semigroup of a ring R is used, where the operation is given by $a \circ b = a + b + ab$ for all $a, b \in R$. All the notions and notation are the same as in [8] which is the first part of this treatment. When referring to result from [8], we write e.g. I.7.22 for [8, 7.22].

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1. Radical rings whose additive and/or adjoint groups have finite torsionfree rank

1.1 Remark. An abelian group G is said to have *torsionfree rank at most n*, n being a non-negative integer, if G has an (at most) n-generated subgroup A such that G/A is torsion; we denote the fact by $\mathrm{rnk}_{\mathrm{Tf}}(G) \le n$ and, moreover, we put $\mathrm{rnk}_{\mathrm{Tf}}(G) = n$ if $\mathrm{rnk}_{\mathrm{Tf}}(G) \le n$ and $n = \min\{k \mid \mathrm{rnk}_{\mathrm{Tf}}(G) \le k\}$. If G has not finite torsionfree rank then we say that G has *infinite torsionfree rank.*

(i) $\mathrm{rnk}_{\mathrm{Tf}}(G) = 0$ if and only if G is torsion.

(ii) If H is a subgroup of G then $\mathrm{rnk}_{\mathrm{Tf}}(G)$ is finite if and only if both $\mathrm{rnk}_{\mathrm{Tf}}(H)$ and $\mathrm{rnk}_{\mathrm{Tf}}(G/H)$ are so.

(iii) $\mathrm{rnk}_{\mathrm{Tf}}(G) = n \ge 0$ if and only if G has a free (abelian) subgroup F of rank n such that G/F is torsion.

(iv) If H is a subgroup of G such that neither H nor G/H is torsion and if $\mathrm{rnk}_{\mathrm{Tf}}(G) = n$ then $1 \le \mathrm{rnk}_{\mathrm{Tf}}(H) < n$ and $1 \le \mathrm{rnk}_{\mathrm{Tf}}(G/H) < n$.

1.2 Example. Consider the radical domain R constructed in I.9.2(iii). Then $R(+)$ is torsionfree and $\mathrm{rnk}_{\mathrm{Tf}}(R(+)) = 1$. On the other hand, $R(\circ) \simeq \mathbb{Z}_2(+) \times$ $\times\ \mathbb{Z}(+)^{(\omega)}$ for $q = 2$ and $R(\circ) \simeq \mathbb{Z}(+)^{(\omega)}$ for $q \ge 3$. Thus $R(\circ)$ has infinite torsionfree rank.

1.3 Proposition. *Let R be a nilpotent ring. Then the additive group $R(+)$ has finite torsionfree rank if and only if the same is true for the adjoint group $R(\circ)$.*

Proof. Use I.7.21. □

1.4 Proposition. *Let R be a nil-ring such that* $\mathrm{rnk}_{\mathrm{Tf}}(R(+))$ *is finite. Then* $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ)) = \mathit{rnk}_{\mathit{Tf}}(R(+))$ *is finite.*

Proof. We proceed by induction on $m = \mathrm{rnk}_{\mathrm{Tf}}(R(+))$. If $m = 0$ then both $R(+)$ and $R(\circ)$ are torsion (I.7.22), and so $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ)) = 0$. Therefore, assume that $m \ge 1$. Then $T \ne R$, T being the torsion part of $R(+)$, both groups $T(+)$ and $T(\circ)$ are torsion and $S = R/T$ is a nil-ring with $\mathrm{rnk}_{\mathrm{Tf}}(S(+)) = m$. If $S^2 = 0$ then $S(+) = S(\circ)$ and $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ)) = \mathrm{rnk}_{\mathrm{Tf}}(S(\circ)) = \mathrm{rnk}_{\mathrm{Tf}}(S(+)) = m$. Now, assume that $S^2 \ne 0$. Since S is a nil-ring, it is not a domain and it follows from I.1.21 that S has a non-zero ideal K such that the additive group $(S/K)(+)$ is not torsion. In particular, $K \ne S$ and we consider the factor-ring $P = S/K$. We have $m = k + l$, where $k = \mathrm{rnk}_{\mathrm{Tf}}(K(+))$ and $l = \mathrm{rnk}_{\mathrm{Tf}}(P(+))$. Using the fact that none of the groups $K(+)$, $P(+)$ is torsion and then the induction hypothesis, we get $\mathrm{rnk}_{\mathrm{Tf}}(K(\circ)) = k \ge 1$ and $\mathrm{rnk}_{\mathrm{Tf}}(P(\circ)) = l \ge 1$. Thus $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ)) =$ $= \mathrm{rnk}_{\mathrm{Tf}}(S(\circ)) = k + l = m$. □

1.5 Proposition. *The following conditions are equivalent for a ring R:*

(i) *R is a radical ring and the adjoint group $R(\circ)$ is torsion.*

(ii) *R is a nil-ring and the additive group $R(+)$ is torsion.*

Proof. (i) ⇒ (ii). If $0 \neq a \in R$ and S is the subring generated by a then S is a radical ring (I.7.5) and S is nilpotent by I.10.4. Consequently, $a \in \mathcal{N}(R)$ and R is a nil-ring. By I.7.22, $R(+)$ is torsion.

(ii) ⇒ (*i*). See I.7.22. □

1.6 Proposition. *Let R be a radical ring such that $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$ is finite. Then R is a nil-ring.*

Proof. Assume, on the contrary, that $\mathcal{N}(R) \neq R$. Then in view of I.1.8, we may assume that R is a domain and we denote by F the field of fractions of R, by P the prime subfield of F and by X a transcendent basis of F over P. Now, F is algebraic over $Q = P(X)$ and $R_1 = R \cap Q \neq 0$. By I.7.24, R_1 is a radical domain and, of course, $R_1(\circ)$ has finite torsionfree rank. On the other hand, $R_1(\circ)$ is isomorphic (via $a \mapsto a + 1$) to a subgroup of Q^* (the multiplicative group of non-zero elements of Q) and the latter group is isomorphic of the product $P^* \times A$, A being a free abelian group. If T denotes the torsion part of $R_1(\circ)$ then $T(\circ)$ is isomorphic to a subgroup of P^*, and consequently $T(\circ)$ is a finite group. Furthermore, $R_1(\circ)/T(\circ)$ is isomorphic to a subgroup of $\mathbb{Z}(+)^{(\omega)} \times A$. Thus $R_1(\circ)/T(\circ)$ is a free abelian group of finite torsionfree rank and it means that the group is finitely generated. We conclude that $R_1(\circ)$ is finitely generated. But then R_1 is nilpotent by I.10.5, a contradiction. □

1.7 Proposition. *Let R be a radical ring such that $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$ is finite. Then $\mathrm{rnk}_{\mathrm{Tf}}(R(+))$ is finite.*

Proof. We proceed by induction on $m = \mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$. If $m = 0$ then $R(\circ)$ is torsion, and hence $R(+)$ is torsion by 1.5. Now, assume that $m \geq 1$ and consider an ideal I of R maximal with respect to the property that $I(\circ)$ is torsion. Then $I \neq R$, $\mathrm{rnk}_{\mathrm{Tf}}(I(+)) = 0 = \mathrm{rnk}_{\mathrm{Tf}}(I(\circ))$, $S = R/I$ is a radical ring and $\mathrm{rnk}_{\mathrm{Tf}}(S(\circ)) = m$. If $S^2 = 0$ then $\mathrm{rnk}_{\mathrm{Tf}}(R(+)) = \mathrm{rnk}_{\mathrm{Tf}}(S(+)) = \mathrm{rnk}_{\mathrm{Tf}}(S(\circ)) = m$. Consequently, assume that $S^2 \neq 0$. By 1.6, S is a nil-ring, and so S is not a domain. Moreover, if T is the ideal of R such that $I \subseteq T$ and T/I is the torsion part of $S(+)$ then $(T/I)(\circ)$ is torsion (1.5), and hence $T(\circ)$ is torsion and $T = I$ due to the maximaliy of I. We have shown that $S(+)$ is torsionfree, and therefore $(S/K)(+)$ is not torsion for a non-zero ideal K of S (I.1.21). Now, both $K(\circ)$ and $(S/K)(\circ)$ are not torsion and it follows easily that both ranks $\mathrm{rnk}_{\mathrm{Tf}}(K(\circ))$ and $\mathrm{rnk}_{\mathrm{Tf}}((S/K)(\circ))$ are lesser than $m = \mathrm{rnk}_{\mathrm{Tf}}(S(\circ))$. By induction, the ranks $\mathrm{rnk}_{\mathrm{Tf}}(K(+))$ and $\mathrm{rnk}_{\mathrm{Tf}}((S/K)(+))$ are finite and then the same is true for $\mathrm{rnk}_{\mathrm{Tf}}(S(+)) = \mathrm{rnk}_{\mathrm{Tf}}(R(+))$. □

1.8 Theorem. *The following conditions are equivalent for a ring R:*

(i) *R is a radical ring and the adjoint group $R(\circ)$ has finite torsionfree rank.*

(ii) *R is a nil-ring and the additive group $R(+)$ has finite torsionfree rank.*

Moreover, if these conditions are satisfied then $\mathrm{rnk}_{\mathrm{Tf}}(R(+)) = \mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$.

Proof. Combine 1.4, 1.6 and 1.7. □

1.9 Corollary. *Let R be a radical ring such that either R is not nil or at least one of the groups $R(+)$ and $R(\circ)$ has infinite torsionfree rank. Then the free (abelian) group $\mathbb{Z}(+)^{(\omega)}$ of infinite countable rank is isomorphic to a subgroup of the adjoint group $R(\circ)$.* □

1.10 Example. Let $S = \mathbb{Z}_2[x]$ be the polynomial ring in one indeterminate x over the two-element field $\mathbb{Z}_2$ and let $Q = \mathbb{Z}_2(x)$ be the field of fractions of S. Then S is a principal ideal domain, we take an irreducible polynomial $q \in S$ and we put $R = \{fg^{-1} \mid f \in Sq,\ q \in S \setminus Sq\}$ (I.9.2(ii)). Then R is a radical domain, $\operatorname{char}(R) = 2$, $R(+)$ is a (torsion) 2-elementary group and $R(\circ) \simeq \mathbb{Z}(+)^{(\omega)}$ is a torsionfree group (of infinite torsionfree rank).

1.11 Example. Consider the radical ring R constructed in I.9.6(ii), where $p = 2$. Then $R(+) \simeq R(\circ)$ are 2-elementary groups, $(0 : R) = 0$, $R^2 = R$ and $a^2 = 0$ for every $a \in R$.

1.12 Example. Consider the radical ring R constructed in I.9.3(ii), where $F = \mathbb{Z}_p$, p being a prime. Then R is a radical domain, $R^2 = R$ and $R(+)$ is a p-elementary group (the adjoint group $R(\circ)$ has infinite torsionfree rank).

1.13 Remark. (cf. 1.11, 1.12) Let R be a radical ring such that $R^2 = R$.

(i) If $R(\circ)$ has finite torsionfree rank then both $R(\circ)$ and $R(+)$ are torsion (see 3.9).

(ii) If $R(+)$ has finite torsionfree rank then $R(+)$ is torsion (see 3.9).

1.14 Remark. Let R be a radical domain. By 1.8, $R(\circ)$ has infinite torsionfree rank. If $\operatorname{rnk}_{\mathrm{Tf}}(R(+))$ is finite then either $\operatorname{rnk}_{\mathrm{Tf}}(R(+)) = 0$ and $R(+)$ is an elementary p-group or $\operatorname{rnk}_{\mathrm{Pr}}(R(+)) = \operatorname{rnk}_{\mathrm{Tf}}(R(+))$ is finite and $R(+))$ is torsionfree (see 3.6).

1.15 Propositin. *Let R be a nil-ring such that the additive group $R(+)$ is torsionfree and has finite torsionfree rank $m = \operatorname{rnk}_{\mathrm{Tf}}(R(+))$. Then R is nilpotent of index at most $m + 1$ (i.e., $R^{m+1} = 0$).*

Proof. We proceed by induction on m. Let $a \in R$ and $I = (0 : a)$. Since R is nil, we have $I \neq 0$ and $\operatorname{rnk}_{\mathrm{Tf}}(I(+)) \geq 1$. If $I = R$ then $Ra = 0$. If $I \neq R$ then $S = R/I$ is a nil-ring and, since $R(+)$ is torsionfree, the same is true for $S(+)$. Moreover, $\operatorname{rnk}_{\mathrm{Tf}}(S(+)) < m$, $S^m = 0$ by induction, and hence $R^m \subseteq I$. Thus $R^m a = 0$. □

1.16 Corollary. *Let R be a nil-ring such that the additive group $R(+)$ has finite torsionfree rank $m = \operatorname{rnk}_{\mathrm{Tf}}(R(+))$. Let T be the torsion part of $R(+)$. Then $R^{m+1} \subseteq T$. In particular, R is nilpotent if and only if T is so.* □

1.17 Corollary. *Let R be a radical ring such that the additive group $R(+)$ has finite torsionfree rank $m = \operatorname{rnk}_{\mathrm{Tf}}(R(+))$. Let T be the torsion part of $\mathcal{N}(R)(+)$. Then $\mathcal{N}(R)^{m+1} \subseteq T$ and, moreover:*

(i) *If* $T = 0$ *then* $\mathcal{N}(R)^{m+1} = 0$.
(ii) *If* $(R/\mathcal{N}(R))(+)$ *is not torsion then* $\mathcal{N}(R)^m \subseteq T$.
(iii) *If* $(R/T)(+)$ *is torsionfree and* $\mathcal{N}(R) \neq R$ *then* $\mathcal{N}(R)^m \subseteq T$.
(iv) *If* $T = 0$ *and* $R/\mathcal{N}(R))(+)$ *is not torsion then* $\mathcal{N}(R)^m = 0$.
(v) $R(+)$ *is torsionfree and* $\mathcal{N}(R) \neq R$ *then* $\mathcal{N}(R)^m = 0$. □

2. Radical rings whose adjoint groups have finite prüfer rank

2.1 Remark. A (possibly non-commutative) group G is said to have *Prüfer rank at most n*, n being a non-negative integer, if every finitely generated subgroup of G is (at most) n-generated; we denote this fact by $\mathrm{rnk}_{\mathrm{Pr}}(G) \leq n$ and, moreover, we put $\mathrm{rnk}_{\mathrm{Pr}}(G) = n$ if G contains at least one finitely generated subgroup that is not $(n-1)$-generated (for $n \geq 1$). If G has not finite Prüfer rank (i.e., for every $n \geq 0$ there exists a finitely generated subgroup that is not generated by n elements) then we say that G has *infinite Prüfer rank.*

(i) $\mathrm{rnk}_{\mathrm{Pr}}(G) = 0$ if and only if G is trivial.

Now, assume that G is abelian.

(ii) If H is a subgroup of G and $\mathrm{rnk}_{\mathrm{Pr}}(G) = n$ then $\mathrm{rnk}_{\mathrm{Pr}}(H) \leq n$ and $\mathrm{rnk}_{\mathrm{Pr}}(G/H) \leq n$.
(iii) If H is a subgroup of G then $\mathrm{rnk}_{\mathrm{Pr}}(G)$ is finite if and only if both ranks $\mathrm{rnk}_{\mathrm{Pr}}(H)$ and $\mathrm{rnk}_{\mathrm{Pr}}(G/H)$ are finite. If so, then $\mathrm{rnk}_{\mathrm{Pr}}(G) \leq \mathrm{rnk}_{\mathrm{Pr}}(H) + + \mathrm{rnk}_{\mathrm{Pr}}(G/H)$.
(iv) $\mathrm{rnk}_{\mathrm{Tf}}(G) \leq \mathrm{rnk}_{\mathrm{Pr}}(G)$, and if G is torsionfree then $\mathrm{rnk}_{\mathrm{Tf}}(G) = \mathrm{rnk}_{\mathrm{Pr}}(G)$.
(v) If T denotes the torsion part of G then $\mathrm{rnk}_{\mathrm{Pr}}(G) = \mathrm{rnk}_{\mathrm{Pr}}(T) + \mathrm{rnk}_{\mathrm{Pr}}(G/T)$. Moreover, $\mathrm{rnk}_{\mathrm{Pr}}(G) = n \geq 0$ if and only if $(\mathrm{rnk}_{\mathrm{Tf}}(G) = \mathrm{rnk}_{\mathrm{Tf}}(G/T) = =)\mathrm{rnk}_{\mathrm{Pr}}(G/T) = m \leq n$, $|\mathrm{Soc}_p(T)| = p^{k_p}$, $0 \leq k_p \leq n$ for every prime p and $n = m + \max(k_p)$.
(vi) If G is a reduced p-group and $\mathrm{rnk}_{\mathrm{Pr}}(G)$ is finite then G is finite.

2.2 Lemma. *Let R be a ring nilpotent of index* $k \geq 2$.
(i) *If* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = r$ *is finite then* $\mathrm{rnk}_{\mathrm{Pr}}(\mathrm{R}(\circ)) \leq (k-1)r$.
(ii) *If* $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = s$ *is finite then* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) \leq (k-1)s$.

Proof. We proceed by induction on k. If $k = 2$ then $R(+) = R(\circ)$ and there is nothing to prove. If $k \geq 3$ then $K = (0:R) \neq R$, $S = R/K$ is nilpotent of index $k-1$, $K(+) = K(\circ)$, $R(+)/K(+) = S(+)$ and $R(\circ)/K(\circ) = S(\circ)$. Now, if r is finite then $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) \leq \mathrm{rnk}_{\mathrm{Pr}}(S(\circ)) + \mathrm{rnk}_{\mathrm{Pr}}(K(\circ)) \leq (k-2)\mathrm{rnk}_{\mathrm{Pr}}(S(+)) + + \mathrm{rnk}_{\mathrm{Pr}}(K(+)) \leq \leq (k-1)r$. Similarly, if s is finite. □

2.3 Corollary. *Let R be a nilpotent ring. Then* $\mathrm{rnk}_{\mathrm{Pr}}(R(+))$ *is finite if and only if* $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ))$ *is finite.* □

2.4 Let R be a finite nil-ring such that $R(+)$ is p-elementary for a prime p, $|R| = p^r$, $r > 1$, $r = \mathrm{rnk}_{\mathrm{Pr}}(R(+))$. Further, let $s = \mathrm{rnk}_{\mathrm{Pr}}(R(\circ))$, $s \geq 1$, k be the nilpotence index of R, $k \geq 2$, and l be the smallest positive integer such that $a^l = 0$ for every $a \in R$.

2.4.1 Lemma. $2 \leq l \leq k \leq r + 1$ *and* $1 \leq s \leq r$.

Proof. Obvious. □

Given $a \in R$ and $t \geq 1$, let $a^{(t)}$ be the t-th power $a \circ a \circ \ldots \circ a$ of the elemment a in the adjoint group $R(\circ)$. Let m be the smallest positive intger such that $a^{\langle p^m \rangle} = 0$ for every $a \in R$.

2.4.2 Lemma. $1 \leq m \leq r$ *ad* $r \leq sm$.

Proof. Obvious. □

2.4.3 Lemma. $p^{m-1} \leq l - 1$.

Proof. Te inequality is clear for $m = 1$ and we assume that $m \geq 2$. There exists $a \in R$ such that $a^{\langle p^{m-1} \rangle} \neq 0$ and, using I.2.3 and the fact that $R(+)$ is p-elementary, we see that $a^{\langle p^{m-1} \rangle} = a^{p^{m-1}}$. Thus $a^{p^{m-1}} \neq 0$ and $p^{m-1} < l$. □

2.4.4 Lemma. $1 \leq p^{m-1} \leq l - 1 \leq k - 1 \leq r \leq sm$.

Proof. Combine the preceding three lemmas. □

2.4.5 Lemma. $m \leq s + 2$.

Proof. Assume, on the contrary, that $s + 2 < m$. Then $4 \leq m$, $s < m - 2$ and $sm < m(m - 2)$. But $m(m - 2) \leq 2^{m-1} \leq p^{m-1}$, and hence $sm < p^{m-1}$, a contradiction with 2.4.4. □

2.4.6 Lemma. $s \leq r \leq s(s + 2)$.

Proof. By 2.4.1, 2.4.4 and 2.4.5. □

2.5 Lemma. *Let R be a finite nil-ring such that $R(+)$ is a p-group for a prime p. If $r = \mathrm{rnk}_{\mathrm{Pr}}(R(+))$ and $s = \mathrm{rnk}_{\mathrm{Pr}}(R(\circ))$ then $r \leq s(s + 2)$.*

Proof. Put $K = \{a \in R \mid pa = 0\}$. Then K is a non=zero ideal of R and $\mathrm{rnk}_{\mathrm{Pr}}(K(+)) = r$. By 2.4.6, we have $r \leq s_1(s_1 + 2) \leq s(s + 2)$, where $s_1 =$ $= \mathrm{rnk}_{\mathrm{Pr}}(K(\circ)) \leq s$. □

2.6 Lemma. *Let R be a radical ring such that $R(\circ)$ is a p-group for a prime p and $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = s$ is finite. Then $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) \leq s(s + 2)$ is finite.*

Proof. By 1.5 and 1.6, R is a nil-ring and $R(+)$ is torsion. Further, it follows from I.7.22 that $R(+)$ is a p-group and we put $K = \{a \in R \mid pa = 0\}$. Then K is a non-zero ideal of R and $K(+)$ is a p-elementary. If K is finite then, by 2.5, $\mathrm{rnk}_{\mathrm{Pr}}(R +)) = \mathrm{rnk}_{\mathrm{Pr}}(K(+)) \leq s_1(s_1 + 2) \leq (s + 2)$, where $s_1 = \mathrm{rnk}_{\mathrm{Pr}}(K(\circ))$. If K is infinite then $K(+)$ contains a finite subgroup $A(+)$ such that $|A| > p^{s(s+2)}$ and we consider the subring S of K generated by A. Then S is a finitely generated nil-ring and S is nilpotent by I.1.12(i). By 2.3, $\mathrm{rnk}_{\mathrm{Pr}}(S(+)) = t$ is finite and, since $S(+)$ is p-elementary, we have $p^t = |S| \geq |A| > p^{s(s+2)}$ and $t > s(s+2)$. On the other hand, $\mathrm{rnk}_{\mathrm{Pr}}(S(\circ)) \leq \mathrm{rnk}_{\mathrm{Pr}}(K(\circ)) = s_1 \leq s$ and $t \leq s(s+2)$ by 2.5, contradiction. □

2.7 Theorem. *Let R be a nil-ring* (e.g., $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$ finite – see 1.6), $p \geq 2$ *be a prime number and $P(+)$ be the p-component of $R(+)$. Then:*

(i) *$P(\circ)$ is the p-component of $R(\circ)$.*
(ii) *$\mathrm{rnk}_{\mathrm{Pr}}(P(+))$ is finite if and only if $\mathrm{rnk}_{\mathrm{Pr}}(P(\circ))$ is finite.*
(iii) *If $\mathrm{rnk}_{\mathrm{Pr}}(P(+))$ is finite and $Q(+)$ is the divisible part of $P(+)$ then $Q(+) = Q(\circ)$ is the divisible part of $P(\circ)$.*
(iv) *If $\mathrm{rnk}_{\mathrm{Pr}}(P(+))$ is finite then P is nilpotent.*

Proof. (i) If $P = R$ then $R(\circ)$ is a p-group by I.7.22, and hence assume that $P \neq R$. Of course, $P(\circ)$ is a p-group and it suffices to show that $R(\circ)/P(\circ)$ has no elements of order p. Let, on the contrary, $a \in R\setminus P$ be such that $a^{\langle p\rangle} = a \circ a \circ \dots \circ a \in P$. Since R is nil and $a \notin P$, $a^k \in P$ and $a^{k-1} \notin P$ for some $k \geq 2$. Now, $pa^{k-1} + \binom{p}{2}a^k + \dots + \binom{p}{p-1}a^{k+p-3} + a^{k+p-2} = ak - 2 \cdot a^{\langle p\rangle} \in P$, and therefore $pa^{k-1} \in P$ and $a^{k-1} \in P$, a contradiction.

(ii), (iii) and (iv). First, assume that $P \neq 0$, $\mathrm{rnk}_{\mathrm{Pr}}(P(+))$ is finite and denote by Q the divisible part of $P(+)$. By I.1.13, Q is and ideal of $P, Q^2 = 0$ and $Q(+) = Q(\circ)$. Then our result is clear for $Q = P$ and we may assume that $Q \neq P$. Now, $T = P/Q$ is a finite nil-ring, and hence T is nilpotent and $\mathrm{rnk}_{\mathrm{Pr}}(T(\circ))$ is finite. Consequently, P is nilpotent, $\mathrm{rnk}_{\mathrm{Pr}}(P(\circ))$ is finite and $Q(\circ)$ is the divisible part of $P(\circ)$.

Conversely, if $\mathrm{rnk}_{\mathrm{Pr}}(P(\circ)) = s$ is finite then $\mathrm{rnk}_{\mathrm{Pr}}(P(+)) \leq s(s+2)$ by 2.6. □

2.8 Theorem. *Let R be a nil-ring* (e.g., $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$ finite – see 1.6) *and T be the torsion part of $R(+)$. Then:*

(i) *$T(\circ)$ is the torsion part of $R(\circ)$.*
(ii) *If $\mathrm{rnk}_{\mathrm{Pr}}(T(\circ)) = s$ is finite then $\mathrm{rnk}_{\mathrm{Pr}}(T(+)) \leq s(s+2)$ is finite.*
(iii) *If $\mathrm{rnk}_{\mathrm{Pr}}(T(+))$ is finite and Q is the divisible part of $T(+)$ then $Q(+) = Q(\circ)$ is the divisible part of $T(\circ)$.*

Proof. (i) T is an ideal of R and $T(\circ)$ is a torsion subgroup of $R(\circ)$ (I.7.22). Then $T \subseteq T_1$, where T_1 is the torsion part of $R(\circ)$ and, by 2.7, every p-component of $T_1(\circ)$ is in T. Thus $T_1 = T$.

(ii) Using (i), the result follows from 2.7(ii).

(iii) Use 2.7(iii) (see the proof of (i)). □

2.9 Theorem. *Let R be a radical ring such that the adjoint group $R(\circ)$ has finite Prüfer rank $s = \mathrm{rnk}_{\mathrm{Pr}}(R(\circ))$. Then:*

(i) *R is a nil-ring.*
(ii) $\mathrm{rnk}_{\mathrm{Tf}}(R(+)) = \mathrm{rnk}_{\mathrm{Tf}}(R(\circ)) = s_1 \leq s$.

(iii) *If* $T(+)$ *is the torsion part of* $R(+)$ *then* $T(\circ)$ *is the torsion part of* $R(\circ)$ *and* $\mathrm{rnk}_{\mathrm{Pr}}(T(\circ)) = s - s_1$.

(iv) *The additive group* $R(+)$ *has finite Prüfer rank* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) \leq s_1 + (s - s_1)(s + 2 - s_1)$.

Proof. (i) and (ii). See 1.8.

(iii) See 2.8.

(iv) We have $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = \mathrm{rnk}_{\mathrm{Pr}}(T(+)) + \mathrm{rnk}_{\mathrm{Pr}}(R(+)/T(+)) \leq (s - s_1)(s + 2 - s_1) + s_1$. □

3. Radical rings whose additive groups have finite Prüfer rank

3.1 Example. Consider the radical domain R from I.9.2(iii), where $q = 2$. Then $R(+)$ is a torsion group of Prüfer rank 1 and $R(\circ)$ is neither torsionfree nor has finite Prüfer rank.

3.2 Theorem. *Let* R *be a radical ring such the additive group* $R(+)$ *is a p-group for a prime* $p \geq 2$ *and the Prüfer rank* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = r$ *is finite. Then:*

(i) *The ring* R *is nilpotent.*

(ii) *The adjoint group* $R(\circ)$ *is a p-group whose Prüfer rank* $rnkp(R(\circ)) = s$ *is finite and* $r \leq s(s + 2)$ *(or* $-1 + \sqrt{r + 1} \leq s$*).*

(iii) *If* Q *is the divisible part of* $R(+)$ *then* $Q \subseteq (0 : R)$, $Q(+) = Q(\circ)$ *is the divisible part of* $R(\circ)$ *and either* $Q = R$ *and* $R^2 = 0$, *or* $Q \neq R$ *and* R/Q *is a finite nilpotent ring.*

Proof. By I.1.13, Q is an ideal of R and $Q \subseteq (0 : R)$. Consequently, $Q(+) = Q(\circ)$ is a divisible subgroup of $R(\circ)$ and we will assume that $Q \neq R$. Then $S = R/Q$ is a finite radical ring, and hence it is nilpotent by I.7.12. Thus R is nilpotent and the rest is clear from 2.7. □

3.3 Theorem. *Let* R *be a radical ring such that the additive group* $R(+)$ *is torsion and has finite Prüfer rank. Then:*

(i) R *is a nil-ring.*

(ii) *The adjoint group* $R(\circ)$ *is torsion.*

(iii) *If* p *is a prime and* $R_p(+)$ *is the p-component of* $R(+)$ *then* $R_p(\circ)$ *is the p-component of* $R(\circ)$.

(iv) *If* $Q(+)$ *is the divisible part of* $R(+)$ *then* $Q(\circ)$ *is the divisible part of* $R(\circ)$.

(v) $R \neq R^2$, $\bigcap_{n \geq 1} R^n = 0$ *and* $\bigcup_{n \geq 1} (0 : R^n) = R$.

(vi) *If* R *is not nilpotent then* $R^n \neq R^{n+1}$ *and* $(0 : R^n)_R \neq (0 : R^{n+1})_R$ *for every* $n \geq 1$.

Proof. The non-zero p-components R_p of R are ideals and R is the ring direct sum of these ideals. The rest follows easily from 3.2. □

3.4 Example. The ring R from I.9.9(ii) is a non-nilpotent nil-ring such that $R(+)$ is torsion and $\text{rnk}_{\text{Pr}}(R(+)) = 1$.

3.5 Theorem. *Let R be a radical ring whose additive group $R(+)$ has finite Prüfer rank. Then:*

(i) $T \subseteq \mathcal{N}(R)$, *where T is the torsion part of $R(+)$.*

(ii) $T(\circ)$ *is a torsion subgroup of the adjoint group* $R(\circ)$.

(iii) $R \neq R^2$.

(iv) *If R is not nilpotent then $R^n \neq R^{n+1}$ for every $n \geq 1$.*

Proof. (i) and (ii). See 3.3.

(iii) We proceed by induction on $r = \text{rnk}_{\text{Tf}}(R(+))(= rnk((R/T)(+)))$. If $r = 0$ then $R(+)$ is torsion and $R \neq R^2$ by 3.3(v). If $r \geq 1$ then $S = R/T$ is a radical ring and $S \neq S^2$, provided that $S^2 = 0$. If $w \in S\backslash(0 : S^2)$ then $K = Sw$ is a non-zero ideal of S, and if $K = S$ then $S \neq S^2$ by I.7.10. On the other hand, if $K \neq S$ then $P = S/K$ is a radical ring, $\text{rnk}_{\text{Tf}}(P(+)) < r$, $P \neq P^2$ by induction and it follows that $R \neq R^2$.

(iv) Use (iii). □

3.6 Theorem. *Let R be a radical domain such that the additive group $R(+)$ has finite Prüfer rank. Then:*

(i) $\text{char}(R) = 0$ *and $R(+)$ is torsionfree.*

(ii) *The field F of fractions of R has finite dimension over its prime subfield Q ($Q \simeq \mathbb{Q}$, the field of rationals).*

(iii) $\zeta(R) \geq 2$ (see I.1.16).

(iv) *$S = R + \mathbb{Z} \cdot 1_F$ is a semilocal domain with unit, R is an ideal of $S, R \subseteq \mathcal{J}(S)$ and $S/R \simeq \mathbb{Z}_{\zeta(R)}$.*

(v) *The additive groups $R(+)$, $S(+)$ $F(+)$ have the same finite Prüfer rank equal to $[F : Q]$.*

(vi) *THe adjoint group $R(\circ)$ has infinite torsionfree rank.*

Proof. Since R is a domain, R is not finite, and hence $\text{char}(R) = 0$ and $R(+)$ is torsionfree (I.1.15). Consequently, $Q \simeq \mathbb{Q}$ and we may assume that $Q = \mathbb{Q}$.

We have $\text{rnk}_{\text{Pr}}(P(+)) = r \geq 1$ and $R(+)$ contains a finitely generated (free) subgroup $A = \langle u_1, \dots, u_r \rangle_{R(+)}$ such that $R(+)/A(+)$ is torsion.

Let $a \in R$, $B = \langle a, a^2, a^3, \dots \rangle_{R(+)}$ and $C = A \cap B$. Then $C(+)$ is a finitely generated subgroup of $B(+)$, and hence $C \subseteq D = \langle a, a^a, \dots, a^m \rangle_{R(+)}$ for some $m \geq 1$. Moreover, $B(+)/C(+) \simeq (A + B)(+)/A(+) \subseteq R(+)/A(+)$ is torsion, and therefore $ka^{m+1} \in C$ for some $k \geq 1$. It follows that $ka^{m+1} = k_1 a + k_2 a^2 + \dots k_m a^m$, so that the element a is algebraic over Q. Consequently, F is algebraic over Q.

Let $a, b \in R$, $a \neq 0$. Then $Q[a]$ is a subfield of F and there exist $l \geq 0$ and rationals $r_0, \dots, r_l$ such that $a^{-1} = r_0 + r_1 a + \dots + r_l a^l$. Now, $ba^{-1} = r_0 b + r_1 ba + \dots r_l ba^l$, b, ba, $\dots$, $ba^l \in R$ and $R(+)/A(+)$ is torsion. Thus, for a positive integer t, all the elements $tb, tba, \dots, tba^l$ are in A and $ba^{-1} = t^{-1}r_0tb + t^{-1}r_1tba + \dots + t^{-1}r_lba^l$. Consequently, $ba^{-1} = q_1u_1 + \dots q_ru_r$, $q_i \in Q$, and we have shown that $F = Qu_1 + \dots + Qu_r$ and $[F : Q] = r$.

Now, take $0 \neq a \in R$. Then $s_0 + s_1a + \dots s_ja^j = 0$ for some integers $j \geq 1$, $s_0, \dots, s_j$, and we assume that j is the smallest one with this property. Since R is a domain and $R(+)$ is torsionfree, we have $s_0 \neq 0$. Of course, $s_0 \in R \cap \mathbb{Z}$ and it means that $\zeta(R) \geq 1$. Since $1_R \notin R$, we have $\zeta(R) \geq 2$.

Finally, since R is not nil, the rank $\mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$ is not finite. □

3.7 Proposition. *Let R be a radical ring such that the additive grop $R(+)$ is not torsion and has finite torsionfree rank. Then $R \neq R^2$ and either $R^n \neq R^{n+1}$ for every $n \geq 1$ or $R^m(+)$ is torsion for some $m \geq 2$.*

Proof. We may assume that $R(+)$ is torsionfree. Then $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = \mathrm{rnk}_{\mathrm{Tf}}(R(+))$ is finite and the result follows from 3.5(ii). □

3.8 Proposition. *Let R be a radical ring such tht $R = R^2$ and the additive group $R(+)$ has finite torsionfree rank. Then:*

(i) *$R(+)$ is torsion.*

(ii) *Both groups $R(+)$ and $R(\circ)$ have infinite Prüfer rank.*

(iii) *Either $R(\circ)$ is torsion and R is a nil-ring, or $R(\circ)$ has infinite torsionfree rank.*

Proof. (i) See 3.8.

(ii) See 2.9(iv) and 3.5(iii).

(iii) If $R(\circ)$ is torsion then R is nil by 1.6. On the other hand, if $R(\circ)$ has finite torsionfree rank then $\mathrm{rnk}_{\mathrm{Tf}}(R(+)) = \mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$ by 1.8 and both $R(+)$ and $R(\circ)$ are torsion by (i). □

3.9 Proposition. *Let R be a nil-ring such that the additive group $R(+)$ is torsionfree and has finite Prüfer rank $m = \mathrm{rnk}_{\mathrm{Pr}}(R(+))$. Then R is nilpotent of index at most $m + 1$ and $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) \leq m^2$.*

Proof. Combine 1.15 and 2.2(i). □

3.10 Corollary. *Let R be a nil-ring such that the additive group $R(+)$ has finite Prüfer rank $m = \mathrm{rnk}_{\mathrm{Pr}}(R(+))$. Let T be the torsion part of $R(+)$.*

(i) *If $T = 0$ then $R^{m+1} = 0$.*

(ii) *If $T \neq 0$ then $R^m \subseteq T$.*

(iii) *R is nilpotent if and only if T is so.*

(iv) *$\mathrm{rnk}_{\mathrm{Pr}}(R(\circ))$ is finite if and only if $\mathrm{rnk}_{\mathrm{Pr}}(T(\circ))$ is so.*

3.11 Corollary. *Let R be a radical ring such that the additive group $R(+)$ has finie Prüfer rank $m = \mathrm{rnk}_{\mathrm{Pr}}(R(+))$. Let T be the torsion part of $\mathcal{N}(R)(+)$. Then $\mathcal{N}(R)^{m+1} \subseteq T$ and, moreover:*

(i) *If $T = 0$ then $\mathcal{N}(R)^{m+1} = 0$.*

(ii) *If* $R/\mathcal{N}(R))(+)$ *is not torsion then* $\mathcal{N}(R)^m \subseteq T$. *If, moreover,* $T \neq 0$ *then* $m \geq 2$ *and* $\mathcal{N}(R)^{m-1} \subseteq T$.

(iii) *If* $(R/T)(+)$ *is torsionfree and* $\mathcal{N}(R) \neq R$ *then* $\mathcal{N}(R)^m \subseteq T$. *If moreover,* $T \neq 0$ *then* $m \geq 2$ *and* $\mathcal{N}(R)^{m-1} \subseteq T$.

(iv) *If* $T = 0$ *and* $R/\mathcal{N}(R))(+)$ *is not torsion then* $\mathcal{N}(R)^m = 0$.

(v) *If* $R(+)$ *is torsionfree and* $\mathcal{N}(R) \neq R$ *then* $\mathcal{N}(R)^m = 0$. □

4. Various examples

4.1 Let $p \geq 2$ be a prime, $1 \leq s \leq r$ be positive integers and $a * b = abp^s (\bmod p^r) \in \mathbb{Z}_{p^r}$ for all $a, b \in Z_{p^r}$.

4.1.1 Proposition. (i) $C = C(p, r, s) = \mathbb{Z}_{p^r}(+,*)$ *is a radical ring.*

(ii) C *is nilpotent of index* q, *where* q *is the smallest positive ineger with* $q \geq 1 + \frac{r}{s}$.

(iii) $\zeta(C) = p^s$.

(iv) $\mathrm{rnk}_{\mathrm{Pr}}(C(+)) = 1$.

Proof. *Easy to check.* □

Consider the adjoint group $C(\circ)$. For all $a \in C$ and $n \geq 1$, the *n-th* power $a \circ a \circ \ldots \circ a$ of a in $C(\circ)$ is denoted by $a^{\langle n \rangle}$.

4.1.2 Lemma. *If* $p \geq 3$ *then* $1^{\langle p^{r-1} \rangle} = p^{r-1}$.

Proof. If $2 \leq i \leq p^{r-1}$ then p^r divides $\binom{p^{r-1}}{i} p^{i-1}$ (clear for $r + 1 \leq i$ and easy to check fo $i \leq r$). Consequently, by I.2.3,

$$1^{\langle p^{r-1} \rangle} = \left(p^{r-1} + \sum_{i=2}^{p^{r-1}} \binom{p^{r-1}}{i} p^{s(i-1)}\right) (\bmod p^r) = p^{r-1}. \qquad \square$$

4.1.3 Lemma. *If* $p = 2$, $r \geq 2$ *and* $s \leq 2$ *then* $1^{\langle 2r-1 \rangle} = 2^{r-1}$.

Proof. Observe that 2^r divides $\binom{2^{r-1}}{i} 2^{i-1}$ for $3 \leq i \leq 2^{r-1}$, and hence $1^{\langle 2^{r-1} \rangle} = (2^{r-1} + 2^{r+s-2} \cdot (2^{r-1} - 1))(\bmod 2^r) = 2^{r-1}$. □

4.1.4 Lemma. *If* $p = 2$, $r \geq 2$ *and* $s = 1$ *ten* $a^{\langle 2^{r-1} \rangle} = 0$ *for every* $a \in C$.

Proof. The result is clear for $r = 2$, and if $r \geq 3$ then $a^{\langle 2^{r-1} \rangle} = 2^{r-1} a(1 + (2^{r-1} - 1)a)$. If a is even then $2^{r-1} a = 0$. If a is odd then $b = 1 + (2^{r-1} - 1)a$ is even and $2^{r-1} b = 0$. □

4.1.5 Lemma. *If* $p = 2$, $r \geq 5$ *and* $s = 1$ *then* $1^{\langle 2^{r-2} \rangle} = 2^{r-1}$.

Proof. If $3 \leq i \leq 2^{r-2}$ and $i \neq 4$ then 2^r divides $\binom{2^{r-2}}{i} 2^{i-1}$. Using this and I.2.3, we see that $1^{\langle 2^{r-2} \rangle} = (2^{r-2} + 2^{r-2}(2^{r-2} - 1) + 2^{r-1} l)(\bmod 2^r)$, where $l = (2^{r-2} - 1)(2^{r-2} - 3)(2^{r-3} - 1)/3$ is odd. Consequently, $2^{r-1}(2^{r-2} + l - 1) \equiv 0 \ (\bmod 2^r)$ and $1^{\langle 2^{r-2} \rangle} = 2^{r-1}$. □

4.1.6 Lemma. (i) *If* $p = 2, r = 4$ *and* $s = 1$ *then* $s = 1$ *then* $1^{\langle 4\rangle} = 8(\neq 2^{r-2})$.
(ii) *If* $p = 2$, $r = 3$ *and* $s = 1$ *then* $1^{\langle 2\rangle} = 4(\neq 2^{r-2})$.

Proof. Easy to check. □

4.1.7 Proposition. (i) *If* $p \geq 3$ *is odd then* $C(\circ) \simeq \mathbb{Z}_{p^r}(+)$ *is cyclic and, moreover,* $\mathrm{rnk}_{\mathrm{Pr}}(C(\circ)) = 1$.
(ii) *If* $p = 2$, $r \geq 2$ *and* $s \geq 2$ *then* $C(\circ) \simeq \mathbb{Z}_{2^r}(+)$ *is cyclic and* $\mathrm{rnk}_{\mathrm{Pr}}(C(\circ)) = 1$.
(iii) *If* $p = 2$, $r = 1$ *then* $C(\circ) \simeq \mathbb{Z}_2(+)$ *is cyclic and* $\mathrm{rnk}_{\mathrm{Pr}}(C(\circ)) = 1$.
(iv) *If* $p = 2$, $r \geq 2$ *and* $s = 1$ *then* $C(\circ) \simeq \mathbb{Z}_{2^{r-1}}(+) \times \mathbb{Z}_2(+)$ *is 2-generated and* $\mathrm{rnk}_{\mathrm{Pr}}(C(\circ)) = 2$.

Proof. (i) and (ii). By 4.1.2 and 4.1.3, resp., the group $C(\circ)$ contains an element of order (at least) p^r. Since C has just p^r elements, the group $C(\circ)$ is cyclic.
(iii) Obvious.
(iv) By 4.1.4, $a^{\langle 2^{r-1}\rangle} = 0$ for every $a \in C$. On the other hand, by 4.1.5, 4.1.6(i) and 4.1.6(ii), resp., the group $C(\circ)$ contains an element of order 2^{r-1}. But $C(\circ)$ is the product of cyclic groups and C has just 2^r elements. □

4.1.8 Remark. If either $p \geq 3$ and $r > s$ or $p = 2$ and $2 \leq s < r$ then $C(+) \neq C(\circ)$ but $C(+) \simeq C(\circ)$.

4.1.9 Remark. Using 4.1.1(iii), it is easy to see that $C(p_1, r_1, s_1) \simeq C(p_2, r_2, s_2)$ if and only if $p_1 = p_2$, $r_1 = r_2$, $s_1 = s_2$ (ad then the rings coincide).

4.2 Let p be a prime and $n \geq 2$. For every $k, 1 \leq k \leq n - 1$, put $R_k = R(p, n, k) = Sp^k$, where $S = \mathbb{Z}_{p^n}$.

4.2.1 Lemma. (i) R_k *is an ideal of the ring* S *and* $R_k \subseteq \mathscr{J}(S)$.
(ii) R_k *is a radical ring and* $|R_k| = p^{n-k}$.
(iii) $R_k(+) \simeq \mathbb{Z}_{p^{n-k}}(+)$.
(iv) $\zeta(R_k) = p^l$, *where* $l = \min(k, n - k)$.

Proof. Easy to check. □

4.2.2 Proposition. $R_k \simeq C(p, n - k, l)$ (see 4.1).

Proof. Define a mapping $\varrho : \mathbb{Z}_{p^{n-k}} \mapsto S$ by $\varrho(a) = p^k a \pmod{p^k}$ for every $0 \leq a < p^{n-k}$. Clearly, $\mathrm{Im}(\varrho) = R_k$, ϱ is a homomorphism of the additive groups and, if $a \in \mathrm{Ker}(\varrho)$ then p^n divides $p^k a$, so that p^{n-k} divides a and $a = 0$ in $\mathbb{Z}_{p^{n-k}}$. Thus ϱ is an isomorphism of $\mathbb{Z}p^{n-k}(+)$ onto $R_k(+)$. Moreover, if $l = k$ (i.e., $k \leq n - k$) then $\varrho(a * b) = \varrho(p^k ab) = p^{2k} ab = p^k a \cdot p^k b = \varrho(a)\varrho(b)$ and we see that ϱ is an isomorphism of the rings. On the other hand, if $l = n - k < k$ then $\varrho(a * b) = \varrho(p^{n-k} ab) = p^n ab = 0 = p^k a p^k b = \varrho(a)\varrho(b)$ and our result is proved. □

4.2.3 Lemma. *The following conditions are equivalent:*

(i) $R(p_1, n_1 m k_1) \simeq R(p_2, n_2, k_2)$.

(ii) $p_1 = p_2$ *and (just) one of the following four cases takes place:*

(ii1) $n_1 = n_2$ *and* $k_1 = k_2$;

(ii2) $n_1 = 2k_1$ *(then* $n_1 \geq 2$ *is even) and* $n_2 = n_1 + t$, $k_2 = k_1 + t$, $t \geq 1$;

(ii3) $n_2 = 2k_2$ *(then* $n_2 \geq 2$ *is even) and* $n_1 = n_2 + t$, $k_1 = k_2 + t$, $t \geq 1$;

(ii4) $n_1 \neq n_2$, $2k_1 > n_1$, $2k_2 > n_2$ *and* $n_1 - k_1 = n_2 - k_2$.

Proof. Combine 4.2.2 and 4.1.9. □

4.3 Proposition. *Let p be a prime. Then:*

(i) $C(p, r, s) \simeq R(p, r + s, s)$ *for all* $1 \leq s < r$.

(ii) $C(p, r, r) \simeq R(p, 2r + j, r + j)$ *for all* $1 \leq r$ *and* $0 \leq j$.

(iii) $R(p, n, k) \simeq C(p, n - k, \min(k, n - 1))$ *for all* $1 \leq k \leq n - 1$.

Proof. See 4.2.2. □

4.4 For a prime p, let $C(p, \infty, \infty)$ be the zero multiplication ring whose additive group is the quasicyclic group $\mathbb{Z}_{p^\infty}$.

4.5 Let $m \geq 2$. Denote by $D = D(m)$ the set of rational numbers $\frac{am}{b}$ where $a, b \in \mathbb{Z}$, $\gcd(a, b) = 1$.

4.5.1 Proposition. (i) *D is a subring of the field $\mathbb{Q}$ of rationals and D is a radical domain.*

(ii) $\zeta(D) = m$.

(iii) $\mathrm{rnk}_{\mathrm{Pr}}(D(+)) = 1$.

(iv) *Given a prime number p, the additive group $D(+)$ is p-divisible if and only if p does not divide m.*

(v) *If* $m = 2$ *then* $D(\circ) \simeq \mathbb{Z}_2(+) \times \mathbb{Z}(+)^{(\omega)}$.

(vi) *If* $m \geq 3$ *then* $D(\circ) \simeq \mathbb{Z}(+)^{(\omega)}$.

(vii) $\mathrm{rnk}_{\mathrm{Pr}}(D(\circ))$ *is infinite.*

Proof. (i), (ii), (iii) and (iv). Easy to check (see I.9.2(iii)).

(v), (vi) and (vii). By 3.6(vi), the group $D(\circ)$ has infinite Prüfer rank. Now, since $D(\circ)$ is isomorphic to a subgroup of $Q^* \simeq \mathbb{Z}_2(+) \times \mathbb{Z}(+)^{(\omega)}$, the result is clear. □

4.5.2 Lemma. $D(m_1) \simeq D(m_2)$ *if and only if* $m_1 = m_2$.

Proof. Use 4.5.1(ii). □

4.5.3 Lemma. $D(m_1) \subseteq D(m_2)$ *if and only if* m_1 *divides* m_2.

Proof. Obvious. □

4.5.4 Lemma. *If* m_1 *divides* m_2 *then* $D(m_1)$ *is an ideal of* $D(m_2)$ *if and only if any prime number dividing* m_2/m_1 *also divides* m_2.

Proof. Easy to check. □

4.6 Let $R = R(+,*)$ be the (uniquely determined) ring defined on $\mathbb{Z}_2(+)^{(2)}$ by $(1,0)*(1,0) = (0,1)$ and $(1,0)*(0,1) = (0,1)*(1,0) = (0,1)*(0,1) = (0,0)$. Then R is nilpotent of index 3, $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = 2$ and $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = 1$.

4.7 (i) For $n \geq 1$, let R_n denote the ring direct sum of n copies of the ring $C(2,2,1)$ (see 4.1). The R_n is nilpotent of index 3, $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = n$ and $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = 2n$.

(ii) For $n \geq 1$, let $R_{(n)}$ denote the ring direct sum of n copies of the ring $R(+,*)$ (see 4.6). Then $R_{(n)}$ is nilpotent of index 3, $\mathrm{rnk}_{\mathrm{P}}(R_{(n)}(+)) = 2n$ and $\mathrm{rnk}_{\mathrm{Pr}}(R_{(n)}(\circ)) = n$.

4.8 Consider the ring $R = R_n$ from I.9.12, where we choose $T = \mathbb{Z}_p$, p being a prime and $n \geq 2$. Then R_n is nilpotent of index n, $|R_n| = p^{n-1}$, $R_n(+) \simeq \mathbb{Z}_p(+)^{(n-1)}$ and $\mathrm{rnk}_{\mathrm{Pr}}(R_n(+)) = n - 1$; put $s = \mathrm{rnk}_{\mathrm{Pr}}(R_n(\circ))$.

(i) Denote by F the set of polynomials $f \in T[x]$ such that $\deg(f) \leq n - 1$ and $f^p \in \mathbb{Z}_p[x]x^n$. It is easy to see that $|F| = p^{\frac{n}{2}}$ for n even and $|F| = p^{\frac{n-1}{2}}$ for n odd. From this, it follows easily that $\mathrm{rnk}_{\mathrm{Pr}}(R_n(\circ)) = \frac{n}{2}$ for n even and $\mathrm{rnk}_{\mathrm{Pr}}(R_n(\circ)) = \frac{n-1}{2}$ for n odd.

(ii) Let $m \geq 0$ be such that $p^m \leq n - 1$. Then (p^m-times)

$$\alpha \circ \alpha \circ \ldots \circ \alpha = \alpha_{p^m} = x^{p^m} + \sum_{i=1}^{p^m - 1} \binom{p^m}{i} x^i + \mathbb{Z}_p[x]x^n \neq 0,$$

and so $R_n(\circ)$ contains a cyclic subgroup of order p^{m+1}.

4.9 Put $R = \coprod R_n$, $n \geq 2$ (see 4.8). Then R is a nil-ring, $R(+)$ is a p-elementary group, $R(\circ)$ is a p-group and $R(\circ)$ is not bounded.

5. Radical rings whose additive groups have Prüfer rank 1 or 2

5.1 Proposition. *Let R be a radical ring such that $R(+)$ is a p-group for a prime p and* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = 1$. *Then either R is finite and $R \simeq C(p,r,s)$ for some $1 \leq s \leq r$ or R is infinite and $R \simeq C(p,\infty,\infty)$* (see 4.1, ..., 4.4). *Moreover, $1 \leq \mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) \leq 2$, and $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = 2$ if and only if $p = 2$, $2 \leq r < \infty$ and $s = 1$.*

Proof. If $R(+)$ is not reduced then $R(+) \simeq \mathbb{Z}_{p^\infty}(+)$ and $R^2 = 0$ (I.1.13), so that $R \simeq C(p,\infty,\infty)$. Consequently, we may assume that $R(+)$ is reduced and, moreover, that $R(+) = \mathbb{Z}_{p^r}(+)$, $r \geq 1$. To avoid confusion, denote the multiplication of the ring R by the symbol $*$. Then, for all $0 \leq m, n \leq p^r - 1$, we have $m * n = mn(1 * 1) = mnz$, $z = 1 * 1 \in \mathbb{Z}_{p^r}$. Since R is a finite radical ring, R is nilpotent and it follows easily that p divides r. Thus $z = p^s w$, $1 \leq s \leq r - 1$,

$w \in \mathbb{Z}_{p^r}$. If $w = 0$ then $R = C(p,r,r)$. If $w \neq 0$ and p does not divide w then p^r divides $wv - 1$ for some $v \in \mathbb{Z}_{p^r}$ and the mapping $a \mapsto va$ is an isomorphism of R onto $C(p,r,s)$. □

5.2 Proposition. *Let R be a radical subring of $\mathbb{Q}$. Then $R = D(m)$ for some $m \geq 2$* (see 4.5) *and R is rd-generated by m* (see I.7.18).

Proof. We have $R \cap \mathbb{Z} \neq 0$ and hence, let m be the smallest positive integer in $R \cap \mathbb{Z}$. Since $1 \notin R$, we have $m \geq 2$. If $b \in \mathbb{Z}$ is such that $\gcd(m,b) = 1$ then $1 = um + vb$ for some $u, v \in \mathbb{Z}$ and, since R is a radical ring, we have $\frac{um}{vb} = \frac{um}{1-um} \in R$ and $\frac{um}{b} = v \cdot \frac{um}{vb} \in R$. Furthermore, $\gcd(u,b) = 1$, $1 = zu + wb$, $\frac{zum}{b} \in R$ and, finally, $\frac{m}{b} = \frac{(zu+wb)m}{b} = \frac{zum}{b} + wm \in R$. Thus $D(m) \subseteq R$. On the other hand, if $\frac{c}{d} \in R$, $c, d \in \mathbb{Z}$, $\gcd(c,d) = 1$, then $c \in R \cap \mathbb{Z}$, m divides c and $\gcd(m,d) = 1$. Then $\frac{c}{d} \in D(m)$ and we get $R = D(m)$. □

5.3 Theorem. *A ring R is a radical ring with* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = 1$ *if and only if at least (and then just) one of the following three cases takes place:*

(1) *R is a nil-ring, $R(+)$ is torsion and, if p is a prime such that p-component $R_p(+)$ of $R(+)$ is non-zero, then either R_p is finite and $R_p \simeq C(p,r,s)$ for some $1 \leq s \leq r$ or R_p is infinite and $R_p \simeq C(p,\infty,\infty)$* (see 4.1, ..., 4.4). *(Then R is the ring direct sum of the p-components, $1 \leq \mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) \leq 2$, and $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = 2$ if and only if $R_2 \neq 0$ and $R_2 \simeq C(2,r,1)$, $2 \leq r$.)*

(2) *R is a zero multiplication ring and the additive group $R(+)$ is isomorphic to a (non-zero) subgroup of $\mathbb{Q}(+)$. (Then $R(+)$ is torsionfree and* $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = 1$.*)*

(3) *R is a domain and $R \simeq D(m)$ for some $m \geq 2$* (see 4.5). *(Then R is isomorphic to a subring of $\mathbb{Q}$, $R(+)$ is torsionfree and* $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ))$ *is infinite.)*

Proof. If $R(+)$ is torsion then R is the ring direct sum of its p-components and we use 5.1 to show (1). If $R(+)$ is not torsion then it is torsionfree and isomorphic to a subgroup of $\mathbb{Q}(+)$. Moreover, $R(+)/A(+)$ is torsion for every non-zero subgroup $A(+)$ of $R(+)$ and, by I.1.21, either $R^2 = 0$ and (2) takes place or R is a domain and we use 5.2 to show (3). □

5.4 Lemma. *Let R be a radical ring such that* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = 2$ *and $R(+)$ is torsion. Then R is nil.*

Proof. See 3.3(i). □

5.5 Lemma. *Let R be a radical ring such that* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = 2$ *and* $0 \neq T \neq$ $\neq R$, *T being the torsion part of $R(+)$. Then:*

(i) $\mathrm{rnk}_{\mathrm{Pr}}(T(+)) = 1$, *$T$ is nil (and as in 5.3(1)).*

(ii) *$S = R/T$ is a radical ring,* $\mathrm{rnk}_{\mathrm{Pr}}(S(+)) = 1$ *and $S(+)$ is torsionfree.*

(iii) *Either $S^2 = 0$, $R^2 \subseteq T$ and R is nil or S is a domain (as in 5.3(3)), T is a prime ideal and $T = \mathcal{N}(R)$.*

Proof. Clearly, $\text{rnk}_{\text{Pr}}(T(+)) = 1 = \text{rnk}_{\text{Pr}}(S(+))$ and it remains to use 5.3. □

5.6 Lemma. *Let R be a radical ring such that $\text{rnk}_{\text{Pr}}(R(+)) = 2$ and $R(+)$ is torsionfree. Let I be an ideal of R, $0 \neq I \neq R$. Then just one of the following three cases takes place:*

(1) $\text{rnk}_{\text{Pr}}(I(+)) = 2$ *and* $(R/I)(+)$ *is torsion;*
(2) $\text{rnk}_{\text{Pr}}(I(+)) = 1$ *and* $I^2 = 0$*;*
(3) $\text{rnk}_{\text{Pr}}(I(+)) = 1$ *and I is a domain.*

Proof. Use 5.3. □

5.7 Lemma. *Let R be a radical ring such that $R(+)$ is torsionfree and $\text{rnk}_{\text{Pr}}(R(+)) = 2$. If $a \in R$ then at least one of the following tree cases takes place:*

(1) $(0 : a) = 0$*;*
(2) $(0 : a)$ *is a prime ideal;*
(3) $R^2a = 0$.

Proof. $(R/(0 : a))(+) \simeq (Ra)(+)$ is torsionfree and the rest is clear from 5.3. □

5.8 Proposition. *Let R be a radical ring such that $\text{rnk}_{\text{Pr}}(R(+)) = 2$ and let T denote the torsion part of $R(+)$. Then just one of the following seven cases takes place:*

(1) *R is a nil-ring and $T = R$ (i.e., $R(+)$ is torsion);*
(2) *R is a nil-ring, $0 \neq T \neq R$, $R^2 \subseteq T$,* $\text{rnk}_{\text{Pr}}(T(+)) = 1$ *and T is a nil-ring of the type 5.3.(1);*
(3) *$0 \neq T = \mathcal{N}(R) \neq R$, T is a prime ideal of R, $rnk_{Pr}(T(+)) = 1$, T is a nil-ring of the type 5.3(1),* $\text{rnk}_{\text{Pr}}((R/T)(+)) = 1$ *and R/T is a radical domain of the type 5.3(3);*
(4) *R is nilpotent of index at most 3 (i.e., $R^3 = 0$) and $T = 0$ (i.e., $R(+)$ is torsionfree and isomorphic to a subgroup of $\mathbb{Q}(+) \times \mathbb{Q}(+)$);*
(5) *$T = 0 \neq \mathcal{N}(R) \neq R$ (i.e., $R(+)$ is torsionfree and isomorphic to a subgroup of $\mathbb{Q}(+) \times \mathbb{Q}(+)$), $\mathcal{N}(R)^2 = 0$, $\mathcal{N}(R)$ is a prime ideal of R and $R/\mathcal{N}(R)$ is a radical domain of the type 5.3(3);*
(6) *$T = 0 = \mathcal{N}(R)$ (i.e., R is semiprime, $R(+)$ is torsionfree and isomorphic to a subgroup of $\mathbb{Q}(+) \times \mathbb{Q}(+)$) and there exist two non-zero prime ideals I and J of R such that $I \cap J = 0$, R is isomorphic o a subring of $R/I \times R/J$ and both R/I and R/J are radical domains of the type 5.3(3);*
(7) *R is a domain.*

Proof. If $T = R$ then R is nil by 3.5(i). If $0 \neq T \neq R$ then $\text{rnk}_{\text{Pr}}(T(+)) = 1 = \text{rnk}_{\text{Pr}}((R/T)(+))$ and either (2) or (3) is true by 5.3. Now, assume that $T = 0$, i.e., $R(+)$ is torsionfree. If R is nil then $R^3 = 0$ by 3.9. If $0 \neq \mathcal{N}(R) \neq R$ then (5) is true by 3.11(iv) and 5.3. Finally, assume that $T = 0 = \mathcal{N}(R)$, i.e., R is semiprime and $R(+)$ is torsionfree, and that R is not a domain. Then

$A = \{a \in R \mid (0:a) \neq 0\} \neq 0$ and (6) is true, provided that $(0:a) \cap (0:b) = 0$ for some $a, b \in A$ (use 5.3). In the opposite case, it is easy to see that A is a non-zero ideal of R. If $A \neq R$ then $\mathrm{rnk}_{\mathrm{Pr}}(A(+)) = 1$ and A is a domain by 5.3, a contradiction. Thus $A = R$.

Now, let $a, b \in R$ be such that $ab \neq 0$. If $c \in (0:a)$ then $ca = 0 \in (0:b)$ and, since $(0:b)$ is prime by 5.7, we have $c \in (0:b)$. Consequently, $(0:a) \not\subseteq (0:b)$ and, in fact, $(0:a) = (0:b)$, the converse inclusion being similar.

Choose $0 \neq u \in R$ and take $0 \neq v \in (0:u)$. If $w \notin (0:u) \cup (0:v)$ then $wu \neq 0 \neq wv$, and hence $(0:u) = (0:w) = (0:v)$, $v \in (0:v)$ and $v^2 = 0$, which is a contradiction with $\mathcal{N}(R) = 0$. We have shown that $(0:u) \cup (0:v) = R$. But this is not possible, since both $(0:u)$ and $(0:v)$ are proper ideals of R. □

5.9 Proposition. *Let R be a radical ring such that* $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = 1$. *Then* $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) \leq 3$ *and just one of the following two cases takes place:*

(1) *R is a nil-ring and both groups $R(+)$ and $R(\circ)$ are torsion;*

(2) *R is a zero multiplication ring (i.e., $R^2 = 0$) and $R(+)$ is isomorphic to a subgroup of $\mathbb{Q}(+)$.*

Proof. First, R is a nil-ring by 2.9(i). If $R(\circ)$ is torsion then $R(+)$ is torsion by 1.5 and $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) \leq 3$ by 2.8(ii). On the other hand, if $R(\circ)$ is not torsion then it is torsionfree, and so $R(+)$ is torsionfree, too (2.9(iii)). Now, $1 = \mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = \mathrm{rnk}_{\mathrm{Tf}}(R(\circ)) = \mathrm{rnk}_{\mathrm{Tf}}(R(+)) = \mathrm{rnk}_{\mathrm{Pr}}(R(+))$, $R(+)$ is isomorphic to a subgroup of $\mathbb{Q}(+)$ and $R^2 = 0$ by 1.15. □

5.10 Example. Consider the four-element ring R from 4.6 (see also 4.8). Then $R^3 = 0$, $\mathrm{rnk}_{\mathrm{Pr}}(R(+)) = 2$ and $\mathrm{rnk}_{\mathrm{Pr}}(R(\circ)) = 1$.

6. Radical rings whose additive and/or adjoint groups have pseudofinite weak Prüfer rank

6.1 Remark. (cf. 2.1(v)) An abelian group G is said to have *pseudofinite weak Prüfer rank* if $\mathrm{rnk}_{\mathrm{Tf}}(G)$ is finite and, moreover, $\mathrm{rnk}_{\mathrm{Pr}}(T_p)$ is finite, where p is any prime number and T_p is the p-component of G; we denote this fact by $\mathrm{rnk}_{\mathrm{Pw}}(G) < \infty$.

(i) If $\mathrm{rnk}_{\mathrm{Pr}}(G)$ is finite then $\mathrm{rnk}_{\mathrm{Pw}}(G) < \infty$.

(ii) If H is a subgroup of G then $\mathrm{rnk}_{\mathrm{Pw}}(G) < \infty$ if and only if $\mathrm{rnk}_{\mathrm{Pw}}(H) < \infty$ and $\mathrm{rnk}_{\mathrm{Pw}}(G/H) < \infty$.

(iii) Put $G = \coprod \mathbb{Z}_p(+)^{(p)}$, where p runs trough an infinite set of prime numbers. Then G is a torsion group, $\mathrm{rnk}_{\mathrm{Pw}} G < \infty$, but G has infinite Prüfer rank.

6.2 Theorem. *The following conditions are equivalent for a ring R:*

(i) *R is a radical ring and* $\mathrm{rnk}_{\mathrm{Pw}}(R(\circ)) < \infty$.

(ii) *R is a nil-ring and* $\mathrm{rnk}_{\mathrm{Pw}}(R(+)) < \infty$.

Proof. (i) implies (ii). R is a nil-ring by 1.6 and we have $\text{rnk}_{\text{Pw}}(R(+)) < \infty$ by 1.7 and 2.7(i), (ii).

(ii) implies (i). By 1.8, $\text{rnk}_{\text{Tf}}(R(\circ))$ is finite and we use 2.7(i),(ii) again. □

6.3 Remark. Let R be a radical ring such that $R = R^2$.

(i) Assume that $R(+)$ is a p-group for a prime p. By 3.8(ii), both groups $R(+)$ and $R(\circ)$ have infinite Prüfer rank. If $R = pR$ then $R(+)$ is divisible and $R^2 = 0$ by I.1.13(iii), a contradiction with $R = R^2$. Thus $R \neq pR$, $S = R/pR$ is a radical ring, $S = S^2$ and $S(+)$ is a p-elementary group with infinite Prüfer rank. Consequently, $S(+)$ is a direct sum of an infinite number of copies of $\mathbb{Z}_p(+)$.

(ii) Assume that $\text{rnk}_{\text{Tf}}(R(+))$ is finite. By 3.8(i), $R(+)$ is torsion. Now, R is the ring direct sum of its p-components R_p, and if $R_p \neq 0$ then R_p is a radical ring $R_p = R_p^2$ and $R_p(+)$ is a p-group (see (i)).

6.4 Proposition. *Let R be a radical ring such that either* $\text{rnk}_{\text{Pw}}(R(+)) < \infty$ *or* $\text{rnk}_{\text{Pw}}(R(\circ)) < \infty$. *Then* $R \neq R^2$.

Proof. See 6.2 and 6.3. □

6.5 Theorem. *Let R be a radical ring such that the additive group $R(+)$ is torsion and* $\text{rnk}_{\text{Pw}}(R(+)) < \infty$. *Then:*

(i) *R is a nil-ring.*

(ii) *The adjoint group $R(\circ)$ is torsion.*

(iii) *If p is a prime and $R_p(+)$ the p-component of $R(+)$ then $R_p(\circ)$ is the p-component of $R(\circ)$.*

(iv) *If $Q(+)$ is the divisible part of $R(+)$ then $Q(\circ)$ is the divisible part of $R(\circ)$.*

(v) $R \neq R^2$, $\bigcap_{n\geq 1} R^n = 0$ *and* $\bigcup_{n\geq 1}(0 : R^n) = R$.

(vi) *If R is not nilpotent then* $R^n \neq R^{n+1}$ *and* $(0 : R^n)_R \neq (0 : R^{n+1})_R$ *for every* $n \geq 1$.

Proof. The same as that of 3.3. □

7. Radical rings whose additive and/or adjoint groups are minimax

7.1 Remark. A (possibly non-commutative) group G is called *minimax* if G contains a normal subgroup H such that H satisfies the maximal condition on subgroup and the factorgroup G/H satisfies the minimal condition on subgroups.

(i) The following conditions are equivalent for an abelian group $G(= G(+))$:

(i1) G is torsion and minimax.

(i2) G satisfies the minimal condition on subgroups.

(i3) $\text{rnk}_{\text{Pr}}(G)$ is finite and G is P-group for a finite set P of primes.

(i4) G is a direct sum of finitely many cyclic or quasicyclic p-groups.

(ii) An abelian group is finite, provided that it is reduced, torsion and minimax.

(iii) An abelian group G is minimax if and only if G contains a finitely generated free subgroup F such that the factorgroup G/F satisfies the equivalent conditions from (i).

(iv) If G is an abelian minimax group then both ranks $\mathrm{rnk}_{\mathrm{Tf}}(G)$ and $\mathrm{rnk}_{\mathrm{Pr}}(G)$ are finite.

(v) The class of abelian minimax groups is closed under taking subgroups, factor-groups and extensions.

(vi) No infinite direct sum or product of non-zero abelian groups is minimax.

(vii) The additive group $\mathbb{Q}_p(+)$ (see I.9.1) is a torsionfree minimax group that is not finitely generated.

(viii) The quasicyclic p-group $\mathbb{Z}_{p^\infty}$ is a torsion minimax group that is not finitely generated.

(ix) The additive group $\mathbb{Q}(+)$ of rationals is a torsionfree group of Prüfer rank 1, but it is not minimax.

(x) The direct sum $\coprod \mathbb{Z}_p(+)$, p running through an infinite set of primes, is a torsion group of rank 1, but it is not minimax.

7.2 Proposition. *Let R be a radical ring such that $R(\circ)$ is minimax. Then $R(+)$ is minimax.*

Proof. By 1.6 and 1.7, R is nil and $r = \mathrm{rnk}_{\mathrm{Tf}}(R(+))$ is finite. Now, we proceed by induction on r.

If $r = 0$ then $R(+)$ is torsion and, by 2.9(iv), $\mathrm{rnk}_{\mathrm{Pr}}(R(+))$ is finite. Further, since $R(\circ)$ is minimax, this group has only finitely many non-zero p-components and, in view of 2.7, the same is true for $R(+)$. Consequently, having finite Prüfer rank, the group $R(+)$ is minimax.

Next, let $r \geq 1$ and let T denote the torsion part of $R(+)$. We have $T \neq R$, $T(+)$ is minimax (as shown above) and we put $S = R/T$. Then $S(+)$ is torsionfree and it suffics to show that the group is also minimax. If $S^2 = 0$ then $S(+) = S(\circ)$ is minimax. Hence assume $S^2 \neq 0$. Since S is nil, it is not a domain, and so $(S/K)(+)$ is not torsion for a non-zero ideal K of S (I.1.21). Clearly, $\mathrm{rnk}_{\mathrm{Tf}}(K(+)) < r$, $\mathrm{rnk}_{\mathrm{Tf}}((S/K)(+)) < r$, and therefore both $K(+)$ and $S(+)/K(+)$ are minimax. Thus $S(+)$ is minimax, too. □

7.3 Proposition. *Let R be a radical ring such that $R(+)$ is torsion and minimax. Then R is nilpotent.*

Proof. The divisible part Q of $R(+)$ is an ideal of R and $Q \subseteq (0 : R)$ by I.1.13. Further, $R(+) = Q(+) \oplus A(+)$, the reduced torsion minimax group $A(+)$ is finite and $mA = 0$ for some $m \in \mathbb{Z}$, $m \geq 1$. The set $I = \{a \in R \mid ma = 0\}$ is an ideal of R, I is finite and $R = Q + I$. Now, I is nilpotent and the same is true for R. □

7.4 Lemma. *Let R be a radical ring such that $R(+)$ is minimax and $(R/I)(+)$ is torsion for every non-zero ideal I of R. Then R is nilpotent.*

Proof. If $(0:R) \neq 0$ then R is nilpotent by 7.3 (consider $R/(0:R)$). If $(0:R) = 0$ then R is a domain by I.1.21. Let F be the field of fractions of R and let Q denote the prime subfield of F. According to 3.6, F has finite dimension over Q and we may assume that $Q = \mathbb{Q}$ is the field of rationals. Consequently, the integral closure V of $\mathbb{Z}$ in F is a Dedekind domain. Further, there are a finitely generated subgroup $A(+)$ of $R(+)$ and a finite set P of prime numbers such that $R(+)/A(+)$ is a torsion P-group. If $W = V[p^{-1} | p \in P] \subseteq F$ then F is a quotient field of both domains V and W and it is easy to see that $R \subseteq W$ and that $R(\circ)$ is isomorphic to a subgroup of W^*. Finally, if $\mathcal{M}$ is the set of maximal ideals I of V such that $Vp \subseteq I$ for at least one $p \in P$ then $\mathcal{M}$ is finite and $\mathcal{M}$ generates a subgroup $\mathcal{G}$ in the group $\mathcal{F}$ of non-zero fractional ideals of V. The mapping $\varphi : w \mapsto Vw$, $w \in W^*$, is a homomorphism of W^* into $\mathcal{G}$ and $\mathrm{Ker}(\varphi) = V^*$. Thus both $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ are finitely generated, too. We have shown that $R(\circ)$ is finitely generated and then R is nilpotent by I.10.5, a contradiction with R being a domain. □

7.5 Proposition. *Let R be a radical ring such that $R(+)$ is minimax. Then R is nilpotent.*

Proof. We proceed by induction on $r = \mathrm{rnk}_{\mathrm{Tf}}(R(+))$. *If* $r = 0$ then $R(+)$ is torsion and R is nilpotent by 7.3. Hence, assume $r \geq 1$ and put $S = R/T$, T being the torsion part of $R(+)$. The ideal T is nilpotent (7.3) and we have to show that S is nilpotent, too. Now, $S(+)$ is a torsionfree minimax group, $\mathrm{rnk}_{\mathrm{Tf}}(S(+)) = r$ and, due to 7.4, we may assume that $P(+) = (S/K)(+)$ is a non-zero torsionfree group for a non-zero ideal K of S. Clearly, $r = \mathrm{rnk}_{\mathrm{Tf}}(P(+)) + \mathrm{rnk}_{\mathrm{Tf}}(K(+))$ and the radical rings P and K are nilpotent by induction. Thus S is nilpotent. □

7.6 Lemma. *Let R be a nilpotent ring. Then $R(+)$ is minimax if and only if $R(\circ)$ is so.*

Proof. Use I.7.21. □

7.7 Theorem. *Let R be a radical ring. Then te additive group $R(+)$ is minimax if and only if the adjoint group $R(\circ)$ is minimax. If these conditions are satisfied then R is nilpotent.*

Proof. Combine 7.2, 7.5 and 7.6. □

7.8 Example. Let R be a zero multiplication ring such that $R(+) \simeq \mathbb{Q}_p(+)$ ($R(+) \simeq \mathbb{Z}_{p^\infty}(+)$, resp.). Then $R(+)$ is a non-finitely generated torsionfree (torsion, resp.) minimax group and R is not finitely id-generated.

7.9 Remark. Let G be an abelian minimax group. Then G contains a finitely generated free subgroup F such that $K = G/F$ satisfies the equivalent conditions of 7.1(i). Now, given a prime number p, the divisible part of the p-component K_p of K is the direct sum of $m_p \geq 0$ copies of $\mathbb{Z}_{p^\infty}(+)$ and we put $\mathrm{rnk}^p_{\mathrm{Mn}}(G) = m_p$. Further, we put $\mathrm{rnk}_{\mathrm{Mm}}(G) = \mathrm{rnk}_{\mathrm{Tf}}(G) + \sum \mathrm{rnk}^p_{\mathrm{Mm}}(G)$, p running through all primes.

(i) $\mathrm{rnk}_{\mathrm{Tf}}(G) \leq \mathrm{rnk}_{\mathrm{Mm}}(G)$, and $\mathrm{rnk}_{\mathrm{Tf}}(G) = \mathrm{rnk}_{\mathrm{Mm}}(G)$ if and only if G is finitely generated.

(ii) If H is a subgroup of G then $\mathrm{rnk}^p_{\mathrm{Mm}}(G) = \mathrm{rnk}^p_{\mathrm{Mm}}(H) + \mathrm{rnk}^p_{\mathrm{Mm}}(G/H)$ for every prime p and $\mathrm{rnk}_{\mathrm{Mm}}(G) = \mathrm{rnk}_{\mathrm{Mm}}(H) + \mathrm{rnk}_{\mathrm{Mm}}(G/H)$.

7.10 Proposition. *Let R be a radical ring such that the additive group $R(+)$ (or the adjoint group $R(\circ)$) is minimax* (see 7.7). *Then* $\mathrm{rnk}^p_{\mathrm{Mm}}(R(+)) = \mathrm{rnk}^p_{\mathrm{Mm}}(R(\circ))$ *for every prime p and* $\mathrm{rnk}_{\mathrm{Mm}}(R(+)) = \mathrm{rnk}_{\mathrm{Mm}}(R(\circ))$.

Proof. By 7.7, R is nilpotent and, by 1.8, $\mathrm{rnk}_{\mathrm{Tf}}(R(+)) = \mathrm{rnk}_{\mathrm{Tf}}(R(\circ))$. We proceed by induction on the nilpotence index n of R to show that $\mathrm{rnk}^p_{\mathrm{Mm}}(R(+)) = \mathrm{rnk}^p_{\mathrm{Mm}}(R(\circ))$. If $n = 2$ then $R(+) = R(\circ)$ and the assertion is trivial. If $n \geq 3$ then $I = (0 : R) \neq R$ and $S = R/I$ is a radical ring nilpotent of index $n - 1$. Now, $I(+) = I(\circ)$, and so $\mathrm{rnk}^p_{\mathrm{Mm}}(I(+)) = \mathrm{rnk}^p_{\mathrm{Mm}}(I(\circ))$. On the other hand, $\mathrm{rnk}^p_{\mathrm{Mm}}(S(+)) = \mathrm{rnk}^s_{\mathrm{Mm}}(S(\circ))$ by induction and the rest is clear. □

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