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Open Mappings on Extremally Disconnected Compact Spaces

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A class of complete Boolean algebras (the "exponent") is introduced, which generalizes the complete Boolean algebras known from Mathias forcing, Prikry forcing and other forcing notions with the same combinatorial property (Prikry property). It is shown that any member of this class with a canonical complete homomorphism on it gives rise to an extremally disconnected compact space and an open mapping on it (via Stone duality) such that the set of fixed points is nowhere dense but not empty. The main theorem states the opposite: Any such topological situation is essentially the result of such a construction.

1. Introduction

Recall the following lemma of four sets from set theory:

Lemma of four sets (de Bruijn and Erdös [dBE], Katětov [K])

Let $f: X \to X$ be a mapping on the set X. There exists a partition (X^0, X^1, X^2, X^3) of X into 4 parts such that $f[X^i] \cap X^i = \emptyset$ for $i < 3$ and $f \upharpoonright X^3 = id \upharpoonright X^3$.

This lemma has many topological versions, one of them is the following:

Theorem (Krawczyk, Steprāns [KS]*) For any zerodimensional compact space X and any continuous mapping $f: X \to X$, there exists a partition (U^0, U^1, U^2) of X into 3 open and closed subsets such that $f[U^i] \cap U^i = \emptyset$ iff f has no fixed points.*

This theorem exludes fixed points. The difficulty to find a substitute for the set of fixed points X^3 in the lemma of four sets was solved for extremally disconnected compact spaces and embeddings by Frolík:

Theorem (Frolík [Fr]*) For any embedding of an extremally disconnected compact space into itself, the set of fixed points is open and closed.*

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In this paper we will consider similar assumptions. The topological situation we are interested in is an extremally disonnected compact space X and an open continuous mapping $f: X \to X$. This is also of interest because of the Boolean dualization — a complete Boolean algebra and a complete homomorphism on it. The question arose whether the set of fixed points must be a clopen set also in the context of open mappings (Abramovich, Arenson, and Kitover [AAK], see also Vermeer [V]). A negative answer to this problem is given by the following example ([Th]): Define a shift operation on the partial order of Mathias forcing and transfer it to the completion of this partial order to a complete Boolean algebra. The topological dual works. The combinatorial kernel of the proof uses the following key property of Mathias forcing (we name it Prikry property, it is also called Pure decision): Given a condition of the forcing and a sentence, there is a smaller condition *with the same stem* which decides the sentence. We should stress here that we do not use the forcing method in this paper, but only its combinatorial patterns. When speaking about a forcing notion, we mean the underlying partial order or its completion to a complete Boolean algebra. The Prikry property reads in Boolean language as follows: Given an element a of the partial order and a partition of unity into two elements of the completion of the partial order, there is a smaller $b \leq a$ *with the same stem as the starting element* a, which is under one element of the partiton of unity. It turns out that the construction of the example goes through with an arbitrary partial order with this Prikry property, for example the partial orders of Mathias forcing with a selective ultrafilter as parameter or Prikry forcing for a measurable cardinal. We could ask whether there is a common shape for those partial orders or its completions. So, we are looking for a class of complete Boolean algebras with the Prikry property. We would like to have on those Boolean algebras a complete homomorphism into itself, which would serve for the construction of an example of our topological situation as in the case of the complete Boolean algebra from Mathias forcing. Paragraph 3 is devoted to the construction of such a class of complete Boolean algebras — the exponent. This is done by combining a given Boolean algebra with a tree structure. After the definition and a list of basic properties in 3.1, paragraph 3.2 examines dense subsets of the exponent. The main interest in dense subsets comes from the understanding of the exponent as a forcing notion. All mentioned and similar forcing notions occur as dense subsets of such an exponent. The remarks about the forcing properties of the exponent are not needed in the following and can be skipped. Paragraph 3.3 gives a different kind of application of the exponent. In paragraph 4 it is shown that any exponent gives rise (via Stone duality) to an example of our situation, i.e. an extremally disconnected compact space with an open mapping such that the set of fixed points is nowhere dense and nonempty. It will follow that we find by this construction for any given extremally disconnected compact space an example of this topological situation such that the set of fixed points is homeomorphic with the given space. The main theorem of this paper is contained in paragraph 5. It states the surprising fact that all examples of our topological situation are essentially obtained from the Stone dual of an exponent.

2. Preliminaries

For all of this work, X will be an extremally disconnected compact Hausdorff space and $f: X \to X$ an open continuous mapping. A topological space is extremally disconnected if the closure of any open set is open. A mapping is said to be open if the image of any open set is open. The dynamical system (X, f) has an algebraic dual via Stone duality – a complete Boolean algebra and a complete homomorphism on it. We will steadily switch between topological and algebraic arguments. Concerning notation and basic facts in the theory of Boolean algebras, topology, and set theory, we refer to [HBA], [BS1] and [BS2], [E], [CN], and [BJ].

We denote by $Clop(X)$ the complete Boolean algebra of open and closed (= clopen) subsets of X. By U, V, W we denote elements of $Clop(X)$. The closure, the interior, and the boundary are denoted by *cl*, *int*, and *bd* resp. If $a \in B$ is an element of a Boolean algebra, *Stone (a)* denotes the corresponding clopen subset in the Stone space. For a Boolean algebra $(B, \vee, \wedge, -, \mathbf{1}_B, \mathbf{0}_B)$, we denote by B^+ the set of nonzero elements. A subset $A \subseteq B^+$ is called dense, if for any $b \in B^+$ there is an $a \in A$ such that $a \leq b$.

We will say that a Boolean algebra is (κ, λ)-distributive ($(<\kappa, <\lambda)$-distributive resp.), if any collection of κ (less that κ resp.) maximal antichains of power at most λ (less that λ resp.) has a common refinement in this algebra. A Boolean algebra, which is not (κ, λ)-distributive in the restriction to any nonzero element, is called nowhere (κ, λ)-distributive. It is said to be κ-distributive if it is (κ, λ)-distributive for arbitrary λ.

We denote by $[A]^{<\omega}$ ($[A]^{\omega}$ resp.) the set of all finite (countable infinite resp.) subsets of the set A. Let $s, t \in [\omega]^{<\omega}$. By $s \sqsubseteq t$ we express the fact that s is an initial part of t, i.e. $s = t$ or $s \subset t$ and $\max(s) < \min(t \backslash s)$.

A set $Z \subseteq X$ is called invariant if $f[Z] \subseteq Z$. Note that the closure of an invariant set is invariant for any continuous f, if f is even open, the same holds true also for the interior. Let the "exit" of $Z \subseteq X$ be defined as $Ex(Z) = Z \backslash f^{-1}[Z]$. That are the points which are leaving Z with f. A subset Z is invariant iff its "exit" is empty.

The following construction is due to Frolík [Fr] and Abramovich, Arenson, and Kitover [AAK]. For arbitrary $Z \subseteq X$ we denote by $Z^{\#}$ the smallest closed superset of Z with a complement invariant under f. For an open mapping f it holds

$$Z^{\#} = cl \bigcup \{f^{-n}[Z] : n < \omega\}.$$

This follows from the following property of open mappings, which we will use throughout the whole paper: $f^{-1}[cl(Z)] = cl[f^{-1}[Z]]$ for arbitrary $Z \subseteq X$.

Lemma 2.1. *Properties of the operator $\#$:*

(i) The operator $\#$ is a closure operator and $Z_1^\# \cup Z_2^\# = (Z_1 \cup Z_2)^\#$.
(ii) If $\mathscr{Z}$ is a family of subsets of X, then $cl \bigcup_{Z\in\mathscr{Z}} Z^\# = (\bigcup_{Z\in\mathscr{Z}} Z)^\#$.
(iii) If Z is open, then $Z^\#$ is open too.
(iv) If Z is invariant, then $Z^\#$ is invariant too. □

Note that for (ii) we need f to be open.

Define the set of fixed points: $Fix = Fix_f(X) = \{x\in X : f(x) = x\}$. The following theorem (true for arbitrary continuous mappings on an extremally disconnected space) is of fundamental importance.

Theorem 2.2. (Frolík [Fr]**).** *Any fixed point has a clopen base consisting of invariant sets.*

Sketch of the proof. Let U be a clopen neighborhood of a fixed point x. Then the set $X\backslash(X\backslash U)^\#$ is the desired invariant clopen neighborhood of x, which is a subset of U. □

Define

$$\begin{aligned} X_1 &= X\backslash(Fix^\#) \\ X_2 &= (int(Fix))^\# \\ X_3 &= (bd(Fix))^\#. \end{aligned}$$

It can be shown that this is a partition of X into three invariant clopen parts such that

in X_1:	$Fix(X_1) = \emptyset$	
in X_2:	$Fix(X_2)$ is clopen	$Fix(X_2)^\# = X_2$
in X_3:	$Fix(X_3)$ is nowhere dense	$Fix(X_3)^\# = X_3$.

The key argument is here the fact that $Fix^\#$ is open [AAK]. The partition is possible for arbitrary continuous f.

A further important result in this area (also without the assumption that f is open) is the following theorem of Vermeer [V]: If the set Fix is nonempty, then it is a retract of X, i.e. there exists a continuous mapping $v : X \to Fix$ such that $v \circ e = id_{Fix}$, where $e : Fix \to X$ is the identical embedding of Fix into X. The proof of the theorem is based on the following facts, which are important also for our purposes:

Lemma 2.3. *Let O_1, O_2, $O \in Clop(Fix)$ and suppose $X = Fix^\#$.*

(i) If $O_1 \cap O_2 = \emptyset$, then $O_1^\# \cap O_2^\# = \emptyset$;
(ii) if $O_1 \cup O_2 = Fix$, then $O_1^\# \cup O_2^\# = X$;
(iii) $O^\# \in Clop(X)$.

Proof. For (i) use Frolík's theorem 2.2 and lemma 2.1. (iii) and (iv). Assertion (ii) follows from $X = Fix^\#$, (iii) is a corollary of (i) and (ii). □

(Vermeer's retract is now the Stone dual of the Boolean homomorphism $O \in Clop(Fix) \mapsto O^{\#} \in Clop(X)$ on $X_2 \cup X_3$ and a constant mapping on X_1.) As a corollary from Vermeer's theorem we get that the set Fix is extremally disconnected.

Our interest concentrates on the part X_3. We ask whether it can be nonempty. We will therefore in the following constellation speak about **"our situation" (topological version):**

X is an extremally disconnected compact Hausdorff space
$f : X \to X$ is an open continuous mapping
(i) Fix is nowhere dense
(ii) Fix is nonempty.

This has a Boolean dualization – **"our situation" (Boolean version)**:

C is a complete Boolean algebra
$F : C \to C$ is a complete homomorphism
(i) for any $a \in C^+$ there is a nonzero $b \leq a$ such that $b \wedge F(b) = \mathbf{0}_C$
(ii) there is no partition of unity (a_0, a_1, a_2) such that $a_i \wedge F(a_i) = \mathbf{0}_C$ for all $i = 0, 1, 2$.

The dualization of (ii) uses the theorem of Krawczyk and Steprāns mentioned in paragraph 1.

The aim of this paper is to find a class of examples of our situation in such a way that any example of our situation contains in a certain manner an example from that class. This means that we are going to find the combinatorial background, which is responsible for the fact that the set of fixed points is nowhere dense and nonempty.

3. The Exponent

3.1. Definition and basic properties

We are going to define the exponent, a complete Boolean algebra with the desired properties as described in the preceding paragraph. The construction of the exponent starts from a complete Boolean algebra B. Besides that there are two other parameters, a regular infinite cardinal κ and a κ-complete homomorphism $r : B^\kappa \to B$. Here B^κ is denoting the Boolean product of κ copies of the algebra B. The mapping r has to be a retract with respect to the natural embedding of B into B^κ, i.e. if $const(b) \in B^\kappa$ is the constant function with value b, then $r(const(b)) = b$. Moreover, the mapping r is supposed to be uniform, i.e. $r(\langle b_\alpha \rangle_{\alpha<\kappa}) = \mathbf{0}_B$ if $|\{\alpha < \kappa : b_\alpha \neq \mathbf{0}_B\}| < \kappa$. The existence of such a κ-complete uniform retract follows for $\kappa = \omega$ and arbitrary complete Boolean algebra B from Sikorski's

Extension theorem. As we shall see later, if κ is uncountable and such a retract r exists, then κ must be even inaccessible.

Consider the Boolean product $B^{[\kappa]^{<\omega}}$. For any element T of this product and $s \in [\kappa]^{<\omega}$, let $T(s) \in B$ be the value in s. We define now a subset of this product.

Definition of the exponent

$$Exp_\kappa(B,r) = \{T \in B^{[\kappa]^{<\omega}} : \forall s \in [\kappa]^{<\omega} : r(\langle T(s \cup \{\alpha\})\rangle_{\alpha<\kappa}) = T(s)\}$$

When it is clear, which κ, B, r we are thinking about, we will omit the parameters and write simply *Exp*. We can look at the elements of the exponents as weighted trees.

The set *Exp* is a κ-complete subalgebra of the product. This follows from the κ-completeness of the homomorphism r. But in itself, it is even a complete Boolean algebra. For proving this, we need the following lemma:

Lemma 3.1.1. *For $\langle b_s\rangle_{s\in[\kappa]^{<\omega}} \in B^{[\kappa]^{<\omega}}$ such that $r(\langle b_{s\cup\{\alpha\}}\rangle_{\alpha<\kappa}) \geq b_s$ there exists a minimal upper bound in $Exp_\kappa(B,r)$.*

Proof. The upper bound is obtained by taking the "closure" with respect to the retract r. Put

$$\begin{aligned} b_s^0 &= b_s \\ b_s^{\beta+1} &= r(\langle b^\beta_{s\cup\{\alpha\}}\rangle_{\alpha<\kappa}) \\ b_s^\beta &= \bigvee_{\gamma<\beta} b_s^\gamma \qquad \text{in the limit step } \beta. \end{aligned}$$

Our assumption implies that the sequence $\langle b_s^\beta\rangle_\beta$ is increasing, so there is a β_0, such that $b_s^{\beta_0} = b_s^{\beta_0+1}$ for all s. It is easy to see that $\langle b_s^{\beta_0}\rangle_{s\in[\kappa]^{<\omega}}$ is an element of *Exp* and that it is the least upper bound of $\langle b_s\rangle_{s\in[\kappa]^{<\omega}}$. □

Lemma 3.1.2. *The Boolean algebra $Exp_\kappa(B,r)$ is complete.*

Proof. Let $T_\delta \in Exp$ be given for $\delta < \tau$ and put $b_s = \bigvee_{\delta<\tau} T_\delta(s)$. Because of $r(\langle b_{s\cup\{\alpha\}}\rangle_{\alpha<\kappa}) = r(\langle \bigvee_{\delta<\tau} T_\delta(s \cup \{\alpha\})\rangle_{\alpha<\kappa}) \geq \bigvee_{\delta<\tau} r(\langle T_\delta(s\cup\{\alpha\})\rangle_{\alpha<\kappa} = \bigvee_{\delta<\tau} T_\delta(s) =$
$= b_s$ we can apply the preceding lemma. □

The properties of the exponent depend to a great extent on the starting algebra B. This is also caused by the following fact:

Lemma 3.1.3. *The Boolean algebra B is a complete subalgebra of $Exp_\kappa(B,r)$.*

Proof. This is witnessed by the natural embedding, which assigns to every element of B the constant function with this element as its value. □

The exponent combines the starting algebra with a tree structure. By this way it gains properties independent from the starting algebra, one of them is expressed by the following lemma. For $s \in [\kappa]^\omega$, let $T^s \in Exp_\kappa(B,r)$ be defined by $T^s(t) = \mathbf{1}_B$ if $t \sqsupseteq s$, otherwise $T^s(t) = \mathbf{0}_B$.

Lemma 3.1.4. *The exponent $Exp_\kappa(B, r)$ is nowhere (ω, κ)-distributive.*

Proof. This is witnessed by $\{\{T^s\}_{|=n}\}_{n<\omega}$. □

The next lemma states the fusion property for the exponent, a property known also in different variants for other tree forcing notions. The lemma will also be useful when dealing with concrete computations in the exponent.

Let $T, S \in Exp$ and $n < \omega$. We write $T \leq_n S$ if $T \leq S$ and $T(s) = S(s)$ for all s with $|s| \leq n$. A sequence $\langle T_n \rangle_{n<\omega}$ is called a fusion sequence if $T_{n+1} \leq_n T_n$ for all $n < \omega$.

Lemma 3.1.5. *(i) If $\langle T_n \rangle_{n<\omega}$ is a sequence such that for any m there is an n_0 such that for all $n \geq n_0$ it holds $T_n(s) = T_{n_0}(s)$ for all s with $|s| \leq m$, then $(\bigwedge_{n<\omega} T_n)(s) = \bigwedge_{n<\omega} T_n(s)$ for any $s \in [\kappa]^{<\omega}$. The same holds true for the dual operation $\bigvee$.*

(ii) (Fusion lemma) If $\langle T_n \rangle_{n<\omega}$ is a fusion sequence and $T_0(\emptyset) \neq \mathbf{0}_B$, then $\bigwedge_{n<\omega} T_n \neq \mathbf{0}_{Exp}$, namely $(\bigwedge_{n<\omega} T_n)(\emptyset) = T_0(\emptyset) \neq \mathbf{0}_B$.

Proof. Realize that under the assumption of (i), $\bigwedge_{n<\omega} T_n(s)$ is in fact a finite intersection. Let n_0 be the number existing by the assumption of (i) for $m = |s| + 1$. It follows

$$\begin{aligned}
r(\langle \bigwedge_{n<\omega} T_n(s \cup \{\alpha\}) \rangle_{\alpha<\kappa}) &= r(\langle \bigwedge_{n\leq n_0} T_n(s \cup \{\alpha\}) \rangle_{\alpha<\kappa}) \\
&= \bigwedge_{n\leq n_0} r(\langle T_n(s \cup \{\alpha\}) \rangle_{\alpha<\kappa}) \\
&= \bigwedge_{n\leq n_0} T_n(s) \\
&= \bigwedge_{n<\omega} T_n(s).
\end{aligned}$$

So, $\langle \bigwedge_{n<\omega} T_n(s) \rangle_{s\in[\kappa]^{<\omega}}$ is already an element of the exponent and (i) follows.

Assertion (ii) is a special case of (i). □

We introduce a notion useful for the analysis of the retract r. Given r with the demanded properties, define $\phi = \phi_r : \mathscr{P}(\kappa) \to B$ by $\phi(A) = r(\chi(A))$, where $\chi(A)$ is the "characteristic function" of A, i.e. $\chi(A) = \langle b_\alpha \rangle_{\alpha<\kappa}$ with $b_\alpha = \mathbf{1}_B$ if $\alpha \in A$ and $b_\alpha = \mathbf{0}_B$ if $\alpha \notin A$. The mapping ϕ is a κ-complete uniform homomorphism. It describes the behaviour of a complete subalgebra under the retract r and is therefore determined by r. Under certain assumptions, the opposite holds true:

Lemma 3.1.6. *If B is $(\kappa, 2)$-distributive, then r is already determined by ϕ_r, namely*

$$r(\langle b_\alpha \rangle_{\alpha<\kappa}) = \bigvee \{a : \phi_r(\{\alpha < \kappa : a \leq b_\alpha\}) \geq a\}.$$

Proof. Let $\langle b_\alpha \rangle_{\alpha<\kappa}$ be given. For $a \in B^+$ let $A_a = \{\alpha < \kappa : a \leq b_\alpha\}$. If $\phi(A_a) \geq a$ then $r(\langle b_\alpha \rangle_{\alpha<\kappa}) \geq r(\chi(A_a) \wedge const(a)) = r(\chi(A_a)) \wedge r(const(a)) =$

$= \phi(A_a) \wedge a = a$. In the same way, we get for $A'_a = \{\alpha < \kappa : a \leq -b_\alpha\}$: If $\phi(A'_a) \geq a$ then $-r(\{b_\alpha\rangle_{\alpha<\kappa}) \leq a$. The assertion of the lemma will follow from the fact that the set $D_1 = \{a \in B^+ : \phi(A_\alpha) \geq a$ or $\phi(A'_a) \geq a\}$ is dense in B^+. But this can be seen as follows: Because of $(\kappa, 2)$-distributivity, the set $D_2 = \{a : \forall\alpha < \kappa : a \leq b_\alpha$ or $a \leq -b_\alpha\}$ is dense in B^+. For an $a \in D_2$ we have $A_a \cup A'_a = \kappa$, therefore $\phi(A_\alpha) \vee \phi(A'_a) = \mathbf{1}_B$. Hence either $a \wedge \phi(A_a)$ or $a \wedge \phi(A'_a)$ is nonzero and therefore an element of D_1. □

The case $\kappa = \omega$ will be our main concern. When looking for examples with uncountable κ, we should keep in mind that κ has to be inaccessible, as we will see from the following two lemmas.

Lemma 3.1.7. *The algebra $\phi[\mathscr{P}(\kappa)]$ is $(<\kappa, <\kappa)$-distributive.*

Proof. Assume on the contrary that $\{a_{\gamma\beta}\}_{\gamma<\lambda,\beta<\lambda_\gamma}$ witnesses that $\phi[\mathscr{P}(\kappa)]$ is not $\{<\kappa, <\kappa)$-distributive, i.e. $a_{\gamma\beta} \in \phi[\mathscr{P}(\kappa)]$, $\{a_{\gamma\beta}\}_{\beta<\lambda_\gamma}$ is a maximal antichain for all $\gamma < \lambda$ and there exists $a \neq \mathbf{0}_B$ such that for all nonzero $a' \leq a$ there is a $\gamma < \lambda$ such that for all $\beta < \lambda_\gamma$ it holds $a' \not\leq a_{\gamma\beta}$. Choose for any $\alpha_{\gamma\beta}$ a representation $A_{\gamma\beta} \subseteq \kappa$ such that $\phi(A_{\gamma\beta}) = a_{\gamma\beta}$. Since ϕ is κ-complete we can assume that $\{A_{\gamma\beta}\}_{\beta<\lambda_\gamma}$ is a partition of κ. Let $\beta(\gamma, \alpha)$ be that ordinal for which $\alpha \in A_{\gamma\beta(\gamma,\alpha)}$ and define $b^\gamma_\alpha = -\phi(A_{\gamma,\beta(\gamma,\alpha)})$. Using the κ-completeness of r, we compute

$$\begin{aligned}
r(\langle\bigvee_{\gamma<\lambda} b^\gamma_\alpha\rangle_{\alpha<\kappa} &= \bigvee_{\gamma<\lambda} r(\langle b^\gamma_\alpha\rangle_{\alpha<\kappa}) \\
&= \bigvee_{\gamma<\lambda} r(\bigvee_{\beta<\lambda_\gamma} (const(-\phi(A_{\gamma,\beta})) \wedge \chi(A_{\gamma,\beta}))) \\
&= \bigvee_{\gamma<\lambda}\bigvee_{\beta<\lambda_\gamma} (-\phi(A_{\gamma,\beta}) \wedge \phi(A_{\gamma,\beta})) \\
&= \mathbf{0}_B.
\end{aligned}$$

On the other hand, it follows from the choice of the matrix $\{A_{\gamma\beta}\}_{\gamma<\lambda,\beta<\lambda_\gamma}$ as a witness of the nondistributivity under a:

$$r(\langle\bigvee_{\gamma<\lambda} b^\gamma_\alpha\rangle_{\alpha<\kappa}) = r((-\bigwedge_{\gamma<\lambda} \phi(A_{\gamma,\beta(\gamma,\alpha)})\rangle_{\alpha<\kappa}) \geq r(const(a)) = a \neq \mathbf{0}_B,$$

a contradiction. □

Lemma 3.1.8. *The cardinal κ is equal to ω or inaccessible.*

Proof. Assume in the contrary that there is a $\lambda < \kappa$ and a 1-1 mapping ϱ from κ into 2^λ. Define $A_{\gamma i} = \{\alpha < \kappa : \varrho(\alpha)(\gamma) = i\}$ for $\gamma < \lambda$ and $i = 0{,}1$. Since ϱ is injective, for any $f \in {}^\lambda 2$ it holds $|\bigcap_{\gamma<\lambda} A_{\gamma f(\gamma)}| \leq 1$. It follows that $\bigwedge_{\gamma<\lambda} \phi(A_{\gamma f(\gamma)}) = \phi(\bigcap_{\gamma<\lambda} A_{\gamma f(\gamma)}) = \mathbf{0}_B$. This means that $\langle A_{\gamma i}\rangle_{\gamma<\lambda,i=0}$ is a witness that the algebra $\phi[\mathscr{P}(\kappa)]$ is not $(<\kappa, <\kappa)$-distributive, a contradiction to the preceding lemma. □

3.2. Dense subsets of the exponent

As noted above, the combinatorial kernel of the exponent, which makes it interesting for the forcing method as well as for our application in dynamics (paragraph 4), is the Prikry property:

Lemma 3.2.1 (Prikry property for the exponent). *For any $T \in Exp^+$ and $s \in [\kappa]^{<\omega}$ such that $T(s) \neq \mathbf{0}_B$ and for any partition of unity (T_1, T_2) of $\mathbf{1}_{Exp}$ there is a $T' \in Exp^+$ such that $T' \leq T$ and $T' \leq T_1$ or $T' \leq T_2$ and moreover $T'(s) \neq \mathbf{0}_B$.*

Proof. Since (T_1, T_2) is a partition of unity of $\mathbf{1}_{Exp}$, $(T_1(s), T_1(s))$ is a partition of unity of $\mathbf{1}_B$. Either $T(s) \wedge T_1(s) \neq \mathbf{0}_B$ or $T(s) \wedge T_2(s) \neq \mathbf{0}_B$. Therefore either $T' = T \wedge T_1$ or $T' = T \wedge T_2$ is as desired. □

We are looking for special dense subsets with a nice description. We will obtain all mentioned forcing notions and many more as such dense subsets. In ths way, a big advantage of the exponent becomes the fact that we have an interesting partial order and simultaneously its completion in an explicit form.

Definition (Prikry dense subsets of the exponent). *We say that a subset $\mathbb{P}$ of Exp^+ is Prikry dense if for any $T \in Exp^+$ and $s \in [\kappa]^{<\omega}$ with $T(s) \neq \mathbf{0}_B$ there is a $T' \in \mathbb{P}$ such that $T' \leq T$ and moreover $T'(s) \neq \mathbf{0}_B$.*

The intention of this definition of a strong form of density is in the fact that Prikry dense subsets obviously preserve the Prikry property of the exponent:

Lemma 3.2.2 (Prikry property for Prikry dense subsets $\mathbb{P}$ of the exponent). *For any $T \in \mathbb{P}$ and $s \in [\kappa]^{<\omega}$ such that $T(s) \neq \mathbf{0}_B$ and for any partition of unity (T_1, T_2) of $\mathbf{1}_{Exp}$ there is a $T' \in \mathbb{P}$ such that $T' \leq T$ and $T' \leq T_1$ or $T' \leq T_1$ and moreover $T'(s) \neq \mathbf{0}_B$.* □

This fact allows us to prove the Prikry property for a given forcing notion by finding parameters κ, B, and r and an embedding into $Exp_\kappa(B, r)$ onto a Prikry dense subset.

We say that an element T of Exp has a stem, denoted by $stem(T)$, if $stem(T) \in [\kappa]^{<\omega}$ and $T(stem(T)) \neq \mathbf{0}_B$ and $T(t) \neq \mathbf{0}_B$ implies $t \sqsupseteq stem(T)$ for every t.

An element T of Exp is said to be invariant, if for any $s \subseteq t$ it holds $T(s) \geq T(t)$. For such an invariant $T \neq \mathbf{0}_{Exp}$ it holds $T(\emptyset) \neq \mathbf{0}_B$. The choice of the notion "invariant" will become clear in the following. After introducing a complete homomorphism on Exp, the "invariant" elements of Exp will become invariant in a strong sense in the topological dual.

For any $T \in Exp$ we can define $T^{inv} \in Exp$ by $T^{inv}(s) = \bigwedge\{T(t) : t \subseteq s\}$. The fact that $T^{inv} \in Exp$ is shown by the following consideration:

$$
\begin{aligned}
r\big(\langle T^{inv}(s \cup \{\alpha\})\rangle_{\alpha<\kappa}\big) &= r\big(\langle \bigwedge_{t \subseteq s} T(t \cup \{\alpha\}) \wedge \bigwedge_{t \subseteq s} T(t)\rangle_{\alpha<\kappa}\big) \\
&= \bigwedge_{t \subseteq s} r\big(\langle T(t \cup \{\alpha\})\rangle_{\alpha<\kappa}\big) \wedge r\big(\langle \bigwedge_{t \subseteq s} T(t)\rangle_{\alpha<\kappa}\big) \\
&= \bigwedge_{t \subseteq s} T(t) \wedge r\big(\langle T^{inv}(s)\rangle_{\alpha<\kappa}\big) \\
&= T^{inv}(s) \wedge T^{inv}(s) = T^{inv}(s).
\end{aligned}
$$

It is evident that T^{inv} is invariant and below T and $T^{inv}(\emptyset) = T(\emptyset)$.

For $s \in [\kappa]^{<\omega}$ and invariant $T \in Exp^+$, we define $[s, T] \in Exp$ by

$$
[s, T](r) = \begin{cases} T(r \setminus s) & \text{if } r \sqsupseteq s \\ \mathbf{0}_B & \text{otherwise.} \end{cases}
$$

Those elements of Exp are called standard. The stem of $[s, T]$ is s. It is clear that an element $T \in Exp$ is standard iff it has a stem and for all $s, t \sqsupseteq stem(T)$ it holds $T(s) \leq T(t)$ whenever $s \setminus stem(T) \supseteq t \setminus stem(T)$.

Lemma 3.2.3. *The set Exp^{stand} of the standard elements is Prikry dense in Exp.*

Proof. For any $T \in Exp^+$ and $s \in [\kappa]^{<\omega}$ with $T(s) \neq \mathbf{0}_B$ define $S \in Exp$ by $S(t) = T(s \cup t)$ and put $T' = (S)^{inv}$. Then $[s, T'] \leq T$ and $[s, T'](s) = = T(s) \neq \mathbf{0}_B$. □

Note that we proved indeed more: For any $T \in Exp^+$ and $s \in [\kappa]^{<\omega}$ with $T(s) \neq \mathbf{0}_B$ we found a smaller standard element with the same value in s.

For special cases, we can find even simpler Prikry dense subsets, which are easier to handle. Say that a standard element of the exponent is two-valued if it has only one nonzero value.

Lemma 3.2.4. *If B is $(\kappa, 2)$-distributive, then the set Exp^{twoval} of the two-valued elements is Prikry dense in Exp.*

Proof. Let $T \in Exp^+$ and $T(s) \neq \emptyset$. We can assume by the preceding lemma that T is standard with stem s. By $(\kappa, 2)$-distributivity of B, we find a nonzero $a \leq T(s)$ such that for all t either $a \leq T(t)$ or $a \leq -T(t)$. If $S_a \in Exp$ is the element of the exponent with constant value a, then $T \wedge S_a$ is the desired two-valued element under T with $(T \wedge S_a)(s) = a \neq \mathbf{0}_B$. □

We will now assume an additional closer connection between the Boolean algebra B and the tree structure of the exponent. Namely, we suppose in the following that the range of the mapping $\phi : B^\kappa \to B$ as defined in the precending paragraph is dense in B^+. Let us mention that in this case B is $(\kappa, 2)$-distributive iff $\mathscr{P}(\kappa)/ker(\phi)$ is κ-distributive. This follows from the known fact that $(\kappa, 2)$-distributivity and $(\kappa, 2^\kappa)$-distributivity are equivalent in complete Boolean algebras. The mapping ϕ is determined by its kernel. This is a uniform, κ-complete ideal on κ. That means that all subsets of κ with power less that κ as well as all unions of less than κ elements of the ideal are again elements of the ideal. We will therefore assume in the following all ideals to be uniform and κ-complete. Such an ideal $\mathscr{I}$ on κ with a κ-distributive quotient $\mathscr{P}(\kappa)/\mathscr{I}$ will be called also κ-distributive.

When looking for ω-distributive ideals on ω, we should keep in mind that not only the ideal of finite subsets of ω and all maximal ideals are of this kind, but also all F_σ-ideals, as follows from a result of Just and Krawczyk [JK].

If $\mathscr{I}$ is a κ-distributive ideal, we can take $B = Compl(\mathscr{P}(\kappa)/\mathscr{I})$ and ϕ the natural mapping from $\mathscr{P}(\kappa)$ into B with kernel $\mathscr{I}$. By this way, we get the exponent $Exp_\kappa(B, r_\phi)$. We write for short $Exp(\mathscr{I})$ and call it the exponent received from the ideal $\mathscr{I}$. This case will be our main application of the exponent. The restriction brings the exponent already very close to the mentioned forcing notions, by which the introduction of the exponent was motivated.

As it will follow from the next lemma, the construction of the exponent from a κ-distributive ideal allows us to take a dense subset of elements, which are already described by the tree structure. The value $T(s)$ of such an element T in a concrete $s \in [\kappa]^{<\omega}$ will be determined by the branching of the tree in this s. A two-valued element is said to be tree-like if for any $s \in [\kappa]^{<\omega}$ it holds $\phi(\{\alpha < \kappa : T(s \cup \{\alpha\}) \neq \mathbf{0}_B\}) = T(s)$. The set of the tree-like elements is denoted by $Exp^{treelike}$.

Lemma 3.2.5. *For a κ-distributive ideal $\mathscr{I}$, the set $Exp^{treelike}$ of the tree-like elements is Prikry dense in $Exp(\mathscr{I})$.*

Proof. This follows easily from the last lemma and from the fusion lemma 3.1.5. (ii). □

We should explain the choice of the notion tree-like. Let us consider trees $\mathscr{T} \subseteq \kappa^{<\omega}\uparrow$ consisting of finite increasing sequences. Such a tree is said to have a stem u if for all $v \in \mathscr{T}$ it holds either $v \subset u$ or $u \subseteq v$. For an element $u \in \mathscr{T}$, we define the set of successors by $succ_{\mathscr{T}}(u) = \{\alpha < \kappa : u^\frown\langle\alpha\rangle \in \mathscr{T}\}$. Under the assumptions of the last lemma, we can define now a mapping $\Theta : Exp^{treelike} \to \to \mathscr{P}(\kappa^{<\omega})$ by $\Theta(T) = \{u \in \kappa^{<\omega}\uparrow : range(u) \sqsubset stem(T) \text{ or } T(range(u)) \neq \mathbf{0}_B\}$. The range of the mapping ϕ consists of trees. The mapping is an order isomorphism onto its range.

We continue with an example for the situation of the last lemma. Let $B = \{\mathbf{0},\mathbf{1}\}$. The kernel $ker(\phi)$ of ϕ will be a maximal (and therefore ω-distributive) ideal $\mathscr{I}$, its complement an ultrafilter $\mathscr{U}$. Consider $Exp(\mathscr{I})$. By the last lemma, the tree-like elements form a Prikry dense subset. The mapping Θ is now an order isomorphism onto the set of all trees with a stem u, which are branching into a set of the ultrafilter $\mathscr{U}$, i.e. for all $v \in \mathscr{T}$, $v \supseteq u$ it holds $succ_{\mathscr{T}}(v) \in \mathscr{U}$. This partial order of trees is known as Laver forcing $\mathbb{L}_{\mathscr{U}}$ with the ultrafilter $\mathscr{U}$.

We insert here a remark for the case $\kappa = \omega$. For any $T \in Exp^{treelike}$, we have the set of all branches of the tree $\Theta(T)$. This is a subset of $\omega^{\omega}\uparrow \cong [\omega]^{\omega}$. The tree-like elements define by this way a family of subsets of $[\omega]^{\omega}$. We could redefine the Ramsey property for subsets of $[\omega]^{\omega}$ with respect to this family and try to generalize the known facts around the Ellentuck theorem. The key tool will be the Prikry property of the exponent. We will not accomplish those ideas in this paper.

Aiming for our original examples of partial orders with the Prikry property, we would like to find even better Prikry dense subsets. Since the mapping Θ identifies the elements of $Exp^{treelike}$ with trees, the following definition is natural: For $T \in Exp^{treelike}$ let $succ_T(s) = \{\alpha > \max(s) : T(s \cup \{\alpha\}) \neq \mathbf{0}_B\}$. Let $\mathscr{I}$ be a κ-distributive ideal on κ. A decreasing family $\langle A_\alpha \rangle_{\alpha \in A}$ i called an *$\mathscr{I}$-tower* if $A_\alpha \subseteq A \in I^+$ and $A \backslash A_\alpha \in \mathscr{I}$ for all $\alpha \in A$. A tree-like element $T \in Exp(\mathscr{I})$ with stem s is called tower-like if there is an $\mathscr{I}$-tower such that $T(t) \neq \mathbf{0}_B$ iff either $t = s$ or $t \sqsupset s$ and $\min(t \backslash s) \in A$ and $t(> \alpha) \subset A_\alpha$ for all $\alpha \in t\backslash s$, where $t(> \alpha)$ is the set of elements of t greater than α. We write $T = [s, \langle A_\alpha \rangle_{\alpha \in A}]$. A tower-like element corresponds with a tree, for which the sets of successors depend only on the last element of the short branch s.

Lemma 3.2.6. *For a κ-distributive ideal $\mathscr{I}$ the set of the tower-like elements is Prikry dense in $Exp(\mathscr{I})$.*

Proof. We begin from lemma 3.2.5. For a given tree-like element T with stem s, put $A = succ_T(s)$ and $A_\alpha = \bigcap \{succ_T(t) : T(t) \neq \mathbf{0}_B \,\&\, \max(t) \leq \alpha\}$. The resulting tower-like element $T' = [s, \langle A_\alpha \rangle_{\alpha \in A}]$ is under T and $T'(s) = T(s)$. $\square$

For finding even simpler Prikry dense subsets, we would like to have ideals $\mathscr{I}$ such that the $\mathscr{I}$-towers have a "diagonal". A set $D \in \mathscr{I}^+$ is said to be a diagonal of the $\mathscr{I}$-tower $\langle A_\alpha \rangle_{\alpha \in A}$ if $D \subseteq A$ and $D(> \alpha) \subseteq A_\alpha$ for all $\alpha \in D$. A tower-like element of the exponent with stem s is called simple if it is obtained from an $\mathscr{I}$-tower $\langle A_\alpha \rangle_{\alpha \in A}$ where $A_\alpha = A(> \alpha)$. This simple element is denoted by $[s, A]$. It is clear that $[s, A](t) = \phi(A)$ if $s \sqsubseteq t$ and $t \backslash s \subset A$, else $[s, A](t) = \mathbf{0}_B$.

Lemma 3.2.7. *For a κ-distributive ideal $\mathscr{I}$ such that there exists a diagonal for any $\mathscr{I}$-tower, the set of simple elements Exp^{simple} is Prikry dense in $Exp(\mathscr{I})$.*

Proof. Starting from lemma 3.2.6. with a tower-like element T defined by the $\mathscr{I}$-tower $\langle A_\alpha \rangle_{\alpha \in A}$, we find a diagonal D of the tower and realize that the resulting simple element $[s, D]$ is under T and $[s, D](s) \neq \mathbf{0}_B$. $\square$

An example of such an ideal is a normal maximal ideal on a measurable cardinal. The resulting exponent, or more exactly the partial order Exp^{simple}, is the well known Prikry forcing [P].

We will now study the case $\kappa = \omega$. An ω-distributive ideal $\mathscr{I}$ such that there exists a diagonal for any $\mathscr{I}$-tower is called semiselective (Farah [F]).

From the last lemma it follows that the set of simple elements $Exp^{simple} = = \{[s,D] : s \in [\omega]^{<\omega} \,\&\, D \in \mathscr{I}^+\}$ has the Prikry property.

If we add the demand that the quotient $\mathscr{P}(\omega)/\mathscr{I}$ has to be even σ-closed, we get the notion of a selective ideal, its coideal also called happy family. The partial order Exp^{simple} is now isomorphic with Mathias forcing $\mathbb{M}_{\mathscr{H}}$ with the happy family $\mathscr{H} = \mathscr{P}(\omega)\backslash\mathscr{I}$ via the order isomorphism Θ. Special cases are Mathias forcing $\mathbb{M}$, where the happy family is the set of all infinite subsets of ω, and Mathias forcing $\mathbb{M}_{\mathscr{U}}$ with a selective ultrafilter $\mathscr{U}$.

In the context of the exponent, the role of selectivity or its variations becomes clearer. When looking on the proofs for the different forcing notions derived from ideals with some kind of selectivity, we note that it goes always the same way. At first a tree is constructed. After that, only the last argument of the proof uses selectivity to find a branch, which is not in the ideal. If we agree to work with trees instead of sets, we can drop the last argument of the proof and therefore also the demand of selectivity. This is the idea of the mentioned forcing with Laver trees. Selectivity has no impact on the Prikry property, but it gives us an even simpler Prikry dense subset, which of course can be important for applications.

We got Mathias forcing as a special form of the exponent. It is now a natural question, to what extent the properties of this forcing generalize to the exponent. The Prikry property holds for the exponent in general. There is another important property of Mathias forcing, which holds in this general frame. Since the starting algebra B is a complete subalgebra of the exponent, it follows from general considerations that forcing with the exponent splits into a two step forcing with the first factor B. Under the assumption that the exponent was received from a ω-distributive ideal, also the second factor has a nice description:

$$Exp(\mathscr{I}) \approx \mathscr{P}(\omega)/\mathscr{I} \bigstar \mathbb{L}_{\mathscr{U}}.$$

Here $\mathscr{U}$ is the generic ultrafilter over $\mathscr{P}(\omega)/\mathscr{I}$. So in this case, the corresponding exponent splits into an iteration of a ω-distributive factor and a σ-centered factor. If the generic ultrafilter U is even selective, Laver forcing with this ultrafilter and Mathias forcing with this ultrafilter coincide and we get the decomposition

$$Exp(\mathscr{I}) \approx \mathscr{P}(\omega)/\mathscr{I} \bigstar \mathbb{M}_{\mathscr{U}}.$$

It is known that for a selective ideal $\mathscr{I}$ also the generic ultrafilter over $\mathscr{P}(\omega)/\mathscr{I}$ is selective (Grigorieff [G]). An opposite example is the summation ideal, an ω-distributive F_σ-ideal, where the corresponding generic ultrafilter will not be selective. Yet there exists the splitting of the exponent in an ω-distributive (even σ-closed) and a σ-centered factor as seen above.

The exponent $Exp_\omega(B,r)$ adds always a dominating real. Mathias forcing does not add a Cohen real. We can never proof a theorem like this for the exponent. This follows from the fact that for any complete Boolean algebra B we find an exponent and this Boolean algebra B will be the first factor in the decomposition.

But if we start from an ω-distributive ideal $\mathscr{I}$ such that the generic ultrafilter on $\mathscr{P}(\omega)/\mathscr{I}$ must be nowhere dense (for definition see Baumgartner [B]), then the resulting exponent $Exp(\mathscr{I})$ does not add a Cohen real. This follows from a result of Brendle (see [W]) stating that Laver forcing $\mathbb{L}_{\mathscr{U}}$ with an ultrafilter $\mathscr{U}$ does not add a Cohen real iff $\mathscr{U}$ is nowhere dense.

For any exponent we can define a generic sequence (generic real for the case $\kappa = \omega$) x_G as the union of the stems of standard elements in the generic ultrafilter G. For uncountable κ, the set x_G is a countable cofinal subset of κ. In case $\kappa = \omega$, the enumeration of x_G is a dominating real. If we construct the exponent from a κ-distributive ideal, the generic ultrafilter can be recovered from the generic sequence. In this context, it holds also that any subset of an Exp-generic sequence is Exp-generic.

3.3. An application of the exponent

In the last paragraph we exposed the importance of additional assumptions on the parameters of the exponent for getting results for the exponent similar to results for Mathias forcing and other known forcing notions. Yet the exponent can be useful without such assumptions on the distributivity of the starting algebra. Just for illustration, we reprove the following theorem. A Boolean algebra C is said to be completely generated by a subset A, if the only complete subalgebra containing the subset A is the algebra C itself. A Boolean algebra satisfies the countable chain condition (ccc) if any set of mutually disjoint elements is at most countable.

Theorem (Martin and Solovay [MS]). *Any complete ccc Boolean algebra B with power at most continuum can be embedded as a complete subalgebra in a complete ccc Boolean algebra C with countable many complete generators.*

Proof. This algebra C will be of course the exponent, and the regular embedded algebra B will be the starting algebra. For $A \subseteq \omega$ let $\chi(A) = \langle b_i \rangle_{i<\omega} \in B^\omega$ be defined by $b_i = \mathbf{1}_B$ if $i \in A$ and $b_i = \mathbf{0}_B$ if $i \notin A$. Choose now an independent modulo finite collection $\mathscr{A}$ of subsets A of ω with the same power as the algebra B. Here a set $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is called independent modulo finite if for any $\mathscr{A}_1$, $\mathscr{A}_2 \in [\mathscr{A}]^{<\omega}$ it holds $|\bigcap \{A : A \in \mathscr{A}_1\} \cap \bigcap \{-A : A \in \mathscr{A}_2\}| = \omega$. It is known that there exists an independent modulo finite collection on ω of power continuum. Fix a bijection ϱ between the collection $\mathscr{A}$ and B. Define $r(\chi_A) = \varrho(A)$. By the independence of the collection $\mathscr{A}$, we can apply Sikorski's Extension theorem to get an r defined on the whole of B^ω, which satisfies the demands on the retract for the construction of the exponent. The algebra B is a complete subalgebra of $C = Exp_\omega(B, r)$ by lemma 3.1.3. The algebra C is complete by lemma 3.1.2. and satisfies ccc as B does. The latter follows from the fact that the exponent is a subalgebra of the product of countable many copies of the ccc algebra B.

Define $T^s \in C = Exp_\omega(B,r)$ by $T^s(t) = \mathbf{1}_B$ if $s \sqsubseteq t$ and $T^s(t) = \mathbf{0}_B$ otherwise. The proof is finished by verifying that $\{T^s\}_{s\in[\omega]^{<\omega}}$ is a system of complete generators for $Exp_\omega(B,r)$.

Given $T \in Exp_\omega(B,r)$. Let A be the complete subalgebra of the exponent completely generated by $\{T^s\}_{s\in[\omega]^{<\omega}}$. For $n < \omega$ and $|s| = n$ put

$$T_n = \bigvee_{|s|=n} \bigvee \{T^{s\cup\{\alpha\}}: \alpha > \max(s)\ \&\ \alpha \in \varrho^{-1}(T(s))\}.$$

It holds $T_n \in A$ and $T_n(s) = T(s)$ for all s with $|s| = n$, and therefore also for all s with $|s| \leq n$. From lemma 3.1.5.(i) we infer $(\bigwedge_{n\geq k} T_n)(s) = \bigwedge_{n\geq k} T_n(s)$, hence $(\bigwedge_{n\geq|s|} T_n)(s) = T(s)$. Also $(\bigwedge_{n\geq k} T_n)(s) \leq T(s)$ since $T_n(s) = T(s)$ for all $n \geq |s|$, i.e. $\bigwedge_{n\geq k} T_n \leq \mathbb{T}$. It follows

$$T = \bigvee_{k<\omega} \bigwedge_{\nu\geq k} T_n \in A. \qquad \square$$

4. The shift operator and its topological dual

In this paragraph we will show how the exponent and a suitable homomorphism on it can serve as examples of our situation as defined in paragraph 2.

If $s \in [\omega]^{<\omega}$ and $s \neq \emptyset$ let $shift(s) = s\backslash\{\min(s)\}$ and $shift(\emptyset) = \emptyset$. Define a mapping sh: $Exp \to Exp$ by $sh(T)(s) = T(shift(s))$. It is easily checked that the mapping sh is correctly defined and that it is a homomorphism. We realize now that the pair (Exp, sh) is indeed an example of our situation.

Theorem 4.1. *Any exponent $Exp_\kappa(B,r)$ equipped with the canonical shift operation sh is an example of a complete Boolean algebra together with a complete homomorphism on it such that*

(i) for any $T \in Exp^+$ there is a nonzero $S \leq T$ such that $S \wedge sh(S) = \mathbf{0}_{Exp}$.

(ii) there is no partition of unity (T_0, T_1, T_2) such that $T_i \wedge sh(T_i) = \mathbf{0}_{Exp}$ for all $i = 0,1,2$.

Proof. We noted already that the exponent is a complete Boolean algebra and that sh is a homomorphism on it. The fact that sh is even a complete homomorphism follows from the construction of the supremum of an arbitrary subset of the exponent in the proof of lemma 3.1.1. The construction of the supremum and the homomorphism sh commute in each step of the construction, therefore $sh(\bigvee_{\delta<\tau} T_\delta) = \bigvee_{\delta<\tau} sh(T_\delta)$.

We prove (i). For any element $S \in Exp$ with a nonempty stem it holds $S \wedge sh(S) = \mathbf{0}_{Exp}$. As shown in paragraph 3.2, those elements are dense in Exp.

Assertion (ii) is a consequence of the Prikry property for the exponent. Let a partition of unity (T_0, T_1, T_2) be given. There is an $i = 0, 1, 2$ such that $T_i(\emptyset) \neq \mathbf{0}_B$.

But then $(T_i \wedge sh(T_i))(\emptyset) = T_i(\emptyset) \wedge sh(T_i)(\emptyset) = T_i(\emptyset) \wedge T_i(\emptyset) = T_i(\emptyset) \neq \mathbf{0}_B$, i.e. $T_i \wedge sh(T_i) \neq \mathbf{0}_{Exp.}$ □

As a corollary of this theorem we get its dual topological form:

Theorem 4.2. *Any exponent $Exp_\kappa(B,r)$ equipped with the canonical shift operation sh gives via the Stone dualization an example of an extremally disconnected compact Hausdorff space and an open continuous mapping on it such that*

(i) Fix is nowhere dense
(ii) Fix is nonempty.

This dualization allows us to apply methods of topological dynamics. Expecially, we will show that the construction from an exponent is essentially the only way to get such a situation.

We are now asking for the concrete shape of the set of fixed points.

Lemma 4.3. *The set of fixed points in the topological dual of the exponent $Exp_\kappa(B,r)$ and its shift operation sh is homeomorphic with the Stone-space of the algebra B.*

Proof. The assertion of the lemma will follow from the fact that $Stone(T) \cap Fix = \emptyset$ iff $T(\emptyset) = \mathbf{0}_B$. For this aim we determine the Boolean dual of the topological notions from the beginning of paragraph 2. If $U \in Clop(X)$ is the topological dual of some $T \in Exp$, we denote by $T^\#$ resp. $Ex(T)$ the Boolean dual of $U^\#$ resp. $Ex(U)$. It is easily checked that $T^\# = \bigvee_{i<\omega} sh(T)$ and therefore $T^\#(s) = \bigvee_{i=0}^{|s|} T(shift^i(s))$ (use lemma 3.1.5. (i)), moreover $Ex(T)(s) = T(s) - sh(T)(s) = T(s) - T(shift(s))$. We compute

$$(*) \qquad Ex(T)^\#(s) = \bigvee_{i=0}^{|s|} (T(shift^i(s)) - T(shift^{i+1}(s))).$$

From lemma 5.1.1. (iii) we get $Stone(T) \cap Fix = \emptyset$ iff $Ex(Stone(T))^\# \supseteq Stone(T)$, i.e. dually iff $Ex(T)^\# \geq T$. We claim $Ex(T)^\# \geq T$ iff $T(\emptyset) = \mathbf{0}_B$. Indeed, if $Ex(T)^\# \geq T$, then also $Ex(T)^\#(\emptyset) \geq T(\emptyset)$, but by (*) $Ex(T)^\#(\emptyset) = T(\emptyset) - T(\emptyset) = \mathbf{0}_B$, hence $T(\emptyset) = \mathbf{0}_B$. For the opposite direction, if $T(\emptyset) = \mathbf{0}_B$ then by (*): $Ex(T)^\#(s) = \bigvee_{i=0}^{|s|} T(shift^i(s)) \geq T(s)$.

We proved $Stone(T) \cap Fix = \emptyset$ iff $T(\emptyset) = \mathbf{0}_B$. This implies now that $Clop(Fix)$ is isomorphic with the factorization of the exponent by the ideal of those elements T, for which $T(\emptyset) = \mathbf{0}_B$, but this factor is isomorphic with B. It follows by Stone duality that Fix is homeomorphic with the Stone-space of B. □

This lemma and theorem 4.2. together with our observation at the beginning of paragraph 3.1., that for $\kappa = \omega$ and for any complete Boolean algebra B there is a retract r with the demanded properties, give now an answer to the question, which spaces can occur as nowhere dense sets of fixed points for open mappings on extremally disconnected compact spaces.

Theorem 4.4. *For any extremally disconnected compact Hausdorff space Z there is another extremally disconnect compact Hausdorff space X together with an open continuous mapping f on it, such that the set of fixed points is nowhere dense and homeomorphic with the space Z.* $\square$

Other than extremally disconnected compact spaces cannot occur as sets of fixed points in extremally disconnected compact spaces. This follows from the theorem of Vermeer mentioned in paragraph 1.

5. The main theorem

We state now the main theorem of this paper. It states that the construction of our topological situation from an exponent as in paragraph 4 is essentially the only way to obtain our topological situation:

Main theorem. *For any extremally disconnected compact Hausdorff space X and any open continuous mapping f on X into itself such that the set of fixed points is nowhere dense and not empty, there are a regular cardinal number κ, a complete Boolean algebra B, and a κ-complete uniform retract $r : B^{\kappa} \to B$ as well as a clopen invariant subspace $\bar{X}$ of X and an open onto-mapping $\sigma : \bar{X} \to Stone(Exp_{\kappa}(B,r))$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\quad f \quad} & \bar{X} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ Stone(Exp_{\kappa}(B,r)) & \xrightarrow{Stone(sh)} & Stone(Exp_{\kappa}(B,r)). \end{array}$$

The proof of the main theorem is contained in the following paragraphs. Since we can replace X by the clopen invariant set $Fix^{\#}$ (see paragraph 2), we will in the following proof suppose $X = Fix^{\#}$.

For an element a of the Boolean algebra C we can consider the Boolean algebra $C \restriction a$ of the elements of C below a with the natural Boolean operations. For a homomorphism $F : C \to C$ and an element $a \in C$ such that $F(a) \geq a$, we define the homorphism $F \restriction a : C \restriction a \to C \restriction a$ by $b \mapsto F(b) \wedge a$.

The main theorem has its dual formulation:

Main theorem (dual formulation). *For any complete Boolean algebra C and any complete homomorphism F on C into it self such that*

(i) For any $a \in C^{+}$ there is a nonzero $b \leq a$ such that $b \wedge F(b) = \mathbf{0}_C$

(ii) there is no partition of unity (a_0, a_1, a_2) such that $a_i \wedge F(a_i) = \mathbf{0}_C$ for all $i = 0, 1, 2$.

There are a regular cardinal number κ, a complete Boolean algebra B, and a κ-complete uniform retract $r : B^\kappa \to B$ as well as an element a of C such that $F(a) \geq a$ and a complete embedding $\varrho : Exp_\kappa(B,r) \to C \restriction a$ such that the following diagram is commutative:

$$\begin{array}{ccc} C \restriction a & \xrightarrow{F \restriction a} & C \restriction a \\ \varrho\uparrow & & \uparrow\varrho \\ Exp_\kappa(B,r) & \xrightarrow{sh} & Exp_\kappa(B,r). \end{array}$$

5.1. Fusion

The first step in the proof of the main theorem is the following fusion lemma. We choose the notion "fusion" since we aim to find "coordinates" describing clopen sets of our space by weighted trees, namely by the elements of the exponent. In this way, the property of being a fusion sequence in our sense will correspond to the classical one for trees.

Before stating the fusion lemma, we will collect some arguments for $W \cap Fix = \emptyset$. Recall that in paragraph 2 the exit of a clopen set was defined by $Ex(W) = W \backslash f^{-1}[W]$.

Lemma 5.1.1. *Let W be clopen and $Fix^\# = X$.*

(i) If $W \cap f[W] = \emptyset$ then $W \cap Fix = \emptyset$.
(ii) If $W \cap f[W] = \emptyset$ then $W^\# \cap Fix = \emptyset$.
(iii) $W \cap Fix = \emptyset$ iff $Ex(W)^\# \supseteq W$.
(iv) If $W \cap Fix = \emptyset$ then $Ex(W)^\# = W^\#$.
(v) If $W \cap Fix = \emptyset$ then $W^\# \cap Fix = \emptyset$.
(vi) $W \cap Fix = \emptyset$ iff there exists V such that $f[V] \cap V = \emptyset$ and $V^\# \supseteq W$.

Proof. (i) evident

(ii) Let $W_0 = W$ and $W_{n+1} = f^{-1}[W_n] \backslash W$. The W_n's are disjoint and $f[W_{n+1}] \subseteq W_n$. It follows $f[\bigcup_{i=0}^\infty W_{2i+1}] \cap \bigcup_{i=0}^\infty W_{2i+1} = \emptyset$ and $f[\bigcup_{i=1}^\infty W_{2i}] \cap \bigcup_{i=1}^\infty W_{2i} = \emptyset$ and $f[W_0] \cap W_0 = \emptyset$ by assumption. From the fact that X is extremally disconnected, f is open and from (i), we infer $cl \bigcup_{i=0}^\infty W_{2i+1} \cap Fix = \emptyset$ and $cl \bigcup_{i=1}^\infty W_{2i} \cap Fix = \emptyset$ and $W_0 \cap Fix = \emptyset$, hence $W^\# \cap Fix = cl \bigcup_{i=0}^\infty W_i \cap Fix = \emptyset$.

(iii) For the implication $(\to)$, note that $W \backslash Ex(W)^\#$ is invariant and open. If it would be non empty, the general assumption $Fix^\# = X$ would imply $W \cap Fix \neq \emptyset$, a contradiction.

For the opposite implication $(\leftarrow)$, use (ii) and replace W with $Ex(W)$.

(iv) immediately from (iii).

(v) follows from (ii) with $Ex(W)$ instead of W and (iv).

(vi) For $(\to)$ put $V = Ex(W)$ and use (iii), for $(\leftarrow)$ apply (ii). □

Definition. *For clopen subsets U and V of X, write $U \supseteq_n V$ if $U \supseteq V$ and $U \cap f^{-n}[Fix] = V \cap f^{-n}[Fix]$. A sequence $\langle U_n \rangle_{n<\omega}$ is called a fusion sequence if $U_0 \supseteq_0 U_1 \supseteq_1 \dots U_n \supseteq_n \dots$.*

Fusion lemma 5.1.2. *For any fusion sequence $\langle U_n \rangle_{n<\omega}$ it holds*

$$int_X \bigcap_{n<\omega} U_n \supseteq U_0 \cap Fix.$$

Proof. We can assume $U_0 \cap Fix \neq 0$. Since we can replace X by the clopen invariant set $(U_0 \cap Fix)^{\#}$, we can assume also without loss of generality $U_0 \supset Fix$, therefore $U_n \supset Fix$ for all $n < \omega$. Define $W_n = U_{2^n} \backslash U_{2^{n+1}}$. We will construct V_n for $n < \omega$ such that

(i) $$\bigcup_{j=0}^{n} V_j^{\#} \supseteq W_n$$

(ii) $$V_n \cap f[V_m] = \emptyset \text{ for all } m, n < \omega$$

(iii) $$f[V_n]^{\#} \cap f^{2^n+1}[V_n] = \emptyset.$$

If we find such V_n for $n < \omega$, we can put $V = \text{cl} \bigcup_{n<\omega} V_n$. From (ii) it follows that $\bigcup_{n<\omega} V_n \cap \bigcup_{n<\omega} f[V_n] = \emptyset$. Since X extremally disconnected and f is open, we get $\text{cl}(\bigcup_{n<\omega} V_n) \cap \text{cl}(\bigcup_{n<\omega} f[V_n]) = \emptyset$, hence $V \cap f[V] = \emptyset$. On the other hand $V^{\#} = (\text{cl} \bigcup_{n<\omega} V_n)^{\#} \supseteq \text{cl} \bigcup_{n<\omega} V_n^{\#} \supseteq \text{cl} \bigcup_{n<\omega} W_n$ by (i). Lemma 5.1.1. (ii) implies now $\text{cl} \bigcup_{n<\omega} W_n \cap Fix \subseteq V^{\#} \cap Fix = \emptyset$. Since $Fix \subset U_1$, we get $\text{int} \bigcap_{n<\omega} U_n = U_1 \backslash \text{cl} \bigcup_{n<\omega} W_n \supseteq Fix$ and the assertion of the fusion lemma follows.

We shall now construct the V_n's by induction. Define

$$V_n = Ex\Big(\bigcup_{i=1}^{2^n} f^{-i}[Ex(f^{2n}[W_n]^{\#})] \backslash \bigcup_{j=0}^{n-1} (V_j^{\#} \cup f[V_j])\Big).$$

We have to verify (i)-(iii). The main problem will cause (i): Define

$$Z = Ex(f^{2^n}[W_n]^{\#}).$$

At first we note that $U_{2^n} \supseteq_{2^n} U_{2^{n+1}}$, therefore $f^{2^n}[W_n] \cap Fix = \emptyset$, hence by lemma 5.1.1. (v) and (iv) $f^{2^n}[W_n]^{\#} = Ex(f^{2^n}[W_n]^{\#})^{\#} = Z^{\#}$. The evident fact $W_n \subseteq f^{2^n}[W_n]^{\#}$ implies now

$$W_n \subseteq Z^{\#}.$$

On the other hand, from the definition of Z it follows $f[Z] \cap Z^{\#} = \emptyset$, hence $f^{i+1} f^{-i}[Z] \cap f^{2^n}[W_n]^{\#} = \emptyset$ for any $i < 2^n$. Since $f^{2^n}[W_n]^{\#} \supseteq f^{i+1}[W_n]$, we obtain $f^{-i}[Z] \cap W_n = \emptyset$ for any $i < 2^n$. But $W_n \subseteq Z^{\#} = \bigcup_{k=0}^{i} f^{-k}[Z] \cup \text{cl} \bigcup_{k=i+1}^{\infty} f^{-k}[Z]$, hence $W_n \subseteq f^{-(i+1)}[Z]^{\#}$. We have therefore

$$W_n \subseteq f^{-i}[Z]^{\#} \text{ for any } i \leq 2^n.$$

We compute

$$\bigcup_{j=0}^{n} V_j^{\#} = V_n^{\#} \cup \bigcup_{j=0}^{n-1} V_j^{\#}$$

(a) $$= Ex\left(\bigcup_{i=1}^{2^n} f^{-i}[Z] \setminus \bigcup_{j=0}^{n-1} (V_j^{\#} \cup f[V_j])\right)^{\#} \cup \bigcup_{j=0}^{n-1} V_j^{\#}$$

(b) $$= \left(\bigcup_{i=1}^{2^n} f^{-i}[Z] \setminus \bigcup_{j=0}^{n-1} (V_j^{\#} \cup f[V_j])\right)^{\#} \cup \bigcup_{j=0}^{n-1} V_j^{\#}$$

(c) $$= \left(\bigcup_{i=1}^{2^n} f^{-i}[Z] \setminus \bigcup_{j=0}^{n-1} (V_j^{\#} \cup f[V_j]) \cup \bigcup_{j=0}^{n-1} V_j^{\#}\right)^{\#}$$

$$\supseteq \left(\bigcup_{i=1}^{2^n} f^{-i}[Z] \setminus \bigcup_{j=0}^{n-1} f[V_j]\right)^{\#}$$

$$\supseteq \left(f^{-i}[Z] \setminus \bigcup_{j=0}^{n-1} f[V_j]\right)^{\#} \quad \text{for any } i = 1 \ldots 2^n$$

Equality (a) follows from the definition of V_n and Z. In (b) we could drop the operator Ex by lemma 5.1.1. (iv) since its argument has an empty intersection with Fix, the latter holds because of $Z \cap Fix = \emptyset$. For (c) we used lemma 2.1. (i).

From this we get $W_n \setminus \bigcup_{j=\emptyset}^{n} V_j^{\#} \subseteq f^{-i}[Z]^{\#} \setminus (f^{-i}[Z] \setminus \bigcup_{j=0}^{n-1} f[V_j])^{\#} \subseteq$ $\subseteq (f^{-i}[Z] \cap \bigcup_{j=0}^{n-1} f[V_j])^{\#}$, the latter holds because of $(U \cup V)^{\#} \setminus V^{\#} \subseteq U^{\#}$ (lemma 2.1. (i)). If (i) does not hold, then $W_n \setminus \bigcup_{j=\emptyset}^{n} V_j^{\#} \neq \emptyset$ and moreover $\bigcup_{k=0}^{\infty} f^{-k}[f^{-i}[Z] \cap \bigcup_{j=0}^{n-1} f[V_j]]$ is open and dense in it for $i = 1 \ldots 2^n$. We find an $x \in W_n$ such that for all $i = 1 \ldots 2^n$ there is a $k_i < \omega$ and a $j_i = 0 \ldots n-1$ such that $f^{k_i}(x) \in f^{-i}[Z] \cap f[V_{j_i}]$. Let $H_j = \{i : j_i = j\}$ for $j = 0 \ldots n-1$. Since $\sum_{j=0}^{n-1} 2^j < 2^n$, there must be a j_0 such that $|H_{j_0}| > 2^{j_0}$. Note that $f^{-i}[Z] \cap$ $\cap f^{-i'}[Z] = \emptyset$ for $i \neq i'$ since $Z = Ex(\ldots^{\#})$. It follows $k_i \neq k_{i'}$. For $k' = \min\{k_i : i \in H_{j_0}\}$ and $k'' = \max\{k_i : i \in H_{j_0}\}$, we have therefore $k'' - k' \geq 2^{j_0}$. From $f^{k'}(x) \in f[V_{j_0}]$ we get

(*) $$f^{k'+2^{j_0}}(x) \in f^{2^{j_0}+1}[V_{j_0}]$$

and from $f^{k''}(x) \in f[V_{j_0}]$ and $k'' \geq k' + 2^{j_0}$ it follows

$$f^{k''-k'-2^{j_0}} f^{k'+2^{j_0}}(x) \in f[V_{j_0}],$$

hence

(**) $$f^{k'+2^{j_0}}(x) \in f[V_{j_0}]^{\#}.$$

From (*) and (**) we get now a contradiction with (iii) for $j_0 < n$. We proved (i).

For proving (ii) for $m = n$ it suffices to realize that $V_n = Ex(\ldots)$.

Condition (ii) for $m < n$ follows from

$$V_n \cap f[V_m] = Ex\left(\ldots \setminus \bigcup_{j=0}^{n-1} (\ldots \cup f[V_j])\right) \cap f[V_m] \subseteq (\ldots \setminus f[V_m]) \cap f[V_m] = \emptyset.$$

Let now $m > n$. From $V_m = Ex\left(...\setminus\bigcup_{j=0}^{m-1}\left(V_j^{\#} \cup ...\right)\right) \subseteq ...\setminus V_n^{\#}$ we infer $V_m \cap \cap V_n^{\#} = \emptyset$. This implies condition (ii) for $m > n$.

Finally condition (iii): By definition of V_n it holds $V_n \subseteq \bigcup_{i=1}^{2^n} f^{-1}[Z]$. This implies on one hand $f[V_n] \subseteq \bigcup_{i=0}^{2^n-1} f^{-1}[Z] \subseteq Z^{\#}$, hence

$$\text{(A)} \qquad f[V_n]^{\#} \subseteq Z^{\#}.$$

On the other hand, we get

$$\text{(B)} \qquad f^{2^n+1}[V_n] \subseteq \bigcup_{i=1}^{2^n} f^i[Z].$$

Since $Z = Ex(...^{\#})$, we have $Z^{\#} \cap f^i[Z] = \emptyset$ for any $i > 0$. From $Z^{\#} \cap \cap \bigcup_{i-1}^{2^n} f^i[Z] = \emptyset$ and (A), (B), it follows now (iii): $f[V_n]^{\#} \cap f^{2^n+1}[V_n] = \emptyset$. □

5.2. The coordinates

A space X and a mapping f as in the assumption of the main theorem are given. Our general assumption is $X = Fix^{\#}$. We continue the proof of the main theorem by looking for clopen subsets U_α of X which could represent the elements T_α of the (until now undefined) exponent, where

$$T_\alpha(s) = \begin{cases} \mathbf{1}_B & s \neq \emptyset \quad \& \quad min(s) = \alpha \\ \mathbf{0}_B & otherwise \end{cases}$$

Since by a suitable choice of the U_α's certain clopen subsets of X will be described by elements of the exponent, we will call those U_α's the coordinates.

Lemma 5.2.1. *Let Y be a clopen invariant subset of X.* For any clopen subset *W of X, which is disjoint from the set Fix of fixed points of f, there is a clopen subset $V = \mathbf{V}(W, Y)$ of Y such that*

$$\begin{aligned} &(i)\ V \cap Fix = \emptyset \\ &(ii)\ V^{\#} \cap Y = V \\ &(iii)\ V \supseteq W \cap Y \\ &(iv)\ V \cup f[V] = Y. \end{aligned}$$

Proof. Define

$$\mathbb{Z} = \{V \text{ clopen subset of Y} : C \text{ satisfies (i), (ii), (iii)}\}$$

and let $\mathbb{Z}$ be ordered by $V_1 \leq V_2$ iff $V_1 \subseteq V_2$ and $Ex(V_2) \cap f[Ex(V_1)] = \emptyset$. Let $\mathscr{V}$ be a chain in $\mathbb{Z}$ and $\bar{V} = cl\bigcup\mathscr{V}$. We show $\bar{V} \in \mathbb{Z}$. Lemma 2.1. (ii) implies (ii) for $\bar{V}$, (iii) is immediately clear. For proving (i) let $V_1 \leq V_2$. Then $Ex(V_2) \cap \cap f[Ex(V_1)] = \emptyset$ and also $Ex(V_1) \cap f[Ex(V_2)] = \emptyset$. Using the fact that X is extremally disconnected, we obtain $cl\bigcup_{V\in\mathscr{V}} Ex(V) \cap cl\bigcup_{V\in\mathscr{V}} f[Ex(V)] = \emptyset$, hence

$cl \bigcup_{V \in \mathcal{V}} Ex(V) \cap Fix = \emptyset$ (lemma 5.1.1. (i)), and therefore $(cl \bigcup_{V \in \mathcal{V}} Ex(V))^{\#} \cap$ $\cap\, Fix = \emptyset$ (lemma 5.1.1. (v)). But now it holds by lemma 2.1. (ii) and lemma 5.1.1. (iii) $(cl \bigcup_{V \in \mathcal{V}} Ex(V))^{\#} = cl \bigcup_{V \in \mathcal{V}} Ex(V)^{\#} \supseteq cl \bigcup_{V \in \mathcal{V}} V = \bar{V}$, and property (i) follows. By Zorn's lemma there is a maximal element $\bar{V}$, which will prove the lemma. We have to show (iv). Aiming for a contradicting, suppose that $\bar{V} \cup f[\bar{V}] \neq Y$ and take a nonempty clopen $W \subseteq Y \setminus (\bar{V} \cup f[\bar{V}] \cup Fix)$. The set $\bar{V} \cup W^{\#} \cap Y$ is an element of $\mathbb{Z}$ greater than $\bar{V}$ in contradiction to the maximality of $\bar{V}$. □

The preceding lemma makes the following inductive construction possible.

Lemma 5.2.2. *There is an ordinal number κ and a family $\{V_\alpha : \alpha < \kappa\}$ of clopen subsets of X such that*

$$(i)\ V_\alpha \cap Fix = \emptyset$$

$$(ii)\ V_\alpha^{\#} \setminus cl \bigcup_{\beta < \alpha} V_\beta = V_\alpha$$

$$(iii)\ cl \bigcup_{\beta < \alpha} V_\beta \cup V_\alpha \cup f[V_\alpha] = X$$

$$(vi)\ cl \bigcup_{\alpha < \kappa} V_\alpha \cap Fix \neq \emptyset.$$

Proof. As long as $cl \bigcup_{\beta < \alpha} V_\beta \cap Fix = \emptyset$, we define

$$V_\alpha = \mathbf{V}(W_\alpha, X \setminus cl \bigcup_{\beta < \alpha} V_\beta)$$

where W_α is an arbitrary clopen subset of X with $W_\alpha \cap Fix = \emptyset$.

Note that $Y = X \setminus cl \bigcup_{\beta < \alpha} V_B$ is invariant by the inductive assumption (ii) and by lemma 2.1. (ii), so the definition of V_α is correct. Properties (i)—(iii) follow from the corresponding properties (i), (ii), (iv) in lemma 5.2.1. There is a κ where the construction terminates, this κ satisfies (iv). □

Depending on the choice of the W_α's and the non unique operator $\mathbf{V}$, the construction can have different outcomes. We fix under all posible constructions one with a minimal κ.

Lemma 5.2.3. *If κ is the minimal possible, then it is equal to the minimal power of a family of clopen sets disjoint from Fix such that the closure of its union contains a fixed point.*

Proof. It is clear that κ is at least equal to that minimal power since each $\{V_\alpha\}_{\alpha < \kappa}$ is such a family. Let $\{W_\alpha\}_{\alpha < \lambda}$ now be such a family and use W_α as a parameter of the operator $\mathbf{V}$ in the construction of V_α. This witnesses the equality. □

Since a minimal power as in lemma 5.2.3. has to be a regular cardinal, we get as an immediate consequence the following lemma:

Lemma 5.2.4. *If κ is the minimal possible, then it is a regular cardinal number.* $\square$

As it will follow from the main theorem and from lemma 3.1.8., such a κ must be in fact ω or even inaccessible.

We will now improve the properties of the family $\{V_\alpha : \alpha < \kappa\}$.

Lemma 5.2.5. *There is a regular cardinal number κ, a clopen invariant subset X' of X, and a family $\{V_\alpha' : \alpha < \kappa\}$ of clopen subsets of X' such that*

$$(i)\ V_\alpha' \cap Fix = \emptyset$$

$$(ii)\ V_\alpha'^{\#} \backslash cl \bigcup_{\beta<\alpha} V_\beta' = V_\alpha'$$

$$(iii)\ cl \bigcup_{\beta<\alpha} V_\beta' \cup V_\alpha' \cup f[V_\alpha'] = X'$$

$$(iv)\ cl \bigcup_{\alpha<\kappa} V_\alpha' = X'$$

$$(v)\ X' = (X' \cap Fix)^{\#}.$$

Proof. We start with a family $\{V_\alpha : \alpha < \kappa\}$ guarantied by lemma 5.2.2. We can assume that κ is a regular cardinal number (lemma 5.2.4.). Put $\overline{Fix} = cl \bigcup_{\alpha<\kappa} V_\alpha \cap \cap Fix$. Define $X' = \overline{Fix}^{\#}$ and $V_\alpha' = V_\alpha \cap X'$. We show that X' and $\{V_\alpha' : \alpha < \kappa\}$ have the demanded properties. The set X' is clopen by lemma 2.3. (iii) and invariant by lemma 2.1. (iv). The same holds for its complement. The properties of V_α' are easily deduced from the corresponding properties of V_α. Property (iv) follows from lemma 2.1. (ii) and $cl \bigcup_{\alpha>\kappa} V_\alpha' = (cl \bigcup_{\alpha<\kappa} V_\alpha)^{\#} \supseteq \overline{Fix}^{\#} = X'$. $\square$

We will improve lemma 5.2.5. further:

Lemma 5.2.6. *There is a regular cardinal number κ, a clopen invariant subset $\bar{X}$ of X, and a family $\{U_\alpha : \alpha < \kappa\}$ of clopen subsets of $\bar{X}$ such that*

$$(i)\ U_\alpha \cap f[U_\alpha] = \emptyset$$

$$(ii)\ U_\alpha^{\#} \backslash cl \bigcup_{\beta<\alpha} U_\beta = U_\alpha$$

$$(iii)\ cl \bigcup_{\beta<\alpha} U_\beta \cup U_\alpha \cup f[U_\alpha] = \bar{X}$$

$$(iv)\ cl \bigcup_{\alpha<\kappa} U_\alpha = \bar{X}$$

$$(v)\ \bar{X} \subseteq (\bar{X} \cap Fix)^{\#}.$$

Proof. We begin with a family $\{V_\alpha' : \alpha < \kappa\}$ and a subset X' of X as obtained in lemma 5.2.5. For any $\alpha < \kappa$ define $V_\alpha'' = V_\alpha' \backslash Ex(V_\alpha')$. We have $f[Ex(V_\alpha'')] \subseteq Ex(V_\alpha')$ and therefore $f[cl \bigcup_{\alpha<\kappa} Exp(V_\alpha'')] \subseteq cl \bigcup_{\alpha<\kappa} Ex(V_\alpha')$. It holds $cl \bigcup_{\alpha<\kappa} Ex(V_\alpha'') \cap$

$\cap\, cl \bigcup_{\alpha<\kappa} Ex(V'_\alpha) = \emptyset$ since X is extremally disconnected, hence $(cl \bigcup_{\alpha<\kappa} Ex(V''_\alpha))^{\#} \cap$ $\cap\, Fix = \emptyset$ (lemma 5.1.1. (i) and (v)). Then $\bar{X} = X \setminus (cl \bigcup_{\alpha<\kappa} Ex(V''_\alpha))^{\#}$ is an invariant clopen set. Put $U_\alpha = V'_\alpha \cap \bar{X}$. Lemma 5.1.1. (iii) and $V''_\alpha \cap Fix = \emptyset$ imply $Ex(V''_\alpha)^{\#} \supseteq V''_\alpha$, hence $U_\alpha \subseteq Ex(V'_\alpha)$ and therefore $U_\alpha \cap f[U_\alpha] = \emptyset$ (i). The other properties are implied by the corresponding properties in the preceding lemma. $\square$

For a simpler notation, we will assume in the following that the operator $\#$ is taken within the invariant subspace $\bar{X}$, i.e. for $Z \subseteq \bar{X}$, by $Z^{\#}$ we denote in fact the set $Z^{\#} \cap \bar{X}$. Also, Fix will be in the following $Fix_f(\bar{X})$. Condition (v) of lemma 5.2.6. gets therefore the form $\bar{X} = Fix^{\#}$.

5.3. Property Δ

The construction of the coordinates as done until now may not be the best. By a more careful construction we will try to improve the coordinates.

First we realize that for any fixed point x it holds $x \in \mathrm{cl}(f^{-1}(x) \setminus \{x\})$. In the opposite case, we could find (by Frolík's theorem 2.2) an invariant clopen neighborhood U of x such that $U \cap f^{-1}(x) \setminus \{x\} = \emptyset$, a contradiction to lemma 5.2.1 applied on U. From our observation it follows immediately that $Fix \subseteq \mathrm{cl}(f^{-1}[Fix] \setminus \{Fix\})$. We could now ask whether this holds also in the slightly stronger form

$$(\Delta) \qquad Fix \subseteq \mathrm{cl}\Big(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha\Big).$$

The answer is in general no. But we can achieve this by repeating the construction of the coordinates and by employing the parameter W of the operator $\mathbf{V}$.

We begin with a lemma.

Lemma 5.3.1. *The set $cl(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha) \cap Fix$ is open in Fix.*

Proof. Let O be a clopen subset of Fix. We say that O refuses Δ everywhere if $\mathrm{cl}(f^{-1}[O] \cap \bigcup_{\alpha<\kappa} U_\alpha) \cap O = \emptyset$. Let $\mathcal{O}$ be an antichain in $Clop(Fix)$ of sets O, which refuse Δ everywhere, and let $\mathcal{O}$ be maximal with this respect. Since every $O \in \mathcal{O}$ refuses Δ everywhere, we find $U(O)$ clopen in $\bar{X}$ such that $U(O) \supset$ $\supset f^{-1}[O] \cap \bigcup_{\alpha<\kappa} U_\alpha$ and $U(O) \cap O = \emptyset$. Since $O^{\#}$ is clopen by lemma 2.3. (iii), we can suppose that $U(O) \subseteq O^{\#}$. Put $\bar{U} = \mathrm{cl} \bigcup \{U(O) : O \in \mathcal{O}\}$ and $\bar{O} = \mathrm{cl} \bigcup \mathcal{O}$. We claim that $\bar{U}$ witnesses that $\bar{O}$ refuses Δ everywhere:

$$\bar{U} \supseteq cl \bigcup_{O \in \mathcal{O}} \Big(f^{-1}[O] \cap \bigcup_{\alpha<\kappa} U_\alpha\Big)$$

$$\supseteq cl \bigcup_{O \in \mathcal{O}} f^{-1}[O] \cap \bigcup_{\alpha<\kappa} U_\alpha$$

$$= f^{-1}\left[cl \bigcup_{O\in\mathcal{O}} O\right] \cap \bigcup_{\alpha<\kappa} U_\alpha$$
$$= f^{-1}[\bar{O}] \cap \bigcup_{\alpha<\kappa} U_\alpha.$$

On the other hand, $\bar{U} \cap O = \bar{U} \cap O^{\#} \cap O = U(O) \cap O = \emptyset$, hence $\bar{U} \cap \bar{O} = \bar{U} \cap cl \bigcup_{O\in\mathcal{O}} O = cl \bigcup_{O\in\mathcal{O}} (\bar{U} \cap O) = \emptyset$.

The assertion of the lemma will follow from the fact that $\mathrm{cl}(f^{-1}[Fix] \cap \bigcup_{\alpha>\kappa} U_\alpha) \cap Fix = Fix \backslash \bar{O}$. Note that $\bar{O}^{\#}\backslash\bar{U}$ is a neighborhood of $\bar{O}$ disjoint from $f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha$. So, the inclusion $(\subseteq)$ is expressed by the fact that $\bar{O}$ refuses Δ everywhere, as shown above. For the other inclusion $(\supseteq)$, let $x \in (Fix\backslash\bar{O})\backslash \mathrm{cl}(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha)$. Then there is a clopen neighborhood O of x in Fix such that $O \cap \bar{O} = \emptyset$ and $O \cap cl(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha) = \emptyset$. But this means that O refuses Δ everywhere, a contradiction with the maximality of $\mathcal{O}$. □

We ask now, whether there exists an open neighborhood U of the set Fix and a number $n < \omega$, such that (1) the set $A = \mathrm{cl}(f^{-(n+1)}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha) \cap f^{-n}[Fix] \cap U$ is not empty and (2) $f^{-n}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha \cap U = \emptyset$.

Case 1: No, such U and n do not exist.

We will derive a contradiction for this case. We construct by induction a fusion sequence $\langle W_n \rangle_{n<\omega}$ such that $f^{-n}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha \cap W_n = \emptyset$. Begin with $W_0 = \bar{X}$ and suppose, that the W_i's were found up to n. The inductive assumption implies that (2) is fulfilled for $U = W_n$ and n. Hence (1) does not hold:

$$\mathrm{cl}\left(f^{-(n+1)}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha\right) \cap f^{-n}[Fix] \cap W_n = \emptyset.$$

We find therefore a clopen neighborhood $W_{n+1} \subseteq W_n$ of $f^{-n}[Fix] \cap W_n$ such that

$$f^{-(n+1)}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha \cap W_{n+1} = \emptyset.$$

From $W_{n+1} \supseteq f^{-n}[Fix] \cap W_n$ it follows now that $W_{n+1} \subseteq_n W_n$. We constructed a fusion sequence $\langle W_n \rangle_{n<\omega}$ such that $f^{-n}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha \cap W_n = \emptyset$. It follows

$$\bigcup_{n<\omega} f^{-n}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha \cap \bigcap_{n<\omega} W_n = \emptyset.$$

But this is impossible since $\bigcup_{n<\omega} f^{-n}[Fix]$ is dense in $\bar{X}$ by the assumption $\bar{X} = Fix^{\#}$ (lemma 5.2.6 (v) – see the remark at the end of paragraph 5.2), $\bigcup_{\alpha<\kappa} U_\alpha$ is open dense in $\bar{X}$ (lemma 5.2.6. (iv)), and $\bigcap_{n<\omega} W_n$ has nonempty interior in $\bar{X}$ because of the fusion lemma 5.1.2.

Case 2: Yes, such U and n exist.

Our assumption (2) implies $f^n[U_\alpha \cap U] \cap Fix = \emptyset$. We can therefore repeat our construction of the coordinates with putting in $W_\alpha = f^n[U_\alpha \cap U]$. As the result we get U'_α. The κ remains unchanged: Since the original κ was minimal under all possible coordinates, we have only to show, that the construction terminates at the latest at κ. This follows from $\mathrm{cl}\left(\bigcup_{\alpha<\kappa} U'_\alpha\right) \cap Fix \supseteq \mathrm{cl}\left(\bigcup_{\alpha<\kappa} W_\alpha\right) \cap \cap Fix = \mathrm{cl}\left(\bigcup_{\alpha<\kappa} f^n[U_\alpha \cap U]\right) \cap Fix = f^n\left[cl\left(\bigcup_{\alpha<\kappa} U_\alpha\right) \cap U\right] \cap Fix = = f^n[U] \cap Fix = Fix$.

It turns out that Δ holds in these new coordinates U'_α: From our assumption

$$\emptyset \neq A = \mathrm{cl}\left(f^{-(n+1)}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha\right) \cap f^{-n}[Fix] \cap U$$

we obtain

$$\begin{aligned}\emptyset \neq f^n[A] &= f^n\left[\mathrm{cl}\left(f^{-(n+1)}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha \cap U\right)\right] \cap Fix \\ &= \mathrm{cl}\left(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} f^n[U_\alpha \cap U]\right) \cap Fix \\ &\subseteq \mathrm{cl}\left(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U'_\alpha\right) \cap Fix.\end{aligned}$$

For $O = \mathrm{cl}\left(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U'_\alpha\right) \cap Fix$ we proved $O \neq \emptyset$. Lemma 5.3.1. implies that O is clopen. By replacing $\bar{X}$ with the invariant clopen subset $O^{\#}$ (lemma 2.1. (iv) and 2.3. (iii)) and U_α with $U'_\alpha \cap O^{\#}$, we see that Δ holds with respect to the new coordinates. We proved the following lemma:

Lemma 5.3.2. *There are an invariant clopen subset $\bar{X}$ of X and coordinates $\{U_\alpha : \alpha < \kappa\}$ in it such that (Δ) and properties (i)–(v) of lemma 5.2.6. hold.* □

5.4. Proof of the main theorem

We conclude the proof of the main theorem. Let $\{U_\alpha : \alpha < \kappa\}$ be as obtained in lemma 5.3.2. Define $U_s = \bigcap_{i=0}^{n-1} f^{-i}[U_{\alpha_i}]$ and $Fix_s = f^{-|s|}[Fix] \cap U_s$ for $s = \{\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1}\} \in [\kappa]^{<\omega}$. The set Fix_s consists of those points of $\bar{X}$, which are moved by f only through the U_{α_i}'s and end up in a fixed point. Here $U_\emptyset = \bar{X}$ and $Fix_\emptyset = Fix$.

Lemma 5.4.1. *If $f^{-n}[Fix] \cap U \neq \emptyset$ for some clopen $U \subseteq \bar{X}$ then there is an s with $|s| = n + 1$ and $Fix_s \cap U \neq \emptyset$.*

Proof. Assume in the contrary that for any $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \kappa$ it holds

$$f^{-(n+1)}[Fix] \cap \bigcap_{i=0}^{n} f^{-i}[U_{\alpha_i}] \cap U = \emptyset$$

hence

$$Fix \cap f^{n+1}\left[\bigcap_{i=0}^{n} f^{-i}[U_{\alpha_i}] \cap U\right] = \emptyset.$$

Fix $\alpha_1 \dots \alpha_n$. By lemma 5.2.3., *Fix* has an empty intersection with the closure of the union of any family of clopen subsets disjoint from *Fix*, provided this family has power less than κ. We conclude that

$$Fix \cap \mathrm{cl} \bigcup_{\alpha_0 < \alpha_1} f^{n+1}\left[\bigcap_{i=0}^{n} f^{-i}[U_{\alpha_i}] \cap U\right] = \emptyset,$$

hence

$$Fix \cap f^{n+1}\left[\mathrm{cl} \bigcup_{\alpha_0 < \alpha_1} U_{\alpha_0} \cap f^{-1}[U_{\alpha_1}] \cap \bigcap_{i=2}^{n} f^{-i}[U_{\alpha_i}] \cap U\right] = \emptyset.$$

But by the construction of the U_α's, it holds $\mathrm{cl} \bigcup_{\alpha_0 < \alpha_1} U_{\alpha_0} \supseteq f^{-1}[U_{\alpha_1}]$. We get

$$Fix \cap f^{n+1}\left[\bigcap_{i=1}^{n} f^{-1}[U_{\alpha_i}] \cap U\right] = \emptyset.$$

Repeating this procedure n times using

$$cl\, f^{-i}\left[\bigcup \{U_{\alpha_i} : i \le \alpha_i < \alpha_{i+1}\}\right] \supseteq f^{-(i+1)}[U_{\alpha_{i+1}}],$$

we obtain

$$Fix \cap f^{n+1}[f^{-n}[U_{\alpha_n}] \cap U] = \emptyset$$

for $n \le \alpha_n < \kappa$, hence

$$f^{-1}[Fix] \cap \bigcup_{\alpha_n \ge n} U_{\alpha_n} \cap f^{n}[U] = \emptyset.$$

But (Δ) implies now $f^n[U] \cap Fix = \emptyset$, a contradiction with the assumption of the lemma. $\square$

The next lemma encloses the role of the sets Fix_s.

Lemma 5.4.2. *The set* $\bigcup_{s \in [\kappa]^{<\omega}} Fix_s$ *is dense in* $\bar{X}$.

Proof. Let U be a clopen subset of $\bar{X}$. Since $Fix^{\#} = \bar{X}$, we find an n such that $f^{-n}[Fix] \cap U \neq \emptyset$. By the preceding lemma, there is an s with $|s| = n + 1$ and $Fix_s \cap U \neq \emptyset$. $\square$

For further use, we describe in the next lemma the impact of the property Δ on the sets Fix_s.

Lemma 5.4.3. *Let W be a clopen subset of $\bar{X}$. It holds*

$$W \cap Fix_s = cl \bigcup_{\alpha < \kappa} (W \cap Fix_{s \cup \{\alpha\}}) \cap Fix_s.$$

Proof. Property (Δ) implies $Fix = cl \bigcup_{\alpha<\kappa} (f^{-1}[Fix] \cap U_\alpha) \cap Fix$, therefore

$$Fix_s = f^{-|s|}[Fix] \cap U_s$$

$$= cl \bigcup_{\alpha<\kappa} (f^{-(|s|+1)}[Fix] \cap f^{-|s|}[U_\alpha] \cap U_s) \cap f^{-|s|}[Fix] \cap U_s$$

$$= cl \bigcup_{\alpha<\kappa} (f^{-(|s|+1)}[Fix] \cap U_{s\cup\{\alpha\}}) \cap Fix_s$$

$$= cl \bigcup_{\alpha<\kappa} Fix_{s\cup\{\alpha\}} \cap Fix_s$$

and the assertion of the lemma follows. □

Although we used the assumption of the main theorem in its topological form, the conclusion will be proved in the Boolean version. We are looking for a complete subalgebra of $Clop(\bar{X})$ isomorphic to some exponent. At first we have to give the parameters of this exponent. The regular cardinal will be the κ obtained in the construction of the coordinates in paragraph 5.2., the complete Boolean algebra will be $Clop(Fix)$. (This is in fact $Clop(Fix_f(\bar{X}))$ – see the remark at the end of paragraph 5.2.) The retract $r: Clop(Fix)^\kappa \to Clop(Fix)$ is defined in the following way:

$$r(\langle O_\alpha\rangle_{\alpha<\kappa}) = cl \bigcup_{\alpha<\kappa} (f^{-1}[O_\alpha] \cap U_\alpha) \cap Fix.$$

Lemma 5.4.4. *The uniform κ-complete retract r is defined correctly.*

Proof. Note that

$$r(\langle O_\alpha\rangle_{\alpha<\kappa}) \subseteq cl \bigcup_{\alpha<\kappa} (f^{-1}[O_\alpha^\#] \cap U_\alpha)$$

and

$$r(\langle Fix\backslash O_\alpha\rangle_{\alpha<\kappa}) \subseteq cl \bigcup_{\alpha<\kappa} (f^{-1}[(Fix\backslash O_\alpha)^\#] \cap U_\alpha).$$

Since O_α and $Fix\backslash O_\alpha$ are disjoint, so are the open sets $O_\alpha^\#$ and $(Fix\backslash O_\alpha)^\#$ (lemma 2.3.). We get $r(\langle O_\alpha\rangle_{\alpha<\kappa}) \cap r(\langle Fix\backslash O_\alpha\rangle_{\alpha<\kappa}) = \emptyset$ since $\bar{X}$ is extremally disconnected. On the other hand

$$\begin{aligned} r(\langle O_\alpha\rangle_{\alpha<\kappa}) \cup r(\langle Fix\backslash O_\alpha\rangle_{\alpha<\kappa}) &= \Big(cl \bigcup_{\alpha<\kappa} (f^{-1}[O_\alpha] \cap U_\alpha) \cap Fix\Big) \cup \\ &\quad \cup \Big(cl \bigcup_{\alpha<\kappa} (f^{-1}[Fix\backslash O_\alpha] \cap U_\alpha) \cap Fix\Big) \\ &= cl\Big(f^{-1}[Fix] \cap \bigcup_{\alpha<\kappa} U_\alpha\Big) \cap Fix \\ &= Fix, \end{aligned}$$

the latter follows from (Δ). Therefore $r(\langle O_\alpha\rangle_{\alpha<\kappa}) \in Clop(Fix)$ and r respects complements. As a byproduct of our considerations we get the following fact.

Fact: $r(\langle O_\alpha\rangle_{\alpha<\kappa}) = cl \bigcup_{\alpha<\kappa} (f^{-1}[O_\alpha^\#] \cap U_\alpha) \cap Fix$

Now it follows from this fact and from lemma 2.1. (ii) and lemma 5.2.3. that r is a κ-complete homomorphism. Let $\tau < \kappa$:

$$\begin{aligned} r\Big(\bigvee_{\delta<\tau} \langle O_\alpha^\delta\rangle_{\alpha<\kappa}\Big) &= cl \bigcup_{\alpha<\kappa} \Big(f^{-1}\Big[\Big(cl \bigcup_{\delta<\tau} O_\alpha^\delta\Big)^{\#}\Big] \cap U_\alpha\Big) \cap Fix \\ &= cl \bigcup_{\delta<\tau} cl \bigcup_{\alpha<\kappa} (f^{-1}[(O_\alpha^\delta)^{\#}] \cap U_\alpha) \cap Fix \\ &= cl \bigcup_{\delta<\tau} \Big(cl \bigcup_{\alpha<\kappa} (f^{-1}[(O_\alpha^\delta)^{\#}] \cap U_\alpha) \cap Fix\Big) \\ &= cl \bigcup_{\delta<\tau} r(\langle O_\alpha^\delta\rangle_{\alpha<\kappa}) \\ &= \bigvee_{\delta<\tau} r(\langle O_\alpha^\delta\rangle_{\alpha<\kappa}). \end{aligned}$$

The fact and lemma 5.2.3. imply also that r is uniform: If $|\{\alpha < \kappa : O_\alpha \neq \emptyset\}| < \kappa$, then $r(\langle O_\alpha\rangle_{\alpha<\kappa}) = cl \bigcup_{\alpha<\kappa}(f^{-1}[O_\alpha^{\#}] \cap U_\alpha) \cap Fix = \emptyset$ since only less than κ elements of the union are nonempty.

Let now $O_\alpha = O$ for $\alpha < \kappa$, we show that $r(\langle O_\alpha\rangle_{\alpha<\kappa}) = O$. The inclusion $\subseteq$ follows from $f^{-1}[O] \cap U_\alpha \subseteq O^{\#}$ and $O^{\#} \cap Fix = O$, and the inclusion $\supseteq$ from (Δ). This means that r is a retract. □

We proved that the mapping $r : Clop(Fix)^\kappa \to Clop(Fix)$ has the desired properties, we can consider $Exp_\kappa(Clop(Fix), r)$. We are looking for a complete embedding $\varrho : Exp_\kappa(Clop(Fix), r) \to Clop(\bar{X})$.

Definition of the embedding $\varrho : Exp_\kappa(Clop(Fix), r) \to Clop(\bar{X})$.

$$\varrho(T) = cl \bigcup_{s\in[\kappa]^{<\omega}} (f^{-|s|}[T(s)] \cap U_s)$$

We have to show that ϱ is defined correctly and is as claimed in the main theorem.

Lemma 5.4.5. *For any $T \in Exp_\kappa(Clop(Fix), r)$, it holds $\varrho(T) \in Clop(\bar{X})$.*

Proof. We will prove

$$(*) \qquad \varrho(T) \cap Fix_s = f^{-|s|}[T(s)] \cap U_s$$

for all $T \in Exp_\kappa(Clop(Fix), r)$. The closed sets $\varrho(T)$ and $\varrho(-T)$ are therefore complements on the dense set $\bigcup_{s\in[\kappa]^{<\omega}} Fix_s$. Since $\bar{X}$ is extremally disconnected, they are complements in the whole space. This proves the lemma.

We prove (*) for more and more general cases:

1. Let T be constant, i.e. $T(s) = O$ for some $O \in Clop(Fix)$ and for all $s \in [\kappa]^{<\omega}$. Then $\varrho(T) = O^{\#}$. The inclusion $\subseteq$ follows directly from the definition, the inclusion $\supseteq$ follows from lemma 5.4.2. by considering the open set $U = O^{\#}\backslash\varrho(T)$. The equality implies the assertion (*) for this special case.

2. Let T have a stem s and be constant equal to some O over the stem, i.e.

$$T(t) = \begin{cases} O & \text{if } t \sqsupseteq s \\ \mathbf{0}_B & \text{otherwise.} \end{cases}$$

By the same arguments as above it holds $\varrho(T) = f^{-|s|}[O^{\#}] \cap U_s$ and the assertion (*) for this case follows.

3. Let T be invariant (see 3.2 for the definition). For $n < \omega$ define

$$T_n(t) = \begin{cases} T(t \upharpoonright n) & \text{if } |t| \geq n \\ T(t) & \text{if } |t| < n, \end{cases}$$

where $t \upharpoonright n = s$ if $s \sqsubseteq t$ and $|s| = n$. Then $T_n \in Exp_\kappa(Clop(Fix), r)$. Define the clopen set $W = cl(\bigcup_{|s|=n} f^{-|s|}[T(s)^{\#}] \cap U_s)$. From the preceding case it is clear that $\varrho(T_n) \cap U_s = W \cap U_s$ for all s with $|s| = n$ and therefore $W \subseteq \varrho(T_n)$ and $W \cap Fix_s = f^{-|s|}[T_n(s)] \cap U_s$ for all s with $|s| \geq n$. Take an s with $|s| = n - 1$. We infer from lemma 5.4.3.:

$$\begin{aligned} W \cap Fix_s &= cl \bigcup_{\alpha<\kappa} (W \cap Fix_{s\cup\{\alpha\}}) \cap Fix_s \\ &= cl \bigcup_{\alpha<\kappa} (f^{-(|s|+1)}[T_n(s \cup \{\alpha\})] \cap U_{s\cup\{\alpha\}}) \cap f^{-|s|}[Fix] \cap U_s \\ &= cl \bigcup_{\alpha<\kappa} (f^{-|s|}[f^{-1}[T_n(s \cup \{\alpha\})] \cap U_\alpha] \cap U_s) \cap f^{-|s|}[Fix] \cap U_s \\ &= f^{-|s|}\left[cl \bigcup_{\alpha<\kappa} (f^{-1}[T_n(s \cup \{\alpha\})] \cap U_\alpha) \cap Fix\right] \cap U_s \\ &= f^{-|s|}[r(\langle T_n(s \cup \{\alpha\})\rangle_{\alpha<\kappa}\}] \cap U_s \\ &= f^{-|s|}[T_n(s)] \cap U_s. \end{aligned}$$

By induction we get $W \cap Fix_s = f^{-|s|}[T_n(s)] \cap U_s$ for all s. This implies immediately $\varrho(T_n) = W$. Fact (*) holds therefore also for T_n.

Since T is invariant, we have $T_n \geq T$. It follows $\varrho(T_n) \supseteq \varrho(T)$. Also $T_n(s) = T(s)$ for $n \geq |s|$ by the definition of T_n and $\varrho(T_n) \cap Fix_s = f^{-|s|}[T_n(s)] \cap U_s$ as shown above. Therefore $\varrho(T) \cap Fix_s \subseteq \bigcap_{n<\omega} \varrho(T_n) \cap Fix_s = f^{-|s|}[T(s)] \cap U_s \subseteq \varrho(T) \cap Fix_s$. We obtained (*) for the case T invariant.

4. Let S be invariant and $T = [s, S]$ standard (see 3.2 for the definition). Fact (*) follows from case 3. and the equation $\varrho(T) = f^{-|s|}[\varrho(S)] \cap U_s$.

5. Let T be arbitrary. By lemma 3.2.3., the standard elements are Prikry dense in the exponent. Hence

$$\begin{aligned} \varrho(T) &\supseteq cl \bigcup \{\varrho(T_1) : T_1 \text{ is standard } \& \ T_1 \leq T\} \supseteq \\ &\supseteq cl \bigcup_{s\in[\kappa]^{<\omega}} (f^{-|s|}[T(s)] \cap U_s = \varrho(T) \end{aligned}$$

and (*) follows. □

We remark that it can be shown that the elements in the range of ϱ are exactly those clopen subsets W of $\bar{X}$ such that for all $s \in [\kappa]^{<\omega}$ and all $x, y \in Fix_s$ with $f^{|s|}(x) = f^{|s|}(y)$ it holds $x \in W$ iff $y \in W$. This implies for the case that $f \restriction U_\alpha$ are homeomorphisms that the mapping σ in the main theorem will be a homeomorphism.

Lemma 5.4.5. proves that the mapping ϱ was defined correctly. We have to show that it is as demanded by the assertion of the main theorem.

Lemma 5.4.6. *The mapping ϱ is a complete embedding.*

Proof. It follows from lemma 5.4.5. and (*) in its proof that ϱ preserves complements. The fact that ϱ preserves finite unions is computed directly from the definition of the mapping. The mapping ϱ is therefore a homomorphism. It is clear that ϱ has a trivial kernel, hence ϱ is an embedding. The completeness of the mapping ϱ will follow from its regularity. A criterion for the regularity is the following. For any $W \in Clop(\bar{X})$ there is a $T \in Exp^+$ such that there is no $T' \in Exp^+$ with $\varrho(T') \subseteq \varrho(T) \setminus W$. We want to apply lemma 3.1.1. to get T as a minimal upper bound in the exponent. Consider $b_s = f^{|s|}[W \cap Fix_s]$ for $s \in [\kappa]^{<\omega}$. We have to prove the assumption of the lemma. Compute using lemma 5.4.3. and 5.2.3.:

$$\begin{aligned} f^{|s|}[W \cap Fix_s] &= f^{|s|}\Big[cl \bigcup_{\alpha<\kappa} (W \cap Fix_{s\cup\{\alpha\}}) \cap Fix_s\Big] \\ &\subseteq cl \bigcup_{\alpha<\kappa} f^{|s|}[W \cap Fix_{s\cup\{\alpha\}}] \cap Fix \\ &\subseteq cl \bigcup_{\alpha<\kappa} (f^{-1} f^{|s|+1}[W \cap Fix_{s\cup\{\alpha\}}] \cap U_\alpha) \cap Fix \\ &= r(\langle f^{|s|+1}[W \cap Fix_{s\cup\{\alpha\}}]\rangle_{\alpha<\kappa}). \end{aligned}$$

It is now clear from (*) of the proof of lemma 5.4.5. that the obtained T proves the regularity from the criterion. □

At last we have to show that the embedding ϱ commutes with the shift operation.

Lemma 5.4.7. *The following diagram is commutative:*

$$\begin{array}{ccc} Clop(\bar{X}) & \xrightarrow{F} & Clop(\bar{X}) \\ \varrho\uparrow & & \uparrow\varrho \\ Exp_\kappa(B, r) & \xrightarrow{sh} & Exp_\kappa(B, r). \end{array}$$

where $F : Clop(\bar{X}) \to Clop(\bar{X})$ is the Boolean dual of the mapping f, i.e. $F(U) = f^{-1}[U] \cap \bar{X}$.

Proof. We compute for $\alpha < \kappa$:

$$\begin{aligned}
\varrho(sh(T)) \cap U_\alpha &= cl \bigcup_{s \in [\kappa]^{<\omega}} (f^{-|s|}[T(shift(s))] \cap U_s) \cap U_\alpha \\
&= cl \bigcup_{\min(t) > \alpha} (f^{-(|t|+1)}[T(t)] \cap U_{\{\alpha\} \cup t}) \\
&= cl \bigcup_{\min(t) > \alpha} (f^{-(|t|+1)}[T(t)] \cap f^{-|t|}[U_t] \cap U_\alpha)] \\
&= f^{-1}\left[cl \bigcup_{t \in [\kappa]^{<\omega}} (f^{-|t|}[T(t)] \cap U_t)\right] \cap U_\alpha \\
&= F(\varrho(T)) \cap U_\alpha.
\end{aligned}$$

Since the clopen sets $F(\varrho(T))$ and $\varrho(sh\, T))$ coincide on the open dense set $\bigcup_{\alpha < \kappa} U_\alpha$, they are equal. □

The mapping ϱ is as asserted. This proves the main theorem.

6. Conclusion

The notion of the exponent should be of interest by its own. It has (with some reasonable restrictions) all nice properties of Mathias forcing and similar forcing notions, yet it needs no selectivity and is of big variability. But our main interest is in applications of the main theorem. For this aim we should look at this theorem as follows: An extremally disconnected compact space admitting an open mapping into itself, which is nowhere trivial (the identity on a clopen set) but has fixed points, is of a very special form (a clopen subset can be mapped by an open mapping onto the Stone space of some exponent). Or dually: A complete Boolean algebra admitting a complete homomorphism into itself, which is nowhere trivial (the identity in the restriction to some element) and for which there does not exist a finite partition of unity, such that each element of this partition is disjoint with its image, is of a very special form (under some element it contains some exponent as a complete subalgebra). Especially, we should regard homogeneous complete Boolean algebras, where homogeneity guarantees the existence of suitable complete homomorphisms.

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