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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 46 (2005), No. 1, 65--75

Persistent URL: http://dml.cz/dmlcz/142745

Commutative Semigroup with Few Fully Invariant Congruences II.

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Praha

Received 24. August 2004

Simple objects in the class of semimodules over a semigroup are studied.

This paper is an immediate continuation of [1] and kind reader is referred to [1] as for terminology, notation, various prerequisities, further references, etc.

10. Examples of simple idempotent semimodules

Example 10.1. Let $M(+)$ be a non-trivial semilattice (i.e., a commutative idempotent semigroup). Then M is a simple S–ip-semimodule, where $S = End(M(+))$. (This follows from the following observation: For all a, b, c, $d \in M$ such that $a \neq b$, $c \neq d$ and $c + d = d$, there exists an endomorphism φ of $M(+)$ such that $\varphi(M) = \{c, d\}$ and $\varphi(a) \neq \varphi(b)$.)

Example 10.2. Let R be a subsemigroup of a semigroup S and put $\mathscr{I}^o = \mathscr{I}_l^o(R, S)$ (see [1]). Then $\mathscr{I}^o$ is a non-trivial semilattice with respect to the operation of intersection, the empty set $\emptyset = o$ is the absorbing element of $\mathscr{I}^o$ and $\mathscr{I}^o$ becomes an S–ip-semimodule, where the S–scalar multiplication is defined by $a * A = (A : a)_r = \{b \in S \mid ba \in A\}$ for all $a \in S$ and $A \in \mathscr{I}^o$ (see 1.11); notice that

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Key words and phrases: Commutative semigroup, fully invariant congruence.

1991 Mathematics Subject Classification. Primary 16Y16, secondary 20M14.

The work is a part of the research project MSM0021620839 financed by MSMT and partly supported by the Grant Agency of the Charles University, grant # 444/2004/B-MAT/MFF.

$S * \emptyset = \{\emptyset\}$. Further, we denote by $\mathcal{K}^o (= \mathcal{K}_l^o(R, S))$ the subsemimodule of $\mathcal{J}^o$ generated by the set $\{\emptyset, R\}$.

Proposition 10.2.1. *(i) Every subsemimodule $\mathcal{A}$ of $\mathcal{J}^o$ such that $\mathcal{K}^o \subseteq \mathcal{A}$ is ideal-simple.*

(ii) Both $\mathcal{K}^o$ and $\mathcal{J}^o$ are ideal-simple.

Proof. Let $\mathcal{B} \neq \{\emptyset\}$ be an ideal of $\mathcal{A}$, $\emptyset \neq A \in \mathcal{B}$ and $a \in A$. Then $R \subseteq a * A \in \mathcal{B}$, and hence $R \in \mathcal{B}$. Now, if $C \in \mathcal{A}$, then $Cc \subseteq R$ and $C \subseteq$ $\subseteq c * R \in \mathcal{B}$, $C \in \mathcal{B}$. Thus $\mathcal{B} = \mathcal{A}$. □

Now, define a relation σ on $\mathcal{J}^o$ by $(A, B) \in \sigma$ iff $\{C \in \mathcal{J}^o \mid Ca \cap A = \emptyset\} =$ $= \{C \in \mathcal{J}^o \mid Ca \cap B = \emptyset\}$ for every $a \in S$. Then σ is a congruence of $\mathcal{J}^o$, we put $\mathcal{M}^o = \mathcal{S}^o/\sigma (\mathcal{M}^o = \mathcal{M}_l^o(R, S))$ and we denote by $\pi : \mathcal{J}^o \to \mathcal{M}^o$ the natural projection. Let $\mathcal{N}^o = \mathcal{N}_l^o(R, S) = \pi(\mathcal{K}^o)$.

Proposition 10.2.2. *$\mathcal{M}^o$ is a simple ip-semimodule.*

Proof. Since $(\emptyset, R) \notin \sigma$, we have $|\mathcal{M}^o| \geq 2$. Now, $\mathcal{M}^o$ is ideal-simple by 10.2.1 (ii) and $\mathcal{M}^o$ is simple by 6.2. □

Lemma 10.2.3. *(i) $Ann(\mathcal{J}^o) = Ann(\mathcal{M}^o) = \{a \in S \mid Sa \cap Rb = \emptyset$ for every $b \in A_l\} \subseteq S \setminus A_l$, $A_l = A_l(R, S)$ (see 1.12).*

(ii) If S is right subcommutative, then $Ann(\mathcal{J}^o) = S \setminus A_l$.

Proof. (i) Clearly, $Ann(\mathcal{J}^o) \subseteq Ann(\mathcal{M}^o)$. If $a \in Ann(\mathcal{M}^o)$, then $(\emptyset, a * A) =$ $= (a * \emptyset, a * A\} \in \sigma$ for every $A \in \mathcal{J}^o$. Consequently, $a * A = \emptyset$, $a \in Ann(\mathcal{J}^o)$ and $Sa \cap Rb = \emptyset$ for every $b \in A_l$. Conversely, if the latter is true and if $c \in a * A$, then $b = ca \in A_l$ and $Sa \cap Rb \neq \emptyset$, a contradiction.

(ii) Let $a \in S \setminus A_l$. Then $R \cap aS = \emptyset$ by 1.12(i) and we will assume, for a moment, that $Sa \cap Rb \neq \emptyset$ for some $b \in A_l$. There is $c \in S$ with $bc \in R$, and so $\emptyset \neq Sac \cap Rbc \subseteq Sac \cap R \subseteq aS \cap R$, a contradiction. Thus $Sa \cap Rb = \emptyset$ for every $b \in A_l$ and $a \in Ann(\mathcal{J}^o)$ by (i). □

Lemma 10.2.4. *Suppose that $|\mathcal{M}^o| = 2$. Then:*

(i) *R is left uniform.*

(ii) *$Ra \cap Rb \neq \emptyset$ for all $a, b \in A_1 = A_1(R, S)$.*

(iii) *If $a \in S$ and $b \in A_l$ are such that $Sa \cap Rb = \emptyset$, then $Sa \cup \{a\} \subseteq S \setminus A_l$.*

Proof. Since $|\mathcal{M}^o| = 2$, the only blocks of the congruence σ are $\{\emptyset\}$and $\mathcal{J}^o \setminus \{\emptyset\}$. It follows that $(A, B) \in \sigma$ for all $A, B \in \mathcal{J}^o$, $A \neq \emptyset \neq B$ and, in particular, $A \cap B \neq \emptyset$, so that the assertions (i) and (ii) are clear. Finally, if $Sa \cap Rb = \emptyset$, then $a * Rb = \emptyset$, $A \in Ann(\mathcal{M}^o)$ and $Sa \cup \{a\} \subseteq S \setminus A_l$ by 10.2.3 (ii). □

Lemma 10.2.5. *If $\mathcal{M}^o$ is a qza-semimodule, then R is left uniform.*

Proof. If I and J are left ideals of R, then $(I \cup J) \cap I \neq \emptyset$, $(I \cup J, J) \in \sigma$ and $(I \cup J) \cup J \neq \emptyset$ implies $I \cap J \neq \emptyset$ (use 6.2). □

Lemma 10.2.6. *Suppose that S is left subcommuutative. Then $\mathcal{M}^o$ is a qza-semimodule if and only if $A \cap B = \emptyset$, whenever $A, B, C \in \mathcal{J}^o$ are such that $A \cap C = \emptyset \neq B \cap C$.*

Proof. First, assume that $\mathcal{M}^o$ is a qza-semimodule. If $A \cap C = \emptyset \neq B \cap C$, then $(A, B) \notin \sigma$ (see 6.2), and therefore $\pi(A) \neq \pi(B)$, $\pi(A \cap B) = o$, $(A \cap B, \emptyset) \in \sigma$ and $A \cap B = \emptyset$. Conversely, if the condition is satisfied and $A, B \in \cap \mathcal{J}^o$ are such that $(A, B) \notin \sigma$, then $(a * A) \cap C = \emptyset \neq (a * B) \cap C$, $a * (A \cap B) = \emptyset$, $a \notin Ann(\mathcal{J}^o)$ (since $a * B \neq \emptyset$) and $A \cap B = \emptyset$ by 5.6. □

Lemma 10.2.7. *Suppose that S is right cancellative and R left uniform. If $A, B, C \in \mathcal{J}^o$ are such that $A \cap C = \emptyset \neq B \cap C$, then $A \cap B = \emptyset$.*

Proof. If $I = Bb \subseteq R$, then $B \cap C \neq \emptyset$ implies $J = I \cap Cb \neq \emptyset$. Further, if $K = Ab \cap I = \emptyset$, then $A \cap B = \emptyset$. On the other hand, if $K \neq \emptyset$, then $\emptyset \neq K \cap J \subseteq Ab \cap Cb = (A \cap C)b = \emptyset$, a contradiction. □

Corollary 10.2.8. *Suppose that S is left subcommutative and right cancellative. Then $\mathcal{M}^o$ is a qza-semimodule if and only if R is left uniform.*

Example 10.3. Let M denote the set of ordered pairs of integers equipped with a semilattice operation $(m, n) \oplus (k, l) = (min(m, k), min(n, l))$. The mappings $\varphi : (m, n) \to (n, m)$ and $\psi : (m, n) \to (m, n + 1)$ are endomorphisms of $M(\oplus)$ and M becomes a simple $S-$ip-semimodule, where S is the endomorphism semigroup of $M(\oplus)$ generated by $\{\varphi, \psi\}$. Notice that S is cancellative and $h-$uniform and that $M(\oplus)$ is not a chain (of course, $M(\oplus)$ is a lattice).

Example 10.4. Let R be a subsemigroup of a semigroup S such that $aS \cap bR \neq \emptyset$ for all $a \in S$ and $b \in R$ (e.g., S a group). The set $\mathcal{J} = \mathcal{J}_r(R, S)$ is a semillatice with respect to the operation of union ($S = o$ becomes the absorbing element) and $\mathcal{J}$ is a semimodule via $(a, A) \to aA$, $a \in S$, $A \in \mathcal{J}$. Now, define a relation ϱ of $\mathcal{J}$ by $(A, B) \in \varrho$ iff $\{C \in \mathcal{J} \mid A \cap C = \emptyset\} = \{C \in \mathcal{J} \mid B \cap C = \emptyset\}$.

Lemma 10.4.1. *(i) ϱ is a congruence of the semimodule $\mathcal{J}$.*
(ii) $(aS, S) \in \varrho$ for every $a \in S$.
(iii) If η is a cngruence of $\mathcal{J}$ such that $\varrho \subseteq \eta$ and $(R, S) \in \eta$, then $\eta = \mathcal{J} \times \mathcal{J}$.
(iv) If $(R, S) \in \varrho$, then $\varrho = \mathcal{J} \times \mathcal{J}$ and R is right uniform.

Proof. Easy. □

Corollary 10.4.2. *Suppose that R is not right uniform. If τ is a congruence of $\mathcal{J}$ maximal with respect to $(R, S) \notin \tau$, then $\mathcal{J}/t$ is a simple ip-semimodule that is not a qza-semimodule. Moreover, there exist $x, y \in \mathcal{J}/\tau$ such that $(x + z, y + z) \neq (x, y)$ for every $z \in \mathcal{J}/\tau$.*

11. Simple idempotent semimodules

In the sequel, an ipa-semimodule (ipn-semimodule, resp.) will be ann ip-semimodule with (without, resp.) an (additively) absorbing element. An ipaa-semimodule will be an ipa-semimodule where $So = o$.

11.1. Let M be a simple ipa-semimodule with $|M| \geq 3$. For $w \in M$, $w \neq o$, put $R_w = \{a \in S \mid w = w + aw\}$ and $\Phi_w(x) = \{a \in S \mid w = w + ax\}$ for every $x \in M$ (thus $R_w = \Phi_w(w)$).

Proposition 11.1.1. *(i) R_w is a subsemigroup of S and Φ_w is a homomorphism of the semimodule M into the semimodule $\mathscr{J}^o = \mathscr{J}_l^o(R_w, S)$ (see 10.2).*

(ii) $\Phi_w(x) \neq \emptyset$ for every $x \neq o$ and $\Phi_w(w) = R_w$.

(iii) If M is an ipaa-semimodule, then $\Phi_w(o) = \emptyset$.

(iv) If $1_S \in S$, then $1_S \in R$.

Proof. Since M is simple and $|M| \geq 3$, we have $Sx \neq o$ for every $x \neq o$. Then $|Sx + M| \geq 2$, so that $Sx + M = M$ and $\Phi(x) \neq \emptyset$. In particular, $R = \Phi(w) \neq \emptyset$ and it is easy to see that R is a subsemigroup of S. Now, if $aw + w = w = bx + w$, then $w = abx + aw + w$ and $ab \in \Phi(x)$. Moreover, $bc \in R$, where $x = cw + z$. Thus $\Phi(x) \in \mathscr{J}^o$ and, in fact, Φ is a semimodule homomorphism. The rest is clear. □

Lemma 11.1.2. *Suppose that $T = \Phi_w(o) \neq \emptyset$. Then:*

(i) *$T \subseteq \Phi_w(x)$ for every $x \in M$.*

(ii) *T is a righht ideal of S.*

(iii) *$T \subseteq R_w$ and T is an ideal of R_w.*

(iv) *If S is right uniform, then $A_l(R_w, S) = S$.*

Proof. We have $o = o + x$, and hence $T = T \cap \Phi(x)$. Further, $T \subseteq \Phi(ao) = \{b \mid ba \in T\}$, so that $Ta \subseteq T$ and $TS \subseteq T$. Since $T \subseteq \Phi(w) = R$, T is an ideal of R. Finally, (iv) follows from (ii) and 1.12. □

Lemma 11.1.3. *Suppose that Φ_w is not injective. Then:*

(i) *R_w is a right ideal of S.*

(ii) *Either, $R_w = S$ or $S \setminus R_w$ is a right ideal of S.*

(iii) *$\mathscr{J}_l^o(S, R_w) = \mathscr{J}_l^o(R_w)$.*

(iv) *If S is right uniform, then $R_w = S$.*

Proof. We have $r = ker(\Phi) \neq id_M$ and, since M is siple, $r = M \times M$, Φ is constant and $\Phi(M) = \{R\}, R = \Phi(w) = \Phi(o) \neq \emptyset$. By 11.1.2, R is a right ideal of S. Finally, $a * R = \Phi(aw) = R$ for every $a \in R$ for every $a \in S$ and the rest is clear. □

Lemma 11.1.4. *If M is an ipaa-semimodule or if 1_S, then Φ_w is injective.*

Proof. Use 11.1.3. □

Lemma 11.1.5. $Ann(\mathscr{J}^o) = Ann(\mathscr{M}^o) \subseteq \{a \in S \mid aM = \{o\}\} \subseteq Ann(M)$. *If S is subcommutative, then* $Ann(\mathscr{J}^o) = S \setminus A_l(R_w, S) = Ann(M)$.

Proof. Firstly, if $ax \neq o$, $a \in S$, $x \in M$, then $w = bax + w$ for some $b \in S$, $ba \in \Phi(x)$ and $cba \in Sa \cap Rba$ for every $c \in R$. Now, by 10.2.3(i), $a \notin Ann(\mathscr{J}^o) = Ann(\mathscr{M}^o)$ and we have proved that $Ann(\mathscr{J}^o) \subseteq \{a \in S \mid aM = o\} \subseteq Ann(M)$. Conversely, if S is right subcommutative and $a \in Ann(M)$, then $aM = o$ by 5.7(iii) and, if $b \in a * A$, $A \in \mathscr{J}^o$, then $ba \in A$, $bac \in R$, $ba = ad$, $w = bacw + w = adcw + w = o + w = o$, a contradiction. Thus $a * A = \emptyset$ and $a \in Ann(\mathscr{J}^o)$.

□

Lemma 11.1.6. *Suppose that the homomorphism $\pi\Phi_w : M \to \mathscr{M}^o$ is not injective, $\pi : \mathscr{J}^o \to \mathscr{M}^o$ being the natural projection. Then:*

(i) $\pi\Phi_w$ is constant and the semimodule $\Phi_w(M)$ is contained in a block of the congruence σ.

(ii) $T = \Phi_w(o) \neq \emptyset$ (see 11.1.2).

(iii) $0 \in \mathscr{M}^o$ and $(\pi\Phi_w)(M) = 0$.

(iv) $Ann(\mathscr{J}^o) = Ann(\mathscr{M}^o) = \emptyset$.

Proof. (i) Since M is simple and $\pi\Phi$ is not injective, it is a constant mapping. It follows that $\Phi(M)$ is contained in a block of σ.

(ii) We have $(\emptyset, R) \in \sigma$, $R = \Phi(w)$ and hence $\Phi(o) \neq \emptyset$.

(iii) By (i) and (ii), $(\pi\Phi)(M) = \{q\} \subseteq \mathscr{M}^o$, $q \neq o$. Of course, $\{q\}$ is a subsemimodule of $\mathscr{M}^o$, and so $Sq = q$ and $Sq + \mathscr{M}^o = q + \mathscr{M}^o$. Now $q + \mathscr{M}^o$ is an ideal of $\mathscr{M}^o$ and we get $q + \mathscr{M}^o = \mathscr{M}^o$ and $q = 0 \in \mathscr{M}^o$.

(iv) For every $a \in S$, $ao = o \neq q = aq$ is true in $\mathscr{M}^o$. □

Lemma 11.1.7. *Suppose that $\pi\Phi_w$ is not injective (see 11.1.6).*

(i) If S is either right uniform or subcommutative, then $A_l(R_w, S) = S$ (i.e., for every $a \in S$ there exists $b \in S$ such that $R_w ab \subseteq R_w$).

(ii) If S is subcommutative, then $Ann(M) = \emptyset$.

(iii) If $\mathscr{M}^o$ is a qza-semimodule, then $|\mathscr{M}^o| = 2$, R_w is left uniform and $R_w a \cap R_w b \neq \emptyset$ for all $a, b \in A_l(R_w, S)$.

Proof. Combine 11.1.6, 11.1.2(iv), 11.1.5 and 10.2.4. Notice that every qza-semimodule with a neutral element contains at most two elements. □

Lemma 11.1.8. *(i) $Sa \cap Rw \neq \emptyset$ for every $a \in S$ such that $aw \neq o$.*

(ii) If R is left subcomutative, then $Sa \cap Rw \neq \emptyset$ for every $a \in S \setminus Ann(M)$.

Proof. We have $M = Saw + M$ and (ii) follows from 5.6. □

Proposition 11.1.9. *Φ_w is injective for at least one $w \in M$, $w \neq o$.*

Proof. Let $v \in M$, $v \neq o$. By 5.8, there is $a \in S$ such that $w = av \neq ao$. If $w = o$, then $ao = a(o + v) = ao + av = ao + o = ao = av$, a contradiction.

Thus $w \neq o$ and $a \in \Phi(v)$. On the other hand, $a \notin \Phi(o)$, since otherwise $av = w = w + ao = a(v + o) = ao$. □

Proposition 11.2. *Let M be a simple ipaa-semimodule with $|M| \geq 3$. Then, for every $w \in M$, $w \neq o$, the semimodule M is isomorphic to a subsemimodule $\mathscr{A}$ of the semimodule $\mathscr{M}_l^o(R_w, S)$ such that $\pi(\mathscr{K}_l^o) \subseteq \mathscr{A}$.*

Proof. See 11.1.6 and 10.2.1. □

Proposition 11.3. *Let M be a simple ipa-semimodule with $|M| \geq 3$.* Then *M is a ipaa-semimodule, provided that at least one of the following three conditions is satisfied:*

(a) S is a group;

(b) S is subcommutative and M finite;

(c) S is subcommutative and $Ann(M) \neq \emptyset$.

Proof. Assume that S is subcommutative. Now, if $Ann(M) \neq \emptyset$, then $So = o$ by 4.2 and 5.4, and hence we can assume that $Ann(M) = \emptyset$. Then, by 5.6, $x \to ax$ is an injective transformation of M for every $a \in S$. Moreover, if S is a group or M is finite, then $x \to ax$ is a permutation of M, and therefore $aw = o$ for some $w \in M$. Now, $ao = a(w + o) = aw + ao = o + ao = o$. □

Theorem 11.4. *Suppose that S is left subcommutative. The following conditions are equivalent:*

(i) Every simple ipaa-semimodule is a qzaa-semimodule.

(ii) S is hl-uniform.

Moreover, if S is right cancellative, then these conditions are equivalent to:

(iii) No subsemigroup of S is a free semigroup of rank (at least) 2.

Proof. Firstly, if (i) is true, then S is hl-uniform by 1é.2.2 and 10.2.5. Now assume that (ii) is true. Let, on the contrary, M be a simple ipaa-semimodule such that $x = y + z \neq o$ for some $x, y, z \in M$, $x \neq y$. By 6.2, $\sigma = id_M$ and there exist $c \in S$ and $u \in M$ such that $cx + u = o \neq v = cy + u$. Clearly, $c \notin Ann(M)$, $x \to cx$ is injective and $cx \neq o$. Put $A = \{a \in S \mid ay = ay + acx\}$ and $B = \{b \in S \mid cy = cy + bv\}$. One checks easily that $AA \cup BA \subseteq A$ and $BB \cup AB \subseteq B$. Further, $N = \bigcup (M + acx)$, $a \in S$, is an ideal of M and, if $N = \{o\}$, then $Scx = o$, a contradiction with $cx \neq o$ and $|M| \geq 3$. Thus $N = M$ and $A \neq \emptyset$. Quite similarly, $B \neq \emptyset$. Since S is hl-uniform, we can find $d \in A \cap B$ and we get $cy = dcx + cy$, $cy = dv + cy$, $o \neq cy = cy + cy = dcx + dv + cy = d(cx + v) + cy = d(cx + cy + u) + cy = do + cy = o$, a contradiction. □

Proposition 11.5. *Suppose that S is cancellative and that no subsemigroup of S is a free semigroup of rank (at least) 2. If M is a simple ipa-semimodule with $So \neq o$, then the semilattice $M(+)$ is upwards-directed (i.e., for all $x, y \in M$ there exists $z \in M$ with $x = x + z$ and $y = y + z$). If M is finite, then $M(+)$ is a lattice (and then $0 \in M$).*

Proof. By 1.3, S is h-uniform. Now, take $w, v \in M$, $w \neq o \neq v$, and put $R = R_w$ (see 11.1). By 11.4, the simple semimodule $\mathscr{M}^o(R, S)$ is a qzaa-semimodule, and hence every non-trivial subsemimodule of $\mathscr{M}^o$ is a qzaa-semimodule, too. Since $So \neq o$, $\pi\Phi$ is not injective and consequently, by 11.1.7(i), (iii), we have $Ra \cap Rb \neq \emptyset$ for all $a, b \in S$. In particular, R is left dense in S, R is right dense in S by 1.9 and $R \cap aR \neq \emptyset$ for every $a \in S$. Now, there is $a \in S$ such that $v = v + aw$, we have $ab = c$, $b, c \in R$, and $w = w + cw$, $w = w + bw$, $v = = v + aw = v + aw + abw = v + cw$. □

11.6. *Let M be a simple ipn-semimodule. Then M is infinite and, for all $w, x \in M$, we put $R_w = \{a \in S \mid w = w + aw\}$ and $\Phi_w(x) = \{a \in S \mid w = w + ax\}$.*

Proposition 11.6.1. *(i) R_w is a subsemigroup of S and Φ_w is a homomorphism of the semimodule M into the semimodule $\mathscr{J}^o = \mathscr{J}_l^o(R_w, S)$ (see 10.2).*

(ii) $\Phi_w(x) \neq \emptyset$ for every $x \in M$ and $\Phi_w(w) = R_w$.

(iii) If $1_S \in S$, then $1_S \in R_w$ and Φ_w is injective.

(iv) Φ_w is injective for at least one $w \in M$.

Proof. We proceed similarly as in the proof of 11.1.1 to show (i) and (ii), (iii) being clear. As for (iv), let $u, v \in M$ be such that $u \neq v$. If $u + v = u$, then we put $x = u$ and $y = v$. If $u + v = v$, then $x = v$ and $y = u$. If $u \neq u + v \neq v$, then $x = u + v$ and $y = v$. Whatever, we get $x \neq y$ and $x + y = x$. ow, by 5.8, $ax \neq ay = w$ for some $a \in S$ and we have $a \in \Phi_w(y)$ and $a \notin \Phi_w(x)$. □

Lemma 11.6.2. *If $\pi\Phi_w : M \to \mathscr{M}^o$ is not injective, then $\pi\Phi_w$ is constant, $0 \in \mathscr{M}^o$, $Ann(\mathscr{J}^o) = Ann(\mathscr{M}^o) \neq \emptyset$ and $(\pi\Phi_w)(M) = 0$.*

Proof. Similar to that of 11.1.6. □

12. Simple idempotent semimodules - summary

Proposition 12.1. *Let M be a simple ip-semimodule with $|M| \geq 3$. The there exists a subsemigroup R of S such that M is isomorphic to a subsemimodule $\mathscr{A}$ of the semimodule $\mathscr{J}_l^o(R, S)$, where $\mathscr{K}^o \subseteq \mathscr{A}$ (see 10.2).*

Proof. Use 11.1.9 and 11.6.1(iv). □

Corollary 12.2. *Let M be a simple ip-semimodule. Then $|M| \leq 2^{|S|}$.*

Theorem 12.3. *Let S be a cancellative semigroup such that $Sa = aS$ for every $a \in S$ and no subsemigroup of S is a free semigroup of rank (at least) 2.*

(i) If M is a simple idempotent semimodule such that $o \in M$, $So = o$ and $|M| \geq 3$, then $x + y = o$ for all $x, y \in M$, $x \neq y$, $A = S \setminus Ann(M)$ is a subsemigroup of S, the mapping $x \to ax$, $x \in M$, is a permutation of M for every $a \in A$, A operates transitively on $M \setminus \{o\}$ and M is a simple A-semimodule.

(ii) If M is a simple idempotent semimodule such that $o \in M$, $So \neq o$ and $|M| \geq 3$, then M is infinite, the mapping $x \to ax$, $x \in M$, is an injective transformation of M for every $a \in S$ and for all x, $y \in M$ there exists $z \in M$ such that $x = x + z$ and $y = y + z$.

(iii) If M is a simple idempotent semimodule such that $o \notin M$, then M is infinite, the mapping $x \to ax$, $x \in M$, is an injective transformation of M for every $a \in S$ and for all x, $y \in M$ there exists $z \in M$ such that $x = x + z$ and $y = y + z$.

Proof. (i) See 11.4, 9.11 and 9.12.

(ii) See 11.3, 5.6 and 11.5.

(iii) Since $o \notin M$, M is infinite. Further, $Ann(M) = \emptyset$ by 5.7(iii) and 5.4 and the transformations $x \to ax$ are injective by 5.6. Finally, using 11.6 and 10.2.3(ii), we may proceed similarly as in the proof of 11.5 to show the rest. □

13. Simple modules

Proposition 13.1. *A non-trivial module M is simple (as an S-semimodule) if and only if 0 and M are the only submodules of M (i.e., M has just two submodules and is simple as a module).*

Proof. If N is a submodule of M, then r is a congruence of M, where $(x, y) \in r$ iff $x - y \in N$. Consequently, if M is a simple semimodule, then either $r = id_M$ and $N = 0$ or $r = M \times M$ and $N = M$. Conversely, if s is a congruence of the semimodule M, then $K = \{x \in M \mid (x, 0) \in r\}$ is a submodule of M and $u - v \in K$ for all $(u, v) \in s$. □

Example 13.2. Let $p \geq 2$ be a prime number.

(i) Define a scalar multiplication on the (abelian simple) group $\mathbb{Z}_p(+)$ of integers modulo p by $ax = 0$ for all $a \in S$ and $x \in \mathbb{Z}_p$. Then we get a simple S-module and we denote it by $\mathbb{Z}^0_{(p,S)}$. Notice that $\mathbb{Z}^0_{p,S}$ is not unitary if $1_S \in S$.

(ii) Define another scalar multiplication on $\mathbb{Z}_p(+)$ by $ax = x$ for all $a \in S$ and $x \in \mathbb{Z}_p$. Again, we get a simple module denoted by $\mathbb{Z}^1_{(p,S)}$.

Remark 13.3. *Suppose that $1_S \in S$ and denote by R the semigroup-ring $\mathbb{Z}S$ of the semigroup S over the ring $\mathbb{Z}$ of integers.*

Let M be a simple S-module. If M is not unitary, then $M \simeq \mathbb{Z}^0_{(p,S)}$ (see 13.2(i)) for a prime p (5.3). On the other hand, if M is unitary, then M is a simple R-module, and hence ${}_RM \simeq R/I$ for a maximal left ideal I of R.

Conversely, if J is a maximal left ideal of R, then the simple R-module R/J is also a simple unitary S-module (13.1).

Remark 13.4. *Suppose that $1_S \notin S$ and put $S_1 = S \cup \{1_{S_1}\}$. If M is a simple S-module, then M is also a unitary simple S_1-module (5.10) and 13.3 applies.*

Proposition 13.5. *Let M be a non-trivial module such that M is isomorphic neither to $\mathbb{Z}^0_{(p,S)}$ nor to $\mathbb{Z}^1_{(p,S)}$, $p \geq 2$ any prime number. Then the following conditions are equivalent:*

(i) M is a simple module.

(ii) 0 and M are the only submodules of M.

(iii) For all x, $y \in M$, $x \neq 0$, there exist a positive integer n and elements a_1, ..., a_n, b_1, ..., $b_n \in S$ such that $a_1x + ... + a_nx = y + b_1x + ... + b_nx$.

Proof. (i) implies (ii). See 13.1.

(ii) implies (iii). Denote by N the set of the elements $y \in M$ such that $a_1x + ... + a_nx = y + b_1x + ... + b_nx, n \geq 1$, a_i, $b_i \in S$. Clearly, N is a submodule of M and (iii) is true for $N = M$. Now, assume that $N \neq M$. Then, by (ii), $N = 0$ and we have $ax = bx$ for all a, $b \in S$. Consequently, $K \neq 0$, $K = \{z \in M \mid az = bz \text{ for all } a, b \in S\} = \{z \in M \mid |Sz| = 1\}$. Of course, K is a submodule of M, $K = M$ and there is an endomorphism φ of $M(+)$ such that $\varphi(z) = az$ for all $a \in S$ and $z \in M$. If $\varphi = 0$, then $M \simeq \mathbb{Z}^0_{(p,S)}$. Thus $\varphi \neq 0$, and therefore $Ker(\varphi) \neq M$. Again, $Ker(\varphi)$ is a submodule of M, $Ker(\varphi) = 0$ and φ is injective. Finally, $\varphi(z) = az = a^2z = a \cdot az = \varphi^2(z)$, $z = \varphi(z)$ for every $z \in M$ and $M \simeq \mathbb{Z}^1_{(p,S)}$.

(iii) implies (i). Let $r \neq id_M$ be a congruence of M. Then $(u, v) \in r$ for some u, $v \in M$, $u \neq v$, and so $(x, 0) \in r$, $x = u - v \neq 0$. Now, it follows from (iii) that $(y, 0) \in r$ for every $y \in M$. Thus $r = M \times M$ and M is simple. □

14. Simple cancellative semimodules

Lemma 14.1. *Let M be a simple cn-semimodule such that M is not a module. Then $0 \notin M$ and $x + y \neq x$ for all x, $y \in M$.*

Proof. Easy (use 5.4). □

Proposition 14.2. *Let M be a non-trivial cn-semimodule such that M is isomorphic neither to $\mathbb{Z}^0_{(p,S)}$ nor to $\mathbb{Z}^1_{(p,S)}$, $p \geq 2$ any prime number. Then the following conditions are equivalent:*

(i) The only cancellative congruences of M are id_M and $M \times M$.

(ii) The difference module $D(M) = \{x - y \mid x, y \in M\}$ of M is simple.

(iii) For all x, y, $z \in M$, $x \neq y$, there exist a positive integer n and elements a_1, ..., a_n, b_1, ..., $b_n \in S$ such that $a_1x + ... + a_nx + b_1y + ... + b_ny = z + a_1y + ... + a_ny + b_1x + ... + b_nx$.

Proof. Let r be a congruence of M and $= \{x - y \mid (x, y) \in r\}$. Then N is a submodule of $D(M)$ and $M \mid r$ is cancellative iff $r = \{(u, v) \mid u - v \in N\}$. The rest is clear from 13.5. □

Theorem 14.3. *Let M be a non-trivial cancellative semimodule. Then M is simple if and only if the following three conditions are satisfied:*

(a) The difference module $D(M)$ is simple (see 13.1 and 14.2);

(b) M is archimedean (i.e., for all $x, y \in M$ there exist a positive integer n and an element $z \in M$ such that $x + z = ny$);

(c) M is ideal-free (equivalently, for all $x, y \in M$ there exist $a \in S$ and $z \in M$ sch that $ax + z = y$).

Proof. (i) Let M be a simple semimodule. If M is a module, then $D(M) = M$ is simple. If M is not a module, then $D(M)$ is simple by 14.2. Thus (a) is true. Further, (b) is true by 5.9 and (c) by 5.2(ii).

(ii) Let the conditions (a), (b) and (c) be satisfied. We may also assume that M is not a module. Firstly, consider a congruence r of M such that $(w, 2w) \in r$ for some $w \in M$. We claim that $r = M \times M$.

Let $\pi : M \to M/r = N$ denote the natural projection. It follows immediately from (b) that $\pi(w)$ is the only idempotent element of $N(+)$. Consequently, $S\pi(w) = \pi(w)$ and $N = N + S\pi(w) = N + \pi(w)$ by (c). Now, using again (b) and (c), we conclude that $\pi(w) = 0$ is a neutral element of N and N is a module. By (a) and 14.2, either $r = id_M$ or $r = M \times M$. But if $r = id_M$, then M is a module, a contradiction.

Now, take $w \in M$ such that $w \neq 2w$ and consider a congruence s of M maximal with respect to $(w, 2w) \notin s$. It follows from the preceding part of the proof that s is a maximal congruence of M, and hence $K = M/s$ is a simple semimodule. Due to (b) and (c), K is ideal-free and not idempotent. Consequently, K is cancellative by 5.1, and therefore $s = id_M$ by (a) and 14.2. Thus M is a simple semimodule. □

Remark 14.4. Let M be a non-trivial cn-semimodule satisfying the conditions 14.3(a),(b) and containing at least one element v such that $M = \{v\} \cup Sv \cup \cup (M + v) \cup (M + Sv)$ (cf. 14.3(c)). That is, M just the ideal generated by $\{v\}$. Finally, assume that M is not simple. Now, due to 14.3, M is not ideal-free and M contains at least one proper ideal. Of course, no proper ideal contains the element v and, if K is an ideal maximal with respect to $v \notin K$, then K is a (proper) maximal ideal of M. Consequently, M/r is an ideal-simple semimodule, where $r = (K \times K) \cup id_M$. If $v = 2v$, then $v = 0$ in M, M is a module and M is simple, a contradiction. Thus $v \neq 2v$ and there exists a congruence s of M maximal with respect to $r \subseteq s$ and $(v, 2v) \notin s$. It is easy to see that s is a maximal congruence of M (see the proof of 14.3), and so M/s is a simple semimodule. Of course, M/s is neither idempotent nor cancellative and consequently, by 5.1, M/s is either a za-semimodule or a zs-semimodule.

If M/s is a za-semimodule, then $M + M \neq M$.

If M/s is a zs-semimodule, then S is neither finite nor hr-uniform (8.2).

In both cases, the mapping $x \to 2x$ is a non-projective injective endomorphism of M.

Remark 14.5. Assume that S is hr-uniform. Taking into account 14.4, we see that 14.3 also remains true when (c) is replaced by any of the following three (formally weaker) conditions:
(c′) $M + M = M$ and M, as an ideal of itself, is generated by a single element;
(c″) $(M + v) \cup (M + Sv) = M$ for at least one $v \in M$;
(c‴) $M + Sv = M$ for at least one $v \in M$.

Proposition 14.6. *If M is a simple cn-semimodule, then $|M| \leq max(|S|, \aleph_0)$. Moreover, if S is finite, then M is a finite module.*

Proof. The inequality follows easily from 13.3 and 13.4 (use the difference module).

Now, assume that S is a finite semigroup. We also may assume that M is a module. If $SM = 0$, then $M \simeq \mathbb{Z}^0_{(p,S)}$ (see 13.2(i)) and M is finite. Now, let $Sw \neq 0$ for some $w \neq 0$ for some $w \in M$. If N is the subgroup of $M(+)$ generated by Sw, then N is a non-zero submodule of M, and hence $N = M$. Thus $M(+)$ is a finitely generated abelian group. Consequently, if $nM = 0$ for some $n \geq 1$, then M is finite. On the other hand, if $nM \neq 0$, the $nM = M$, since nM is a submodule of M, $M(+)$ is a divisible abelian group, a contradiction with the fact than $M(+)$ is finitely generated. □

Remark 14.7. Let M be a simple cn-semimodule such that M is not a module. Define a relation $\leq$ on the difference module $D(M)$ by $u \leq v$ iff $v - u \in M \cup \{0\}$. Then $\leq$ is an order on $D(M)$ and this order is compatible with respect to both the addition and the scalar S−multiplication. Moreover, $M = \{u \in D(M) \mid u > 0\}$ is just the cone of positive elements, M is upwards cofinal in $D(M)$ and, by 14.3(b), for all $x \in M$ and $u \in D(M)$ there is $n \geq 1$ with $u \leq x$, i.e., every element from M is an order unit in $D(M)$.

Clearly, the order $\leq$ is linear if and only if the semimodule M (or the semigroup $M(+)$) is semisubstractive. That is, if $x, y \in N$, then either $x \in M + y$ or $y \in M + x$.

References

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