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The BGG Diagram For Contact Orthogonal Geometry of Even Dimension

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BGG sequences are sequences of invariant differential operators acting on sections of vector bundles associated to a principal bundle locally modeled by G/P, where G is a simple Lie group, P its parabolic subgroup. They contain a large and important class of invariant differential operators in parabolic geometries. BGG diagram contains the representation – theoretical information on the BGG sequence. We study its structure for $G = Spin(2n, \mathbb{C})$ and its real forms, when P is given by crossing the second root in the Dynkin or Satake diagram of G. We show that for certain real forms and certain representations the shape of the BGG diagram differs from the shape for the complex case.

1. Introduction

A lot of attention was paid recently to a study of invariant operators on manifolds with a given geometric structure. In particular, it is true for the so called parabolic geometries (see [Slo92], [BasE89]): that invariant differential operators on manifolds with a specified parabolic structure are relatively rare. Part of them (the so called standard operators) appears in sequences called Bernstein–Gelfand– Gelfand (BGG) sequences (see [CSS01], [CalD01]). Combinatorial structure of the (split) BGG sequence is recorded in the Hasse diagram and – for a given sequence – it is necessary to know highest weights of inducing modules, that appear in the sequence (let us call the Hasse diagram with these highest weights the BGG diagram).

Hasse diagram and BGG diagrams are known for some parabolic geometries (e.g. conformal, quaternionic or CR geometries). There is one Hasse diagram and

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a set of BGG diagrams parameterized by the set of dominant weights for each complex parabolic geometry. If we want to consider real parabolic geometries, it is necessary to study the corresponding several real versions.

This paper is devoted to contact orthogonal geometry in even dimensions. The homogenous model for this geometry is $M = G/P$, where G is a complex Lie group $Spin(2n, \mathbb{C})$, $n \geq 4$ and P is a parabolic subgroup specified by the second crossed node in the corresponding Dynkin diagram. We shall describe first the Hasse diagram in the complex case (Sect. 2). In the Sect. 3, a description of the full BGG diagram for the complex case is given. Finally, in the last section, we discuss all possible real cases of this parabolic geometry and we show that for some particular real cases and specific highest weights, real Hasse diagrams degenerate in certain points and we describe its form in these situation.

The main tool used for the computation of the Hasse diagram is the calculus of saturated sets (they describe vertices of the diagram). The computation of all arrows of the diagram uses a suitable induction with respect to the rank of the algebra. For a given irreducible G-module, we are able to compute the full BGG diagram. To treat real cases, we need to use detailed information concerning classification of real reductive Lie algebras (description of autocorrelative representations together with formulae for their indices).

2. Hasse diagram

We shall study the Hasse diagram for the standard contact gradation of D_n, $n \geq 4$, which is given by crossing out the second root. In the complex case we can choose any regular matrix as the matrix of the defining quadratic form of $\mathfrak{so}(2n) \equiv D_n$. According to [Yam93],

$$Q = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}, \tag{1}$$

is a suitable choice, where K is the matrix of rank n with 1's on the antidiagonal and zeros elsewhere. The Lie algebra $\mathfrak{so}(2n)$ then consists of matrices X satisfying $X^T Q + QX = 0$. Such a matrix can be written in a block diagonal form

$$X = \begin{pmatrix} A & B \\ C & -A' \end{pmatrix} \quad \text{where} \quad B = -B', C = -C', \tag{2}$$

where the apostrophe means transposition with respect to the antidiagonal.

As a Cartan subalgebra $\mathfrak{h}$, the subalgebra of diagonal matrices can be chosen. Let $\lambda_i \in \mathfrak{h}^*$ take a value a_i on $\operatorname{diag}(a_1, a_2, \ldots, a_n, -a_n, \ldots, -a_2, -a_1) \in \mathfrak{h}$. The positive roots of $\mathfrak{so}(2n)$ are $\lambda_i - \lambda_j$ and $\lambda_i + \lambda_j$, $1 \leq i < j \leq n$. The corresponding root vectors are antidiagonal skew matrices with only one nonzero entry above the antidiagonal. The position of the nonzero entry is as follows:

	1 − 2	**1 − 3**	**1 − 4**	...	**1 − n**	**1 + n**	...	**1 + 4**	**1 + 3**	*1 + 2*	
		2 − 3	**2 − 4**	...	**2 − n**	**2 + n**	...	**2 + 4**	**2 + 3**		
			3 − 4	...	3 − n	3 + n	...	3 + 4			
				⋱			⋰				
					(n − 1) − n	(n − 1) + n					

where $2 + 5$ is a shorthand for $\lambda_2 + \lambda_5$ etc. This is the upper part of the matrix $X \in \mathfrak{so}(2n)$, the white entries on the left belong to the Cartan subalgebra, the ones on the right are zeros because X is skew with respect to the antidiagonal. Thus the denoted entries contain complete information about elements of the maximal nilpotent subalgebra $\mathfrak{n}$ that is a complement of the standard Borel subalgebra. The boldfaced root vectors are elements of the first graded part $\mathfrak{g}_1$ of the chosen parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $1 + 2$ written in italic is the only root in the second graded part $\mathfrak{g}_2$. Let us denote by Δ^+ the set of positive roots of $\mathfrak{g} \equiv \mathfrak{so}(2n)$ and by W the Weyl group of $\mathfrak{g}$.

Definition 1. (Saturated set). A set $\Phi \subset \Delta^+$ is called a saturated set iff there exists $w \in W$ such that $\Phi = \Phi_w \equiv \Delta^+ \cap w(\Delta^-)$. For $w = \sigma_\alpha$ we denote $\Phi_\alpha = \Phi_w$.

Lemma 1. (Properties of saturated sets). *A set $\Phi \subset \Delta^+$ is saturated iff*

(1) *If $\alpha_1, \alpha_2 \in \Phi$ and $\alpha_1 + \alpha_2 \in \Delta^+$, then $\alpha_1 + \alpha_2 \in \Phi$.*

(2) *If $\alpha \in \Phi$ and $\alpha = \alpha_1 + \alpha_2$ for $\alpha_1, \alpha_2 \in \Delta^+$ then $\alpha_1 \in \Phi$ or $\alpha_2 \in \Phi$.*

Saturated sets are in $1 - 1$ correspondence with elements of W [GoodW98], $|\Phi_w| = l(w)$, the minimum number of simple reflections σ_i such that $w = \sigma_1\sigma_2 \dots \sigma_k$ [Humph72]. We can form a so-called directed graph of W. Its set of vertices is W and there is an arrow from w to w' iff $l(w') = l(w) + 1$ and $w' = \sigma_\alpha w$ for some $\alpha \in \Delta^+$. There is a rule that allows us to determine which arrows α can go from a vertex w:

Lemma 2. (Čap criterion, [Čap]). *Let $\alpha \in \Delta^+$ and Φ_w be a saturated set. There is an arrow labeled by α from Φ_w iff there is a $k \in \mathbb{Z}_0^+$, $|\Phi_\alpha| = 2k + 1$, $|\Phi_w \cap \Phi_\alpha| = k$. The endpoint of this arrow is the set $\Phi_{w'} = \{\alpha\} \cup (\Phi_w \cap \Phi_\alpha) \cup \sigma_\alpha(\Phi_w \setminus \Phi_\alpha)$, where $w' = \sigma_\alpha w$.*

Proof. Since σ_α is a reflection, its expression in terms of simple reflections must contain an odd number of terms. Thus a set $\Phi'_\alpha := \Phi_\alpha \setminus \{\alpha\}$ has $2k$ elements for some k. If $w' = \sigma_\alpha w$, then $w'^{-1} = w\sigma_\alpha$. This means that $w'^{-1}\alpha = -w^{-1}\alpha$, i.e. $\alpha \in \Phi_{w'} \Leftrightarrow \alpha \notin \Phi_w$. Thus we can assume $\alpha \notin \Phi_w$. Δ^+ splits into five disjoint subsets: $\Phi^{++} := \Phi_w \cap \Phi'_\alpha$, $\Phi^{+-} := \Phi_w \cap \Phi^c_\alpha$, $\Phi^{-+} := \Phi^c_w \cap \Phi'_\alpha$, $\Phi^{--} := \Phi^c_w \cap \Phi^c_\alpha$ and $\{\alpha\}$, where the superscript c means the complement.

First we observe that $-\sigma_\alpha(\Phi'_\alpha) \subset \Phi'_\alpha$ and $\sigma_\alpha(\Phi^c_\alpha) \subset \Phi^c_\alpha$. Since σ_α is injective, it is an automorphism of Φ^c_α. Let $\beta \in \Phi^c_\alpha$, $\beta' := \sigma_\alpha\beta$, then $w^{-1}\beta = w^{-1}\sigma_\alpha\beta' = w'^{-1}\beta'$ and hence $\beta \in \Phi_w \Leftrightarrow \beta' \in \Phi_{w'}$ or otherwise stated $\Phi_{w'} \cap \Phi^c_\alpha = \sigma_\alpha(\Phi^{+-})$.

On the other hand, if $\beta \in \Phi'_\alpha$, first assume that $\beta \in \Phi_w$ and consider $w'^{-1}\beta = w^{-1}\sigma_\alpha\beta = w^{-1}\beta - \langle\beta,\alpha\rangle\, w^{-1}\alpha$. Since $\alpha, \beta \in \Delta^+$ and $\sigma_\alpha\beta = \beta - \langle\beta,\alpha\rangle\, \alpha \in \Delta^-$, the number $\langle\beta,\alpha\rangle$ must be positive. Because $\alpha\notin\Phi_w$ by assumption, we see that $w'^{-1}\beta\in\Delta^-$, i.e. $\Phi^{++}\subset\Phi_{w'}$. Moreover it follows that for any $\beta\in\Phi^{++}$ $w^{-1}(\sigma_\alpha(\beta)) = -w'^{-1}\beta\in\Delta^+$ and thus $-\sigma_\alpha(\Phi^{++})\subset\Phi^{-+}$. This means that $|\Phi^{++}|\leq|\Phi^{-+}|$. But the union of these two sets is Φ'_α, thus the former must have $k-e$ elements and the latter $k+e$ elements for some non-negative integer e.

We see that Φ^{-+} can be written as a union of two sets: $-\sigma_\alpha(\Phi^{++})$ and its complement which is invariant with respect to $-\sigma_\alpha$. For $\beta\in-\sigma(\Phi^{++})$ thus $-w'^{-1}\beta = w^{-1}(-\sigma_\alpha\beta)\in\Delta^-$ and thus $\beta\in\Phi^c_{w'}$. On the other hand $\beta\notin-\sigma_\alpha(\Phi^{++})$ implies $\beta\in\Phi_{w'}$. Hence $\Phi_{w'}\cap\Phi'_\alpha = (\Phi^{-+}\setminus-\sigma_\alpha(\Phi^{++}))\cup\Phi^{++}$.

Finally, $\alpha\in\Phi_{w'}$ by assumption. Bringing all this together we get

$$\Phi_{w'} = \sigma_\alpha(\Phi^{+-})\cup(\Phi^{-+}\setminus-\sigma_\alpha(\Phi^{++}))\cup\Phi^{++}\cup\{\alpha\}.$$

which is a union of disjoint sets. Clearly $|\Phi_w| = |\Phi^{+-}\cup\Phi^{++}| = |\sigma_\alpha(\Phi^{+-})\cup\Phi^{++}|$.

Since $|-\sigma_\alpha(\Phi^{++})| = k-e$, $|\Phi^{-+}\setminus-\sigma_\alpha(\Phi^{++})|$ must be $2e$, thus $|\Phi_{w'}| = |\Phi_w|+2e+1$. Since e is non-negative, we see that the length $l(w):=|\Phi_w|$ must increase going from w to w' and it will increase only by one iff $e=0$. But this means that $|\Phi^{++}| = k$ as stated in the lemma and the expression for $\Phi_{w'}$ is given by omiting the second term from the expression upstairs. $\square$

Definition 2 (Hasse diagram). Hasse diagram of a parabolic subalgebra $\mathfrak{p}\subset\mathfrak{g}$ is a labeled oriented graph. The set of its vertices is the set of all saturated subsets of $\Delta^+_\mathfrak{p}$, the set of all positive roots of $\mathfrak{p}$. There is an arrow labelled by $\alpha\in\Delta^+_\mathfrak{p}$ from Φ_w to $\Phi_{w'}$ iff $w' = \sigma_\alpha w$.

Hasse diagram is isomorphic to a subgraph of the directed graph of $\mathfrak{g}$ consisting of vertices w such that $\Phi_w\subset\Delta^+_\mathfrak{p}$ and of all arrows between such vertices. We clearly see from Čap criterion that for any $\alpha\in\mathfrak{p}$ and $\Phi_w\subset\Delta^+_\mathfrak{p}$, $\Phi_{w'}$ where $w'=\sigma_\alpha w$ is a subset of $\Delta^+_\mathfrak{p}$ too. On the other hand, an arrow between any two saturated subsets of $\Delta^+_\mathfrak{p}$ must be labelled by a root $\alpha\in\mathfrak{p}$, since the target saturated set contains α. All $w\in W$, $\Phi_w\subset\Delta^+_\mathfrak{p}$ can be thus expressed as a composition of reflection with respect to roots in $\mathfrak{p}$, i.e. there is a directed path from $\emptyset$ to any other vertex of the Hasse graph.

There are two kinds of roots in $\mathfrak{so}(2n)$, the "plus" and "minus" ones. A root $\beta=\lambda_k\pm\lambda_l$ is in $\Phi_\alpha\equiv\Phi(\alpha)=\Phi(\lambda_l\pm\lambda_j)$ iff

$$\sigma_\alpha\beta = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \lambda_k\pm\lambda_l - (\lambda_k\pm\lambda_l,\lambda_i\pm\lambda_j)(\lambda_i\pm\lambda_j) \tag{3}$$

is a negative root.

Lemma 3. *The sets Φ_α, $\alpha\in\Delta^+$ of the standard contact gradation of $\mathfrak{so}(2n)$ are*

$$\begin{aligned}\Phi(\lambda_i-\lambda_j) &= \{\lambda_i-\lambda_j\}\\ &\cup\{\lambda_i-\lambda_m\,|\,i<m<j\}\\ &\cup\{\lambda_m-\lambda_j\,|\,i<m<j\}\end{aligned}\qquad\begin{aligned}\Phi(\lambda_i+\lambda_j) &= \{\lambda_i+\lambda_j\}\\ &\cup\{\lambda_i-\lambda_m\,|\,i<m<j\}\\ &\cup\{\lambda_m+\lambda_j\,|\,i<m<j\}\\ &\cup\{\lambda_j\pm\lambda_m\,|\,j<m\}\\ &\cup\{\lambda_i\pm\lambda_m\,|\,j<m\}\end{aligned}$$

and thus $\forall \alpha \in \mathfrak{g}_m |\Phi_\alpha \cap \Delta_\mathfrak{p}^+ \setminus \{\alpha\}| = m \cdot q$, *where* $|\Phi_\alpha| = 2q + 1$, $m = 1, 2$.

Proof. One has to check all the possible orderings of the values of k, l, i, j. If $\{k, l\} \cap \{i, j\} = \emptyset$ then the reflection is trivial. The remaining orderings fall into one of the following cases

$$\begin{array}{lcccc} (k, l) = & & (x, i)_{x<i} & (x, j)_{x<i} & (x, j)_{i<x<j} \\ & (i, j) & (i, x)_{j<x} & (j, x)_{j<x} & (i, x)_{i<x<j} \end{array}$$

and one has to check them all for the four different combinations of signs in $\alpha = \lambda_i \pm \lambda_j$, $\beta = \lambda_k \pm \lambda_l$ using the orthogonality of λ_i's.

The second part follows from considering the three cases: $\mathfrak{g}_2 \equiv \{\lambda_1 + \lambda_2\}$, $\mathfrak{g}_1^- \equiv \{\lambda_i - \lambda_j | j \geq 3\}$ and $\mathfrak{g}_1^+ \equiv \{\lambda_i + \lambda_j | j \geq 3\}$. Denote $\Phi_\mathfrak{p}(\alpha) := \Phi_\alpha \cap \Delta_\mathfrak{p}^+ \setminus \{\alpha\}$, then we can visualise the result by

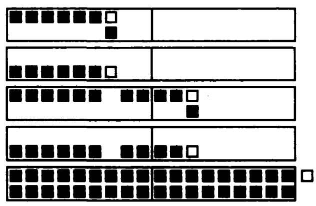
$\Phi_\mathfrak{p}(\lambda_1 - \lambda_j) =$

$\Phi_\mathfrak{p}(\lambda_2 - \lambda_j) =$

$\Phi_\mathfrak{p}(\lambda_1 + \lambda_j) =$

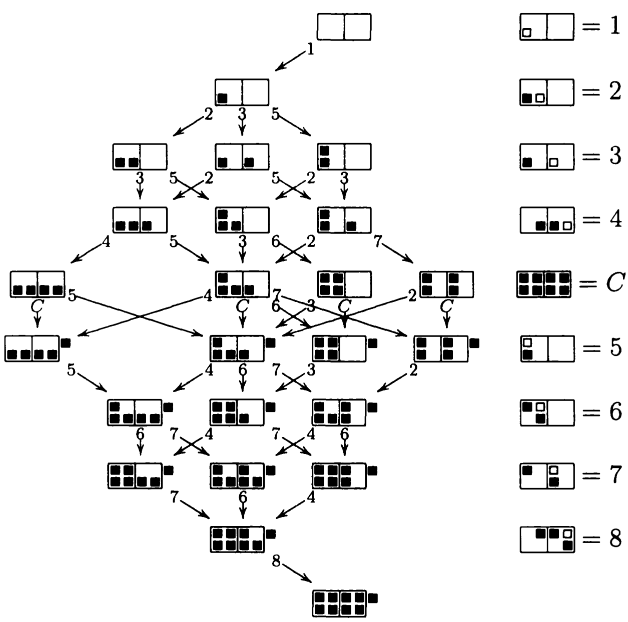
$\Phi_\mathfrak{p}(\lambda_2 + \lambda_j) =$

$\Phi_\mathfrak{p}(\lambda_1 + \lambda_2) =$



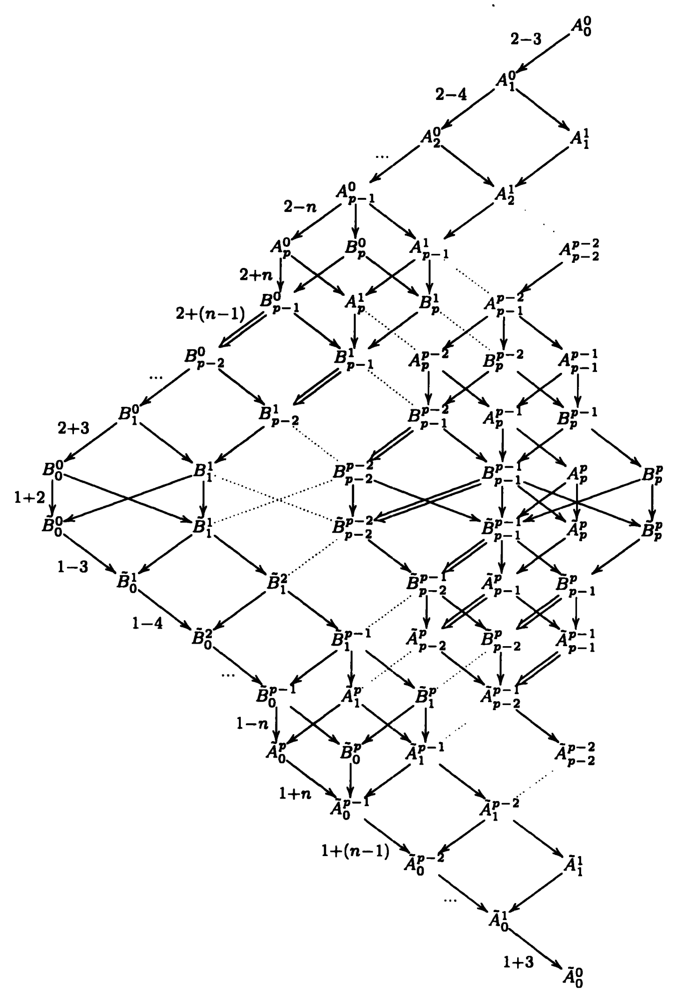
where the rectangle denotes the positions in a general matrix from $\mathfrak{so}(2n)$ that were previously boldfaced, the white square is at the position of the defining root α and the black squares denote the other elements of Φ_α.

Now we have enough information to write down the Hasse diagram of $D_4 \equiv \mathfrak{so}(8)$:



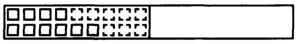
The reader can readily check that in an arbitrary vertex the outgoing arrows correspond precisely to the inscribed roots. We have written the $\Phi_p(\alpha)$'s on the right side for convenience. The rectangles representing saturated sets should be understood in an obvious way — black rectangles represent elements of the corresponding saturated set.

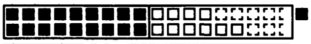
Theorem 1. *Hasse diagram for the standard contact gradation of D_n has the shape*



where the number of vertices in the middle two rows in $n \equiv p + 2$. Two arrows in the same rectangle which are parallel correspond to the same root, thus all the arrows fall into $4(n - 2) + 1$ disjoint families similar to the one denoted by double arrows and labeled the same as its unique member from the "outer belt".

Remark 1. The vertices are denoted according to the type of their respective saturated set:

$A_j^i =$ 

$\tilde{A}_i^j =$ 

$B_j^i =$ 

$\tilde{B}_i^j =$ 

where $0 \leq i \leq j \leq n$. Black squares are roots that are in every set of the corresponding type, white places are ones that are in none of them. Pictures for the type A_j^i contain i framed squares in the first row and j in the second, for $\tilde{A}_i^j$ j *dashed* squares in the first and i in the second, for B_j^i i framed in the first and j dashed in the second and for $\tilde{B}_i^j$ j framed in the first and i dashed in the second. This is because the roots in the right half of the table are numbered in the opposite direction than in the left and also for the sake of symmetry: A_j^i and $\tilde{A}_i^j$ are related by operation of complement and reflection with respect to the center of the table and similarly for B's. The framed squares are roots that are in the respective set for i, j and dashed are ones that aren't.

We do not know yet that these sets are saturated but still we can apply the Čap criterion and decide which outgoing arrows would they allow, if they were saturated. For the root $1 + 2$ we see from lemma 3 that there must be exactly $2p$ roots in the saturated set and this is satisfied by $B_0^0, B_1^1, \ldots, B_p^p, A_p^p$. Similarly for every root α from the first graded part we see that there must be all but one element of $\Phi_p(\alpha)$ in the saturated set where the corresponding arrow starts. Going carefully through the list of the roots and through our four types of sets, we arrive at the following classification.

Lemma 4. *If the sets $A_j^i, B_j^i, \tilde{A}_i^j, \tilde{B}_i^j$ are saturated then they allow arrows of the following type, ordered according to their labels.*[1]

The "generic" roots ($1 \leq k \leq p - 1$):

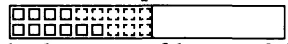
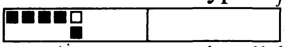
$$\begin{array}{llll}
\lambda_1 - \lambda_{k+2}: & A_l^{k-1} \to A_l^k\,|_{l \geq k} & \lambda_2 + \lambda_{k+2}: & \tilde{A}_k^l \to \tilde{A}_{k-1}^l\,|_{l \geq k} \\
& B_l^{k-1} \to B_l^k\,|_{l \geq k} & & \tilde{B}_k^l \to \tilde{B}_{k-1}^l\,|_{l \geq k} \\
& B_{k-1}^{k-1} \to B_k^k & & B_k^k \to B_{k-1}^{k-1} \\
& \tilde{B}_l^{k-1} \to \tilde{B}_l^k\,|_{l<k} & & B_k^l \to B_{k-1}^l\,|_{l<k} \\
\lambda_2 - \lambda_{k+2}: & A_{k-1}^l \to A_k^l\,|_{l<k} & \lambda_1 + \lambda_{k+2}: & \tilde{A}_l^k \to \tilde{A}_l^{k-1}\,|_{l<k}
\end{array}$$

[1] Note that you can get the content of the second column of the following tables out of the first by inverting the root with respect to the center point of the table, transposing and (un)tilding both the corresponding sets and reversing the arrow. This symmetry is partly a result of a symmetry of the root system $(\lambda_1 + \lambda_j) + (\lambda_2 - \lambda_j) = \lambda_1 + \lambda_2$ and partly of the chosen notation.

The "middle" roots:

$$\begin{array}{rll@{\qquad}rll}
\lambda_1 - \lambda_{k+2}: & A_p^{p-1} \to A_p^p & & \lambda_2 + \lambda_{k+2}: & \tilde{A}_p^p \to \tilde{A}_{p-1}^p & \\
& \tilde{B}_l^{p-1} \to \tilde{A}_l^p & \big|_{l<p} & & A_p^l \to B_{p-1}^l & \big|_{l<p} \\
& B_{p-1}^{p-1} \to \tilde{A}_p^p & & & A_p^p \to \tilde{B}_{p-1}^{p-1} & \\
& \tilde{B}_l^p \to \tilde{A}_l^{p-1} & \big|_{l<p} & & A_{p-1}^l \to B_p^l & \big|_{l<p} \\
\lambda_2 - \lambda_{p+2}: & A_{p-1}^l \to A_p^l & \big|_{l<p} & \lambda_1 + \lambda_{p+2}: & \tilde{A}_l^p \to \tilde{A}_l^{p-1} & \big|_{l<p} \\
& B_p^l \to B_{p-1}^l & \big|_{l<p} & & \tilde{B}_l^{p-1} \to \tilde{B}_l^p & \big|_{p<l} \\
& B_p^p \to \tilde{B}_{p-1}^{p-1} & & & B_{p-1}^{p-1} \to \tilde{B}_p^p & \\
& \tilde{B}_p^p \to \tilde{B}_{p-1}^p & & & B_p^{p-1} \to B_p^p &
\end{array}$$

and the root $\lambda_1 + \lambda_2$ labels arrows $B_k^k \to \tilde{B}_k^k$, $0 \le k \le p$ and $A_p^p \to \tilde{A}_p^p$.

Proof. We have together nine types of roots and four types of sets A_j^i, B_j^i, $\tilde{B}_i^j$, $\tilde{A}_i^j$ and we must go through all these 37 cases. Take for example the root $\alpha = \lambda_1 - \lambda_{k+2}$ and a set of the type A_j^i. The set looks like  and $\Phi_p(\alpha) =$ . Čap criterion and the last part of lemma 3 imply that the set A_j^i must contain all but one element of $\Phi_p(\alpha)$. Since there is no set A that can contain the white square of $\Phi_p(\alpha)$ and not contain the black one under it, we see that a candidate for the starting point of the seeked arrow must contain all the black squares of $\Phi_p(\alpha)$. The only such sets are A_l^{k-1} where $l > k - 1$. Čap criterion gives us the shape of the endpoint—it is A_l^k. The proof for other roots from $\mathfrak{g}_1$ and other sets goes the same way and we will omit it.

For $\lambda_1 + \lambda_2 \in \mathfrak{g}_2$ we see from lemma 3 that the starting set must have $2p$ roots from $\mathfrak{g}_1$ and not contain $\lambda_1 + \lambda_2$. This is satisfied by the sets A_p^p and B_k^k, $0 \le k \le p$. The endpoints are result of Čap criterion again. □

Proof of Theorem. We have already calculated the Hasse diagram for D_4, $H(D_4)$ for briefness. We can easily check that the sets and arrows are precisely the ones given in the statement. Assume that we know all the specified data for D_n, $n \le 4$. We will construct $H(D_{n+1})$ as the union of $H(D_n)$ and the "outer belt" and fill in the missing arrows.

First, let Φ be a saturated set of D_n. We define an inclusion $i : \Delta_p^+(D_l) \to \Delta_p^+(D_{l+1})$ by $i(\lambda_k \pm \lambda_j) = \lambda_k \pm \lambda_{j+1}$, $k = 1, 2, 3 \le j \le p$ and $i(\lambda_1 + \lambda_2) = \lambda_1 + \lambda_2$. We claim that $\Phi' \equiv i(\Phi) \cup \{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3)$ is a saturated set for D_{l+1} of the same type.

If $\Phi = \emptyset$, then $\Phi' = \{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3\}$, which is clearly saturated, because it is connected with the origin of $H(D_{n+1})$ (the empty set) by consecutive arrows $\lambda_2 - \lambda_3$ and $\lambda_1 - \lambda_3$, as can be simply checked by Čap criterion. Assume that for every $\Phi \in H(D_n)$, $|\Phi| = q$, the set Φ' is saturated. Take an arbitrary such Φ and arbitrary root $\alpha \in \mathfrak{g}_1$. We see from lemma 3 and Čap criterion that α is a label of an outgoing arrow from Φ iff $|\Phi_p(\alpha)| - |\Phi_p(\alpha) \cap \Phi| = 1$. If α is a root of the first row, then $\Phi_p i(\alpha)) = i(\Phi_p(\alpha)) \cup \{\lambda_1 - \lambda_3\}$ and if it is of the second row, $\Phi_p i(\alpha)) =$

$i(\Phi_p(\alpha)) \cup \{\lambda_2 - \lambda_3\}$. Hence $|\Phi_p i(\alpha))| - |\Phi_p i(\alpha)) \cap \Phi'| = |\Phi_p(\alpha)| + 1 - (|\Phi_p(\alpha) \cap \Phi| + 1)$ and thus α is an arrow from Φ iff $i(\alpha)$ is an arrow from Φ'. If $\alpha = \lambda_1 + \lambda_2$ then it is an arrow iff $|\Phi| = 2p$ but this happens if and only if $|\Phi'| = 2(p+1)$. Traversing between Φ to Φ' does not change the type of the set $(A, B, \tilde{A}, \tilde{B})$, only adds one to both the indices, therefore the arrow Φ, α is of the same type from the list of lemma 4 as the arrow $\Phi', i(\alpha)$ and the endpoint of the latter is $\Psi' \equiv i(\Psi) \cup \{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3\}$, where Ψ is the endpoint of the former. Hence we get an inclusion of $H(D_n)$ into $H(D_{n+1})$.

Now we claim and prove the shape of the "outer belt" of $H(D_{n+1})$. It consists of four series of sets $(\emptyset, \{\lambda_2 - \lambda_3\}, \{\lambda_2 - \lambda_3, \lambda_2 - \lambda_4\}, \ldots, \Delta_p^+) = (A_0^0, A_1^0, A_2^0, \ldots, A_{p+1}^0, B_{p+1}^0, B_p^0, \ldots, B_0^0, \tilde{B}_0^0, \tilde{B}_0^1, \ldots, \tilde{B}_0^{p+1}, \tilde{A}_0^{p+1}, \tilde{A}_0^p, \ldots, \tilde{A}_0^0)$. A_0^0 is clearly saturated and allows only the arrow $\lambda_2 - \lambda_3$ of Δ_{p+1}^+. We see from lemma 4 that if A_{i-1}^0, $0 < i - 1 < p$ is saturated, it allows an arrow $\lambda_2 - \lambda_{i+2}$ to A_i^0, thus all the members of the series are saturated sets. Moreover, for $0 < i \leq p + 1$ there is an arrow $\lambda_1 - \lambda_3$ from A_i^0 to $A_i^1 = \{1 - 3, 2 - 3\} \cup i(A_{i-1}^0)$ and also there are two arrows $\lambda_2 + \lambda_{p+3}$ from A_p^0 to B_{p+1}^0 and from A_{p+1}^0 to B_p^0. No other arrows exist from the A_i^0-series of sets. We see that B_{p+1}^0 is also a saturated set of $H(D_{n+1})$ and similarly we can show that the whole series $B_{p+1}^0, \ldots, B_0^0$ is in $H(D_{n+1})$ and that only the connecting arrows $\lambda_2 + \lambda_{i+2}$, the mutually parallel arrows $\lambda_1 - \lambda_3$ and the arrow $\lambda_1 + \lambda_2$ from B_0^0 to $\tilde{B}_0^0$ exist. Sets in the series $\tilde{B}_0^0, \ldots, \tilde{B}_0^{p+1}$ and $\tilde{A}_0^{p+1}, \ldots, \tilde{A}_0^0$ allow no other but the connecting arrows.

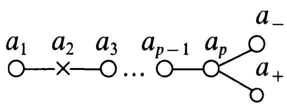
We have now a full control over arrows inside $i(H(D_n))$ labeled by $i(\alpha)$, $\alpha \in \Delta_p^+$ all arrows inside the "outer belt" and all arrows going from the "outer belt" to $i(H(D_n))$. The only arrows that we could have left unnoticed are ones labeled by $\{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}$ that are leaving $i(H(D_n))$. These roots are "generic with $k = 1$" in the language of lemma 4. We see that all the initial sets in the left column of first table of the lemma and the last two in the right column have at least one index zero, so they are not in $i(H(D_n))$. What leaves are arrows $\lambda_2 + \lambda_3 : B_1^1 \to \tilde{B}_0^0, \tilde{B}_1^1 \to \tilde{B}_0^1, \ldots, \tilde{B}_1^{p+1} \to \tilde{B}_0^{p+1}, \tilde{A}_1^{p+1} \to \tilde{A}_0^{p+1}, \ldots, \tilde{A}_1^1 \to \tilde{A}_0^1$ that all lead into sets of the "outer belt".

We have shown that no arrows leave the union of $i(H(D_n))$ and the "outer belt", can there be ones that enter it from a yet unknown saturated set outside the union? According to the remark after definition 2 any set of $H(D_{n+1})$ must be connected with $\emptyset$ by a series of arrows labeled by elements of p. Since we know all arrows leaving $\emptyset$ this series would have to leave the union somewhere which we have proved is not possible. $\square$

3. The BGG diagram

The Hasse diagram gives us first part of the information hidden in the BGG diagram that encodes the structure of invariant differential operators in the appropriate parabolic geometry. Every point of the Hasse diagram correspond to an irreducible representation and what remains to be determined are the highest weights of these representations. They can be calculated from the highest weight of the representation corresponding to $\emptyset$ by means of this well known recipe:

Recipe 1. *Let $p = n - 2$ and*

$\hat{\lambda} =$ 

be the weight of a representation represented by the node $\Phi \in H(D_n)$ expressed by its coordinates in the basis of fundamental weights increased by 1. Let $\Phi \xrightarrow{\alpha} \Phi'$ be an outgoing arrow in the direction of $\alpha \in \Delta_p^+$. Then the representation represented by the node Φ' is given by the affine Weyl action, i.e. its coordinates increased by 1 are given by reflection of $\hat{\lambda}$ with respect to α.

We introduce a notation that allows to write the weights occuring in the BGG diagram in a compact way $(i \leq j)$:

$$\left|{}^{j}_{i}\right| = \sum_{k=1}^{j} a_k$$

$$\left|{}^{j}_{i}\right|_+ = \left|{}^{j}_{i}\right| + a_+ \qquad \left|{}^{j}_{i}\right|_- = \left|{}^{j}_{i}\right| + a_- \qquad \left|{}^{j}_{i}\right|_\pm = \left|{}^{j}_{i}\right| + a_+ + a_-$$

$$+ = a_+ \qquad - = a_- \qquad \pm = a_+ + a_-$$

and $\left|{}^{k,l}_{i,j}\right|_\bullet = \left|{}^{k}_{i}\right| + \left|{}^{l}_{j}\right|_\bullet$, where $\bullet$ stands for $+$, $-$ or $\pm$. When $i = j$, we will write also simply $i, i_+, i_-, i_\pm$. We will also not write full Dynkin diagrams but only the numbers over the nodes, i.e. $1\ 2\ 3 \dots p-1\ p\ {\substack{-\\+}}$ is a shorthand for the labeled diagram in the recipe.

Theorem 2. *If $1\ 2\ 3 \dots p-1\ p\ {\substack{-\\+}}$ is over the node A_0^0 in $H(D_n)$, then*

$$\begin{aligned}
A_j^0 &= \left|{}^{j+1}_{1}\right| - \left|{}^{j+1}_{2}\right|\ 2 \dots j\ \left|{}^{j+2}_{j+1}\right|\ j+3 \dots p-1\ p\ {\substack{-\\+}}\\
A_{p-1}^0 &= \left|{}^{p}_{1}\right| - \left|{}^{p}_{2}\right|\ 2 \dots p-2\ p-1\ {\substack{p_-\\p_+}}\\
A_p^0 &= \left|{}^{p}_{1}\right|_- - \left|{}^{p}_{2}\right|\ 2 \dots p-2\ p-1\ {\substack{p\\p_\pm}}\\
A_j^i &= \left|{}^{j+1}_{i+1}\right| - \left|{}^{j+1}_{1}\right|\ 1 \dots i-1\ \left|{}^{i+1}_{i}\right|\ i+2 \dots j\ \left|{}^{j+2}_{j+1}\right|\ j+3 \dots p-1\ p\ {\substack{-\\+}}\\
A_j^j &= j+1 - \left|{}^{p}_{1}\right|\ 1 \dots j-1\ \left|{}^{j+2}_{j}\right|\ j+3 \dots p-1\ p\ {\substack{-\\+}}\\
A_{p-1}^i &= \left|{}^{p}_{i+1}\right| - \left|{}^{p}_{1}\right|\ 1 \dots i-1\ \left|{}^{i+1}_{i}\right|\ i+2 \dots p-2\ p-1\ {\substack{p_-\\p_+}}\\
A_{p-1}^{p-1} &= p - \left|{}^{p}_{1}\right|\ 1 \dots p-3\ p-2\ {\substack{\left|{}^{p}_{p-1}\right|_-\\ \left|{}^{p}_{p-1}\right|_+}}\\
A_p^i &= \left|{}^{p}_{i+1}\right|_- - \left|{}^{p}_{1}\right|_-\ 1 \dots i-1\ \left|{}^{i+1}_{i}\right|\ i+2 \dots p-2\ p-1\ {\substack{p\\p_\pm}}\\
A_p^{p-1} &= p_- - \left|{}^{p}_{1}\right|_-\ 1 \dots p-3\ p-2\ {\substack{\left|{}^{p}_{p-1}\right|\\ \left|{}^{p}_{p-1}\right|_\pm}}
\end{aligned}$$

$$A^p_p = -\ -\left|{}^{p}_{1}\right|_- \ 1 \dots\ p-3\ p-2 \genfrac{}{}{0pt}{}{(p-1)_\pm}{\left|{}^{p,\,p}_{p,\,p-1}\right|}$$

$$B^0_0 = \left|{}^{p,\,p}_{1,\,2}\right|_\pm \ -\left|{}^{p,\,p}_{2,\,3}\right|_\pm \ 3\ 4 \dots\ p-1\ p \genfrac{}{}{0pt}{}{+}{-}$$

$$B^0_j = \left|{}^{p,\,p}_{1,\,j+2}\right|_\pm \ -\left|{}^{p,\,p}_{2,\,j+2}\right|_\pm \ 2 \dots\ j \left|{}^{j+2}_{j+1}\right| j+3 \dots\ p-1\ p \genfrac{}{}{0pt}{}{+}{-}$$

$$B^0_{p-1} = \left|{}^{p}_{1}\right|_\pm \ -\left|{}^{p}_{2}\right|_\pm \ 2 \dots\ p-2\ p-1 \genfrac{}{}{0pt}{}{p_+}{p_-}$$

$$B^0_p = \left|{}^{p}_{1}\right|_+ \ -\left|{}^{p}_{2}\right|_+ \ 2 \dots\ p-2\ p-1 \genfrac{}{}{0pt}{}{p_\pm}{p}$$

$$B^i_j = \left|{}^{p,\,p}_{i+1,\,j+2}\right|_\pm \ -\left|{}^{p,\,p}_{1,\,j+2}\right|_\pm \ 1 \dots\ i-1 \left|{}^{i+1}_{i}\right| i+2 \dots\ j \left|{}^{j+2}_{j+1}\right| j+3 \dots\ p-1\ p \genfrac{}{}{0pt}{}{+}{-}$$

$$B^j_j = \left|{}^{p,\,p}_{j+1,\,j+2}\right|_\pm \ -\left|{}^{p,\,p}_{1,\,j+2}\right|_\pm \ 1 \dots\ j-1 \left|{}^{j+1}_{i}\right| i+3 \dots\ p-1\ p \genfrac{}{}{0pt}{}{+}{-}$$

$$B^i_{p-1} = \left|{}^{p}_{i+1}\right|_\pm \ -\left|{}^{p}_{1}\right|_\pm \ 1 \dots\ i-1 \left|{}^{i+1}_{i}\right| i+2 \dots\ p-2\ p-1 \genfrac{}{}{0pt}{}{p_+}{p_-}$$

$$B^{p-1}_{p-1} = p_\pm \ -\left|{}^{p}_{1}\right|_\pm \ 1 \dots\ p-3\ p-2 \genfrac{}{}{0pt}{}{\left|{}^{p}_{p-1}\right|_+}{\left|{}^{p}_{p-1}\right|_-}$$

$$B^i_p = \left|{}^{p}_{i+1}\right|_+ \ -\left|{}^{p}_{1}\right|_+ \ 1 \dots\ i-1 \left|{}^{i+1}_{i}\right| i+2 \dots\ p-2\ p-1 \genfrac{}{}{0pt}{}{p_\pm}{p}$$

$$B^{p-1}_p = p_+ \ -\left|{}^{p}_{1}\right|_+ \ 1 \dots\ p-3\ p-2 \genfrac{}{}{0pt}{}{\left|{}^{p}_{p-1}\right|_\pm}{\left|{}^{p}_{p-1}\right|}$$

$$B^p_p = -\ -\left|{}^{p}_{1}\right|_+ \ 1 \dots\ p-3\ p-2 \genfrac{}{}{0pt}{}{\left|{}^{p,\,p}_{p,\,p-1}\right|}{(p-1)_\pm}$$

where $0 < i < j < p - 1$. The representation over $\tilde{A}^j_i$ differs from the one over A^i_j only in the number over the second node and similarly for $\tilde{B}^j_i$ and B^i_j. This number equals

$$-\sum_1^p a_k - \sum_3^p a_k + a_+ + a_- \qquad \textit{for} \quad \tilde{B}^0_0,\ \tilde{A}^0_0$$

$$-\sum_1^p a_k - \sum_{j+1}^p a_k + a_+ + a_- \qquad \textit{for} \quad \tilde{A}^j_0,\ \tilde{B}^j_0 \qquad 0 < j \le p$$

$$-\sum_1^p a_k - \sum_{i+1}^p a_k + a_+ + a_- \qquad \textit{for} \quad \tilde{A}^j_i,\ \tilde{B}^j_i \qquad 0 < i < j < p - 1$$

$$-\sum_1^p a_k + a_+ + a_- \qquad \textit{for} \quad \tilde{A}^p_p,\ \tilde{B}^p_p$$

Proof. To determine the reflection $\sigma_\alpha \lambda = \lambda - \langle \lambda, \alpha \rangle\, \alpha$ we must first know the scalar products $\langle \lambda, \alpha \rangle = (\lambda, \alpha)$ and express α in the basis of fundamental weights. It is a straightforward calculation to show how the roots in Δ^+_p look in bases of simple roots and of fundamental weights:

$110..0^0_0\ 1110..0^0_0 \dots 11..1^0_0\ 11..1^1_0$	$11..1^0_1\ 11..1^1_1\ 11..2^1_1 \dots 111..2^1_1$	$12..2^1_1$
$010..0^0_0\ 0110..0^0_0 \dots 01..1^0_0\ 01..1^1_0$	$01..1^0_1\ 01..1^1_1\ 01..2^1_1 \dots 012..2^1_1$	

$11\text{-}10..0^0_0\ \ 101\text{-}10..0^0_0 \dots\ \ 10..01^{-1}_{-1}\ \ 10..0^{\,1}_{-1}$	$10..0^{-1}_{1}\ \ 10..0\text{-}1^1_1\ \ 10..0\text{-}11^0_0 \dots 1\text{-}110..0^0_0$	$010..0^0_0$
$\text{-}12\text{-}10..0^0_0\ \text{-}111\text{-}10..0^0_0 \dots \text{-}110..01^{-1}_{-1}\ \text{-}110..0^{\,1}_{-1}$	$\text{-}110..0^{-1}_{1}\ \text{-}110..0\text{-}1^1_1\ \text{-}110..0\text{-}11^0_0 \dots \text{-}1010..0^0_0$	

Since the expression of $\lambda_1 + \lambda_2$ in fundamental weights has the only nonzero number over the crossed root, the weights over B_0^0 and $\tilde{B}_0^0$ can differ only in the number over the crossed node. Since $(\lambda_1 - \lambda_k) + (\lambda_2 + \lambda_k) = (\lambda_1 + \lambda_k) + (\lambda_2 - \lambda_k) = \lambda_1 + \lambda_2$ we see that for every arrow in the left part of $H(D_n)$ its counterpart in the right part differs only over the crossed root. We conclude that for every B_j^i and $\tilde{B}_i^j$, the weights differ only over the crossed node and similarly for A's. Hence it suffices to determine the weights in the left part and the coeffitients over crossed nodes in the right part.

What follows is a mere calculation. We can join every saturated set of the left part with the empty set with a path that contains first x arrows of the outer belt $2 - 3$, $2 - 4$, ..., ${}^{2-n}_{2+n}$, ${}^{2+n}_{2-n}$, $2 + (n - 1)$, ..., $2 + 3$ and then it turns up and goes along the arrows labeled $1 - 3$, $1 - 4$, ..., $1 - n$. What bothers us is the variety of cases that must be verified, namely $(n = p + 2)$

$$\begin{array}{ll}
A_0^0 \xrightarrow{2-3} A_1^0 & \\
A_j^0 \xrightarrow{2-(j+3)} A_{j+1}^0 & 0 < j < p - 2 \\
A_{p-2}^0 \xrightarrow{2-(p+1)} A_{p-1}^0 & \\
A_{p-1}^0 \xrightarrow{2-(p+2)} A_p^0 & \\
A_j^0 \xrightarrow{1-3} A_j^1 & 0 < j < p - 1 \\
A_j^i \xrightarrow{1-(i+3)} A_j^{i+1} & 0 < i < j + 1 < p \\
A_i^{i-1} \xrightarrow{1-(i+2)} A_i^i & 0 < i < p - 1 \\
A_{p-1}^0 \xrightarrow{1-3} A_{p-1}^1 & \\
A_{p-1}^i \xrightarrow{1-(i+3)} A_{p-1}^{i+1} & 0 < i < p - 2 \\
A_{p-1}^{p-2} \xrightarrow{1-(p+1)} A_{p-1}^{p-1} & \\
A_p^0 \xrightarrow{1-3} A_p^1 & \\
A_p^i \xrightarrow{1-(i+3)} A_p^{i+1} & 0 < i < p - 1 \\
A_p^{p-1} \xrightarrow{1-(p+2)} A_p^p &
\end{array}$$

The number of cases reflects the number of different "kinds" of simple roots, i.e. the most left one and the two most right with only one neighbor, the series of ones with two neighbors and the ramifying one.

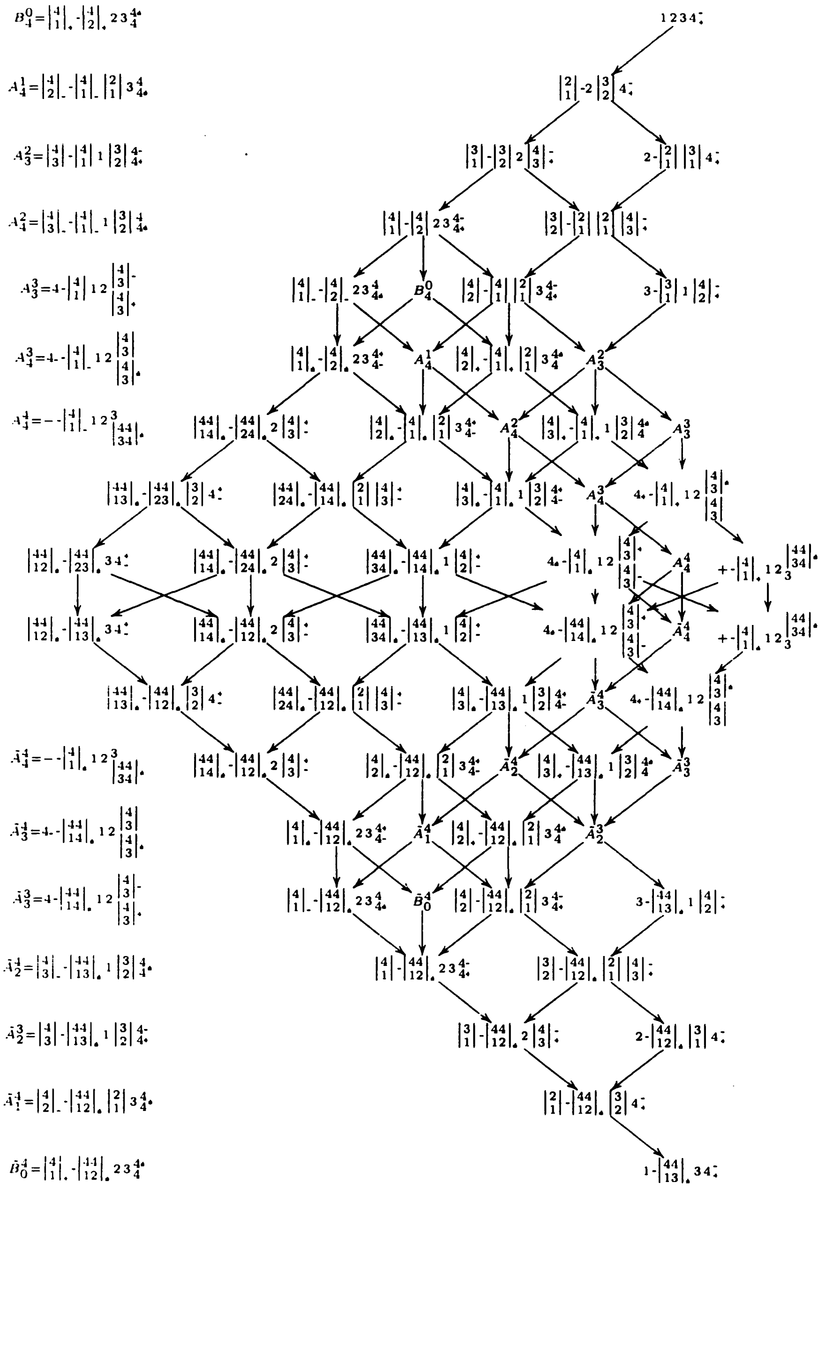
Assume that the weight over A_j^0 is equal to $\hat{\lambda} = \left|{}^{j+1}_{1}\right| - \left|{}^{j+1}_{2}\right| 2 \ldots j \left|{}^{j+2}_{j+1}\right| j+3 \ldots p-1\, p\, {}^{-}_{+}$. The scalar product of this weight with the root $\alpha = 2 - (j + 3) = 011..10..0_0^0$ (there is $j + 1$ nodes with number 1) is given as the ordinary scalar product

$$(\hat{\lambda}, \alpha) = \sum_i \hat{\lambda}_i \alpha_i = -\sum_{k=2}^{j+1} a_k + a_2 + \ldots + a_j + \sum_{k=j+1}^{j+2} a_k = a_{j+2}$$

because the basis of fundamental weights (in which $\hat{\lambda}$ is expressed) is dual to the basis of simple roots (in which α is written). Then the weight over A_{j+1}^0 equals

$$\left(\begin{vmatrix} j+1 \\ 1\end{vmatrix} - \begin{vmatrix} j+1\\ 2\end{vmatrix} 2\ldots j \begin{vmatrix} j+2\\ j+1\end{vmatrix} j+3 \ldots p \,{}^{-}_{+}\right) - \left(-(j+2)(j+2)\,0\ldots 0(j+2) - (j+2)\,0\ldots 0\,{}^{0}_{0}\right) =$$
$$\begin{vmatrix} j+2\\ 1\end{vmatrix} - \begin{vmatrix} j+2\\ 2\end{vmatrix} 2\ldots j+1 \begin{vmatrix} j+3\\ j+2\end{vmatrix} j+4\ldots p\,{}^{-}_{+}$$

where the zeros in the second term mean of course actual zeros, not (a nonexisting) a_0 and there is $j - 1$ of them between the two $j + 2$'s. This is precisely what we supposed. We omit the other cases.



From these arrows we can establish a path leading to ever node of $H(D_n)$ of the A kind. In the same way we can calculate what the $2+n$ does to the weight over A^0_{p-1} and get access to the nodes of the B kind. Since this proceeds mechanically we omit it as well.

Finally we address the question of numbers over crossed nodes in the right part of $H(D_n)$. Since the roots $1-4$, $1-5, \dots, 1+5$, $1+4$ and $2+3$ have zero at the second position if expressed in fundamental weights and therefore do not change the number over the crossed node, we see that excepting the miscellaneous cases $\tilde{A}^p_p$, $\tilde{B}^p_p$, $\tilde{A}^0_0$, $\tilde{B}^p_p$ the number over $\tilde{A}^j_i$, $\tilde{B}^j_i$ does not depend upon i nor the kind $\tilde{A}$, $\tilde{B}$. Thus we can calculate all but $\tilde{A}^0_0$ just by reflecting weights over B^j_j, $0 \le j \le p$ and A^p_p with respect to $1+2$. Finally we determine $\tilde{A}^0_0$ by reflecting $\tilde{A}^1_0$. □

To become more comfortable with what happens we shall draw an example of a BGG diagram for D_6. Although we use the most compact notation, still some labels had to be written aside from the diagram.

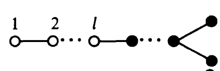
4. Real forms

A real form of a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is a real Lie algebra $\mathfrak{g}$ such that its complexification is $\mathfrak{g}_{\mathbb{C}}$. We shall first summarize some facts concerning real irreducible representations of the real Lie algebra $\mathfrak{g}$ needed in the sequel.

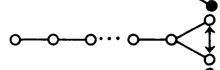
Given a simple complex Lie algebra, its real forms are up to isomorpism in a one to one corresondence with Cartan involutions, i.e. involutive automorphisms θ such that $B(\cdot, \theta\cdot)$ is a negative definite bilinear form, where B is the Killing form of $\mathfrak{g}_{\mathbb{C}}$. The real algebra $\mathfrak{g}$ splits as a direct sum of ± 1 eigenspaces of θ, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ has a centralizer in $\mathfrak{k}$, denoted $\mathfrak{m}$. The real forms are classified by Satake diagrams. Satake diagram is a Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$, where black nodes correspond to the simple root system of $\mathfrak{m}$, the other roots are white and a pair of white roots α_i, α_j is connected by an arrow iff $\alpha_i = \nu\tau(\alpha_j)$, where τ is the automorphism of the root system induced by θ and $\nu = -w_0$, where w_0 is the unique element of the Weyl group such that $w_0(\mathscr{D}) = -\mathscr{D}$, where $\mathscr{D}$ is the dominant Weyl chamber.

The complete list of real forms and Satake diagrams for D_n is given below:

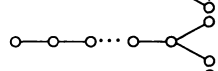
1. $\mathfrak{so}(l, 2n-l)$ $\quad 1 \le l \le n-2$



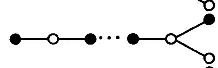
2. $\mathfrak{so}(n-1, n+1)$



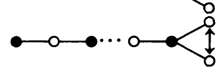
3. $\mathfrak{so}(n, n)$



4. $\mathfrak{u}^*(n, \mathbb{H})$ $\quad n = 2k$



5. $\mathfrak{u}^*(n, \mathbb{H})$ $\quad n = 2k+1$



and the compact form $\mathfrak{so}(2n)$ which does not have a Satake diagram or could be written with all nodes black.

Real representations of real semisimple Lie algebras are classified by means of their complexifications. We shall collect now facts concerning this classification needed in the sequel following the appendix of [OniV88]. First we may distinguish two different classes of real irreducible representations of $\mathfrak{g}$.

Definition 3.

(I) Let V be an irreducible complex representation of $\mathfrak{g}_{\mathbb{C}}$ such that there exists a real structure, i.e. an antilinear involution J commuting with the action of $\mathfrak{g}$. Then the fixed-point set $V_{\mathbb{R}} = \{v \in V \mid Jv = v\}$ of J is an irreducible representation of $\mathfrak{g}$ and V is its complexification. Such representations are called representations of the *real* type.

(II) If V admits no real structure commuting with the action of $\mathfrak{g}$, then V, taken as a real vector space $V^{\mathbb{R}}$, is an irreducible real representation of $\mathfrak{g}$ and its complexification $V^{\mathbb{R}} \otimes \mathbb{C}$ is a sum of two irreducible modules, namely $V^{\mathbb{R}} \otimes \mathbb{C} = W \oplus \overline{W}$. These representations can be divided into two subclasses.

If V admits a quaternionic structure, i.e. an antilinear map $J, J^2 = -\mathrm{Id}$, commuting with the action of $\mathfrak{g}$, then $W \sim \overline{W}$ and the representation is said to be of the *quaternionic* type, otherwise it is of the *complex* type.

Knowing the highest weights of its complexification, it is possible to compute the type of a given real irreducible representation.

Theorem 3. *Let V_ρ be a complex irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ with the highest weight ρ.*

(i) *V_ρ is a complexification of a representation of the real type iff $\rho = \nu\tau\rho$ and the index of ρ (see* [OniV88]*) is even.*

(ii) *ρ is a component of a complexification of a representation of the quaternionic type iff $\rho = \nu\tau\rho$ and the index of ρ is odd.*

(iii) *ρ is a component of a complexification of a representation of the complex type iff $\rho \neq \nu\tau\rho$.*

Moreover, for (ii) and (iii), $V_{\rho^{\mathbb{R}}} \otimes \mathbb{C} \sim V_\rho \oplus V_{\nu\tau\rho}$.

Proof. See [OniV88] and references therein or [ZhiDa82]. We will follows the terminology of [ZhiDa82] and call ρ and $\tau\nu\rho$ correlative representations, if $\rho = \tau\nu\rho$, then ρ is called autocorrelative. □

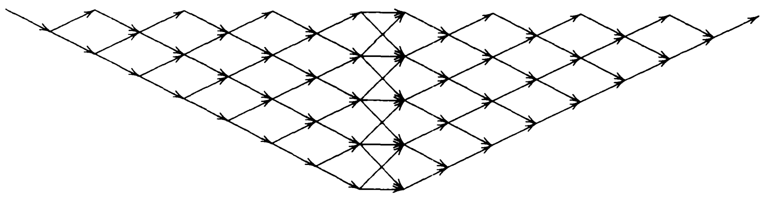
Real (standard) contact gradations are given by crossing a $\nu\tau$-invariant subset of white roots [Yam93]. This excludes from our considerations the real forms $\mathfrak{so}(2n)$ and $\mathfrak{so}(1, 2n-1)$ and the semisimple part of the parabolic subalgebra in other cases is a sum of a real form of $\mathfrak{sl}(2, \mathbb{C})$ and of $\mathfrak{so}(2n-2, \mathbb{C})$. The condition $\rho = \nu\tau\rho$ is nontrivial only for the second and fifth Satake diagram from our list and possibly for the first, since the arrow does not encode information about the action of $\nu\tau$ on the black nodes. According to [GoGro78] (Theorem (8.6.6) and

Proposition (8.7.2)) and [OniV88], τ is nontrivial only for $\mathfrak{so}(2i+1, 2n-(2i+1))$, $i < n$ and ν is nontrivial only for $n = 2j + 1$. Thus $\nu\tau$ is nontrivial for $\mathfrak{so}(i, 2n - i)$ iff the number of black nodes is odd.

To compute the indices we can refer again to [OniV88], [GoodW98] (sect 5.1.7, 5.1.8) and [ZhiDa82]. For $\mathfrak{u}^*(n, \mathbb{H})$ the index of a representation $\rho = (\lambda_1, \ldots, \lambda_n)$ expressed in the basis of fundamental weights is $(-1)^{\sum_\bullet \lambda_i}$ (sum over the set of black nodes). For $\mathfrak{so}(i, 2n - i)$ it is $(-1)^{r(\lambda_{n-1}+\lambda_n)}$, where $r = 1$ if $n - i \bmod 4 = 2$ or 3 and $r = 0$ otherwise.

Definition 4. Let $\mathfrak{g}$ be a real simple Lie algebra with a given parabolic subalgebra $\mathfrak{p}$. Then the BGG diagram for an irreducible representation V of $\mathfrak{g}$ is defined as follows. Vertices of order k are irreducible pieces of the cohomology group $H^k(\mathfrak{g}_-, V)$. There is an arrow from an irreducible piece $V \subset H^k(\mathfrak{g}_-, V)$ to an irreducible piece $W \subset H^{k+1}(\mathfrak{g}_-, V)$ iff there is an arrow in the corresponding complex BGG sequence from an irreducible component of $V_{\mathbb{C}}$ to an irreducible component of $W_{\mathbb{C}}$.

Theorem 4. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a highest weight of an irreducible representation of $\mathfrak{so}(2n, \mathbb{C})$ expressed in the basis of fundamental weights. Let $\mathfrak{g}$ be a real form of $\mathfrak{so}(2n, \mathbb{C})$. Suppose $\lambda_{n-1} = \lambda_n$ and one of the following cases occurs:*
(i) $\mathfrak{g} = \mathfrak{so}(i, 2n - i)$ *for $n - i$ odd*
(ii) $\mathfrak{g} = \mathfrak{so}(n - 1, n + 1)$
(iii) $\mathfrak{g} = \mathfrak{u}^*(2k + 1, \mathbb{H})$ *with $\sum_{i=0}^{k-1} \lambda_{2i+1}$ even*
Then the real BGG sequence has the form



that was constructed from the complex case by merging the nodes A_p^i and B_p^i for $i = 0 \ldots p$. It looks the same as in the complex case otherwise.

Proof. Complex BGG sequences are determined by the Dynkin diagram and the highest weight λ of the representation V. Given the weight λ several cases may occur with different Satake diagrams.

Let us take the real forms with $\tau\nu = \mathrm{Id}$. Then all representations in the BGG are either of the real or quaternionic type. Moreover, the index of a representation is preserved by any arrow. This is because the endpoint $\hat{\lambda}'$ of an arrow α differs from the starting point $\hat{\lambda}$ by an integral multiple of α and all α's expressed in the basis of fundamentgal weights have, $\alpha_n + \alpha_{n-1}$ and $\sum_{i=0}^{k-1} \alpha_{2i+1}$ even, hence do not change the index for $\mathfrak{so}(i, 2n-i)$, $2 \leq i \leq n-2$ and $\mathfrak{u}^*(2k, \mathbb{H})$ respectively. Thus for both cases the real BGG sequence has the same shape as its complexification, and it is isomorphic to the complexification or to a direct sum of two copies of the complexification in the real and quaternionic cases, respectively.

If $\tau\nu \neq \mathrm{Id}$, then V can be of the complex, quaternionic or real type. If V is of the complex type, then it is a component of a complexification of a real representation, and the correlative representation is $\tau\nu(\lambda)$. We clearly see from the structure of the complex BGG that for any $w \in W_{\mathfrak{p}}$, $(w \cdot \tau\nu\lambda)_n = w \mid \lambda_{n-1}$ everywhere except over A_p^i, B_p^i, $\tilde{A}_i^p$, $\tilde{B}_i^p$, $0 \leq i \leq p$, where labels of (A_p^i, B_p^i) and $(\tilde{A}_i^p, \tilde{B}_p^i)$ are pairs of correlative representations. The direct sum of the BGGs for λ and for $\tau\nu(\lambda)$ is thus a complexification of a real BGG diagram. If V is of a real or quaternionic type, then $\hat{\lambda}_{n-1} = \hat{\lambda}_n$. The index is preserved for the same reason as above for $\mathfrak{so}(i, 2n-i)$, $2 \leq i \leq n-1$, but for $\mathfrak{u}^*(n, \mathbb{H})$, $n = 2k+1$ the arrows $1+n$, $1-n$, $2+n$, $2-n$ multiply the index by $(-1)^{a_+} = (-1)^{a_-}$, as we see from the BGG diagram. Thus if V is of a real or quaternionic type, all representations corresponding to B_j^i and $\tilde{B}_i^j$, $j < p$ are of the same type as V since the index of the label of B_{p-1}^i is $(-1)^{a_+ + a_-}$ times the index of A_{p-1}^i. For the same reason the representations corresponding to $\tilde{A}_i^p$ and $\tilde{B}_i^p$ are correlative and are of the same type as V over $\tilde{A}_i^j$, $j < p$. This gives the shape of the BGG stated in the theorem for V of the real type and the same shape as in the complex case for V of the quaternionic type. The index calculated for $\mathfrak{so}(i, 2n-i)$, $2 \leq i \leq n-1$ (the first and the second of the Satake diagrams) is 1 regardless of λ and for $\mathfrak{u}^*(2k+1, \mathbb{H})$ it is 1 iff $\sum_\bullet \lambda_i$ is even, hence V is real precisely in these cases.

Finally we have to verify that for a representation of a quaternionic or complex type we cannot pair it with a different representation than the one we have found. But this is clear since a BGG diagram is an orbit of the Weyl group $W_{\mathfrak{p}}$ and thus each weight λ sits in a unique place and in a unique BGG. □

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