Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 45 (2004), No. 1, 67--77

Persistent URL: `http://dml.cz/dmlcz/142736`

BGG Diagrams for Contact Graded Odd Dimensional Orthogonal Geometries

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Received 20. October 2003

This paper is devoted to the structure of Bernstein–Gelfand–Gelfand resolutions for contact graded orthogonal parabolic geometries. We give an explicit description of the BGG diagram for contact graded odd dimensional orthogonal algebras. The shape of the BGG diagram for this gradation of orthogonal algebra is explicitely computed. The methods used includes the description of Konstant's Lie algebra cohomology based on saturated sets. The treatment of real cases uses a detailed description of real forms of Lie algebras and their real representations. We show that the BGG diagram for every real form of any contact graded odd dimensional orthogonal algebra equals to the one for the complex version.

1. Introduction

Invariant differential operators on manifolds with a given geometric structure were studied intensively for a long time. Recently, a lot of attention is devoted to the case of manifolds with a given parabolic structure (see Slovák [15], Čáp, Schichl [5]). In this case, many invariant differential operators appear in the so called Bernstein–Gelfand–Gelfand (BGG) sequences/diagrams, see Bernstein, Gelfand, Gelfand [1, 2]. In homogeneous case (i.e. if $M = G/P$, G simple and P parabolic), these sequences form complexes analogous to the de Rham complex.

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Key words: Bernstein–Gelfand–Gelfand resolutions, contact graded simple Lie algebras, Lie algebra cohomology, real forms of simple Lie algebras.

AMS classification: 17B70, 17B20, 17B05.

The author acknowledges support by the grants GAČR 201/02/1390 and MSM113200007. Thanks are also due to my supervisor prof. Vladimír Souček.

Their curved analogues were constructed by Čáp, Slovák, Souček [6, 7, 8, 9], Calderbank, Diemer [3], Calderbank, Diemmer and Souček [4]. Operators in the sequence are acting on sections of assiociated (natural) bundles, which can be often decomposed further into irreducible pieces. Each such piece is characterized by an irreducible representation of P, which can be identified with a suitable highest weight for the corresponding reductive Levi factor.

The structure of such decomposition is given by the so called Hasse diagram. For every particular case of parabolic geometry, there is an infinite number of BGG sequences, labelled by irreducible G-modules. For a chosen G-module, weights for individual pieces in the decomposition can be computed explicitly in particular cases. This gives then an explicit description of sources and targets of individual operators.

The BGG sequences were constructed both in complex and real cases. In differential geometry, the real versions are much more common and useful. In most cases, the structure of real and complex BGG diagram (the Hasse diagram) is identical but in general it is not so. Hence it is worthwhile to understand the structure of the Hasse diagram in both complex and real cases.

Parabolic subalgebras of a given simple Lie algebra are in one to one correspondence with decompositions of this algebra into graded Lie algebras. The $|1|$-graded case is studied in much more details than the other cases. Among $|2|$-graded Lie algebras, there is a distinguished class of so called contact simple graded Lie algebras, characterized by the fact that the part with grade 2 is one-dimensional.

The paper is devoted to a special case of a contact graded Lie algebra, corresponding to the odd-dimensional orthogonal Lie algebra. We compute here the structure of the corresponding Hasse diagram and write down explicit expressions for weights in BGG diagram for a given G-module with the highest weight λ. Moreover, it is shown that the structure of the BGG diagram is the same as in the complex case for all real forms of the considered Lie algebra.

2. Some general results on contact graded algebras, BGG and Hasse diagrams

2.1 Contact graded Lie algebras. The contact graded simple Lie algebras were classified in Yamaguchi [16]. The *contact graded simple Lie algebras* are graded simple Lie algebras of length 2, i.e.

$$\mathfrak{g} = \bigoplus_{k=-2}^{2} \mathfrak{g}_k = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ ,$$

such that $\dim \mathfrak{g}_2 = 1$. He showed that for each complex simple Lie algebra, there is unique contact gradation. We shall use the standard notation (see Yamaguchi [16]) denoting the gradation by specifying a subset Σ of the so called "crossed nodes" in the set of simple roots.

2.2 Basic facts on Hasse diagrams and Bernstein–Gelfand–Gelfand sequences. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{p}$ its parabolic subalgebra. A subset Q of the system of positive roots, $Q \subseteq \Phi^+$ of the algebra $\mathfrak{g}$ is called *saturated set* if there is an element w of the Weyl group of $\mathfrak{g}$ such that $Q = Q(w)$, where $Q(w) := \{\alpha \in \Phi^+; w^{-1}\alpha \in -\Phi^+\}$. A subset A of the positive root system is called *saturated set for* $(\mathfrak{p}, \mathfrak{g})$ iff it is a saturated set and $A \subseteq \Phi_{\mathfrak{p}_+}$, where $\Phi_{\mathfrak{p}_+}$ is the set of all positive roots α for which the corresponding root spaces belong to $\mathfrak{g}_+$. We shall use the notation $\Phi_\alpha := Q(\sigma_\alpha)$, where σ_α is the reflection in the root α. It is well known that $\sharp Q(w) = |w|$, where $|w|$ is the reduced length of a Weyl group element w.

It is also well known that a subset $A \subseteq \Phi_{\mathfrak{p}_+}$ is a saturated set for the pair $(\mathfrak{p}, \mathfrak{g})$ iff

- (R1) if $\alpha, \beta \in A$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in A$ and
- (R2) if $\gamma \in A$ and $\gamma = \alpha + \beta$ where $\alpha, \beta \in \Phi^+$ then $\alpha \in A$ or $\beta \in A$

holds.

In what follows, we shall recall some basic facts on the Hasse diagram and BGG diagram in general and then (in the third part of this paper) compute their shapes in the particular case of contact graded odd dimensional orthogonal Lie algebras.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{p}$ its parabolic subalgebra. Hasse diagram for the pair $(\mathfrak{p}, \mathfrak{g})$ is defined as follows. Vertices are saturated sets for $(\mathfrak{p}, \mathfrak{g})$ and there is a labelled oriented arrow $Q(w_1) \xrightarrow{\alpha} Q(w_2)$, $w_1, w_2 \in W_{\mathfrak{g}}$ in the diagram if and only if α is a positive root in the parabolic part such that $w_2 = \sigma_\alpha w_1$ and $|w_2| = |w_1| + 1$.

Let $\mathbb{V}_\lambda$ be the irreducible $\mathfrak{g}$-module with the highest weight λ. Denote by $\mathbb{H}^j(\lambda)$ the Lie algebra cohomology $H^j(\mathfrak{g}_-, \mathbb{V}_\lambda)$. The Kostant theorem (see Konstant [13]) shows how the cohomologies $\mathbb{H}^j(\lambda)$ decompose into irreducible $\mathfrak{p}$-modules: $\mathbb{H}^j(\lambda) = \bigoplus_{k=1}^{m_j} \mathbb{H}^j(\lambda)_k$. There is a one-one correspondence between the modules $\mathbb{H}^j_1(\lambda), \ldots, \mathbb{H}^j_{m_j}(\lambda)$ and the vertices $Q(w^j_1), \ldots, Q(w^j_{m_j})$ of the Hasse diagram for $(\mathfrak{p}, \mathfrak{g})$ with $|w^j_r| = j$.

Let us denote by $H^j(\lambda)$ the homogenous vector bundle associated to the $\mathfrak{p}$-module $\mathbb{H}^j(\lambda) = H^j(\mathfrak{g}_-, \mathbb{V}_\lambda)$. This bundle decomposes in the same way as the Lie algebra cohomologies mentioned above. Let us denote the weights of the corresponding irrducible summands in such a decomposition by λ^j_k, $k = 1, \ldots, m_j$.

The *Bernstein–Gelfand–Gelfand (BGG) sequence for a weight* λ consists of invariant differential operators acting between the spaces $H^j(\lambda)$, $D : H^j(\lambda) \to H^{j+1}(\lambda)$.

Let us define the BGG diagram. *Bernstein–Gelfand–Gelfand (BGG) diagram for* λ is an oriented labelled graph. Its vertices are the weights λ^j_k mentioned above and there is a labelled oriented arrow $\lambda^j_r \xrightarrow{\alpha} \lambda^{j+1}_s$ iff α is a positive root in the parabolic part of the root system of $\mathfrak{g}$ and

$$\lambda^{j+1}_s = \lambda^j_r - 2\frac{(\lambda^j_r, \alpha)}{(\alpha, \alpha)}\alpha.$$

The BGG diagram (for any λ) and the Hasse diagram are isomorphic (see Krump, Souček [12]). For a more general setting, see Lepowsky [14].

3. The case of the contact graded odd dimensional orthogonal algebras

To fix a notation, let us recall some basic facts on the structure theory of odd dimensional orthogonal algebras $B_l = \mathfrak{so}(2l + 1, \mathbb{C})$. Let us choose an arbitrary Cartan subalgebra $\mathfrak{h}$ of the algebra B_l. Let the set $\{\varepsilon_1, \dots, \varepsilon_l\}$ be the standard basis of the dual of $\mathfrak{h}$. We assume that this is an orthogonal basis with respect to the Killing-Cartan form $(,) : \mathfrak{h} \to \mathbb{C}$. Then it is possible to choose the system of simple roots in the following way:

(1) $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, l - 1,$

(2) $\alpha_l = \varepsilon_l$.

Now we give a description of the root spaces of the parabolic subalgebra associated to the contact graded algebra $(B_l, \{\alpha_2\})$ in terms of matrices.

We shall use following symbols: let $i - j$, k, $i + j$ denote the roots $\varepsilon_i - \varepsilon_j$, ε_k, $\varepsilon_i + \varepsilon_j$ respectively for suitable i, j, k.

It is possible to choose the defining bilinear form of the algebra B_l in such a way that the root spaces are placed in the matrix as it is shown at the following picture:

$1 - 3$	$1 - 4$	...	$1 - l$	1	$1 + l$	...	$1 + 3$	$1 + 2$
$2 - 3$	$2 - 4$	...	$2 - l$	2	$2 + l$	...	$2 + 3$	★

As a shorthand, we shall use the corresponding notation:

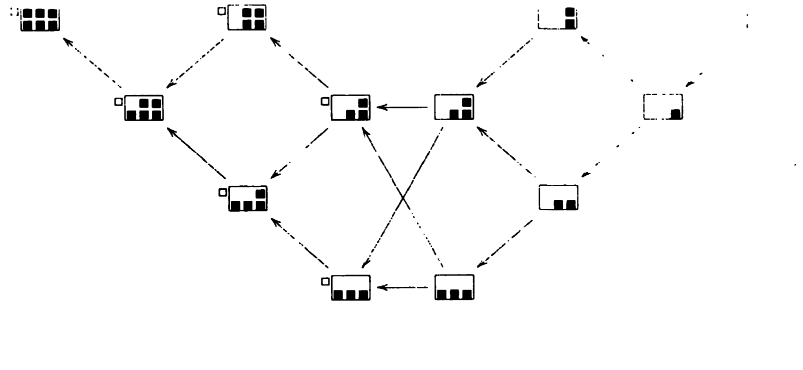
(1) $a_i := \varepsilon_1 - \varepsilon_{i+1}, 1 \le i \le l - 1; a_l := \varepsilon_1; a_{l+i} := \varepsilon_1 + \varepsilon_{l-i+1}, 1 \le i \le l - 1;$

(2) $b_i := \varepsilon_2 - \varepsilon_{i+1}, 1 \le i \le l - 1; b_l := \varepsilon_2; b_{l+i} := \varepsilon_2 + \varepsilon_{l-i+1}, 1 \le i \le l - 2.$

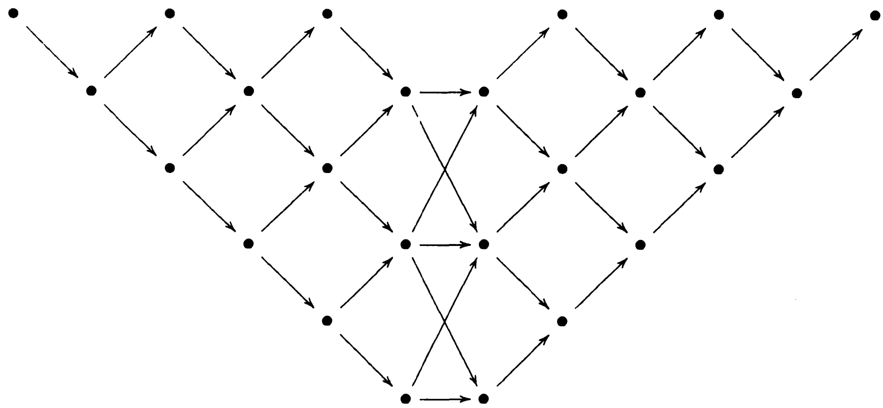
It is easy to compute the sets Φ_{a_i} and Φ_{b_i} (recall that $\Phi_\alpha = Q(\sigma_\alpha)$).

3.1 Structure of saturated sets and the Hasse diagram for $(B_l, \{\alpha_2\})$. We shall start drawing some Hasse diagrams for odd dimensional orthogonal algebras with the contact gradation. One can derive these examples by the definition of the Hasse diagram given above. Then we shall prove a theorem on the Hasse diagram for general odd dimensional contact graded orthogonal algebra.

Example. Let us write the Hasse diagram for the pair $(B_3, \{\alpha_2\})$ (boxes in the diagram indicate a form of corresponding saturated set, described in the Lemma below).



There is the Hasse diagram for $(B_4, \{\alpha_2\})$ at the following picture.



Remark. The examples of the Hasse diagrams for B_3 and B_4 make it easy to see how the shape of the Hasse diagram for B_l is changing with increasing l.

The following lemma describes the form of the saturated sets in general situation (see above the special cases for illustration).

Lemma. *(Saturated sets for $(B_l, \{\alpha_2\})$):* A set A is a saturated set for the graded algebra $(B_l, \{\alpha_2\})$ with a chosen root system Φ and the choice Φ^+ of the positive one if and only if A is one of the two following types:

- $A^l_{1,ij} := (\{a_k; k = 1, ..., i\} \cup \{b_k; k = 1, ..., j\}) \cap \Phi_{p+}$, for $1 \le i \le j \le 2l - 2$ and $i + j \le 2l - 1$,
- $A^l_{2,ij} := (\{a_k; k = 1, ..., i\} \cup \{b_k; k = 1, ..., j\} \cup \{\varepsilon_1 + \varepsilon_2\}) \cap \Phi_{p+}$, for $1 \le i \le j \le 2l - 2$ and $i + j \ge 2l - 1$.

Proof. Firstly, we shall prove that each set of type $A^l_{k,ij}$ for i, j, k satisfying the conditions in the formulation of the lemma, is a saturated set. To check (R1), take $\alpha, \beta \in A^l_{1,ij}$ and $\alpha + \beta \in \Phi$. We should make a conclusion that $\alpha + \beta \in A^l_{1,ij}$. Because of $\alpha + \beta$ never belongs to Φ, the implication is trivially satisfied. Now we do the same for the sets of type $A^l_{2,ij}$. The only pairs $\alpha, \beta \in A^l_{r,ij}$ for which $\alpha + \beta \in \Phi$, are those for which $\alpha + \beta = \varepsilon_1 + \varepsilon_2$. But the sum $\varepsilon_1 + \varepsilon_2$ is contained in $A^l_{2,ij}$ for each i, j. We shall show that the condition (R2) holds. We shall go through the list of all decompositions of an element of $A^l_{1,ij}$ or $A^l_{2,ij}$ into two positive roots.

- $\varepsilon_1 - \varepsilon_k = (\varepsilon_1 - \varepsilon_m) + (\varepsilon_m - \varepsilon_k)$; the element $\varepsilon_m - \varepsilon_k$ is a positive root, iff the relation $m < k$ is satisfied. From the definition of $A^l_{r,ij}$ it follows that if $\varepsilon_1 - \varepsilon_k \in A^l_{r,ij}$ and $m < k$ then $\varepsilon_1 - \varepsilon_m \in A^l_{r,ij}$.
- $\varepsilon_2 - \varepsilon_k = (\varepsilon_2 - \varepsilon_m) + (\varepsilon_m - \varepsilon_k)$; the element $\varepsilon_m - \varepsilon_k$ is a positive root, iff the relation $m < k$ is satisfied. From the definition of $A^l_{r,ij}$ it follows that if $\varepsilon_2 - \varepsilon_k \in A^l_{r,ij}$ and $m < k$, then $\varepsilon_2 - \varepsilon_m \in A^l_{r,ij}$.
- $\varepsilon_1 = (\varepsilon_1 - \varepsilon_m) + \varepsilon_m$, it is obvious that $\varepsilon_1 - \varepsilon_m \in A^l_{r,ij}$, if $\varepsilon_1 \in A^l_{r,ij}$.

- $\varepsilon_2 = (\varepsilon_2 - \varepsilon_m) + \varepsilon_m$, it is obvious that $\varepsilon_2 - \varepsilon_m \in A^l_{r,ij}$, if $\varepsilon_2 \in A^l_{r,ij}$.
- $\varepsilon_1 + \varepsilon_k = (\varepsilon_1 + \varepsilon_m) + (\varepsilon_k - \varepsilon_m)$, $m > k$; if $\varepsilon_1 + \varepsilon_k \in A^l_{r,ij}$ then $\varepsilon_1 + \varepsilon_m \in A^l_{r,ij}$.
- $\varepsilon_2 + \varepsilon_k = (\varepsilon_2 + \varepsilon_m) + (\varepsilon_k - \varepsilon_m)$, $m > k$; if $\varepsilon_2 + \varepsilon_k \in A^l_{r,ij}$ then $\varepsilon_2 + \varepsilon_m \in A^l_{r,ij}$.
- The only remaining root is $\varepsilon_1 + \varepsilon_2 \in A^l_{2,ij}$. We can decompose it into two sums: $\varepsilon_1 + \varepsilon_2$ or $(\varepsilon_1 - \varepsilon_k) + (\varepsilon_2 + \varepsilon_k)$. In each decomposition, there is a summand which belongs to $A^l_{2,ij}$.

Secondly, we are going to check, that there are no other saturated sets then those that are written in the formulation of this lemma. Let A be a saturated set for $(B_l, \{\alpha_2\})$.

- First, let us suppose that $\varepsilon_1 + \varepsilon_2 \notin A$.
  - If $\varepsilon_1 - \varepsilon_k \in A$ then the decomposition $\varepsilon_1 - \varepsilon_k = (\varepsilon_1 - \varepsilon_j) + (\varepsilon_j - \varepsilon_k)$ for $2 < j < k$ implies $\varepsilon_1 - \varepsilon_j \in A$.
  - If $\varepsilon_1 \in A$ then the decomposition $\varepsilon_1 = (\varepsilon_1 - \varepsilon_j) + \varepsilon_j$ implies $\varepsilon_1 - \varepsilon_j \in A$ for $j = 3, \ldots, l$.
  - If $\varepsilon_1 + \varepsilon_k \in A$ for $k \geq 3$ then $\varepsilon_1 \in A$. The decomposition, $\varepsilon_1 + \varepsilon_k = (\varepsilon_1 - \varepsilon_j) + (\varepsilon_j + \varepsilon_k)$ implies $\varepsilon_1 - \varepsilon_j \in A$ for $j = 3, \ldots, k - 1$.
  - An analogical inspection could be done for "b-roots".
  - The following decomposition $\varepsilon_1 + \varepsilon_k = (\varepsilon_1 - \varepsilon_2) + (\varepsilon_2 + \varepsilon_k)$ implies that if $\varepsilon_1 + \varepsilon_k \in A$ then $\varepsilon_2 + \varepsilon_k \in A$. This implies the inequality $i \leq j$ in the formulation of this lemma.
  - We shall show that the inequality $i + j \leq 2l - 1$ holds. Suppose $i + j \geq 2l$ to get a contradiction. Thus if $\varepsilon_2 + \varepsilon_k \in A$ and $\varepsilon_2 - \varepsilon_{k+1} \notin A$, then $\varepsilon_1 - \varepsilon_k \in A$ and consequently $\varepsilon_1 + \varepsilon_2 \in A$, a contradiction. If $\varepsilon_2 \in A$ and $\varepsilon_2 + \varepsilon_j \notin A$, $j = 3, \ldots, l$, then $\varepsilon_1 \in A$ and consequently $\varepsilon_1 + \varepsilon_2 \in A$, a contradiction again. Because there is no other possibility different from those we have discussed, the claim of this item is proved.
- The case $\varepsilon_1 + \varepsilon_2 \in A$ is quite similar. We shall comment only the inequality $i + j \geq 2l - 1$. If $\varepsilon_1 + \varepsilon_2 \in A$ then $\varepsilon_2 \in A$ or $\varepsilon_1 \in A$. Because $\varepsilon_1 \in A$ implies, $\varepsilon_2 \in A$ (due to the inequality $i \leq j$), the root ε_2 is in A in the case of that $\varepsilon_1 + \varepsilon_2 \in A$. Therefore $j \geq l$ and we shall discuss two cases. First, suppose that $\varepsilon_2 + \varepsilon_k \in A$ and $\varepsilon_2 + \varepsilon_{k+1} \notin A$. From the decomposition $\varepsilon_1 + \varepsilon_2 = (\varepsilon_2 + \varepsilon_{k+1}) + (\varepsilon_1 - \varepsilon_{k+1})$, it follows that $\varepsilon_1 - \varepsilon_{k+1} \in A$. Thus $i + j \geq 2l - 1$. Second, suppose that $\varepsilon_2 \in A$ and $\varepsilon_2 + \varepsilon_k \notin A$ for $k = 3, \ldots, l$. From the decomposition $\varepsilon_1 + \varepsilon_2 = (\varepsilon_2 + \varepsilon_l) + (\varepsilon_1 - \varepsilon_l)$, it follows that $\varepsilon_1 - \varepsilon_l \in A$ and therefore $i + j \geq 2l - 1$. □

To describe the structure of arrows in the Hasse diagram, we shall need the following lemma.

Lemma. There is an oriented labelled arrow $\Phi_{w_1} \xrightarrow{\alpha} \Phi_{w_2}$ in the Hasse diagram for $(\mathfrak{p}, \mathfrak{g})$ iff there is a positive integer $k \in \mathbb{N}$ such that $|\Phi_{w_2}| - |\Phi_{w_1}| = k\alpha$, where

$$|\Phi_w| := \sum_{\beta \in \Phi_w \cap \Phi_{\mathfrak{p}_+}} \beta.$$

Proof. See Krump, Souček [12]. □

Theorem *(Hasse diagram for $(B_l, \{\alpha_2\})$).* The Hasse diagram for the parabolic algebra $(B_l, \{\alpha_2\})$ has the following structure: The set of all vertices V^l may be identified with the set $\{A^l_{1,ij}\} \cup \{A^l_{2,ij}\}$, for i, j described in the lemma on saturated sets above. There are following arrows in the Hasse diagram:

(1) $A^l_{1,ij} \xrightarrow{b_{j+1}} A^l_{1,i(j+1)}$, where $i + j < 2l - 1, 1 \leq i \leq j \leq 2l - 2$,
(2) $A^l_{1,ij} \xrightarrow{a_{i+1}} A^l_{1,(i+1)j}$, where $i + j < 2l - 1, 1 \leq i \leq j \leq 2l - 2$,
(3) $A^l_{1,ij} \xrightarrow{a_{i+1}} A^l_{2,(i+1)(j-1)}$, where $i + j = 2l - 1, i = 1, \ldots, 2l - 2$,
(4) $A^l_{1,ij} \xrightarrow{a_{2l-1}} A^l_{2,ij}$, where $i + j = 2l - 1, i = 1, \ldots, 2l - 2$,
(5) $A^l_{1,ij} \xrightarrow{b_{j+1}} A^l_{2,(i-1)(j+1)}$, where $i + j = 2l - 1, i = 1, \ldots, 2l - 2$,
(6) $A^l_{2,ij} \xrightarrow{b_{j+1}} A^l_{2,i(j+1)}$, where $i + j > 2l - 1, 1 \leq i \leq j \leq 2l - 2$,
(7) $A^l_{2,ij} \xrightarrow{b_{j+1}} A^l_{2,(i+1)j}$, where $i + j > 2l - 1, 1 \leq i \leq j \leq 2l - 2$.

Proof. We use the previous lemma. It is easy to compute the first two coordinates of $|A^l_{k,ij}|$ in the $\{\varepsilon_r\}^l_{r=1}$ basis. Let us denote the projection onto the s-th coordinate of an element of the subalgebra $\mathfrak{h}$ by $pr_s : \mathfrak{h} \to \mathbb{C}$, $1 \leq s \leq l$. Regarding the structure of saturated sets, we obtain

- $pr_1(|A^l_{1,ij}|) = i - 1$; $pr_2(|A^l_{1,ij}|) = j - 1$
- $pr_1(|A^l_{2,ij}|) = i$; $pr_2(|A^l_{2,ij}|) = j$, for all admissible i, j.

Clearly $pr_s(A^l_{r,ij}) \in \{0, \pm 1, \pm 2\}$, for $s > 2$ and all admissible r, i, j. We already know that the only possible labels of arrows in the Hasse diagram, are $\varepsilon_1 \pm \varepsilon_k$, $\varepsilon_2 \pm \varepsilon_k$, ε_1, ε_2. Second we know that the number of elements of sets which are joined by an arrow differs exactly by one. If we add a a-root or a b-root, respectively the first or the second coordinate, respectively of $|A^l_{k,ij}|$ increases of one. Other coordinates of $|A^l_{k,ij}|$ either decrease (if the root $a_j, j < 1$ was added into the set) or increase (if the root $a_j, j > l$ was added into the set) or do not change. From that, it is clear that all arrows occurring in the Hasse diagram must be exactly those described in the formulation of this theorem. The inspection of this fact is very simple and it suffers only a carefully handling with indices and using the facts on projections written above.

The only problem could be caused by handling with the root $\varepsilon_1 + \varepsilon_2$. An arrow labelled by this root could be placed only between those saturated sets A, B for which the coordinates of $|A|$ and $|B|$ either differ by 1 at the two first positions or are the same on the other one. Thus these sets differs only by the element $\varepsilon_1 + \varepsilon_2$. From this it is evident that these arrows are exactly of the fourth case mentioned in the formulation of this theorem. □

3.2 BGG diagrams for the real forms of $(B_l, \{\alpha_2\})$. We shall consider how the BGG diagram of a real form of the algebra B_l behaves according to the behaviour of its complexification. At the first part, we shall deal with the complex case, in the second part of this subsection with the real one. In that part part we shall use some basic facts on representation theory of real simple Lie algebras, see Zhida, Dagan [17]. Their approach to a real representation is based on deriving its structure from the so called *weighted Satake diagram.*

3.2.1 *BGG diagrams – the complex case.* Let us fix a notation for writing of weights which occur in the BGG diagram for $(\mathfrak{p}, \mathfrak{g})$. For a vector (in fact, we shall use this notation only for $\mathfrak{p}$-dominant weights) $\beta = (c_1, \dots, c_l)$ written in terms of fundamental weights basis, let us define the following vectors ("γ-vectors"):

For $1 \le i \le j \le l$,

$$\gamma_{1,ij} := \left(\sum_{k=i}^{j} c_k, \; -\sum_{k=1}^{j} c_k, c_1, \dots, c_l\right).$$

For $1 \le i \le j \le 2l - 2$ and $j > l$,

$$\gamma_{1,ij} := \left(\sum_{k=i}^{l} c_k + \sum_{k=i}^{j-1} c_{l-k}, \; -\left(\sum_{k=i}^{l} c_k + \sum_{k=i}^{j-l} c_{l-k}\right), c_1, \dots, c_l\right).$$

Now we define the following vectors ("β-vectors")

1. $$\beta^l_{1,ij} := (\gamma^1_{ij}, \dots, \gamma^i_{1,ij}, \gamma^{i+1}_{1,ij} + \gamma^{i+2}_{1,ij}, \gamma^{i+3}_{1,ij}, \dots, \gamma^{j+1}_{1,ij}, \gamma^{j+2}_{1,ij} + \gamma^{j+3}_{1,ij}, \gamma^{j+4}_{1,ij}, \dots, \gamma^{l+2}_{1,ij}),$$
$$1 \le i \le j \le 2l - 2 \text{ and } j < l - 1.$$

2. $$\beta^l_{1,ij} := (\gamma^1_{ij}, \dots, \gamma^i_{1,ij}, \gamma^{i+1}_{1,ij} + \gamma^{i+2}_{1,ij}, \gamma^{i+3}_{1,ij}, \dots, \gamma^{2l-j}_{1,ij}, \gamma^{2l-j+1}_{1,ij} + \gamma^{2l-j+2}_{1,ij}, \gamma^{2l-j+3}_{1,ij}, \dots, \gamma^{l+2}_{1,ij}),$$
$$1 \le i \le j \le 2l - 2 \text{ and } j > l.$$

3. $$\beta^l_{1,ij} := (\gamma^1_{ij}, \dots, \gamma^i_{1,ij}, \gamma^{i+1}_{1,ij} + \gamma^{i+2}_{1,ij}, \gamma^{i+3}_{1,ij}, \dots, \gamma^l_{1,ij}, 2\gamma^{l+1}_{1,ij} + \gamma^{l+2}_{1,ij}),$$
$$i = 1, \dots, l - 2, j = l - 1, l.$$

4. $$\beta^l_{1,(l-1)j} := (\gamma^1_{ij}, \dots, \gamma^{l-1}_{1,ij}, 2\gamma^l_{1,ij} + 2\gamma^{l+1}_{1,ij} + \gamma^{l+2}_{1,ij}), \; j = l - 1, l.$$

Let us define $\beta^l_{2,ij}$. In the case of $i + j \ge 2l - 1$, the vector $\gamma_{2,ij}$ is equal to $\gamma_{1,ij}$, only with the exception of its second coordinate. The second coordinate of this "γ-vector" is equal to $-(\sum_{k=1}^{l} c_k + \sum_{k=2}^{l-1} c_k)$. The vector $\beta^l_{2,ij}$ is contructed in the same way as in the case of $\beta_{1,(2l-j+1)(2l-i+1)}$ but using the vector $\gamma_{2,(2l-j+1)(2l-i+1)}$ defined above.

Thus the betas are symmetric w.r. to the vertical symmetry axis of the Hasse diagram with the only exception, namely the coefficient above the crossed node $\{\alpha_2\}$.

Let us denote the weight in the BGG diagram for β placed at the position corresponding to the saturated set $A^l_{k,ij}$ in the Hasse diagram by $A^l_{k,ij}(\beta)$.

With help of this notation we can formulate the following theorem on the complex BGG-diagram.

Theorem *(BGG diagram for $(B_l, \{\alpha_2\}, \beta)$ – the complex case).* Let $\beta = \sum_{k=1}^{l} c_k \tilde{\omega}_k$ be a $\mathfrak{g}$-dominant weight.

- If $1 \le i \le j \le 2l - 2$, $i + j \le 2l - 1$ then $A^l_{1,ij}(\beta) = \beta^l_{1,ij}$,
- If $1 \le i \le j \le 2l - 2$, $i + j \ge 2l - 1$ then $A^l_{2,ij}(\beta) = \beta^l_{2,ij}$,

Proof. We prove this theorem for the case 1 only. The second case could be done in a similar way. We shall use an induction on $k = i + j$. I. The first step $k = 2$ is clear, $A^l_{1,11} = \beta = \beta^l_{1,ij}$. II. Let us suppose that the theorem is true for all i, j, s.t. $i + j = k$. From the structure of the Hasse diagram which is isomorphic to the BGG diagram, it follows that it suffices to prove that $\sigma_{\varepsilon_1 - \varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta_{1,(i+1)j}$ and $\sigma_{\varepsilon_2 - \varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta_{1,i(j+1)}$. Let us compute

$$\sigma_{\varepsilon_1 - \varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta_{1,ij} - 2\frac{(\varepsilon_1 - \varepsilon_{i+2}, \beta^l_{1,ij})}{(\varepsilon_1 - \varepsilon_{i+2}, \varepsilon_1 - \varepsilon_{i+2})}(\varepsilon_1 - \varepsilon_{i+2}).$$

From the structure theory of the odd orthogonal algebras, it is well known that

- $\tilde{\omega}_i = \varepsilon_1 + \ldots + \varepsilon_i$, $i = 1, \ldots, l - 1$; $\tilde{\omega}_l = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_l)$,

By substituting that relations into the previous formula for the reflection (using the fact that $\{\varepsilon_i\}_{i=1}^l$ is an orthogonal basis), we obtain $\sigma_{\varepsilon_1 - \varepsilon_{i+2}}(\beta^l_{1,ij}) = \beta_{1,(i+1)j}$ after a straightforward computation regarding the recipe of the construction of $\beta^l_{r,ij}$ described above. In an analogical way, one can prove the remaining induction step, from (i, j) to $(i, j + 1)$. □

3.2.2 *BGG diagrams – the real case.* We shall denote a *Satake diagram* of type BI (see [10]) by $(B_l, \{\alpha_2\}, p)$ if there are l nodes in the diagram and the last p nodes are black. A *weighted Satake diagram* $(B_l, \{\alpha_2\}, p)$ with p black nodes will be denoted by $(\lambda_1, \ldots, \lambda_l | p)$ if the numbers $\lambda_1, \ldots, \lambda_l$ are written above the nodes of such a diagram.

From the structure theory of simple real Lie algebras, it follows that the only admissible real forms of odd dimensional orthogonal algebras admitting the contact structure are represented by Satake diagrams of the type BI with the property that their first two nodes are white, see Yamaguchi [16].

We shall denote the real representation by $\mathbb{V}_\beta$, and call it *real representation with the highest weight* β if its complexification contains a complex irreducible representation with the highest weight β. We shall use the same notation as for a weighted Satake diagram $\beta = (\lambda_1, \ldots, \lambda_l | p)$ as for the highest weight of the corresponding real representation.

The *Maltsev height* $m(\beta)$ of the representation $\mathbb{V}_\beta$ with the highest weight $\beta = (\lambda_1, \ldots, \lambda_l | p)$ is equal to

$$m(\beta) = \sum_{k=p}^{l-1} i(2l + 1 - i)\,\lambda_k + \frac{l(l+1)}{2}\lambda_l,$$

(see Goodman, Wallach [10], 5.1.8 Exercises).

Corollary. The complexification $\mathbb{V}_\beta \otimes_{\mathbb{R}} \mathbb{C}$ of the real representation $\mathbb{V}_\beta$ of the corresponding real form $(B_l, \{\alpha_2\}, p)$ splits into two correlative representation iff the complexification $\mathbb{V}_{\beta'} \otimes \mathbb{C}$ of the representation $\mathbb{V}_{\beta'}$ with the highest weight $\beta' = A^l_{k,ij}(\beta)$ does.

Proof. This is an easy consequence of the theorem on real representations of Zhida, Dagan [17]. From the previous theorem on the complex BGG diagram, we know that the number λ_l' above the last node in the weighted Dynkin diagram of the complex representation $\mathbb{V}_{\beta'} \otimes_{\mathbb{R}} \mathbb{C}$ of B_l (which is placed at the position $A^l_{k,ij}$) differs from the number λ_l over the last node of the Dynkin diagram at the position $A^l_{1,11}$ in the BGG diagram only by an even number, say $2k$ (see the last components in the γ-vectors defined above). From the form of the Maltsev height for the algebra (B_l, p) given above, it follows that $m(\beta) = \frac{l(l+1)}{2}\lambda_l \bmod 2$. The Maltsev height of any representation in the BGG diagram (not only of that at the first position $A^l_{1,11}$) is equal mod 2 to $m(A^l_{k,ij}(\beta)) = \frac{l(l+1)}{2}\lambda_l' \bmod 2 = \frac{l(l+1)}{2}(2k + \lambda_l) \bmod 2 = \frac{l(l+1)}{2}\lambda_l \bmod 2$ (recall that $\lambda_l' = \lambda_l + 2k$), because $l(l+1)$ is an even number. The statement of this corollary follows from the well known classification theorem which is based on the parity of the Maltsev height of the corresponding representation, see Zhida, Dagan [17]. □

Theorem *(BGG diagram for $(B_l, \{\alpha_2\}, \beta)$ – the real case).* The BGG diagram for a real form of algebra $(B_l, \{\alpha_2\})$ has the same form as the BGG diagram in the complex case (for the same weight).

Proof. If we complexify each representation in the real BGG diagram, we should get either a sum of two correlative complex irreducible representations or a complex irreducible representation. According to the previous corollary, we get the mentioned sum at the first position $A_{1,11}$, iff we get it at any position of the BGG diagram. Analogously for the irreducible complexification. □

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