Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 45 (2004), No. 1, 55--66

Persistent URL: `http://dml.cz/dmlcz/142735`

Quasigroups Which Are Unions of Three Proper Subquasigroups

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Received 27. September 2003

Quasigroups that are unions of three proper subquasigroups are characterized.[1]

1. Quasigroups

A groupoid is a non-empty set equipped with a binary operation (usually denoted multiplicatively). A groupoid Q is said to be a quasigroup if for all $a, b \in Q$ there exist uniquely determined elements $u, v \in Q$ such that $au = b = va$.

Proposition 1.1 *Let $A_1, \dots, A_n$, $n \geq 2$, be proper subquasigroups of a quasigroup Q. If $Q = A_1 \cup \dots \cup A_n$, then Q is not one-generated.*

Proof. Let, on the contrary, Q be generated by a single element, say a. Then $a \in A_i$ for at least one i, $1 \leq i \leq n$, and hence $A_i = Q$, a contradiction. □

Proposition 1.2 *Let Q be a non-trivial finitely generated quasigroup. Then:*
(i) Every proper subquasigroup of Q is contained in (at least one) (proper) maximal subquasigroup of Q.
(ii) Q has no maximal subquasigroups if and only if Q has no proper subquasigroups at all.

Proof. The set of proper subquasigroups is upwards inductive and the rest is clear. □

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[1] This research was supported by the grant GAUK 269/2001 and by the institutional grant MSM 113200007.

Remark 1.3 *Clearly, if Q is a quasigroup possessing no proper subquasigroups, then Q is generated by any of its elements and, in particular, Q is countable. On the other hand by [1, Corollary 7], if P is a countable quasigroup containing at least three elements, then P is isotopic to a quasigroup Q such that Q has no proper subquasigroups.*

Proposition 1.4 *(cf. 1.2) Let Q be any non-trivial finitely generated quasigroup such that Q has only finitely many maximal subquasigroups, say $A_1, \dots, A_n$, $n \geq 0$. The following conditions are equivalent:*

(i) Q is not one-generated.
(ii) $n \geq 3$ and $Q = A_1 \cup \dots \cup A_n$.
(iii) $n \geq 1$ and $Q = A_1 \cup \dots \cup A_n$.

Proof. (i) implies (ii). Since Q is not one-generated, every element generates a proper subquasigroup, and hence every element is contained in a maximal subquasigroup (1.2(i)). Consequently, $n \neq 0$ and $Q = A_1 \cup \dots \cup A_n$. Then, clearly, $n \geq 2$ and the inequality $n \geq 3$ is also easily seen (2.1).
(ii) implies (iii). Trivial.
(iii) implies (i). Every element of Q is contained in at least one of the proper subquasigroups $A_1, \dots, A_n$. □

Proposition 1.5 *(cf. 1.2 and 1.4) Assume that there exist finitely many proper subquasigroups $A_1, \dots, A_n$, $n \geq 0$, of a quasigroup Q such that every proper subquasigroup of Q is contained in at least one of $A_1, \dots, A_n$. Then Q is a finitely generated quasigroup and Q has only finitely many maximal subquasigroups.*

Proof. If $n = 0$, then Q has no proper subquasigroups and the assertion is clear. If $n \geq 1$, $a_i \in Q \setminus A_i$ and $S = \{a_i; 1 \leq i \leq n\}$, then Q is generated by S. □

Example 1.6 *Consider the following three-element quasigroup Π:*

Π	α	β	γ
α	α	γ	β
β	γ	β	α
γ	β	α	γ

Then $\{\alpha\}, \{\beta\}$ and $\{\gamma\}$ are maximal subquasigroups of Π and $\Pi = \{\alpha\} \cup \{\beta\} \cup \cup \{\gamma\}$.

Example 1.7 *We may also consider the four-element 2-elementary abelian group $G(+) = \mathbb{Z}_2(+)^{(2)}$ ($\mathbb{Z}_2(+) = \{0,1\}$ is the two-element additive group of integers modulo 2). Then $G = A \cup B \cup C$ and $0 = A \cap B \cap C$, where $A = \{(0,0),(0,1)\}$, $B = \{(0,0),(1,0)\}$ and $C = \{(0,0),(1,1)\}$ are proper subgroups of $G(+)$ (notice that 0, A, B, C and G are the only subgroups of $G(+)$).*

2. The case of two subquasigroups

Proposition 2.1 *Let A and B be subquasigroups of a quasigroup Q such that $Q = A \cup B$. Then either $A = Q$ or $B = Q$.*

Proof. Assume that $A \nsubseteq B$. If $a \in A \setminus B$ and $b \in B$, then $ab \notin B$, and hence $ab \in A$ and $b \in A$. Thus $B \subseteq A$ and consequently, $A = Q$. □

3. The case of three subquasigroups (a)

Throughout this section, let A, B and C be proper subquasigroups of a quasigroup Q such that $Q = A \cup B \cup C$.

Lemma 3.1 *(i) $A \neq B \neq C \neq A$.*
(ii) $Q \neq A \cup B, Q \neq A \cup C$ and $Q \neq B \cup C$.
(iii) $A \nsubseteq B \cup C, B \nsubseteq A \cup C$ and $C \nsubseteq A \cup B$.
(iv) $Q \setminus (A \cup B) \subseteq C, Q \setminus (A \cup C) \subseteq B$ and $Q \setminus (B \cup c) \subseteq A$.

Proof. Easy (use 2.1). □

Lemma 3.2 $A \cap B = A \cap C = B \cap C = A \cap B \cap C$.

Proof. If $a \in (A \cap B) \setminus C$ and $c \in C$, then $ac \notin C$, and so either $ac \in A$ and $c \in A$ or $ac \in B$ and $c \in B$. Thus $C \subseteq A \cup B$, a contradiction with 3.1(iii). We have shown that $A \cap B \subseteq C$ and the remaining inclusions are similar. □

Lemma 3.3 *(i) $(A \setminus B)(B \setminus A) \cup (B \setminus A)(A \setminus B) \subseteq C \setminus (A \cup B)$.*
(ii) $(A \setminus C)(C \setminus A) \cup (C \setminus A)(A \setminus C) \subseteq B \setminus (A \cup C)$.
(iii) $(C \setminus B)(B \setminus C) \cup (B \setminus C)(C \setminus B) \subseteq A \setminus (C \cup B)$.

Proof. If $a \in A \setminus B$ and $b \in B \setminus A$, then $ab \notin A \cup B$, and hence $ab \in Q \setminus (A \cup B) = C \setminus (A \cup B)$. The rest is similar. □

Proposition 3.4 *Assume that $A \cap B \cap C = \emptyset$. Then:*
(i) $\varrho = (A \times A) \cup (B \times B) \cup (C \times C)$ is a congruence of Q and $Q/\varrho \cong \Pi$ (see 1.6.).
(ii) A, B and C are normal maximal subquasigroups of Q.

Proof. (i) By 3.2, the subquasigroups A, B and C are pairwise disjoint, and hence ϱ is an equivalence (defined on Q). Further, by 3.3, $AB \cup BA \subseteq C$, $AC \cup CA \subseteq B$ and $BC \cup CB \subseteq A$. Consequently, ϱ is a (groupoid) congruence of Q and $Q/\varrho \cong \Pi$.
(ii) This follows immediately from (i). □

In the remaining part of this section, let $D = A \cap B \cap C$ (then either $D = \emptyset$ or $D \neq \emptyset$ is a subquasigroup of Q) and $A^* = A \setminus D$, $B^* = B \setminus D$ and $C^* = C \setminus D$.

Lemma 3.5 *(i)* $A \cap B = A \cap C = B \cap C = D$.
(ii) $A^*B^* \cup B^*A^* \subseteq C^*$, $A^*C^* \cup C^*A^* \subseteq B^*$ *and* $B^*C^* \cup C^*B^* \subseteq A^*$.

Proof. See 3.2 and 3.3. □

Lemma 3.6 *(i) For all* $a \in A^*$ *and* $c \in C^*$ *there exist uniquely determined* $b_1, b_2 \in B^*$ *such that* $ab_1 = c = b_2a$.
(ii) For all $b \in B^*$ *and* $c \in C^*$ *there exist uniquely determined* $a_1, a_2 \in A^*$ *such that* $a_1b = c = ba_2$.

Proof. There exasts a uniquely determined $x \in Q$ such that $ax = c$. Since $c \notin D$ and $a \notin D$, we have $x \notin A \cup C$. Thuťs $x \in B^*$. The rest is clear. □

Lemma 3.7 *(i) For all* $a \in A^*$ *and* $b \in B^*$ *there exist uniquely determined* $c_1, c_2 \in C^*$ *such that* $ac_1 = b = c_2a$.
(ii) For all $c \in C^*$ *and* $b \in B^*$ *there exist uniquely determined* $a_1, a_2 \in A^*$ *such that* $a_1c = b = ca_2$.

Proof. Similar to that of 3.6. □

Lemma 3.8 *(i) For all* $b \in B^*$ *and* $a \in A^*$ *there exist uniquely determined* $c_1, c_2 \in C^*$ *such that* $bc_1 = a = c_2b$.
(ii) For all $c \in C^*$ *and* $a \in A^*$ *there exist uniquely determined* $b_1, b_2 \in B^*$ *such that* $b_1c = a = cb_2$.

Proof. Similar to that of 3.6. □

Corollary 3.9 $|A^*| = |B^*| = |C^*|$ *and* $|A| = |B| = |C|$. *If at least one of* A, B *or* C *is finite then so is* Q.

Corollary 3.10 *If* Q *is finite, then* $|Q| = 3m + n = 3k - 2n$, $m = |A^*|$, $n = |D|$ *and* $k = m + n = |A|$.

Proposition 3.11 *Each of the subquasigroups* A, B, C *is a maximal subquasigroup of* Q.

Proof. Let E be a subquasigroup of Q such that $A \subseteq E$ and $A \neq E$. Then either $E \cap B^* \neq \emptyset$ or $E \cap C^* = \emptyset$ and, since $A^* \subseteq E$, we get $E \cap B^* \neq \emptyset \neq E \cap C^*$ by 3.5(ii). Now, if $e \in B \cap B^*$, then $xe \in A^*$ for some $x \in Q$ and we have $x \in E \cap C^*$. If $b \in B^*$, then $xb \in A^* \subseteq E$ and it follows that $b \in E$. Thus $B \subseteq E$ and, quite similarly, $C \subseteq E$. □

Proposition 3.12 *If* $D \neq \emptyset$ *then the following conditions are equivalent:*
(i) A *is normal in* Q.
(ii) $B^*B^* \cup C^*C^* \subseteq D$.
(iii) D *is normal in both* B *and* C *and* $B/D \cong \mathbb{Z}_2(+) \cong C/D$.
(iv) A *is normal in* Q *and* $Q/A \cong \mathbb{Z}_2(+)$.

Proof. (i) implies (ii). Let, on the contrary, $xv \in B^*$ for some $x, v \in B^*$. If $d \in D$, then $v = yd$ and $zd = x \cdot yd$ for some $y, z \in Q$. Clearly, $y, z \in B^*$ and, choosing $a \in A^*$, we have $za = x \cdot yw$. Now, $w \in A$, since A is normal in Q. On the other hand, $za \in C^*$, and hence $yw \in A^*$ and $w \in C^*$, a contradiction.

(ii) is equivalent to (iii). Easy to see.

(ii) implies (iv). The relation $\varrho = (A \times A) \cup ((Q \setminus A) \times (Q \setminus A))$ is a congruence of Q and $Q/\varrho \cong \mathbb{Z}_2(+)$. □

Proposition 3.13 *If $D \neq \emptyset$, then the following conditions are equivalent:*

(i) At least two of the subquasigroups A, B, C, D are normal in Q.

(ii) All four of the subquasigroups A, B, C, D are normal in Q.

(iii) $A^*A^* \cup B^*B^* \cup C^*C^* \subseteq D$.

(iv) D is normal in Q and $Q/D \cong \mathbb{Z}_2(+)^{(2)}$.

(v) D is normal in Q and Q/D is hamiltonian.

(vi) D is normal in all three of the subquasigroups A, B, C and $A/D \cong B/D \cong C/D \cong \mathbb{Z}_2(+)$.

Moreover if these equivalent conditions are satisfied, then $Q/A \cong Q/B \cong Q/C \cong A/D \cong B/D \cong C/D \cong \mathbb{Z}_2(+)$.

Proof. (i) implies (ii) and (iv). If any two of the subquasigroups, A, B, C are normal in Q, then $D = A \cap B = B \cap C = C \cap A$ is normal in Q. Now, let us assume that A, D are normal in Q. By 3.9 and 3.12, we have $|A/D| = |B/D| = |C/D| = 2$, and hence $|Q/D| = 4$ (3.10). We have $Q/D = A/D \cup B/D \cup C/D$ and the three subquasigroups are two element groups. Thus Q/D is a loop and it is easy to see that $Q/D \cong \mathbb{Z}_2(+)^{(2)}$.

The remaining implications are clear (use 3.12). □

Corollary 3.14 *If at least one of the subquasigroups A, B, C is normal in Q and $|D| = 1$, then $Q \cong \mathbb{Z}_2(+)^{(2)}$.*

Proposition 3.15 *Assume that Q is finite and that $k = |A|$ $(= |B| = |C|)$ divides $|Q|$ (e.g., at least one of A, B, C is normal in Q).*

(i) All of the three subquasigroups A, B, C are normal in Q.

(ii) If $D \neq \emptyset$, then D is normal in Q and $Q/D \cong \mathbb{Z}_2(+)^{(2)}$.

(iii) If $D = \emptyset$, then $|Q| = 3k$.

(iv) If $D \neq \emptyset$ and $n = |D|$, then $k = 2n$ and $|Q| = 4n$.

Proof. In view of 3.4, we may assume that $D \neq \emptyset$. Now, $k = 2n$ by 3.10, and hence (i) is true. The rest is clear from 3.13. □

Corollary 3.16 *If Q is finite and $|A|$ $(= |B| = |C|)$ divides $|Q|$, then either 3 or 4 divides $|Q|$.*

3.17 Choose bijections $\sigma^* : A^* \to B^*$ and $\tau^* : A^* \to C^*$ (see 3.9) and define six binary operations on the set A^* by $a_1 \circ a_2 = \tau^{*-1}(a_1\sigma^*(a_2))$, $a_1 \bullet a_2 =$

$\tau^{*^{-1}}(\sigma^*(a_1)\, a_2)$, $a_1 \lhd a_2 = \sigma^{*^{-1}}(a_1 \tau^*(a_2))$, $a_1 \rhd a_2 = \sigma^{*^{-1}}(\tau^*(a_1)\, a_2)$, $a_1 * a_2 = \sigma^*(a_1)\, \tau^*(a_2)$ and $a_1 \bigstar a_2 = \tau^*(a_1)\, \sigma^*(a_2)$ for all $a_1, a_2 \in A^*$.

Lemma 3.17.1 *All the six groupoids $A^*(\circ)$, $A^*(\bullet)$, $A^*(\lhd)$, $A^*(\rhd)$, $A^*(*)$, $A^*(\bigstar)$ are quasigroups.*

Proof. This can be checked easily. □

Lemma 3.17.2 *$ab = \tau^*(a \circ \sigma^{*^{-1}}(b))$, $ba = \tau^*(\sigma^{*^{-1}}(b) \bullet a)$, $ac = \sigma^*(a \lhd \tau^{*^{-1}}(c))$, $ca = \sigma^*(\tau^{*^{-1}}(c) \rhd a)$, $bc = \sigma^{*^{-1}}(b) * \tau^{*^{-1}}(c)$ and $cb = \tau^{*^{-1}}(c) \bigstar \sigma^{*^{-1}}(b)$ for all $a \in A^*$, $b \in B^*$ and $c \in C^*$.*

Proof. Obvious. □

Let $\sigma = \sigma^* \cup id_D$, $\tau = \tau^* \cup id_D$, and define three binary operations $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ on A by $a_1 \underline{\alpha} a_2 = a_1 a_2$, $a_1 \underline{\beta} a_2 = \sigma^{-1}(\sigma(a_1)\, \sigma(a_2))$ and $a_1 \underline{\gamma} a_2 = \tau^{-1}(\tau(a_1)\, \tau(a_2))$.

Lemma 3.17.3 *$A(\underline{\alpha})$, $A(\underline{\beta})$, and $A(\underline{\gamma})$ are quasigroups and the bijections $id_A : A(\underline{\alpha}) \to A$, $\sigma : A(\underline{\beta}) \to B$, and $\tau : A(\underline{\gamma}) \to C$ are quasigroup isomorphisms.*

Proof Obvious. □

Remark 3.18 *Assume that $D \neq \emptyset$, put $Q^* = A^* \cup B^* \cup C^* = Q \setminus D$, $W = \{(x, y);\ x, y \in Q^*, \{x, y\} \not\subseteq A^*, \{x, y\} \not\subseteq B^*, \{x, y\} \not\subseteq C^*\}$ and choose (arbitrarily) quasigroup operations $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ defined on A^*, B^* and C^*, resp. Now, define an operation $\circ$ on Q^* in the following way:*

1. *$A^*(\underline{\alpha}$, $B^*(\underline{\beta})$ and $C^*(\underline{\gamma})$ are subgroupoids of $Q^*(\circ)$;*
2. *$x \circ y = xy$ for every $(x, y) \in W$.*

Then $Q^*(\circ)$ is a quasigroup that is the disjoint union of the three subquasigroups $A^*(\circ)$, $B^*(\circ)$ and $C^*(\circ)$. Moreover, $Q = Q^* \cup D$ and $xy = x \circ y$ for every pair $(x, y) \in W$.

4. The case of three subquasigroups (b)

Construction 4.1 Let R be a non-empty set supplied with nine binary quasigroup operations denoted by the symbols $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, $\circ$, $\bullet$, $\lhd$, $\rhd$, $*$, and $\bigstar$, resp. Put $Q = R \times \{1, 2, 3\}$ and define a multiplication on Q by means of the following rules:

1. $(u, 1)(v, 1) = (u\underline{\alpha}v, 1)$ for all $u, v \in R$;
2. $(u, 2)(v, 2) = (u\underline{\beta}v, 2)$ for all $u, v \in R$;
3. $(u, 3)(v, 3) = (u\underline{\gamma}v, 3)$ for all $u, v \in R$;
4. $(u, 1)(v, 2) = (u \circ v, 3)$ for all $u, v \in R$;
5. $(u, 2)(v, 1) = (u \bullet v, 3)$ for all $u, v \in R$;
6. $(u, 1)(v, 3) = (u \lhd v, 2)$ for all $u, v \in R$;
7. $(u, 3)(v, 1) = (u \rhd v, 2)$ for all $u, v \in R$;

8. $(u, 2)(v, 3) = (u * v, 1)$ for all $u, v \in R$;
9. $(u, 3)(v, 2) = (u \star v, 1)$ for all $u, v \in R$.

Put $A = R \times \{1\}, B = R \times \{2\}, C = R \times \{3\}, \sigma(u, 1) = (u, 2)$ and $\tau(u, 1) = (u, 3)$, $u \in R$.

Lemma 4.1.1 $(u \circ v, 1) = \tau^{-1}((u, 1)\,\sigma(v, 1))$, $(u \bullet v, 1) = \tau^{-1}(\sigma(u, 1)\,(v, 1))$, $(u \lhd v, 1) = \sigma^{-1}((u, 1)\,\tau(v, 1))$, $(v \rhd v) = \sigma^{-1}(\tau(u, 1), (v, 1))$, $(u * v, 1) = \sigma(u, 1)\,\tau(v, 1)$ *and* $(u \star v, 1) = \tau(u, 1)\,\sigma(v, 1)$ *for all* $u, v \in R$.

Proof. Obvious from the definitions of the operations. □

Lemma 4.1.2 *A is a subquasigroup of Q and the mapping $u \mapsto (u, 1)$ is an isomorphism of $R(\underline{\alpha})$ onto A.*

Proof. Easy. □

Lemma 4.1.3 *B is a subquasigroup of Q and the mapping $u \mapsto (u, 2)$ is an isomorphism of $R(\underline{\beta})$ onto B.*

Proof. Easy. □

Lemma 4.1.4 *C is a subquasigroup of Q and the mapping $u \mapsto (u, 3)$ is an isomorphism of $R(\underline{\gamma})$ onto C.*

Proof. Easy. □

Proposition 4.1.5 *Q is a quasigroup, A, B and C are proper subquasigroups of Q, $A \cup B \cup C = Q$ and $A \cap B \cap C = \emptyset$.*

Proof. Easy (use 4.1.1, ..., 4.1.4). □

Theorem 4.2 *Let Q be a quasigroup. The following conditions are equivalent:*

(i) There exist proper subquasigroups A, B, C of Q such that $A \cup B \cup C = Q$ and $A \cap B \cap C = \emptyset$.

(ii) The three-element quasigroup Π (see 1.6) is a homomorphic image of Q.

(iii) Q (or an isomorphic copy of Q) is constructed in the way described in 4.1.

Proof. (i) implies (ii). See 3.4.

(ii) implies (i). Let $\pi : Q \to \Pi$ be a homomorphism of Q onto Π. For the completion of the proof it suffices to put $A = \pi^{-1}(\alpha)$, $B = \pi^{-1}(\beta)$ and $C = \pi^{-1}(\gamma)$.

(i) is equivalent to (iii). Combine 3.17 and 4.1. □

Example 4.3 *(cf. 3.4) In 4.1, let us choose three pair-wise non-isomorphic quasigroups $R(\underline{\alpha})$, $R(\underline{\beta})$ and $R(\underline{\gamma})$. Then $Q = A \cup B \cup C$, where A, B and C are pair-wise non-isomorphic and $A \cap B \cap C = \emptyset$.*

5. The case of three subquasigroups (c)

Construction 5.1 Let R be a non-empty set supplied with three binary quasigroup operations denoted by the symbols $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$, resp., and let S be a proper non-empty subset of R such that S is a subquasigroup of all the three quasigroups and $x\underline{\alpha}y = x\underline{\beta}y = x\underline{\gamma}y$ for all $x, y \in S$. Further, let $T = R \setminus S$ (we have $T \neq \emptyset$) and let $\circ$, $\bullet$, $\lhd$, $\rhd$, $*$, and $\bigstar$ be six quasigroup operations defined on T. Put $Q = (T \times \{1,2,3\}) \cup S$ (we assume $(T \times \{1,2,3\}) \cap S = \emptyset$) and define a multiplication on Q by means of the following rules:

1. $xy = x\underline{\alpha}y$ $(= x\underline{\beta}y = x\underline{\gamma}y)$ for all $x, y \in S$;
2. $x(u, 1) = (x\underline{\alpha}u, 1)$ and $(u, 1)\,x = (u\underline{\alpha}x, 1)$ for all $x \in S$ and $u \in T$;
3. $(u, 1)\,(v, 1) = u\underline{\alpha}v$ for all $u, v \in T$ such that $u\underline{\alpha}v \in S$;
4. $(u, 1)\,(v, 1) = (u\underline{\alpha}v, 1)$ for all $u, v \in T$ such that $u\underline{\alpha}v \in T$;
5. $x(u, 2) = (x\underline{\beta}u, 2)$ and $(u, 2)\,x = (u\underline{\beta}x, 2)$ for all $x \in S$ and $u \in T$;
6. $(u, 2)\,(v, 2) = u\underline{\beta}v$ for all $u, v \in T$ such that $u\underline{\beta}v \in S$;
7. $(u, 2)\,(v, 2) = (u\underline{\beta}v, 1)$ for all $u, v \in T$ such that $u\underline{\beta}v \in T$;
8. $x(u, 3) = (x\underline{\gamma}u, 3)$ and $(u, 3)\,x = (u\underline{\gamma}x, 3)$ for all $x \in S$ and $u \in T$;
9. $(u, 3)\,(v, 3) = u\underline{\gamma}v$ for all $u, v \in T$ such that $u\underline{\gamma}v \in S$;
10. $(u, 3)\,(v, 3) = (u\underline{\gamma}v, 1)$ for all $u, v \in T$ such that $u\underline{\gamma}v \in T$;
11. $(u, 1)\,(v, 2) = (u \circ v, 3)$ for all $u, v \in T$;
12. $(u, 2)\,(v, 1) = (u \bullet v, 3)$ for all $u, v \in T$;
13. $(u, 1)\,(v, 3) = (u \lhd v, 2)$ for all $u, v \in T$;
14. $(u, 3)\,(v, 1) = (u \rhd v, 2)$ for all $u, v \in T$;
15. $(u, 2)\,(v, 3) = (u * v, 1)$ for all $u, v \in T$;
16. $(u, 3)\,(v, 2) = (u \bigstar v, 1)$ for all $u, v \in T$.

Put $A^* = T \times \{1\}$, $B^* = T \times \{2\}$, $C^* = T \times \{3\}$, $A = A^* \cup S$, $B = B^* \cup S$, $C = C^* \cup S$, $\sigma^*(u, 1) = (u, 2)$, $\tau^*(u, 1) = (u, 3)$ for all $u \in T$, $\sigma = \sigma^* \cup id_S$, $\tau = \tau^* \cup id_S$ and $D = S$.

Lemma 5.1.1 $(u \circ v) = \tau^{*^{-1}}((u, 1)\,\sigma^*(v, 1))$, $(u \bullet v, 1) = \tau^{*^{-1}}(\sigma^*(u, 1)\,(v, 1))$, $(u \lhd v, 1) = \sigma^{*^{-1}}((u, 1)\,\tau^*(v, 1))$, $(u \rhd v, 1) = \sigma^{*^{-1}}(\tau^*(u, 1)\,(v, 1))$, $(u * v, 1) = \sigma^*(u, 1)\,\tau^*(v, 1)$ *and* $(u \bigstar v, 1) = \tau^*(u, 1)\,\sigma^*(v, 1)$ *for all* $u, v \in T$.

Proof. Obvious from the definitions of the operations. □

Lemma 5.1.2 *A is a subquasigroup of Q and the mapping $x \mapsto x$, $u \mapsto (u, 1)$, $x \in S$, $u \in T$, is an isomorphism of $R(\underline{\alpha})$ onto A.*

Proof. Easy. □

Lemma 5.1.3 *B is a subquasigroup of Q and the mapping $x \mapsto x$, $u \mapsto (u, 2)$, $x \in S$, $u \in T$, is an isomorphism of $R(\underline{\beta})$ onto B.*

Proof. Easy. □

Lemma 5.1.4 *C is a subquasigroup of Q and the mapping $x \mapsto x$, $u \mapsto (u, 3)$, $x \in S$, $u \in T$, is an isomorphism of $R(\underline{\gamma})$ onto C.*

Proof. Easy. □

Lemma 5.1.5 *D is a subquasigroup of Q and the mapping $x \mapsto x$, $x \in S$, is an isomorphism of $S(\underline{\alpha})$ $(= S(\underline{\beta}) = S(\underline{\gamma}))$ onto D.*

Proof. Obvious. □

Proposition 5.1.6 *Q is a quasigroup, A, B and C are proper subquasigroups of Q, $Q = A \cup B \cup C$ and $D = A \cap B \cap C$.*

Proof. Easy (use 5.1.1, ..., 5.1.5). □

Lemma 5.1.7 *A is normal in Q if and only if $S(\underline{\beta})$ is normal in $R(\underline{\beta})$, $S(\underline{\gamma})$ in $R(\underline{\gamma})$ and $|R(\underline{\beta})/S(\underline{\beta})| = 2 = |R(\underline{\gamma})/S(\underline{\gamma})|$.*

Proof. Combine 5.1.6 and 3.12. □

Lemma 5.1.8 *All three of the subquasigroups A, B, C are normal in Q if and only if $S(\underline{\delta})$ is normal in $R(\underline{\delta})$ and $|R(\underline{\delta})/S(\underline{\delta})| = 2$ for every $\underline{\delta} \in \{\underline{\alpha}, \underline{\beta}, \underline{\gamma}\}$.*

Proof. Use 5.1.7. □

Theorem 5.2 *Let Q be a quasigroup. Then there exist proper subquasigroups A, B and C of Q such that $A \cup B \cup C = Q$ and $A \cap B \cap C = D \neq \emptyset$ if and only if Q (or an isomorphic copy of Q) is constructed in the way described in 5.1.*

Proof. Combine 3.17 and 5.1. □

Theorem 5.3 *Let Q be a quasigroup. The following conditions are equivalent:*

(i) There exist proper normal subquasigroups A, B, C of Q such that $A \cup B \cup C = Q$ and $A \cap B \cap C \neq \emptyset$.

(ii) The four-element 2-elementary group $\mathbb{Z}_2(+)^{(2)}$ is a homomorphic image of Q.

(iii) Q (or an isomorphic copy of Q) is constructed in the way described in 5.1 where $S(\underline{\delta})$ is normal in $R(\underline{\delta})$ and $|R(\underline{\delta})/S(\underline{\delta})| = 2$ for every $\underline{\delta} \in \{\underline{\alpha}, \underline{\beta}, \underline{\gamma}\}$.

Proof. Combine 5.1.5.2 and 3.13. □

Corollary 5.4 *Let Q be a quasigroup. The following conditions are equivalent:*

(i) There exist proper normal subquasigroups A, B, C of Q such that $A \cup B \cup C = Q$.

(ii) Either the three element quasigroup Π or the four-element group $\mathbb{Z}_2(+)^{(2)}$ is a homomorphic image of Q.

Example 5.5 *In 5.1, choose $R(\underline{\beta}) = R(\underline{\gamma}) = \mathbb{Z}(+)$ (the additive group of integers), $S = \mathbb{Z}2$ and $T(\circ) = T(\bullet) = T(\lhd) = T(\rhd) = T(*) = T(\bigstar)$ any commutative quasigroup defined on $T = \mathbb{Z} \setminus \{0\}$. Further choose a commutative loop* *operation $\underline{\alpha}$ defined on $\mathbb{Z}$ such that $a\underline{\alpha}b = a + b$ for all $a, b \in \mathbb{Z}2$ and consider the corresponding commutative loop Q (see 5.1 again).*

(i) A, B, C are proper subloops of Q, $A \cup B \cup C = Q$ and $A \cap B \cap C = \mathbb{Z}2$ is an infinite cyclic group.

(ii) A is normal in Q.

(iii) B (or C) is normal in Q if and only if $D = \mathbb{Z}2$ is normal in $\mathbb{Z}(\underline{\alpha})$ and $\mathbb{Z}(\underline{\alpha})/D \cong \mathbb{Z}_2(+)$.

Notice that we may define $\underline{\alpha}$ on $\mathbb{Z}$ in such a way that $\mathbb{Z}(\underline{\alpha})$ becomes an infinite cyclic group and $|\mathbb{Z}(\underline{\alpha})/D| = 3$. Then $A \cong B \cong C \cong \mathbb{Z}(+)$, A is normal in Q, D is normal in A, B, C and D, B, C are not normal in Q. Moreover, $A/D \cong \mathbb{Z}_3(+)$ and $B/D \cong \mathbb{Z}_2(+) \cong C/D$.

Example 5.6 *In 5.1, choose $R(\underline{\alpha}) = R(\underline{\beta}) = R(\underline{\gamma}) = \mathbb{Z}(+)$ (the additive group of integers), $S = \{0\}$ and $T(\circ) = T(\bullet) = T(\lhd) = T(\rhd) = T(*) = T(\bigstar)$ any commutative quasigroup defined on $T = \mathbb{Z}\setminus\{0\}$.*

Then Q becomes a commutative loop, $Q = A \cup B \cup C$, where A, B, C are subloops isomorphic to $\mathbb{Z}(+)$ and $A \cap B \cap C$ is the unit subloop of Q. Notice that neither A nor B nor C is a normal subloop of Q.

Example 5.7 *Consider the following loop L:*

L	1	a_1	a_2	b_1	b_2	c_1	c_2
1	1	a_1	a_2	b_1	b_2	c_1	c_2
a_1	a_1	a_2	1	c_1	c_2	b_1	b_2
a_2	a_2	1	a_1	c_2	c_1	b_2	b_1
b_1	b_1	c_1	c_2	b_2	1	a_1	a_2
b_2	b_2	c_2	c_1	1	b_1	a_2	a_1
c_1	c_1	b_1	b_2	a_1	a_2	c_2	1
c_2	c_2	b_2	b_1	a_2	a_1	1	c_1

Then Q is a simple commutative loop, $Q = A \cup B \cup C$, where $A = \{1, a_1, a_2\}$, $B = \{1, b_1, b_2\}$, $C = \{1, c_1, c_2\}$ are subloops of Q and $A \cap B \cap C = 1$.

Example 5.8 *In 5.1, choose three pair-wise non-isomorphic loops $R(\underline{\alpha})$, $R(\underline{\beta})$ and $R(\underline{\gamma})$ possesing the same neutral element 1 and put $S = \{1\}$. The quasigroups defined on $T = R\setminus\{1\}$ may be chosen arbitgrarily. Then we get a loop Q such that $Q = A \cup B \cup C$, where A, B and C are pair-wise non-isomorhic proper subloops and $A \cap B \cap C = 1$.*

Remark 5.9 *(cf. 3.18) Let $Q^*(\circ)$ be a quasigroup that is the disjoint union of three proper subquasigroups, say $A^*(\circ)$, $B^*(\circ)$, $C^*(\circ)$ (see 4.2) and let $D(\bullet)$ be a quasigroup such that $D \cap Q^* = \emptyset$. Now, put $A = A^* \cup D$, $B = B^* \cup D$, $C = C^* \cup D$ and choose some quasigroup operations $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ defined on A, B and C, resp., in such a way that $D(\bullet)$ is a subquasigroup of all the three quasigroups.*

Finally put $Q = Q^* \cup D$ and define a multipication on Q as follows:
1. $A(\underline{\alpha})$, $B(\underline{\beta})$ and $C(\underline{\gamma})$ are subquasigroups of Q;
2. $xy = x \circ y$ for all $x, y \in Q$ such that $\{x,y\} \nsubseteq A$, $\{x,y\} \nsubseteq B$, $\{x,y\} \nsubseteq C$.
Then Q is a quasigroup, A, B, C and D are its subquasigroups, $A \cup B \cup C = Q$ and $A \cap B = B \cap C = C \cap A = D$.

6. The case of three subgroups

Proposition 6.1 *([2]) Let A, B, C be proper subgroups of a group G such that $A \cup B \cup C = G$ and $A \cap B \cap C = 1$. Then $G \cong \mathbb{Z}_2(+)^{(2)}$ (see 1.7).*

Proof. By 3.2, $A \cap B = A \cap C = B \cap C = 1$. If $a \in A^*$, $b \in B^*$ and $c \in C^*$, then $ab \in C^*$, $bc \in A^*$ and hence $abc \in A \cap C = 1$, $a = c^{-1}b^{-1}$. It follows that $|A^*| = |B^*| = |C^*| = 1$, and so $|A| = |B| = |C| = 2$ and $|G| = 4$. Finally, since $G = A \cup B \cup C$, we have $x^2 = 1$ for every $x \in G$ and the rest is clear. □

Proposition 6.2 *([2]) Let A, B, C be proper subgroups of a group G such that $G = A \cup B \cup C$. Then each of A, B, C is a normal maximal subgroup of G, $G/A \cong G/B \cong G/C \cong \mathbb{Z}_2(+)$, $D = A \cap B \cap C$ is a normal subgroup of G and $G/D \cong \mathbb{Z}_2(+)^{(2)}$.*

Proof. If $a \in A^*$, $b \in B^*$ and $c \in C^*$, then $abc \in D$ and $a \in Dc^{-1}b^{-1} \subseteq A$. Now it is clear that $[A:D] = 2$, and hence D is a normal subgroup of A and $A \subseteq \mathbb{N}_G(D)$ (the normalizer). Quite similarly, $B \cup C \subseteq \mathbb{N}_G(D)$ and consequently, $\mathbb{N}_G(D) = G$ and D is normal in G. Then $G/D = G_1 = A_1 \cup B_1 \cup C_1$, $A_1 = A/D$, $B_1 = B/D$, $C_1 = C/D$, $A_1 \cap B_1 \cap C_1 = 1$ and the result follows from 6.1. □

Theorem 6.3 *([2]) The following conditions are equivalent for a group G:*
(i) There exist proper subgroups A, B and C of G such that $A \cup B \cup C = G$.
(ii) The group $\mathbb{Z}_2(+)^{(2)}$ is a homomorphic image of G.
(iii) If H denotes the subgroup of G generated by the set $\{x^2;\ x \in G\}$, then the factor-group G/H is not cyclic (clearly, H is normal in G).

Proof. (i) implies (ii). See 6.2.
(ii) implies (iii). If K is a normal subgroup of G with $G/K \cong \mathbb{Z}_2(+)^{(2)}$, then $H \subseteq K$, and so G/H is not cyclic.
(iii) implies (ii). G/H is a direct sum of at least two copies of $\mathbb{Z}_2(+)$.
(ii) implies (i). Use 1.7. □

Example 6.4 *Let $G = \mathbb{S}_3$ (the symmetric group on three letters). Then G contains just four non-trivial proper subgroups, say A, B, C and D, where A is the alternating group, $|A| = 3$, and $|B| = |C| = |D| = 2$. Clearly, all the subgroups are maximal, A is normal in G, B, C and D are not normal in G, $G = A \cup B \cup C \cup D$ and $A \cap B = A \cap C = A \cap D = B \cap C = B \cap D = C \cap D = 1$.*

References

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