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Unions of Subquasigroups

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Quasigroups (loops, groups) that are unions of finitely many proper subquasigroups (subloops, subgroups) are studied.

Studují se kvazigrupy (lupy, grupy), které jsou sjednocením konečně mnoha vlastních podkvazigrup (podlup, podgrup).

The present short note is of expository character and collects various results on quasigroups (loops, groups) available as the set theoretic union of finitely many proper subquasigroups (subloops, subgroups). A reader is referred to [1]–[15] for further information.

1. Quasigroups

A quasigroup Q is a non-empty set supplied with a binary operation (usually denoted multiplicatively) such that for all $a, b \in Q$ there exist uniquely determined $c, d \in Q$ with $ac = b = da$. A quasigroup with a neutral (or unit) element is called a loop. A group is an associative quasigroup (loop).

Proposition 1.1 *Let P and R be subquasigroups of a quasigroup Q such that $Q = P \cup R$. Then either $P = Q$ or $R = Q$.*

Proof. If $a \in P \setminus R$ and $b \in R$, then $ab \notin R$, $ab \in P$ and $b \in P$. Thus $R \subseteq P$ and $P = Q$. □

Corollary 1.2 *Let $P_1, \ldots, P_n$, $n \geq 1$, be proper subquasigroups of a quasigroup Q such that $Q = P_1 \cup \ldots \cup P_n$. Then $n \geq 3$.*

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Remark 1.3 (i) Let Q be a finite quasigroup such that Q is not one-generated. Then $\langle a \rangle \neq Q$ for every $a \in Q$ and, of course, $Q = \bigcup_{a \in Q} \langle a \rangle$. Thus Q is the union of finitely many proper subquasigroups. Moreover if $P = \bigcap_{a \in Q} \langle a \rangle$ is non-empty, then P is the smallest subquasigroup of Q.
(ii) Let Q be a quasigroup and r a congruence of Q such that $P = Q/r$ is a finite quasigroup, not one-generated. If π denotes the natural projection of Q onto P, then $P_u = \pi^{-1}(\langle u \rangle) \neq Q$ for every $u \in P$ and we have $Q = \bigcup_{u \in P} P_u$. Thus Q is the union of finitely many proper subquasigroups.

Example 1.4 Let Q be a quasigroup possesing a congruence r such that Q/r is a non-trivial finite idempotent quasigroup. By 1.3, Q is the union of finitely many proper subquasigroups.

Lemma 1.5 *Let $P_1, \ldots, P_n$, $n \geq 1$, be subquasigroups of a quasigroup $Q = P_1 \cup \ldots \cup P_n$. If $1 \leq j \leq n$ is such that $Q \neq \bigcup_{i=1, i \neq j}^{n} P_i$, then $P_1 \cap \ldots \cap P_n = \bigcap_{i=1, i \neq j}^{n} P_i$.*

Proof. If $a \in \bigcap_{i \neq j} P_i$ and $b \in Q \setminus \bigcup_{i \neq j} P_i$, then $ab \notin \bigcup_{i \neq j} P_i$, and so $ab \in P_j$. But $b \in P_j$ and consequently, $a \in P_j$. □

Example 1.6 If $S = \{a, b\}$ is a two-element semilattice, then both $\{a\}$ and $\{b\}$ are subsemigroups of S, $S = \{a\} \cup \{b\}$ and $\{a\} \cap \{b\} = \emptyset$.

Example 1.7 If $\mathbb{N}$ denotes the additive semigroup of non-negative integers, then $\mathbb{N}$ is a cancellative semigroup, both $\{0\}$ and $M = \{n \in \mathbb{N}; n \geq 1\}$ are subsemigroups of $\mathbb{N}$, $\mathbb{N} = \{0\} \cup M$ and $\{0\} \cap M = \emptyset$.

Example 1.8 If $\mathbb{Z}$ denotes the additive group of integers, then both $K = \{n \in \mathbb{Z}; n < 0\}$ and $N = \{n \in \mathbb{Z}; n \geq 0\}$ are subsemigroups of $\mathbb{Z}$, $\mathbb{Z} = K \cup N$ and $K \cap N = \emptyset$.

Remark 1.9 Let G be a group such that $G = S \cup H$, where S is a subsemigroup of G and H a subgroup of G. Then either $S = G$ or $H = G$.

Indeed, if $H \neq G$, then $S \nsubseteq H$, $T = S \setminus H$ is non-empty and $a^{-1} \in T$ for every $a \in T$. Consequently, $1 \in S$ and $T \subseteq K = \{a \in S; a^{-1} \in S\}$. Now, K is a subgroup of $G = K \cup H$ and 1.1 applies.

Remark 1.10 Let Q be a quasigroup such that $Q = A \cup P$, where A is a subgroupoid of Q and P a subquasigroup of Q. Then either $A = Q$ or $P = Q$.

Indeed, if $a \in A \setminus P$ and $x \in P \setminus A$ then $aq = x$ for some $q \in Q$ and, if $q \in A$, then $x \in A$, a contradiction. Thus $q \in P$ and $a \in P$ (since both Q and P are quasigroups), again a contradiction.

Remark 1.11 Quasigroups that are unions of three proper subquasigroups are described in [10]. Among others, the following are proved: Let Q be a quasigroup such that $Q = P_1 \cup P_2 \cup P_3$, where P_1, P_2, P_3 are proper subquasigroups, and let $R = P_1 \cap P_2 \cap P_3$. If $R = \emptyset$, then P_1, P_2, P_3 are normal maximal subquasigroups of Q, $r = (P_1 \times P_1) \cup (P_2 \times P_2) \cup (P_3 \times P_3)$ is a congruence of Q and Q/r $(=Q/P_1 = Q/P_2 = Q/P_3)$ is a three-element idempotent quasigroup. If $R \neq \emptyset$ and at least two of P_1, P_2, P_3, R are normal in Q, then P_1, P_2, P_3 are normal maximal subquasigroups of Q, R is normal in Q and Q/R is a four-element 2-elementary (abelian) group.

2. Loops

Proposition 2.1 *Let $P_1, \dots, P_n$, $n \geq 1$, be subloops of an infinite loop Q such that the following two conditions are satisfied:*

(1) $P_1 \cup \dots \cup P_n = Q$;

(2) If $n \geq 2$ and $1 \leq j \leq n$, then there exists $a \in Q$ such that $a \notin P_i$ for every $1 \leq i \leq n$, $i \neq j$, and $xy \cdot a = x \cdot ya$ ($a \cdot xy = ax \cdot y$, resp.) for all $x, y \in Q$ (i.e., $(P_j \cap (N_l(Q) \cup N_r(Q)) \nsubseteq \bigcup_{i \neq j} P_i)$).

Then the intersection $P = P_1 \cap \dots \cap P_n$ is an infinite subloop of Q.

Proof. Proceeding by induction on m, we will prove the following assertion:

For every $1 \leq m \leq n$ there exist indices $1 \leq k_1 < \dots < k_m \leq n$ such that the intersection $P_{k_1} \cap \dots \cap P_{k_m}$ is infinite.

Since Q is infinite, it follows from (1) that our assertion is true for $m = 1$. Now, let $1 \leq m < n$ be such that $R = P_1 \cap \dots \cap P_m$ is infinite. By (2), we find $a \in Q$ such that $a \notin P_1 \cup \dots \cup P_m$ and $xy \cdot a = x \cdot ya$ for all $x, y \in Q$. Then $Ra \cap (P_1 \cup \dots \cup P_m) = \emptyset$ and hence $Ra \subset P_{m+1} \cup \dots \cup P_n$. Since R is infinite, we may assume without loss of generality that $Sa \subseteq P_{m+1}$ for an infinite subset S of R. If $w \in S$ is fixed, then for every $u \in S$ there is $v_u \in Q$ such that $v_u \cdot ua = wa$. Clearly, $v_{u_1} \neq v_{u_2}$ for $u_1 \neq u_2$ and $v_u \in P_{m+1}$. On the other hand, $v_u u \cdot a = v_u \cdot ua = wa$, $v_u u = w$ and $v_u \in R$. Thus $V = \{v_u; u \in S\}$ is an infinite subset of $R \cap P_{m+1} = P_1 \cap \dots \cap P_{m+1}$. □

Corollary 2.2 (cf. [7, 6.3]) *Let $P_1, \dots, P_n$, $n \geq 1$, be subloops of a loop Q such that the following three conditions are satisfied:*

(1) $P_1 \cup \dots \cup P_n = Q$;

(2) $P_1 \cap \dots \cap P_n$ is finite (in particular, $P_1 \cap \dots \cap P_n = 1$);

(3) If $n \geq 2$ and $1 \leq j \leq n$, then there exists $a \in Q$ such that $a \notin P_i$ for every $1 \leq i \leq n$, $i \neq j$, and $xy \cdot a = x \cdot ya$ ($a \cdot xy = ax \cdot y$, resp.) for all $x, y \in Q$.

Then Q is a finite loop.

Example 2.3 ([10]) Put $T = \mathbb{Z} \setminus \{0\}$, $\mathbb{Z}$ being the ring of integers, define a bijection $\alpha : T \to \mathbb{Z}$ by $\alpha(m) = m$ for every $m \in \mathbb{Z}$, $m < 0$, $\alpha(n) = n - 1$ for every $n \in \mathbb{Z}$, $0 < n$, and an opertation $*$ on T by $u * v = \alpha^{-1}(\alpha(u) + \alpha(v))$ for all $u, v \in T$ (clearly, $T(*) \simeq \mathbb{Z}(+)$). Further, put $Q = (T \times \{1,2,3\}) \cup \{0\}$ and define a multiplication on Q in the following way:

(*a*) $0 \cdot 0 = 0$;

(*b*) $0 \cdot (u, i) = (u, i) = (u, i) \cdot 0$ for all $u \in T$ and $1 \leq i \leq 3$;

(c) $(u,i)(v,i) = (u+v,i)$ for all $u, v \in T$, $u \neq -v$, and $1 \leq i \leq 3$;
(d) $(u,i)(-u,i) = 0$ for all $u \in T$ and $1 \leq i \leq 3$;
$(u,i)(v,j) = (u * v, k)$ for all $u, v \in T$, $1 \leq i,j,k \leq 3$, $i \neq j \neq k \neq i$.
Now, Q becomes an infinite commutative loop, 0 is its neutral element and $A = (T \times \{1\}) \cup \{0\}$, $B = (T \times \{2\}) \cup \{0\}$, $C = (T \times \{3\}) \cup \{0\}$ are its subgroups, $A \simeq B \simeq C \simeq \mathbb{Z}(+)$. Moreover, $Q = A \cup B \cup C$ and $0 = A \cap B \cap C$.

3. Groups (a)

Proposition 3.1 ([11]) *Let $H_1, \ldots, H_n$, $n \geq 1$, be subgroups of a group G such that $G = H_1 \cup \ldots \cup H_n$ but $G \neq \bigcup_{i=1, i \neq j}^{n} H_i$ for every $1 \leq j \leq n$. Then*
(i) All the subgroups $H_1, \ldots, H_n$ have finite index in G.
(ii) $K = H_1 \cap \ldots \cap H_n$ is a subgroup of finite index in G and $[G:K] \leq n!$.
(iii) $\mathrm{Core}_G(K)$ is a normal subgroup of finite index in G.

Proof. It is divided into two parts.
(a) Firstly, we are going to show that the index $[G:K]$ is finite. Since $G = \bigcup H_i$, it suffices to show that all indices $[H_i : K]$ are finite. Proceeding by contradiction, let us assume that this is not true and, using induction on m, let us prove the following assertion:
For every $1 \leq m \leq n$ there are indices $1 \leq k_1 < \ldots < k_m \leq n$ such that the index $[H_{k_1} \cap \ldots \cap H_{k_m} : K]$ is infinite.
According to our assumption, the assertion is true for $m = 1$ and, now, let $1 \leq m < n$ be such that the index $[H_1 \cap \ldots \cap H_m : K]$ is infinite. Then there is an infinite subset M of $H_1 \cap \ldots \cap H_m$ such that $xy^{-1} \notin K$ for all $x, y \in M$, $x \neq y$. Take $a \in G \setminus (H_1 \cup \ldots \cup H_m)$. Then $Ma \cap (H_1 \cup \ldots \cup H_m) = \emptyset$, and hence $Ma \subseteq H_{m+1} \cup \ldots \cup H_n$. Further, since Ma is infinite, we can assume without loss of generality that $Na \subset H_{m+1}$ for an infinite subset $N \subseteq M$. If $u, v \in N$, then $uv^{-1} = uaa^{-1}v^{-1} \in H_{m+1}$ and, if $u \neq v$, then $uv^{-1} \notin K$. Clearly, $uv^{-1} \in H_1 \cap \ldots \cap H_{m+1}$, and therefore, fixing an element $w \in N$, we get an infinite subset $R = \{uw^{-1}; u \in N\}$ of $H_1 \cap \ldots \cap H_{m+1}$ such that $(uw^{-1})(vw^{-1})^{-1} = uv^{-1} \notin K$ for all $u, v \in N$, $u \neq v$. Consequently, the index $[H_1 \cap \ldots \cap H_{m+1} : K]$ is infinite.
Now, by induction, the index $[H_1 \cap \ldots \cap H_n : K] = [K:K] = 1$, is infinite, a contradiction.
(b) It follows from (a) that the index $[G:L]$, $L = \mathrm{Core}_G(K)$, is finite. Now, considering the factor G/L we may assume without loss of generality that G is a finite group.
For $k = 1, 2, \ldots, n$, put $m_k = \max\{|\bigcap_{j \in J} H_j|;\ J \subseteq \{1, \ldots, n\}, |J| = k\}$; clearly, $m_1 = \max\{|H_i|; 1 \leq i \leq n\}$ and $m_n = |K|$. We are going to show that $m_k \leq (n-k)\, m_{k+1}$ for every $1 \leq k \leq n$.

Assume, on the contrary, the $|F| > (n-k)m_{k+1}$, $F = H_1 \cap \ldots \cap H_k$, for some k, $1 \le k \le n$, and take $a \in G \setminus (H_1 \cup \ldots \cup H_k)$. Then $aF \subseteq H_{k+1} \cup \ldots \cup H_n$, and hence there is l such that $k+1 \le l \le n$ and $|M| > m_{k+1}$, where $M = H_l \cap aF$. If $w \in M$, then $w^{-1}M \subseteq H_1 \cap \ldots \cap H_k \cap H_l$, and therefore $|H_1 \cap \ldots \cap H_k \cap H_l| > m_{k+1}$, a contradiction.

We have shown that $m_k \le (n-k)m_{k+1}$ for every $k = 1, \ldots, n-1$, and so $m_1 \le (n-1)!\, m_n = (n-1)!\,|K|$. Finally, $|G| \le nm_1$ and we get $|G| \le n!|K|$ and $[G:K] \le n!$ □

Corollary 3.2 ([11]) *Let $H_1, \ldots, H_n$, $n \ge 1$, be subgroups of a group G such that $G = H_1 \cup \ldots \cup H_n$, $1 = H_1 \cap \ldots \cap H_n$ and $G \neq \bigcup_{i=1, i \neq j}^{n} H_i$ for every $1 \le j \le n$. Then G is a finite group and $|G| \le n!$*

Remark 3.3 Let $H_1, \ldots, H_n$, $n \ge 1$, be proper subgroups of a group G such that $G = H_1 \cup \ldots \cup H_n$. Then $n \ge 3$ and there exist indices $1 \le i_1 < i_2 < \ldots < i_m \le n$ such that $m \ge 3$, $G = H_{i_1} \cup \ldots \cup H_{i_m}$ and $G \neq \bigcup_{j=1, j \neq k}^{m} H_{i_j}$ for every $1 \le k \le m$.

(i) By 3.1, $K = H_{i_1} \cap \ldots \cap H_{i_m}$ is a subgroup of finite index in G and $[G:K] \le m!$.

(ii) If G is infinite, then all the subgroups $H_{i_1}, \ldots, H_{i_m}$ are infinite and K is infinite, too.

(iii) If K is finite, then G is finite and $|G| \le m!|K|$.

Remark 3.4 ([11]) Let $H_1, \ldots, H_n$, $n \ge 1$, be subgroups of a group G such that $G = H_1 \cup \ldots \cup H_n$ and $G \neq \bigcup_{i=1, i \neq j}^{n} H_i$ for every $j, 1 \le j \le n$. Put $L = \mathrm{Core}_G(H_1 \cap \ldots \cap H_n)$. By 3.1, $[G:L]$ is finite, and hence $\overline{G} = G/L$ is finite group. Moreover $\overline{G} = \overline{H}_1 \cup \ldots \cup \overline{H}_n$, $\overline{H}_i = H_i/L$, and $\overline{G} \neq \bigcup_{i=1, i \neq j}^{n} \overline{H}_i$, $1 \le j \le n$. Clearly, $\mathrm{Core}_{\overline{G}}(\overline{H}_1 \cap \ldots \cap \overline{H}_n) = 1$. If $n \ge 2$, then $n \ge 3$ and $\overline{G}$ is not cyclic.

Theorem 3.5 ([11]) *A group G is the union of finitely many proper subgroups if and only if at least one among the factorgroups of G is a non-cyclic finite group.*

Proof. The direct implication is clear from 3.3 and 3.4. Conversely, if G/N is a non-cyclic finite group for a normal subgroup N of G, then there are finitely many elements $a_1, \ldots, a_n$ in G such that $G = H_1 \cup \ldots \cup H_n$, $H_i = \langle N, a_i \rangle \neq G$. □

Remark 3.6 The following result is easy (see [14]):

Let H, K, L be proper subgroups of a group G such that $G = H \cup K \cup L$. Then each of H, K, L is a normal maximal subgroup of G, $G/H \simeq G/K \simeq G/L \simeq \mathbb{Z}_2(+)$ and $G/(H \cap K \cap L) \simeq (\mathbb{Z}_2(+))^2$ (we have $H \cap K = H \cap L = K \cap L = H \cap K \cap L$).

Example 3.7 Let G be a non-commutative finite simple group. Then $G = \bigcup \langle a \rangle$, $a \in G \setminus \{1\}$, and none of the cyclic subgroups $\langle a \rangle$ is normal in G.

Example 3.8 Let a group G be the semidirect product of a nine-element group H and a two-element group $K = \{1, w\}$ such that $a^3 = 1$ and $waw = a^2$ for every $a \in H$. Then G is metabelian, $Z(G) = 1$, $|G| = 18$, every subgroup of H is normal in G (and hence H is the only Sylow 3-subgroup of G), G has nine Sylow 2-subgroups and elements of G have orders 1, 2 or 3. Moreover, if $a, x \in G$ are such that $a^3 = 1 = x^2$, $x \neq 1$, then $xax = a^2$. Now, there are four (cyclic) three-element subgroups T_1, T_2, T_3 and T_4 of H such that $\bigcup_{i=1}^{4} T_i = H \neq \bigcup_{i=1, i \neq j}^{4} T_i$ for every $1 \leq j \leq 4$ and $T_k \cap T_l = 1$ for all $1 \leq k < l \leq 4$. For $1 \leq i \leq 4$, put $S_i = T_i K$. Then S_1, S_2, S_3 and S_4 are non-normal non-commutative six-element subgroups of G, $\bigcup_{i=1}^{4} S_i = G \neq \bigcup_{i=1, i \neq j}^{4} S_i$ for every $1 \leq j \leq 4$ and $\bigcap_{i=1}^{4} S_i = K = S_k \cap S_l$ for all $1 \leq k < l \leq 4$. In particular, $\bigcap S_i$ is not normal in G (cf. 3.1 and 3.6). Finally, notice that S_1, S_2, S_3, S_4 and H are (the only ones) maximal subgroups of G and that $G \neq H \cup \left(\bigcup_{i=1, i \neq j}^{4} S_i\right)$ for every $1 \leq j \leq 4$.

Remark 3.9 Let a group G be the semidirect product of subgroups H and K, where $H \lhd G$. Assume further that $H = H_1 \cup \ldots \cup H_n$, where $n \geq 3$, $H_1, \ldots, H_n$ are proper subgroups of H, all of them are normal in G, $H_1 \cap \ldots \cap H_n = 1$ and $\bigcup_{i \neq j} H_i \neq H$ for every $1 \leq j \leq n$. Now, put $K_i = H_i K$ for every $1 \leq i \leq n$. Clearly, K_i are proper subgroups of G, $\bigcup K_i = G \neq \bigcup_{i \neq j} K_i$ for every $1 \leq j \leq n$ and $\bigcap K_i = K$.

Example 3.10 Let a group G be the semidirect product of subgroups H and $K = \{1, w\}$ such that $H = (\mathbb{Z}_2(+))^3$ and $w(a, b, c)\, w = (b, c, a)$ for all $a, b, c = \mathbb{Z}_2$. Put $H_1 = \{(0,0,0), (1,0,0), (0,1,0), (1,1,0)\}$, $H_2 = \{(0,0,0), (1,0,1), (0,1,1), (1,1,0)\}$, $H_3 = \{(0,0,0), (0,0,1)\}$, and $H_4 = \{(0,0,0), (1,1,1)\}$. Then all H_1, H_2, H_3, H_4 are nomal subgroups of G and $\bigcup H_i = H \neq \bigcup_{i \neq j} H_i$, $1 \leq j \leq 4$, and $\bigcap H_i = 1$. Now, if $G_i = H_i K$, then $\bigcup G_i = G \neq \bigcup_{i \neq j} G_i$, $1 \leq j \leq 4$, and $K = \bigcap G_i$ is not normal in G. Notice, finally, that $|G| = 16$, and hence G is nilpotent.

Remark 3.11 Let G be a finite group nilpotent of class $m \geq 2$. Then there exist proper subgroups $G_1, \ldots, G_n$ of G, $n \geq 3$, such that $\bigcup G_i = G \neq \bigcup_{i \neq j} G_i$, $1 \leq j \leq n$, and $Z_{m-1}(G) \subseteq K = \bigcap G_i \trianglelefteq G$ (then $K \neq 1$.) (To show this, it is enough to observe that $\bar{G} = G/Z_{m-1}(G)$ is non-cyclic abelian and to use 3.5.)

Remark 3.12 A finite group G is the union of its Sylow subgroups if and only if the orders of elements are powers of prime numbers. Moreover, if G is not a $p-$group, then the intersection of the Sylow subgroups equals 1.

4. Groups (b)

Let G be a non-trivial group. We say that G satisfies the condition (MS) if every proper subgroup of G is contained in a (proper) maximal subgroup. If, moreover, G possesses only finitely many maximal subgroups, then we say that G satisfies (MSF).

Lemma 4.1 *If a non-trivial group G satisfies (MS) ((MSF), resp.), then every non-trivial factorgroup of G does so.*

Proof. Obvious. $\square$

Proposition 4.2 *A group G is finitely generated if and only if unions of non-empty chains of proper subgroups of G are proper.*

Proof. Assume that G is not finitely generated, $\kappa = |G|$ and $G = \{a_\alpha; 0 \le \alpha < \kappa\}$. Now, $G_\alpha = \langle a_\beta; 0 \le \beta \le \alpha \rangle$, $0 \le \alpha < \kappa$, is a chain of proper subgroups of $G = \bigcup G_\alpha$. $\square$

Corollary 4.3 *Every non-trivial finitely generated group satisfies (MS).*

Example 4.4 The infinite direct sum $\mathbb{Z}_2^{(\omega)}$ is a non-finitely generated group satisfying (MS). Moreover, every non-trivial subgroup of this group satisfies (MS) as well.

For a group G, let $F(G)$ denote the Frattini subgroup of G. That is, $F(G)$ is the intersection of all maximal (proper) subgroups of G, provided that at least one such a subgroup exists, and $F(G) = G$ otherwise. Clearly, $F(G)$ is an invariant subgroup of G, and so $F(G)$ is normal in G and $F(G/F(G)) = 1$.

Proposition 4.5 *Let G be a non-cyclic group satisfying (MSF) and let $H_1, \dots, H_n$, $n \ge 1$, be all pair-wise different maximal subgroups of G. Then:*
(i) $n \ge 3$, $G = H_1 \cup \dots \cup H_n$ and $F(G) = H_1 \cap \dots \cap H_n$.
(ii) If $a_i \in G \setminus H_i$ for every $1 \le i \le n$, then G is generated by the elements $a_1, \dots, a_n$.
(iii) G is finitely generated.

Proof. Obvious. $\square$

Proposition 4.6 *A non-trivial group G satisfies (MSF) if and only if G is finitely generated and the Frattini subgroup $F(G)$ has finite index in G.*

Proof. Assume that G satisfies the condition (MSF). Then G is finitely generated by 4.5 (iii) and, if H is a maximal subgroup of G, then every subgroup conjugate to H is maximal, and hence there are only finitely many conjugates to H. It follows that the normalizer $N(H)$ of H has finite index in G. Now, if $H \ne N(H)$, then $N(H) = G$, H is normal in G and $G/H \simeq \mathbb{Z}_p(+)$ for a prime p. We have shown that the index $[G:H]$ is finite and the rest is clear. $\square$

Remark 4.7 The following conditions are equivalent for a group G:
(1) Every subgroup of G is finitely generated.
(2) If $G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$ is an infinite chain of subgroups of G, then $G_m = G_{m+1} = \dots$ for some $m \ge 0$.
(3) If $\mathscr{S}$ is a non-empty set of subgroups of G, then there exists at least one subgroup $H \in \mathscr{S}$ such that $H = K$, whenever $K \in \mathscr{S}$ and $H \subseteq K$.

If these equivalent conditions are satisfied, then we say that G satisfies maximal condition on subgroups.

Remark 4.8 Let G be a non-trivial group such that every non-trivial subgroup of G satisfies (MSF). Then every subgroup of G is finitely generated (4.6, .7) and

G is torsion. Futhermore, if G is not finite, $F_0 = G$ and $F_{m+1}(G) = F(F_m(G))$ for every $m \geq 0$, then $\ldots \subseteq F_3 \subseteq F_2 \subseteq F_1 \subseteq F_0$ is an infinite strictly decreasing chain of normal subgroups of G and all of them have finite index in G (use 4.6).

5. Groups (c)

Let G be a non-cyclic group. We say that G satisfies the condition (MCS) if every element of G is contained in a maximal cyclic subgroup of G (i.e., in a cyclic subgroup that is a maximal element of the ordered set of cyclic subgroups). If, moreover, G possesses only finitely many maximal cyclic subgroups, then we say that G satisfies (MCSF).

5.1 Let G be a non-cyclic group satisfying (MCS) and let C_i, $i \in I$, be all pair-wise different maximal cyclic subgroups of G.

Lemma 5.1.1 *(i)* $|I| \geq 3$.
(ii) $C_i \neq G$ *for every* $i \in I$.
(iii) $\bigcup C_i = G$.
(iv) $\bigcup_{i \neq j} C_i \neq G$ *for every* $j \in I$.

Proof. Easy. □

Lemma 5.1.2 *(i)* $C = \bigcap C_i \subseteq Z(G)$ *(the centre of G).*
(ii) C is a normal cyclic subgroup of G.
(iii) If $C \neq 1$, *then* G/C *is torsion.*
(iv) If C is non-trivial finite, then G is torsion.
(v) If C is infinite, then G is torsion-free.

Proof. (i) is clear, (ii) follows from (i) and (iv) from (iii).
(iii) If $a \in G$, then $a \in C_i = \langle b \rangle$ for some $i \in I$. Now, $a = b^m$ and $b^n \in C$ for an integer m and a positive integer n. Consequently, $a^n = (b^n)^m \in C$.
(v) If $a \in G$, $a \neq 1$, then $a \in C_i$ for some $i \in I$. Since $C \subseteq C_i$, C_i is an infinite cyclic group and the same is true for $\langle a \rangle$. □

Lemma 5.1.3 *The following conditions are equivalent:*
(i) $C = 1$.
(ii) There are subgroups G_j, $j \in J$, $J \neq \emptyset$, *of G such that* $\bigcap G_j = 1$ *and* $\bigcup G_j = G \neq \bigcup_{j \neq k} G_j$ *for every* $k \in J$.

Proof. (ii) implies (i). There is a mapping $f : I \to J$ such that $C_i \subseteq G_{f(i)}$ for every $i \in I$. Clearly, $\bigcup G_{f(i)} = G$, and hence $f(I) = J$ and $\bigcap C_i \subseteq G_{f(i)} = \bigcap G_j = 1$. □

Lemma 5.1.4 *Put* $\overline{G} = G/C$. *Then:*
(i) $\overline{C_i} = C_i/C$, $i \in I$, *are maximal cyclic subgroups of* $\overline{G}$.
(ii) $\overline{C_j} \neq \overline{C_k}$ *for all* $j, k \in I, j \neq k$.

Proof. (i) Let D/C be a cyclic subgroup of $\overline{G}$ such that $C_i \subseteq D$. Then $D = \langle C, a \rangle$ for some $a \in G$ and $a \in C_j$ for some $j \in I$. Now, $C \subseteq C_j$ and $a \in C_j$ implies $D \subseteq C_j$, and therefore $C_i \subseteq C_j$, $i = j$, $C_i = C_j$ and, finally, $C_i = D$.
(ii) Obvious. □

Corollary 5.1.5 *(i) The group $\overline{G}$ is a non-cyclic group satisfying (MCS).*
(ii) $\overline{C_i}$, $i \in I$, are all pair-wise different maximal cyclic subgroups of $\overline{G}$.
(iii) $\bigcup \overline{C_i} = \overline{G}$ and $\bigcap \overline{C_i} = 1$.
(iv) $\bigcup_{i \neq j} \overline{C_i} \neq \overline{G}$ for every $j \in I$.

Proposition 5.2 *The following conditions are equivalent for a group G:*
(i) If $\mathscr{C}$ is a non-empty set of cyclic subgroups of G, then there exists at least one cyclic subgroup $C \in \mathscr{C}$ such that $C = D$, whenever $D \in \mathscr{C}$ and $C \subseteq D$ (i.e., G satisfies maximal condition on cyclic subgroups).
(ii) If $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ is an infinite chain of cyclic subgroups of G, then $C_m = C_{m+1} = \ldots$ for some $m \geq 0$.
(iii) None of the following (abelian) groups is isomorphic to a subgroup of G:
(iii1) The quasicyclic Prüfer $p-$group $\mathbb{Z}_{p\infty}(+)$, p a prime.
(iii2) The direct sum $\coprod \mathbb{Z}_p(+)$, $p \in P$, P an infinite set of primes.
(iii3) A non-cyclic subgroup of the additive group $\mathbb{Q}(+)$ of rationals.

Proof. Standard and easy. □

Proposition 5.3 *Every non-cyclic group satisfying the maximal condition on cyclic subgroups (see 5.2) satisfies the condition (MSC).*

Proof. Obvious. □

Proposition 5.4 *A non-cyclic group G satisfies (MCSF) if and only if G is finite.*

Proof. Let G satisfy (MCSF) and let $C_1, \ldots, C_n$, $n \geq 1$, be all maximal cyclic subgroups of G. By 5.1.2(i), we have $C = C_1 \cap \ldots \cap C_n \subseteq Z(G)$ and, by 3.2, $|G/C| \leq n! = m$. Now, G is finite, provided that C is so, and hence we assume that C is infinite. The mapping $\varphi : x \to x^m$ is a homomorphism of G into C and $\mathrm{Ker}(\varphi) = 1$, since G is torsionfree by 5.1.2(v). Then $G \simeq \varphi(G)$ is cyclic, a contradiction. □

5.5 Let G be a non-cyclic finite group and let $C_1, \ldots, C_n$, $n \geq 1$, be all pair-wise different maximal cyclic subgroups of G (see 5.1 and 5.4).

Lemma 5.5.1 *(i) $n \geq 3$.*
(ii) $C_i \neq G$ for every $1 \leq i \leq n$.
(iii) $\bigcup C_i = G$.
(iv) $\bigcup_{i \neq m} C_i \neq G$ for every $1 \leq m \leq n$.
(v) $C = \bigcap C_i \subseteq Z(G)$ and C is a normal cyclic subgroup of G.

Proof. See 5.1.1. □

Lemma 5.5.2 *The following conditions are equivalent:*
(i) $C = 1$.
(ii) There are subgroups $G_1, \ldots, G_m$, $m \geq 1$, *of* G *such that* $\bigcap G_j = G \neq \bigcup_{j \neq k} G_j$ *for every* $1 \leq k \leq m$.
Moreover, if these equivalent conditions are satisfied, then $3 \leq m \leq n$.

Proof. See 5.1.3. □

Lemma 5.5.3 *(i)* $\overline{G} = G/C$ *is a non-cyclic group and* $|\overline{G}| \leq n!$.
(ii) $\overline{C_i} = C_i/C$, $1 \leq i \leq n$, *are all pair-wise different maximal cyclic subgroups of* $\overline{G}$.
(iii) $\bigcap \overline{C_i} = 1$.

Proof. See 5.1.4. □

Lemma 5.5.4 *Every subgroup of* G *is at most* $(n-1)$*−generated (and, consequently,* G *has Prüfer rank at most* $n-1$*).*

Proof. If H is a subgroup of G, then $H \cap C_i = \langle a_i \rangle$ for some $a_i \in H$ and we put $K = \langle a_1, \ldots, a_{n-1} \rangle$. Clearly, $H = K \cup (H \cap C_n)$, and so either $H = K$ or $H \subseteq C_n$ (1.1). □

For every prime p, let S_p be a Sylow p-subgroup of G. Let Q denote the set of prime numbers q such that either S_q is non-trivial cyclic or $q = 2$ and S_q is dicyclic. Then Q is finite and we put $D = 1$ if $Q = \emptyset$ and $D = \prod_{q \in Q}(S_q \cap Z(G))$ if $Q \neq \emptyset$.

Lemma 5.5.5 $D = C$.

Proof. Firstly, take $q \in Q$, put $R_q = S_q \cap Z(G)$ and let $1 \leq i \leq n$ and $C_i = E \cdot F$, where E is the Sylow q-subgroup of C_i and q does not divide $|F|$. The subgroup $E \cdot R_q$ is a q-group, and hence $E \cdot R_q \subseteq P$ for a Sylow q-subgroup P of G. Futhermore, $E \cdot R_q$ is abelian and, since $q \in Q$, it follows easily that $E \cdot R_q$ is cyclic. Consequently, either $R_q \subseteq E$ or $E \subseteq R_q$. If $R_q \subseteq E$, then $R_q \subseteq C_i$. On the other hand, if $E \subseteq R_q$, then $R_q \cdot F$ is a cyclic group and $C_i \subseteq R_q \cdot F$ implies $C_i = R_q \cdot F$, $E = R_q$ and $R_q \subseteq C_i$ again. Thus $R_q \subseteq C_i$ for every $q \in Q$, and therefore $D \subseteq C_i$ for every i. It follows that $D \subseteq C$.

Now, conversely, take a prime p *such that* C contains a p-element subgroup A and let $V_p \subseteq T$, where V_p is the Sylow p-subgroup of C and T a Sylow p-subgroup of G. Assume, for a moment, that $A \cap B = 1$ for a p-element subgroup B of T and let $1 \leq k, l \leq n$ be such that $A \subseteq C_k$ and $B \subseteq C_l$. Then $k \neq l$, and hence $A \nsubseteq C_l$. On the other hand, $A \subseteq C \subseteq C_l$, a contradiction. We have shown that A is contained in every non-trivial subgroup of T and consequently, $p \in Q$. Thus $V_p \subseteq T \cap Z(G) = S_p \cap Z(G) \subseteq D$. It follows that $C \subseteq D$. □

Theorem 5.6 *The following conditions are equivalent for a non-trivial group* G:
(i) There exist subgroups $H_1, \ldots, H_m$, $m \geq 1$, *of* G *such that* $\bigcap_{i=1}^{m} H_i = 1$ *and*

$\bigcup_{i=1}^{m} H_i = G \neq \bigcup_{i=1, i \neq j}^{m} H_i$ *for every* $1 \leq j \leq m$ *(then* $m \geq 3$, G *is finite and* $|G| \leq m!$*).*

(ii) *G is non-cyclic finite and* $\bigcap_{i=1}^{n} C_i = 1$, *where* $C_1, \ldots, C_n$ *are all maximal cyclic subgroups of* G *(then* $3 \leq m \leq n$*).*

(iii) *G is finite and if P is a Sylow p-subgroup of G such that* $P \cap Z(G) \neq 1$, *then either P is not cyclic or* $p = 2$ *and P is not dicyclic.*

(iv) *G is finite and if p is a prime dividing* $|Z(G)|$, *then G contains at least two subgroups of order p.*

Proof. See 5.5. □

Example 5.7 Consider the group G from 3.8. Then G satisfies the conditions from 5.6 (notice that G has just thirteen maximal cyclic subgroups of order 3 and nine of order 2). On the other hand, the subgroups H_i (see 5.5(i)) cannot be chosen to be all maximal (G has just five maximal subgroups, one of order 9 and four of order 6, and $F(G) = 1$).

Example 5.8 (i) The symmetric group S on three letters satisfies 5.6(iii) (all Sylow subgroups of S are cyclic, but $Z(S) = 1$) and the eight-element quaternion group H does not satisfy 5.6(iii).

(ii) Put $G = S \times \mathbb{Z}_3(+)$, S being the symmetric group on three letters and $\mathbb{Z}_3(+)$ the additive group of integers modulo 3. Then $|G| = 18$, $|Z(G)| = 3$, $|A| = 9$, where A is the (unique) Sylow 3-subgroup of G and $|B| = 2$, where B is any Sylow 2-subgroup of G. Moreover, $A \cap Z(G) \neq 1$ and A is not cyclic, B is cyclic and $B \cap Z(G) = 1$. Thus G satisfies 5.6(iii), and hence all equivalent conditions of 5.6.

(iii) The groups $S \times H$, $S \times \mathbb{Z}_2(+)$ and $H \times \mathbb{Z}_2(+)$ satisfy 5.5(iii).

(iv) The groups $S \times \mathbb{Z}_5(+)$, $H \times \mathbb{Z}_3(+)$ and $H \times \mathbb{Z}_2(+) \times \mathbb{Z}_3(+)$ do not satisfy 5.6(iii).

(v) The groups $\mathbb{Z}_4(+) \times \mathbb{Z}_2(+)$ and $H \times \mathbb{Z}_4(+)$ satisfy 5.6(iii). On the other hand, these groups have non-trivial Frattini subgroups.

(vi) The alternating group A_5 on five letters contains 31 non-trivial cyclic subgroups, each of them is a maximal cyclic subgroup and, pair-wise, they have trivial intersections. Among these cyclic subgroups we find 15 two-element subgroups, 10 three-element subgroups and, finally, 6 five-element subgroups. Consequently, the elements of A_5 have orders 1, 2, 3 and 5, resp.

Remark 5.9 Let G be a finite non-cyclic group, whose every non-identity element has prime order. Then every non-trivial cyclic subgroup of G is both a maximal and minimal cyclic subgroup, G is the union of cyclic subgroups and $A \cap B = 1$ for all cyclic subgroups A, B, $A \neq B$.

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