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On Convex Constructions in Lattices

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V tomto článku studujeme konvexní rozklady distributivních, modulárních, Brouwerových a pseudokomplementárních svazů.

Dans cet article nous étudions les décompositions convexes de treillis distributifs, modulaires, Brouweriens et pseudocomplementés.

In this paper we investigate oriented convex decompositions of distributive, modular, Brouwerian and pseudo-complemented lattices.

1. Introduction

A couple (L_1, L_2) is called a *decomposition* of a lattice $(L, \leq)$ if

(1.1) L_1 and L_2 are proper sublattices of L;
(1.2) $L_1 \cap L_2 \neq \emptyset$ and L_1 and $L_1 \cup L_2 = L$;
(1.3) for any i, j with $1 \leq i \neq j \leq 2$, any $v \in L_i \setminus L_j$ and any $w \in L_j \setminus L_i$ such that $v \leq w$ there exists $x_0 \in L_1 \cap L_2$ satisfying $v \leq x_0 \leq w$.

This definition is based upon a more general study of amalgams in the theory of ordered sets [1].

If moreover

(1.4) the set $L_1 \cap L_2$ is a convex subset in $(L, \leq)$,

we call (L_1, L_2) a *convex decomposition* of L (written $L = cd(L_1, L_2)$). This

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corresponds to the notion of an amalgam with a pasted convex subset in [1]. See also [2, Chap. IV].

In the special case where L_1 and L_2 possess the unit elements 1_1, 1_2 and the zero elements $0_1, 0_2$, the convex decomposition (L_1, L_2) of a lattice L coincides with a well known construction of Hall and Dilworth [6]. We propose to speak about a *Hall-Dilworth decomposition* (L_1, L_2) of L in this case. For some related ideas see also [7] and [5].

The next result [3] explains the notation we use in what follows.

(SP) *Let L be a lattice and let $L = cd(L_1, L_2)$. Then*

$$L_1 = (L_1 \cap L_2] \quad \& \quad L_2 = [L_1 \cap L_2) \tag{1.5}$$

under a suitable relabeling of indices 1 and 2.

Here $(L_1 \cap L_2] := \{x \in L; \exists y \in L_1 \cap L_2\ x \leq y\}$ and $[L_1 \cap L_2)$ is defined dually.

A convex decomposition (L_1, L_2) of a lattice L is said to be an *oriented convex decomposition* of L (written $L = \overrightarrow{cd}(L_1, L_2)$) if also (1.5) is true. If (L_1, L_2) is a Hall-Dilworth decomposition of a lattice L satisfying (1.5), it will be called an *oriented Hall-Dilworth decomposition.*

Before proceeding, it is convenient to introduce a useful convention: We will write $x \in \bullet$ (or $\bullet \ni x$) to indicate that $x \in L_1 \cap L_2$.

Let us start with the formulae for joins and meets taken from [3] in the case where (L_1, L_2) is an oriented convex decomposition of a lattice $(L, \vee, \wedge)$ provided $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ are the corresponding sublattices with explicitly described operations.

(1*) If $a \in L_1$ and $b \in L_2$, then $a \wedge b = a \wedge_1 (a^* \wedge_2 b)$ and $a \vee b = (a \vee_1 b_+) \vee_2 b$ where a^* is any element such that $a \leq a^* \in \bullet$ and where b_+ is any element such that $b \leq b_+ \in \bullet$.

(2*) If a and b belong to L_i where i is either 1 or 2, then $a \wedge b = a \wedge_i b$ and $a \vee b = a \vee_i b$.

(3*) If $a \in L_1$, $b \in L_2$ and $c \in \bullet$, then $a \vee c \in \bullet$ and $b \wedge c \in \bullet$.

Throughout the paper, L always denotes a lattice and L_1, L_2 its sublattices. For the terminology and also for all necessary properties of lattices see the book [4].

2. Distributive and modular lattices

In this section we will apply our formulas (see [3]) to prove that $L = \overrightarrow{cd}(L_1, L_2)$ is distributive (or modular) provided L_1 and L_2 are distributive (or modular). We emphasize that our elementary and computational approach is independent of any general theory for these two classes of lattices.

Theorem 2.1 *Let $L = \overrightarrow{cd}(L_1, L_2)$. If L_1 and L_2 are distributive, then L is also distributive.*

Proof. We will show that

$$(2.1) \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

is true for any a, b, c of L.

(2.2) *Observe that (2.1) is evident whenever*

$$a \in L_i \quad \& \quad b \in L_i \quad \& \quad c \in L_i$$

where $i \in \{1,2\}$.

Let us distinguish the following cases:

Case I: $a \in L_1$, $b \in L_2$ *and* $c \in L_2$. Applying (1*)−(3*) and (2.2) we have that

$$\begin{aligned} a \vee (b \wedge c) &= [a \vee_1 (b \wedge c)_+] \vee_2 (b \wedge_2 c) = \\ &= \{[a \vee_1 (b \wedge c)_+) \vee_2 b\} \wedge_2 \{[a \vee_1 (b \wedge c)_+] \vee_2 c\} = \\ &= \{[a \vee (b \wedge c)_+) \vee b\} \wedge \{[a \vee (b \wedge c)_+] \vee c\} = (a \vee b) \wedge (a \vee c). \end{aligned}$$

Case II: $a \in L_1$, $b \in L_2$ *and* $c \in L_1$. Using (1*)−(3*) and (2.2) we get

$$a \vee (b \wedge c) = a \vee_1 [c \wedge_1 (c^* \wedge_2 b)] = (a \vee_1 c) \wedge_1 [a \vee_1 (c^* \wedge_2 b)].$$

However, $a \in L_1$, $c^* \in L_2$ and $b \in L_2$. Thus by Case I,

$$a \vee_1 (c^* \wedge_2 b) = a \vee (c^* \wedge_2 b) = a \vee (c^* \wedge b) = (a \vee c^*) \wedge (a \vee b).$$

Therefore,

$$a \vee (b \wedge c) = (a \vee c) \wedge (a \vee c^*) \wedge (a \vee b) = (a \vee b) \wedge (a \vee c).$$

Case III: $a \in L_2$, $b \in L_1$ *and* $c \in L_1$. Using (1*)−(3*) and (2.2) repeatedly we obtain

$$\begin{aligned} a \vee (b \wedge c) &= [(b \wedge_1 c) \vee_1 a_+] \vee_2 a = \\ &= [(b \wedge c) \vee a_+] \vee_2 a = [(b \vee a_+) \wedge (c \vee a_+)] \vee_2 a = \\ &= [(b \vee a_+) \vee_2 a] \wedge [(c \vee a_+) \vee_2 a] = (a \vee b) \wedge (a \vee c). \end{aligned}$$

Case IV: $a \in L_2$, $b \in L_1$ *and* $c \in L_2$. By (1*)−(3*) we first find that

$$\begin{aligned} w := a \vee (b \wedge c) &= [b \wedge c) \vee_1 a_+] \vee_2 a = \\ &= \{[b \wedge_1 (b^* \wedge_2 c)] \vee_1 a_+\} \vee_2 a. \end{aligned}$$

Since $a_+ \in L_1 \cap L_2 \subset L_1$, $b^* \wedge_2 c \in L_1 \cap L_2 \subset L_1$ and $b \in L_1$, we can use (2.2) and so

$$w = \{(b \vee a_+) \wedge_1 [(b^* \wedge_2 c) \vee_1 a_+]\} \vee_2 a.$$

From (3*) we can see that $b \vee a_+ \in L_1 \cap L_2 \subset L_2$ and $(b^* \wedge_2 c) \vee_1 a_+ \in L_1 \cap L_2 \subset \subset L_2$. Since $a \in L_2$,

$$w = \{(b \vee a_+) \wedge_2 [(b^* \wedge_2 c) \vee_1 a_+]\} \vee_2 a$$

and, by the distributivity of L_2, we infer that

$$w = [(b \vee a_+) \vee_2 a] \wedge_2 \{[(b^* \wedge_2 c) \vee_1 a_+] \vee_2 a\} = (b \vee a) \wedge [(b^* \wedge_2 c) \vee_2 a].$$

However, b^*, c and a belong to L_2. The distributivity of L_2 implies that

$$w = (b \vee a) \wedge (b^* \vee_2 a) \wedge_2 (c \vee_2 a) = (b \vee a) \wedge (b^* \vee a) \wedge (c \vee a) =$$
$$= (b \vee a) \wedge (c \vee a).$$

Case V: $a \in L_1$, $b \in L_1$ *and* $c \in L_2$. Interchanging the roles of b and c we have Case II.

Case VI: $a \in L_2$, $b \in L_2$ *and* $c \in L_1$. Similarly, replacing b by c and vice versa we get Case IV. □

Theorem 2.2 *Let* $L = \vec{cd}(L_1, L_2)$. *If* L_1 *and* L_2 *are modular,* L *is also modular.*

Proof. We will establish that

$$(2.3) \qquad (a \wedge c) \vee [b \wedge (a \vee c)] = [(a \wedge c) \vee b] \wedge (a \vee c)$$

is true for any $a, b, c \in L$.

The modular identity (2.3) holds if a, b and c belong to the same lattice L_i $(i \in \{1,2\})$.

In the remaining situations we distinguish six cases:

Case I: $a \in L_1$, $b \in L_2$ *and* $c \in L_2$. By $(1^*)-(3^*)$,

$$v := (a \wedge c) \vee [b \wedge (a \vee c)] = (a \wedge c) \vee [b \wedge_2 (a \vee c)] =$$
$$= \{(a \wedge c) \vee_1 [b \wedge_2 (a \vee c)]_+\} \vee_2 [b \wedge_2 (a \vee c)].$$

Here $[b \wedge_2 (a \vee c)]_+ \in \bullet$. From (2*) and (3*) it follows that

$$(a \wedge c) \vee_1 [b \wedge_2 (a \vee c)]_+ = (a \wedge c) \vee [b \wedge_2 (a \vee c)]_+ \in L_1 \cap L_2 \subset L_2$$

and, moreover, $(a \wedge c) \vee [b \wedge_2 (a \vee c)]_+ \le a \vee c$. Now $b \in L_2$, and $a \vee c \in L_2$. Since L_2 is modular,

$$v = (\{(a \wedge c) \vee_1 [b \wedge_2 (a \vee c)]_+\} \vee_2 b) \wedge_2 (a \vee c).$$

Note that $\bullet \ni [b \wedge_2 (a \vee c)]_+ \le b$. This, together with (1*) and (2*) implies that

$$v = [(a \wedge c) \vee b] \wedge_2 (a \vee c) = [(a \wedge c) \vee b] \wedge (a \vee c).$$

Case II: $a \in L_1$, $b \in L_2$ *and* $c \in L_1$. Again, by $(1^*)-(3^*)$,

$$s := (a \wedge c) \vee [b \wedge (a \vee c)] = (a \wedge c) \vee \{(a \vee_1 c) \wedge_1 [(a \vee_1 c)^* \wedge_2 b]\}.$$

Since $a \wedge c \le a \vee_1 c$ and since, from (3*), $(a \vee_1 c)^* \wedge_2 b \in L_1 \cap L_2 \subset L_1$, it follows from the modularity of L_1 that

$$s = \{(a \wedge c) \vee_1 [(a \vee_1 c)^* \wedge_2 b]\} \wedge_1 (a \vee_1 c).$$

Then in view of $(a \vee_1 c)^* \wedge_2 b \in L_1 \cap L_2 \subset L_2$, $a \wedge c \in L_1$ and (1*) we have

$$t := (a \wedge c) \vee_1 [(a \vee_1 c)^* \wedge_2 b] = \{(a \wedge c) \vee_1 [(a \vee_1 c)^* \wedge_2 b]_+\} \vee_2 [(a \vee_1 c)^* \wedge_2 b].$$

Clearly, $\bullet \ni (a \wedge c) \vee_1 [(a \vee_1 c)^* \wedge_2 b]_+ \leq (a \vee_1 c)^* \in L_2$. Consequently, it follows by the modularity of L_2 that

$$t = (a \vee_1 c)^* \wedge_2 (\{(a \wedge c) \vee_1 [(a \vee_1 c)^* \wedge_2 b]_+\} \vee_2 b) =$$
$$= (a \vee c)^* \wedge \{(a \wedge c) \vee [(a \vee_1 c)^* \wedge b]_+ \vee b\} = (a \vee c)^* \wedge \{(a \wedge c) \vee b\}$$

and, therefore,

$$s = (a \vee c)^* \wedge \{(a \wedge c) \vee b\} \wedge (a \vee c) = \{(a \wedge c) \vee b\} \wedge (a \vee c).$$

Case III: $a \in L_2$, $b \in L_1$ *and* $c \in L_1$. From (1*)—(3*) and from the modularity of L_1 it follows that

$$p := (a \wedge c) \vee [b \wedge (a \vee c)] = (a \wedge c) \vee_1 \{b \wedge_1 [b^* \wedge_2 (a \vee c)]\}.$$

If b^* is such that $a \wedge c \leq b^* \in \bullet$, then $(a \wedge c) \vee_1 b \leq b^*$. By the modularity of L_1,

$$p = [(a \wedge c) \vee_1 b] \wedge_1 [b^* \wedge_2 (a \vee c)] = [(a \wedge c) \vee b] \wedge [b^* \wedge (a \vee c)] =$$
$$= [(a \wedge c) \vee b] \wedge (a \vee c).$$

Case IV: $a \in L_2$, $b \in L_1$ *and* $c \in L_2$. By (1*)—(3*),

$$r := (a \wedge c) \vee [b \wedge (a \vee c)] = (a \wedge c) \vee \{b \wedge_1 [b^* \wedge_2 (a \vee c)]\}.$$

Here $b \wedge_1 [b^* \wedge (a \vee c)] \in L_1$ and (1*) shows that

$$r = (\{b \wedge_1 [b^* \wedge_2 (a \vee c)]\} \vee_1 (a \wedge c)_+) \vee_2 (a \wedge c).$$

Since we can suppose that $b^* \geq (a \wedge c)_+$, $b^* \wedge (a \vee c) \geq (a \wedge c)_+$. Hence (taking the modularity of L_1 into account),

$$r = \{[b \vee_1 (a \wedge c)_+] \wedge_1 [b^* \wedge_2 (a \vee c)]\} \vee_2 (a \wedge c) =$$
$$= \{[b \vee (a \wedge c)_+] \wedge b^* \wedge (a \vee c)\} \vee_2 (a \wedge c) =$$
$$= \{[b \vee_1 (a \wedge c)_+] \wedge (a \vee c)\} \vee (a \wedge c).$$

Now $b \vee_1 (a \wedge c)_+ \in L_1 \cap L_2 \subset L_2$, $a \vee c \in L_2$ and $a \wedge c \in L_2$. Therefore, by the modularity of L_2,

$$r = \{[b \vee_1 (a \wedge c)_+] \wedge_2 (a \vee c)\} \vee_2 (a \wedge c) =$$
$$= \{[b \vee_1 (a \wedge c)_+] \vee_2 (a \wedge c)\} \wedge_2 (a \vee c) = [b \vee (a \wedge c)] \wedge (a \vee c).$$

Now, interchanging a and c, it is straightforward to check that Case V ($a \in L_1$, $b \in L_1$ and $c \in L_2$) and Case VI ($a \in L_2$, $b \in L_2$ and $c \in L_1$) can be treated as Case III and Case I, respectively. □

3. Decompositions of Brouwerian lattices

A lattice L is called *Brouwerian* [4, p. 45] if, for any $a, b \in L$, the set $\{x \in L; a \wedge x \leq b\}$ contains a greatest element denoted by $b :_L a$ (or simply by $b : a$) which is called the *relative pseudo-complement* of a in b. Note that any Brouwerian lattice is distributive and it possesses the greatest element. By definition,

$$(3.1) \qquad a \wedge (b : a) = a \wedge b \quad \& \quad b \leq b : a .$$

whenever a and b belong to a Brouwerian lattice L.

Theorem 3.1 *Let* $L = \overrightarrow{cd}(L_1, L_2)$. *If* L_1 *and* L_2 *are Brouwerian, then* L *is also Brouwerian.*

Proof. Let u denote the greatest element in L_1, let 1 denote the greatest element in L_2 (so that 1 is the greatest element in L) and let 0 denote the least element in L_1 (so that 0 is the zero element in L). Note that $u \in \bullet$, by (1.2) and (3*).

We will distinguish between four cases.

Case I: $a \in L_1$ and $b \in L_1$. Let $e := b :_{L_1} a$, i.e., $e \in L_1$ and e is the greatest element in L_1 such that $e \wedge a \leq b$.

I - 1: There is no $d \in L_2 \setminus L_1$ such that $e < d$ and $d \wedge a \leq b$. Then $e = b :_L a$.

I - 2: There exists at least one element $d' \in L_2 \setminus L_1$ such that $e < d'$ and $d' \wedge a \leq b$. By (1.3), there exists $e_0 \in \bullet$ such that $e \leq e_0 \leq d'$. Consequently, $e_0 \wedge a \leq d' \wedge a \leq b$. Hence $e = e_0 \in \bullet$. It follows from (3*) that $a \vee e \in \bullet$. Let $d := e :_{L_2} (a \vee e)$. Then $d \wedge (a \vee e) \leq e$ and so $d \wedge a \leq e \wedge a \leq b$. Let $d_1 \geq d$ be such that $d_1 \wedge a \leq b$. By Therorem 2.1, L is distributive. From (3.1) we see that

$$(a \vee e) \wedge d_1 = (a \wedge d_1) \vee (e \wedge d_1) \leq b \vee (e \wedge d_1) \leq b \vee e = e .$$

This together with the choice of d implies that $d = d_1$. Therefore, $d = b :_L a$.

Case II: $a \in L_1$ and $b \in L_2 \setminus L_1$. Then $a \vee b \in L_2$. In view of (3*) we can see that $u \wedge (a \vee b) \in \bullet$ and that $u \wedge b \in \bullet$.

Let $d := (u \wedge b) :_{L_2} [u \wedge (a \vee b)]$. Hence $[u \wedge (a \vee b)] \wedge d \leq u \wedge b$ and $b \leq d$. We want to show that $d = b :_L a$.

Evidently, $d \wedge a \leq d \wedge u \wedge (a \vee b) \leq u \wedge b \leq b$.

If $d_1 \geq d$ is such that $d_1 \wedge a \leq b$, then $d_1 \wedge a \leq b$ and from $b \leq d \leq d_1$ we infer that $a \wedge b \leq a \wedge d_1$ and so $a \wedge d_1 = a \wedge b$. Using the distributivity of L together with $a \leq u$ and $b \leq d_1$, we get

$$\begin{aligned} [u \wedge (a \vee b)] \wedge d_1 &= (u \wedge a \wedge d_1) \vee (u \wedge b \wedge d_1) = \\ &= (a \wedge d_1) \vee (u \wedge b) = (a \wedge b) \vee (u \wedge b) = u \wedge b . \end{aligned}$$

By the choice of d we therefore have $d_1 \leq d$, i.e., $d = d_1$.

Case III: $a \in L_2 \setminus L_1$ and $b \in L_2$. Put $d := b :_{L_2} a$. Our aim is to prove that $d = b :_L a$. Suppose there exists d_1 such that $d < d_1$ and $d_1 \wedge a \leq b$. Since d is the greatest element in L_2 with respect to the considered property, $d_1 \in L_1 \setminus L_2$. By (1.5), there exist $d_{10}, d_{20} \in \bullet$ such that $d_{10} \leq d < d_1 \leq d_{20}$. Using (1.4) we deduce that $d_1 \in \bullet$, a contradiction.

Case IV: $a \in L_2 \setminus L_1$ and $b \in L_1 \setminus L_2$. From (3*), it follows that $u \wedge a \in \bullet$. Put $d := b :_{L_1} (u \wedge a)$ so that $(u \wedge a) \wedge d \leq b$. By (3.1), $b \leq d$.

We claim that $d = b :_L a$. Taking (1*) and (2*) into account, we get

$$d \wedge a = d \wedge_1 (u \wedge_2 a) = (u \wedge a) \wedge d \leq b.$$

Suppose there exists $d_1 \in L$ such that $d < d_1$ and $d_1 \wedge a \leq b$.

IV - 1: $d_1 \in L_1$. Then

$$(u \wedge a) \wedge d_1 = u \wedge (a \wedge d_1) \leq u \wedge b = b,$$

contradicting the choice of d.

IV - 2: $d_1 \in L_2 \setminus L_1$. From (1.5) it follows that there exist $b_0, c_0 \in \bullet$ such that $b_0 \leq d_1 \wedge a \leq b \leq c_0$. By (1.4), $b \in \bullet$, contradicting the hypothesis $b \in L_1 \setminus L_2$. □

A complete lattice is said to be *completely distributive on meets* (cf. [4, p. 128]), if $a \wedge \bigvee x_\alpha = \bigvee (a \wedge x_\alpha)$ for any set $\{x_\alpha\}$.

Corollary 3.2 *Let (L_1, L_2) be an oriented Hall–Dilworth decomposition of a lattice L where L_1 and L_2 are complete lattices which are completely distributive on meets. Then L is a complete lattice which is completely distributive on meets.*

Proof. The lattice L is complete by [3]. The remainder follows from [4, Thm 24, p. 128]. □

Corollary 3.3 *Let $L = \overrightarrow{cd}(L_1, L_2)$. If L_1 is a Brouwerian lattice, then also $L_1 \cap L_2$ is a Brouwerian lattice.*

Proof. Choose $a, b \in L_1 \cap L_2$ and put $d := b :_{L_1} a$. By (3.1), $b \leq d$. From (1.5) it follows that $d \in L_2$ and so $d \in \bullet$. Thus $d = b :_{L_1 \cap L_2} a$. □

Theorem 3.4 *Let $L = \overrightarrow{cd}(L_1, L_2)$ be a Brouwerian lattice and let L_1 possess the greatest element. Then L_1 and L_2 are Brouwerian lattices.*

Proof. Let u denote the greatest element in L_1. Choose $a, b \in L_1$ and put $d := b :_L a$. If $d \in L_1$, then $d = b :_{L_1} a$. Now suppose $d \in L_2 \setminus L_1$. We claim that $u \wedge d = b :_{L_1} a$. Indeed, if $d' \in L_1$ is such that $a \wedge d' \leq b$ and $u \wedge d \leq d'$, then $d' \leq d$ and so $u \wedge d = u \wedge d' = d'$.

Finally, let $c, d \in L_2$ and let $e := c :_L d$. From $c \leq e$ and $c \in L_2$ it follows that $e \in L_2$. Hence $e = c :_{L_2} d$ and we see that L_2 is also Brouwerian. □

Theorem 3.5 *Let $L = \overrightarrow{cd}(L_1, L_2)$. The following requirements are equivalent.*

(i) *The lattices L_1 and L_2 are Brouwerian.*

(ii) *The lattice L is Brouwerian and L_1 possesses the greatest element.*

(iii) *The lattices L and $L_1 \cap L_2$ are Brouwerian.*

Proof. (i) ⇒ (iii) Use Theorem 3.1 and Corollary 3.3.

(iii) ⇒ (ii) The greatest element of $L_1 \cap L_2$ is the greatest element of L_1.

(ii) ⇒ (i) Apply Theorem 3.4. □

4. Decompositions of pseudo-complemented lattices

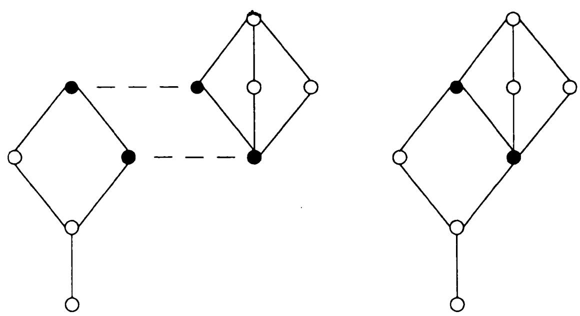
A lattice L which has the least element 0 is called *pseudo-complemented* [4, p. 46] if it has the following property: For any $a \in L$, the set $\{y \in L;\ y \wedge a = 0\}$ has the greatest element a^* called *pseudo-complement* of a in L. Note that 0^* is the greatest element 1 of L.

Theorem 4.1 *Let* $L = \overrightarrow{cd}(L_1, L_2)$ *be a pseudo-complemented lattice. If there exists the greatest element in* L_1, *then* L_1 *is also pseudo-complemented.*

Proof. Let $a \in L_1$. If $a^* \in L_1$, then it is immediate that the pseudo-complement a^{*1} of a in L_1 is equal to a^*.

If $a^* \in L_2$ and if u denotes the greatest element in L_1, it is easy to see that $a^{*1} = u \wedge a^*$. □

Remark 4.2 Under the hypotheses of Theorem 4.1, the lattice L_2 may not be pseudo-complemented, as shown in Figure 1. (The shaded small circles represent the pasted elements.)



L_1 L_2 L

Figure 1

There exist lattices L of the form $L = \overrightarrow{cd}(L_1, L_2)$ which are not pseudo-complemented but where L_1 and L_2 are pseudo-complemented (see Figure 2).

Theorem 4.3 *Let* (L_1, L_2) *be an oriented Hall–Dilworth decomposition of a lattice* L, *let* L_1 *be pseudo-complemented and let* L_2 *be a Boolean lattice. Then* L *is pseudo-complemented.*

Proof. Let o denote the least element in L_2 and let u be the greatest element in L_1. It is easily checked that $o, u \in \bullet$.

We will consider two cases.

Case I: $a \in L_1$.

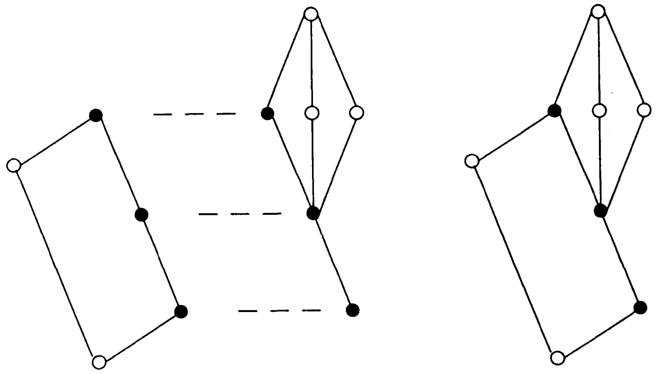


Figure 2

I – 1: There exists $h \in L_2 \setminus L_1$ such that $a \wedge h = 0$.

Let a^{*1} denote the pseudo-complement of a in L_1. Since $a \wedge h = 0$, $a \wedge o = 0$ and so $o \leq a^{*1}$. From $\bullet \ni o \leq a^{*1} \leq u \in \bullet$ we conclude that $a^{*1} \in \bullet$.

Let d denote the relative complement of u in the interval $[a^{*1}, 1]$.

We will now show that $d = a^*$.

First it is clear that $a \wedge d \leq u \wedge d = a^{*1}$. Hence $a \wedge d \leq a \wedge a^{*1} = 0$.

Next, let $b \in L$ be such that $a \wedge b = 0$. If $b \in L_1$, then $b \leq a^{*1} \leq d$.

If $b \in L_2 \setminus L_1$, then by (1*),

$$(4.1) \qquad 0 = a \wedge b = a \wedge_1 (u \wedge_2 b).$$

At the same time, if follows from (2*) and (3*) that $u \wedge b = u \wedge_2 b \in \bullet$. Therefore, from (4.1), we obtain $u \wedge b \leq a^{*1} = u \wedge d$. It is clear that $u \vee (d \vee b) = (u \vee d) \vee b = 1$. On the other hand, the distributivity of L_2 guarantees that

$$u \wedge (d \vee b) = (u \wedge d) \vee (u \wedge b) = u \wedge d = a^{*1}.$$

It then follows from the distributivity of L_2 that $d \vee b = d$. Hence $b \leq d$ and we can see that $d = a^*$.

I – 2: For any $h \in L$, $a \wedge h = 0$ implies $h \in L_1$. Then it is immediate that $a^* = a^{*1}$.

Case II: $a \in L_2 \setminus L_1$. Then, by (3*), $u \wedge a \in \bullet$. Put $c := (u \wedge a)^{*1}$. We claim that $c = a^*$. Using (1*), we get $c \wedge a = c \wedge_1 (u \wedge_2 a) = 0$.

Now let $h \in L$ be such that $0 = h \wedge a$. First we have $h \in L_1 \setminus L_2$. Indeed, suppose $h \in L_2$. Then $0 = a \wedge h \in L_2$ and (1.5) shows that $L_2 = L$, a contradiction. Thus $h \in L_1 \setminus L_2$. From (1*) we deduce that $0 = h \wedge a = h \wedge_1 (u \wedge_2 a)$. Consequently, $h \leq (u \wedge_2 a)^{*1} = c$. □

Theorem 4.4 *Let (L_1, L_2) be an oriented Hall–Dilworth decomposition of a lattice L, let L_1 be a Boolean lattice and let L_2 be pseudo-complemented. Then L is pseudo-complemented.*

Proof. Let o and u be defined in the same way as in the proof of Theorem 4.3. For any $x \in L_1$, let x' denote its complement in L_1.

Let us distinguish two cases:

Case I: $a \in L_1$.

I – 1: There exists $h \in L_2 \setminus L_1$ such that $h \wedge a = 0$. Then, by (1*), $0 = a \wedge h = a \wedge_1 (u \wedge_2 h)$. Using (3*), we get $u \wedge_2 h \leq a'$. From (3*) we conclude that $a \vee o \in \bullet$. Let t denote the pseudo-complement $(a \vee o)^{*2}$ of $a \vee o$ in L_2. Then

$$a \wedge t \leq (a \vee o) \wedge t = (a \vee o) \wedge (a \vee o)^{*2} = o. \tag{4.2}$$

It follows from (2*) and (3*) that

$$\bullet \ni u \wedge h = u \wedge_2 h \leq a' \leq u \in \bullet .$$

Thus, by (1.4), $a' \in \bullet$. Now, referring to (4.2), we see that $t \wedge a \leq o \wedge a \leq a' \wedge a = 0$, i.e., $t \wedge a = 0$.

Finally, we show that $t = a^*$. Since $a \in L_1$, $o \in L_1$ and $a' \in L_1$, the distributivity of L_1 implies that $(a \vee o) \wedge a' = o$. Hence $a' \leq (a \vee o)^{*2} = t$.

Now let $h \in L_1$ be such that $a \wedge h = 0$. Then it is clear that $h \leq a' \leq t$.

Next let $h \in L_2 \setminus L_1$ be such that $a \wedge h = 0$. Then, by the distributivity of L_1, (1*) and by the fact that $u \wedge_2 h \in \bullet$, we have

$$\begin{aligned}(a \vee o) \wedge h &= (a \vee o) \wedge_1 (u \wedge_2 h) = [a \wedge_1 (u \wedge_2 h)] \vee [o \wedge_1 (u \wedge_2 h)] = \\ &= [a \wedge_1 (u \wedge_2 h)] \vee o = (a \wedge h) \vee o = 0 \vee o = o .\end{aligned}$$

Consequently, $h \leq (a \vee o)^{*2} = t$. In Case I - 1 we therefore have $a^* = t$.

I – 2: For any $h \in L$, $h \wedge a = 0$ implies that $h \in L_1$. In this case $a^* = a'$.

Case II: $a \in L_2 \setminus L_1$. Then $h \wedge a = 0$ implies $h \in L_1 \setminus L_2$. As above, $u \wedge a \in \bullet$. From (1*) it is seen that

$$0 = h \wedge a = h \wedge_1 (u \wedge_2 a) \Leftrightarrow h \leq (u \wedge_2 a)' .$$

Therefore, here $a^* = (u \wedge_2 a)'$. □

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