The Period of Oscillations in Non-Linear Systems

Z. MORÁVEK and J. FIALA

Praha

Received 10. June 2003

In this paper we propose a new approach to the determination of the period of oscillations in general non-linear systems. The method is based on determination of the length of the orbit in a generalised phase space and its relation to the time variable. Since these quantities always exist for an arbitrary oscillating system, this approach is not restricted to systems with special properties, for instance to Hamiltonian systems. Knowledge of the period determined theoretically gives more information about the system's behaviour, especially how its dynamics changes when dynamical parameters are varied.

1. Introduction

The most important characteristic of an oscillating system is its period of oscillations. In case of simple systems, the period can be derived in an analytical form either from the solution or from the evolution equations appropriate to such system. Nevertheless, in case of more complicated systems, the derivation becomes difficult or even impossible. At the same time, the knowledge of the period for such systems and its dependence on dynamical parameters of the system helps significantly in understanding and predicting the systems' behaviour.

The usual way of determining the period is based on the theory of adiabatic invariants [1]. Therein a general formula is derived for the period T

$$T = \oint \mathrm{d}q \, \frac{\partial p}{\partial H} . \tag{1}$$

However, the validity of Eq. (1) is restricted to systems for which the Hamilton function can be found. On the other hand, in case of thermodynamically opened

Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic

systems, whose important examples are biological oscillating systems, the Hamilton functions is not known. For this reason, we propose here a different approach to the problem.

2. General theory

In our approach, the derivation is based on the fact that the period of oscillations is related to the length of a trajectory in the phase space (or in a generalised phase space, if the coordinate q and momentum, p, cannot be used). The evolution equations

$$\begin{aligned} \dot{x} &= g_1(x, y), \\ \dot{y} &= g_2(x, y), \end{aligned} \tag{2}$$

define an ordinary differential equation $\mathrm{d}x/\mathrm{d}y = g_1(x, y)/g_2(x, y)$, for which we can always find a solution $x = f(y)$. This solution describes the trajectory as a parametric dependence of x on y.

Provided that the function f is known, the period can be obtained by integrating the first of Eqs. (2). Such procedure requires that an inverse function to f is determined analytically in order to express y in terms of x. Only then the integral of Eq. (2) may be resolved. Since this is not often possible, another parameterisation have to be used. In general, we can write

$$\begin{aligned} x &= f_1^{-1}(u), \\ y &= f_2^{-1}(u), \end{aligned} \tag{3}$$

where u is the new variable. The parameterisation defined by Eq. (3) may be chosen arbitrarily, but a particular choice affects the possibility of whether the inverse functions f_1^{-1}, f_2^{-1} can be found and this way how far an analytical calculation can go.

For an arbitrary oscillating system a first quadrant period can be defined as $T^I = X^{-1}(x_M) - X^{-1}(x_0)$, where the function X^{-1} is the inverse function to the solution of Eq. (2), $x = X(t)$, and x_0, x_M represent points where the trajectory enters and leaves the first quadrant. For the new parameterisation we define in analogy with the first quadrant period a quantity

$$T_u^I = f_1(x_M) = f_2(x_0). \tag{4}$$

This quantity can be assigned to the real period, if the relation $u = u(t)$ is known. The left-hand side of the first of Eqs. (2) may be expanded as $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t}$ so that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g_1(x, y)\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{5}$$

Further, we express Eq. (5) in terms of u using $x = f_1^{-1}(u)$, $y = f_2^{-1}(u)$ and $u = f_1(x)$ as follows from Eq. (3). Then, an equation defining the searched relation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g_1(f_1^{-1}(u),\ f_2^{-1}(u))\frac{\mathrm{d}f_1}{\mathrm{d}x}(f_1^{-1}(u)). \tag{6}$$

The integration of Eq. (6) enables us to express period in a closed from

$$T^I = \int_{u_i}^{u_f} \frac{\mathrm{d}u}{g_1(f_1^{-1}(u), f_2^{-1}(u))\dfrac{\mathrm{d}f_1}{\mathrm{d}x}(f_1^{-1}(u))}, \tag{7}$$

which, in the worst case, can be solved numerically. In general, the period T is neither a multiple of the time required to cross the quadrant, T^I, nor it changes in the same manner in all quadrants, and Eq. (7) must be solved in each quadrant separately. Other possibility is to find u_i, u_f as the minimum resp. maximum of the function $u(t)$. Then T^I in Eq. (7) represents one half of the real period. Nevertheless, it must be assured that there are no other extrema of $u(t)$ within the period. Otherwise, the period is equal to the sum of Eqs. (7) with u_i being a particular minimum and u_f as the following maximum.

3. Examples

3.1 Linear harmonic oscillator

The evolution equations of the linear harmonic oscillator (LHO) assume that $g_1(x, y) = y$, $g_2(x, y) = -\omega^2 x$ is substituted to Eq. (2). It follows from Eq. (2) that the trajectory in the phase space is represented by an ellipse

$$\omega^2 x^2 + y^2 = E^2. \tag{8}$$

Here, the constant E is the energy of the system. Natural choice of the transformation in Eq. (3) in this case is

$$\begin{aligned} x &= \frac{E}{\omega}\sin u, \\ y &= E\cos u. \end{aligned} \tag{9}$$

Then we get $u = \arcsin(\omega x)/E$ and $T_u^I = u(E/\omega) - u(0) = \pi/2$. Relation between u and the real period can be obtained from Eq. (6), and it yields $T = 2\pi/\omega$.

To demonstrate the arbitrariness of the choice of transformation in Eq. (2), another possibility is discussed. Let Eq. (2) be

$$\begin{aligned} x &= \frac{u}{\omega}, \\ y &= \sqrt{E^2 - u^2} \end{aligned} \tag{10}$$

satisfying Eq. (8) as well. Then we have $T_u^I = u(E/\omega) - \theta(0) = E$, and $t = \frac{1}{\omega}\arcsin u/E$. Again, we arrive at $T = 2\pi/\omega$.

3.2 Duffing oscillator

The Duffing oscillator is a non-linear extension of LHO in which the definitions of functions in Eq. (2) are assumed to fulfill $g_1(x, y) = y$, $g_2(x, y) = -\omega^2 x - \alpha^2 x^2$. Another type of the Duffing oscillator uses the parameter α with a positive sign. However, the following discussion is restricted to the case with a negative sign.

The trajectory in the phase space is now represented by the curve

$$\omega^2 x^2 + \frac{\alpha^2}{2} + y^2 = 2E^2, \tag{11}$$

and a simple transformation

$$\begin{aligned} x &= \sqrt{u}, \\ y &= \sqrt{2E^2 - \omega^2 u - \frac{\alpha^2}{2}u^2} \end{aligned} \tag{12}$$

can be introduced. In such a case we get

$$T_u^I = u\left(\sqrt{\frac{\sqrt{\omega^4 + 4E^2\alpha^2} - a}{\alpha^2}}\right) - u(0) = \frac{\sqrt{\omega^4 + 4E^2\alpha^2} - a}{\alpha^2}. \tag{13}$$

According to Eq. (6), the relation between T_u^I and T^I is given by the equation

$$\frac{\mathrm{d}u}{\sqrt{u(2E^2 - \omega^2 u - \alpha^2 u^2/2)}} = \mathrm{d}t, \tag{14}$$

which has a solution that cannot be expressed by elementary functions. However, using the elliptic functions of the first order, EllipticE(), one arrives at

$$\begin{aligned} t - t_0 = \sqrt{\frac{2}{\omega^2 + \sqrt{\omega^4 + 4E^2\alpha^2}}}\,\mathrm{EllipticF}\left(u\sqrt{\frac{\omega^2 + \sqrt{\omega^4 + 4E^2\alpha^2}}{4E^2}},\right. \\ \left. \mathrm{i}\sqrt{\frac{2E^2\alpha^2 + \omega^2(\omega^2 - \sqrt{\omega^4 + 4E^2\alpha^2})}{2E^2\alpha^2}}\right). \end{aligned} \tag{15}$$

Substituting Eq. (13) into Eq. (15), the period of the Duffing oscillator yields

$$T = 4\sqrt{\frac{2}{\omega^2 + \sqrt{\omega^4 + 4E^2\alpha^2}}}\,\mathrm{EllipticF}\left(1, \mathrm{i}\sqrt{\frac{2E^2\alpha^2 + \omega^2(\omega^2 - \sqrt{\omega^4 + 4E^2\alpha^2})}{2E^2\alpha^2}}\right). \tag{16}$$

Table 1 shows the comparison of Eq. (16) and the numerical solution of Eq. (2) for the case of the Duffing oscillator.

Parameters	$\omega^2 = 1$ $\alpha^2 = 1$ $E^2 = 3/4$	$\omega^2 = 4$ $\alpha^2 = 16$ $E^2 = 6$	$\omega^2 = 4$ $\alpha^2 = 2$ $E^2 = 5/2$
T(theo)	4.7680	1.5899	2.6833
T(num)	4.768	1.588	2.684

Table 1: Comparison of the theoretically predicted (Eq. (16)) and numerically calculated (using Eq. (2)) values of the period of the Duffing oscillator for different values of the dynamical parameters. Numerical calculations were done with precision ± 0.002.

3.3 Lotka-Volterra system

For the previous examples the Hamilton function is known and therefore Eq. (1) could be used in principle. Nevertheless, this is not possible in the case of the Lotka-Volterra (L-V) systems. L-V system is defined by

$$\begin{aligned} \dot{x} &= ax - kxy, \\ \dot{y} &= kxy - by, \end{aligned} \tag{17}$$

where a, k, and b are positive constants. L-V system describes the interaction between a prey population x, and a predator population y. The evolution of this system leads to sustained oscillations. Originally, Eq. (17) was used to describe the population dynamics, later it proved to be useful in modelling of biochemical reactions [2].

One can solve Eq. (17) for the $x = x(y)$ dependence, but in general this dependence cannot be expressed analytically. It follows from Eq. (17) that

$$e^{\frac{c-k(x+y)}{b}} x y^{\frac{a}{b}} = 1, \tag{18}$$

where c is a constant defining a particular trajectory by analogy with the energy in the previous examples. A suitable parameterisation allows us to find functions $x(u)$, $y(u)$ in a closed form. Let us use the substitution

$$\begin{aligned} u &= x y^{\frac{a}{b}}, \\ u^{-1} &= e^{\frac{c-k(x+y)}{b}}. \end{aligned} \tag{19}$$

Such parameterisation implies

$$\frac{k}{b}\left[x + \left(\frac{u}{x}\right)^{b/a}\right] = \frac{c}{b} + \ln u. \tag{20}$$

In general, Eq. (20) can be resolved only if it leads to a polynomial equation of the order less that 5. Thus, we can take $b/a = 1/3, 1/2, 1, 2, 3$. For the sake of simplicity $b = a$ is used in what follows.

According to Eq. (6), du/dt may be expressed as

$$\frac{du}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = \frac{ku[(x(u))^2 - u]}{x(u)}. \tag{21}$$

This equation defines the integral in Eq. (7). In case of L-V system, it cannot be solved analytically, and a numerical computation is required. Before discussing it, some general features of L-V system with $a = b$ will be mentioned. The cycle may be dissected to four regions, each defining a quadrant period T^i. Let us use x_0, y_0 as the coordinates of the fixed point. In case of $a = b$ we get $x_0 = y_0$, and X_M as a defining value for a particular trajectory.

The first region starts with the maximum value of the prey, $x_A = x_0 + X_M$, and $y_A = y_0$. It ends when the predator reaches its maximum, $x_B = x_0$, y_B. It follows from Eq. (17) that $y_B = x_A$. In the second region, the prey reaches minimum, x_C, $y_C = y_0$. It follows Eq. (17) that the time necessary to cross this region is equal to that necessary to cross the last region, where the prey reaches maximum value, x_A, y_0. In the third region, the predator population reaches the minimum value, $x_D = x_0$, $y_D = x_C$.

In order to solve Eq. (7) we must find the extrema of u within the time period. These extrema define the ranges of the integration in Eq. (7). In L-V systems the interspecies interactions lead to a single increase of one species to its maximum, while the other one is decreasing simultaneously. Due to this property it is assured that there is a single maximum and minimum within one period. They can be found by solving the equation

$$\frac{du}{dt} = ku[x(u) - y(u)] = 0 \tag{22}$$

numerically. For this reason values shown in Tab. 2 are computed approximately. The error in determination of the period from Eq. (7) depends mostly on the accuracy with which Eq. (22) is solved.

k	1	1	1
b	0.5	1	1.5
c	2.143841	2.306853	2.017366
X_M	1	1	1
u_i	0.026269	0.295133	0.966424031
u_f	1.289186	2.757637	4.719770107
T	14.51	6.60	4.56
T_{num}	14.52	6.59	4.59

Table 2: Lotka-Volterra system: comparison between predicted period and period obtained from the numerical solution of Eq. (17). Numerical solution was done with precision ± 0.02.

Using this algorithm, the dependence of the period on dynamical parameters b and k can be computed. Functions $T(b)$, $T(k)$ are shown in fig. 1a–b.

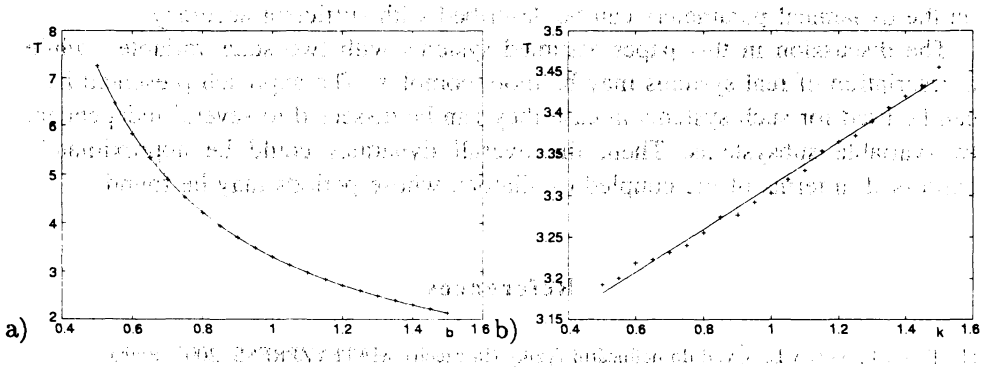

Figure 1: Dependence of the period on dynamical parameters for Lotka-Volterra system. a) Birth/extension rates are varied while $k = 1$, b) interaction rate is varied while $b = 1$. Solid lines represent fitted curves.

Curves were fitted using the least square method with the results

$$\begin{aligned} T &= \frac{1}{A_1 + A_2 b}, & A_1 &= -0.026 \\ & & A_2 &= 0.329 \\ & & R &= 0.99999 \\ T &= A_1 + A_2 k, & A_1 &= 3.053 \\ & & A_2 &= 0.259 \\ & & R &= 0.997 \end{aligned} \tag{23}$$

It is obvious that Lotka-Volterra system is more sensitive to changes of the birth/extinction rates and it probably remains valid also for $a \neq b$.

4. Conclusions

A system of differential equations describing an oscillating system can often be solved in terms of the mutual dependence of the state variables. This dependence allows us to use a formal parameter, which defines the trajectory in the phase space to find the period of oscillations.

Such formal parameter can be chosen arbitrarily and the most natural choice in simple systems is the time. Period is then determined by the initial and final values of the state variable. However, this choice makes usualy the computations difficult, and other choices are more convenient. To get the period, a relation between the formal parameter and time must be then found. It could be achieved using one of the evolution equations of the system.

This approach is not restricted to Hamiltonian systems, as it was demonstrated in case of the Lotka-Volterra system. In general case, some parts of this algorithm must be solved numerically. Despite this fact, the oscillations and their dependence on the dynamical parameters can be described with sufficient accuracy.

The discussion in this paper assumed systems with two state variables, while a description of real systems may be more complex. The approach presented here can be used for such systems in case they can be dissected to several independent two-variable subsystems. Then, the overall dynamics could be approximately expressed in terms of the coupled oscillators, whose periods may be found.

References

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