Otto Bernard

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On Contexts of Direct Products, Ordinal Sums and Ordinal Products

OTTO BERNARD

Praha

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Given regular contexts of lattices L_1 and L_2, we describe how to construct regular contexts of their direct product. We show that the regular contexts of ordinal sums and products of two lattices can be constructed from the contexts determined by L_1 and L_2 in the case of $L_1 \oplus L_2$ and from the order matrix of L_1 and from the context determined by L_2 in the case of $L_1 \odot L_2$.

1. Introduction

A *formal context* is a triple (G, M, I) where G and M are sets and I is a binary relation between G and M. The elements of G are referred to as *objects* and the elements of M as *attributes*. Let $H \subseteq G$ and $N \subseteq M$. Then the definitions

$$H^{\uparrow} := \{m \in M; \forall g \in H \ (g, m) \in I\}$$

and

$$N^{\downarrow} := \{g \in G; \forall m \in N \ (g, m) \in I\}$$

together with the mappings $\varphi : P(G) \to P(M)$, $\varphi : H \mapsto H^{\uparrow}$ and $\psi : P(M) \to P(G)$, $\psi : N \mapsto N^{\downarrow}$ establish a Galois correspondence between the power sets $P(G)$ and $P(M)$ of G and M.

A pair (H, N) where $H \subseteq G$ and $N \subseteq M$ is called a *formal concept* if $H^{\uparrow} = N$ and $N^{\downarrow} = H$. In this case we also have $H^{\uparrow\downarrow} = H$ and $N^{\downarrow\uparrow} = N$. The set H is

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Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8 - Karlín, Czech Republic

Department of Urology, UVN, U vojenské nemocnice 1200, 169 02 Praha 6, Czech Republic.

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called the *extent* of the formal context (H, N) and N is said to be the *intent* of (H, N). On the set of all formal concepts of (G, M, I) one can define an ordering by

$$(H_1, N_1) \leq (H_2, N_2) \Leftrightarrow H_1 \subseteq H_2 .$$

In this way we obtain a complete lattice which is called the *concept lattice* of the context $K = (G, M, I)$. It will be denoted by $B(G, M, I)$ or $B(K)$. For general facts about concept lattices and their applications, the reader may refer to [4], [10] and [8].

Let us consider the formal context $K_1 := (G, M, I)$ where $G := \{g_1, g_2, ..., g_6\}$ and $M := \{m_1, m_2, ..., m_3\}$ and where the relation I is given in Figure 1.

m_1	m_2	m_3	m_4	m_5	m_6
g_1	1	1	1	0	0
g_2	1	1	0	0	0
g_3	1	0	1	1	0
g_4	1	1	0	1	0
g_5	0	0	1	1	1
g_6	0	0	1	0	0

Fig. 1

Here, for example, $(g_1, m_3) \in I$ and $(g_1, m_4) \notin I$.

Notice that we can identify "the interior" of Figure 1 with the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

of $\mathbb{Z}_2^{6\times 5}$; in what follows we often use a similar approach.

By the definitions given above or by some other methods (see, e.g., [6], [7], [9], [5]; we have used the program ConImp of P. Burmeister (TU Darmstadt)) one can find that the considered context K_1 has the following formal concepts:

$$k_1 := (G; \emptyset), k_2 := (\{g_3, g_4, g_5\}, \{m_4\}), k_3 := (\{g_1, g_3, g_5\}, \{m_3\}),$$
$$k_4 := (\{g_1, g_2, g_3, g_4), \{m_1\}), k_5 := (\{g_3, g_5\}, \{m_3, m_4\}),$$
$$k_6 := (\{g_5, g_6), \{m_3, m_5\}), k_7 := (\{g_1, g_3\}, \{m_1, m_3\}),$$
$$k_8 := (\{g_1, g_2, g_4\}, \{m_1, m_2\}), k_9 := (\{g_5\}, \{m_3, m_4, m_5\}),$$
$$k_{10} := (\{g_3, g_4\}, \{m_1, m_4\}), k_{11} := (\{g_1\}, \{m_1, m_2, m_3\}),$$
$$k_{12} := (\{g_3\}, \{m_1, m_3, m_4\}), k_{13} := (\{g_4\}, \{m_1, m_2, m_4\}), k_{14} := (\emptyset, M).$$

The concept lattice $B(K_1)$ can be visualized by the diagram shown in Figure 2.

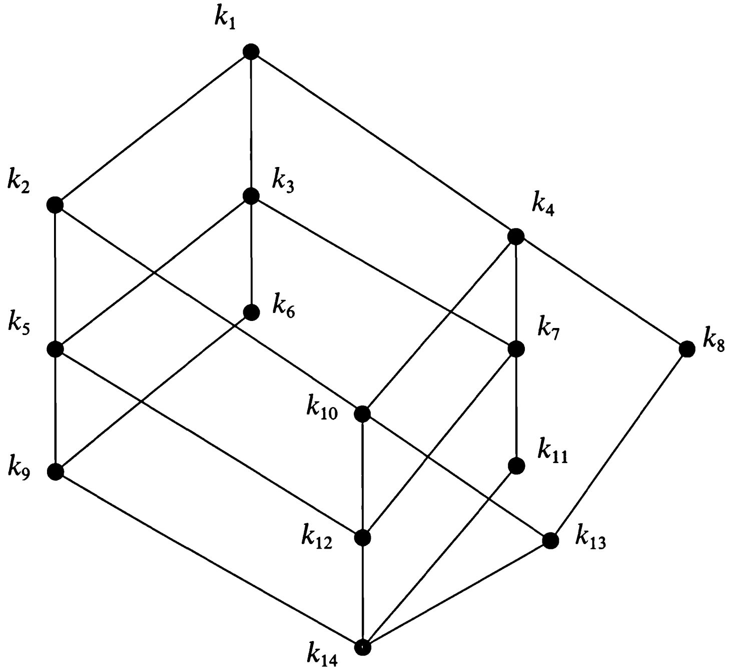

Fig. 2

The formal context K_1 was derived from data concerning men with localized prostate cancer (see also [1] and [2]).

In this paper, "lattice" will mean "finite lattice having at least two elements" and its unit and zero element will be denoted by 1 and 0.

An element r of a lattice L is said to be *meet-irreducible*, if $r \neq 1$ and if the implication

$$p \wedge q = r \Rightarrow (p = t \text{ or } q = r)$$

holds for any $p, q \in L$. The set of all meet-irreducible elements in L will be denoted by $M(L)$.

An element s of L is called *join-irreducible* if $s \neq 0$ and if

$$p \vee q = s \Rightarrow (p = s \text{ or } q = s)$$

is true for any p, q or L. The set of all join-irreducible elements in L will be denoted by $J(L)$.

Notice that

$$(1.1) \qquad 1 \notin M(L) \quad \& \quad (\notin J(L) .$$

As usual, $a \parallel b$ will be used to denote the fact that a and b are incomparable, i.e., $a \not\leq b$ and $b \not\leq a$.

Remark 1.1. Any atom of a lattice L is join-irreducible and any dual atom of L is meet-irreducible. Hence $M(L) \neq \emptyset$ and $J(L) \neq \emptyset$.

Suppose that a lattice L has exactly t elements, i.e., $\sharp L = t$, and write $L = \{c_1, c_2, \ldots, c_t\}$. Next suppose that

$$(1.2) \qquad 1 = c_1 \rightarrow c_2 \rightarrow \ldots \rightarrow c_t = 0$$

is a fixed sequence of the elements belonging to L where "$\rightarrow$" denotes either "$>$" or "$\parallel$".

A sequence $c_{i_1}, c_{i_2}, \ldots, c_{i_u}$ of elements in L will be said to be *regular* with respect to (1.2), if

$$1 \leq i_u < \ldots < i_2 < i_1 \leq t\,.$$

Let $M(L) = \{r_1, r_2, \ldots, r_p\}$ where the sequence $r_1, r_2, \ldots, r_p$ is regular with respect to (1.2), and, similarly, let $J(L) = \{s_1, s_2, \ldots, s_q\}$ where $s_1, s_2, \ldots, s_q$ is a sequence regular with respect to (1.2) so that $\sharp M(L) = p$ and $\sharp J(L) = q$. By a *J/M-context* of L *regular* with respect to (1.2) we mean the matrix $C = (c_{ij}) \in \mathbb{Z}_2^{q \times p}$ where $c_{ij} = 1$ if and only if $s_i \leq r_j$ and $c_{ij} = 0$ otherwise.

Given any lattice L, it is natural to ask whether there exists a formal context $K_L = (G, M, I)$ with $B(K_L)$ isomorphic to L. It can be shown that it is possible to choose any J/M-context of L for K_L, the ordering of the elements in L being quite irrelevant (see [4, Hilfsatz 12, p. 27]).

2. Direct products

In this section we suppose that L_1 and L_2 are two lattices, $\sharp L_1 = m$ and $\sharp L_2 = n$ with

$$L_1 = \{a_1, a_2, \ldots, a_m\}, L_2 = \{b_1, b_2, \ldots, b_n\}$$

and

$$(2.1) \qquad m \geq 2 \quad \& \quad n \geq 2\,.$$

Moreover, we suppose that there are given two fixed ordering of the elements in L_1 and in L_2

$$(2.2) \qquad 1_1 = a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_{m-1} \rightarrow a_m = 0_1$$

and

$$(2.3) \qquad 1_2 = b_1 \rightarrow b_2 \rightarrow \ldots \rightarrow b_{n-1} \rightarrow b_n = 0_2$$

where "$\rightarrow$" denotes either "$>$" or "$\parallel$".

Lemma 2.1. *Let $L_3 := L_1 \otimes L_2$ be the direct product of the lattices L_1 and L_2. Then the following assertions hold for any $a \in L_1$ and any $b \in L_2$.*

(i) $(1_1, b) \in M(L_3) \Leftrightarrow b \in M(L_2)$;
(ii) $(a, 1_2) \in M(L_3) \Leftrightarrow a \in M(L_1)$;
(iii) $(a, 0_2) \in J(L_3) \Leftrightarrow a \in J(L_1)$;
(iv) $(0_1, b) \in J(L_3) \Leftrightarrow b \in J(L_2)$;
(v) *if* $a \neq 1_1$ *and* $b \neq 1_2$, *then* $(a, b) \notin M(L_3)$;
(vi) *if* $a \neq 0_1$ *and* $b \neq 0_2$, *then* $(a, b) \notin J(L_3)$.

Proof. (i) Let $(1_1, b) \in M(L_3)$. By (1.1), $b \neq 1_2$. Suppose on the contrary that $b \notin M(L_2)$. Then $b = b_u \wedge b_v$ where $b < b_u$ and $b < b_v$. It follows that

$$(1_1, b) = (1_1, b_u) \wedge (1_1, b_v)$$

and

$$(1_1, b) < (1, b_v) \quad \& \quad (1_1, b) < (1_1, b_v)$$

a contradiction of the fact that $(1, b) \in M(L_3)$.

Now suppose that $b \in M(L_2)$. We want to show that $(1_1, b) \in M(L_3)$. By (1.1), $b \neq 1_2$ and so $(1_1, b) \neq (1_1\ 1_2)$. If $(1_1, b) = (a_x, b_u) \wedge (a_y, b_v)$, then $a_x = a_y = 1_1$ and $b = b_u \wedge b_v$. Since $b \in M(L_2)$, we can assume that $b = b_u$. Then $(1_1, b) = (1_1, b_u)$ and it follows that $(1_1, b) \in M(L_3)$.

The assertions (ii) – (iv) can be proved similarly.

(v) From $a \neq 1_1$ we conclude that there exists $a^* \in L_1$ such that a^* covers a. Analogously, since $b \neq 1_2$, there exists b^* covering b. Then $(a, b) = (a^*, b) \wedge (a, b^*)$ with $(a, b) < (a^*, b)$ and $(a, b) < (a, b^*)$. Thus $(a, b) \notin M(L_3)$.

A similar reasoning appeals to the assertion (vi). □

Let

$$(2.4) \qquad c_1 = (1_1, 1_2) = (a_1, b_1) \to (a_1, b_2) \to \dots \to (a_m, b_1) \to \dots$$
$$\dots \to (a_m, b_n) = (0_1, 0_2) = c_t$$

be the lexicographic ordering of the elements in $L_3 = L_1 \otimes L_2$ with respect to (2.2) and (2.3). By a *J/M-context* of L_3 *regular* with respect to (2.2) and (2.3) we mean the J/M-context of L_3 regular with respect to (2.4).

A matrix $P = (p_{ij}) \in \mathbb{Z}_2^{m \times n}$ is said to be a *1-matrix*, if $p_{ij} = 1$ for any $i = 1, 2, \dots, m$ and any $j = 1, 2, \dots, n$.

Theorem 2.2. *If $L_3 = L_1 \otimes L_2$, then the J/M-context of L_3 regular with respect to (2.2) and (2.3) is equal to the matrix $\mathbb{A}$ partitioned into blocks U_1, K_1, K_2, U_2 of the form*

$$\mathbb{A} = \begin{pmatrix} U_1 & K_1 \\ K_2 & U_2 \end{pmatrix}$$

where U_1 and U_2 are 1-matrices, K_1 is the J/M-context of the lattice L_1 regular with respect to (2.2) and K_2 is the J/M-context of L_2 regular with respect to (2.3). The matrix K_i ($i = 1, 2$) is of the type $(\#J(L_i), \#M(L_i))$.

Proof. Let

$$M(L_1) = \{p_1, p_2, \ldots, p_t\}, M(L_2) = \{r_1, r_2, \ldots, r_s\} \tag{2.5}$$

and

$$J(L_1) = \{v_1, v_2, \ldots, v_a\}, J(L_2) = \{w_1, w_2, \ldots, w_b\} \tag{2.6}$$

where $p_1, p_2, \ldots, p_t$ and $v_1, v_2, \ldots, v_a$ are sequences regular with respect to (2.2) and where $r_1, r_2, \ldots, r_s$ and $w_1, w_2, \ldots, w_b$ are sequences regular with respect to (2.3). Using the description of the sets $M(L_3)$ and $J(L_3)$ given in Lemma 2.1 we can see that the J/M-context of L_3 mentioned in our Theorem can be formed by using Figure 3.

	$(1_1, r_1)$	$\ldots$	$(1_1, r_s)$	$(p_1, 1_2)$	$\ldots$	$(p_t, 1_2)$
$(v_1, 0_2)$	1	$\ldots$	1			
$\ldots$	$\ldots$	$\ldots$	$\ldots$		K_1	
$(v_a, 0_2)$	1	$\ldots$	1			
$(0_1, w_1)$				1	$\ldots$	1
$\ldots$		K_2		$\ldots$	$\ldots$	$\ldots$
$(0_1, w_b)$				1	$\ldots$	1

□

Fig. 3

3. Ordinal sums

Given lattices $L_1 := \{a_1, a_2, \ldots, a_m\}$ and $L_2 := \{b_1, b_2, \ldots, b_n\}$, in what follows we suppose that

$$m \geq 2 \quad \& \quad n \geq 2. \tag{3.1}$$

Further, we suppose that there are fixed orderings

$$1_1 = a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_{m-1} \rightarrow a_m = 0_1 \tag{3.2}$$

and

$$1_2 = b_1 \rightarrow b_2 \rightarrow \ldots \rightarrow b_{n-1} \rightarrow b_n = 0_2 \tag{3.3}$$

where "$\rightarrow$" denotes either "$>$" or "$\|$".

We recall that

$$0 \notin J(L) \quad \& \quad 1 \notin M(L). \tag{3.4}$$

The *ordinal sum* $L_1 \oplus L_2$ of lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$ satisfying $L_1 \cap L_2 = \emptyset$ can be defined (cf. [3, p. 198 and p. 201, ex. 10]) as a lattice $(L_3, \leq)$ such that $L_3 = L_1 \cup L_2$ and where "$\leq$" is determined as follows: For any $x \in L_1$ and any $y \in L_2$ one has $x < y$; furthermore, if $x, y \in L_1$, then $x \leq y$ if and only if $x \leq_1 y$ and, similarly, if $z, v \in L_2$, then $z \leq v$ if and only if $z \leq_2 v$.

Let

(3.5) $1_2 = b_1 \to b_2 \to \dots \to b_{n-1} \to b_n = 0_2 > a_1 = 1_1 \to \dots a_{m-1} \to a_m = 0_1$

be an ordering of the elements in L_3 obtained by the concatenation of the orderings in (3.2) and (3.3).

Theorem 3.1. *The J/M-context of $L_3 = L_1 \oplus L_2$ regular with respect to (3.5) is the matrix*

$$\begin{pmatrix} K_2 & O_1 & O_2 \\ U_1 & O_3 & O_4 \\ U_2 & U_3 & K_1 \end{pmatrix}$$

where the block K_1 is the J/M-context of L_1 regular with respect to (3.2), K_2 is the J/M-context of L_2 regular with respect to (3.3), O_i ($i = 1, 2, 3, 4$) are zero matrices and U_j ($j = 1, 2, 3$) are 1-magtrices. The matrices U_1 and O_4 are row matrices, O_1 and U_3 are column matrices and $O_3 = (0) \in \mathbb{Z}_2^{1\times 1}$. The type of K_q ($q = 1, 2$) is $(\#J(L_q), \#M(L_q))$.

Proof. Let

$$M(L_1) = \{p_1, \dots, p_t\}, M(L_2) = \{r_1, \dots, r_s\}$$

and

$$J(L_1) = \{v_1, \dots, v_a\}, J(L_2) = \{w_1, \dots, w_b\}$$

where $p_1, \dots, p_t$ and $v_1, \dots, v_a$ are regular with respect to (3.2) and where $r_1, \dots, r_s$ and $w_1, \dots, w_b$ are regular with respect to (3.3).

Then it is easily seen that

$$M(L_3) = \{r_1, \dots, r_s, 1_1, p_1, \dots, p_t\}$$

and

$$J(L_3) = \{w_1, \dots, w_b, 0_2, v_1, \dots, v_a\}.$$

It is evident that the both sequences $r_1, \dots, r_s, 1_1, p_1, \dots, p_t$ and $w_1, \dots, 0_2, v_1, \dots, v_a$ are regular with respect to (3.5). Therefore, the J/M-context of L_3 regular with respect to (3.5) can be deduced from Figure 4.

	r_1	...	r_s	1_1	p_1	...	p_t
w_1				0	0	...	0
...		K_2		...	...	...	...
w_b				0	0	...	0
0_2	1	...	1	0	0	...	0
v_1	1	...	1	1			
...	...	...	...	...		K_1	
v_a	1	...	1	1			

□

Fig. 4

4. Ordinal products

Ordinal product $L_1 \odot L_2$ of lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$ (see [2, loc. cit.]) is the lattice $(L_1 \times L_2, \leq)$ where "$\leq$" denotes the lexicographic order, i.e., $(a, b) < (a', b')$ if either $a <_1 a'$ or if $a = a'$ and $b <_2 b'$.

Lemma 4.1. *Let* (a, b) *and* (c, d) *be elements of the ordinal product* $L_3 := L_1 \odot L_2$. *Then* $(a, b) \wedge (c, d)$ *is equal to*
(i) $(a, b \wedge d)$ *whenever* $a = c$;
(ii) (a, b) *whenever* $a < c$;
(iii) (c, d) *whenever* $c < a$;
(iv) $(a \wedge c, 1_2)$ *whenever* $a \parallel c$.
Dually, the element $(a, b) \vee (c, d)$ *is equal to*
(i) $(a, b \vee d)$ *whenever* $a = c$;
(ii) (c, d) *whenever* $a < c$;
(iii) (a, b) *whenever* $c < a$;
(iv) $(a \vee c, 0_2)$ *whenever* $a \parallel c$.

Proof. This is an easy consequency of the definition of order in L_3. □

Let $a \in L_1$. The sequence

$$(a, *)^M := (a, b'_1), (a, b'_2), \ldots, (a, b'_p)$$

where $\{(a, b'_1), (a, b'_2), \ldots, (a, b'_p)\}$ denotes the set consisting of all the elements of $M(L_1 \odot L_2)$ having the first component equal to a and where $\{b'_1, b'_2, \ldots, b'_p\}$ is a sequence regular with respect to (3.3) will be called an *M-tract* determined by a. A *J-tract* determined by a is defined dually as the sequence

$$(a, *)^J := (a, b''_1), (a, b''_2), \ldots, (a, b''_q)$$

$\{(a, b''_1), (a, b''_2), \ldots, (a, b''_q)\}$ is the set formed by all the elements of $J(L_1 \odot L_2)$ having a as the first component and where $b''_1, b''_2, \ldots, b''_q$ is a sequence regular with respect to (3.3). The number p will be called the *length* of the M-tract $(a, *)^M$ and q will be denoted as the *length* of the J-tract $(a, *)^J$.
It follows from Remark 1.1 that $(a, *)^M \neq \emptyset$, $(a, *)^J \neq \emptyset$, for arbitrary $a \in L_1$.

Lemma 4.2. *Let* $L_3 = L_1 \odot L_2$ *and let* $a \in L_1$. *Then*
(i) $(a, 1_2) \in M(L_3) \Leftrightarrow a \in M(L_1)$;
(ii) $(a, 0_2) \in M(L_3) \Leftrightarrow 0_2 \in M(L_2)$;
(iii) $(a, 1_2) \in J(L_3) \Leftrightarrow 1_2 \in J(L_2)$;
(iv) $(a, 0_2) \in J(L_3) \Leftrightarrow a \in J(L_1)$.
If $b \in L_2$ *does not belong to* $\{0_2, 1_2\}$ *and* $a \in L_1$, *then*
(v) $(a, b) \in M(L_3) \Leftrightarrow b \in M(L_2)$;
(vi) $(a, b) \in J(L_3) \Leftrightarrow b \in J(L_2)$.

Proof. (i) Let $(a, 1_2) \in M(L_3)$. By (3.4), $a \neq 1_1$. Suppose $a = a' \wedge a''$ where $a < a'$ and $a < a''$. Then $a' \parallel a''$. It follows from Lemma 4.1 that $(a, 1_2) = (a', 1_2) \wedge (a'', 1_2)$. This together with $(a, 1_2) < (a', 1_2)$ and $(a, 1_2) < (a'', 1_2)$ gives a contradiction. Hence $a \in M(L_1)$.

Conversely, suppose that $a \in M(L_1)$. From (3.4) we can see that $a \neq 1_1$ and, therefore, $(a, 1_2) \neq (1_1, 1_2)$. Suppose, by way of contradiction, that

$$(4.1) \qquad (a, 1_2) = (c, d) \wedge (e, f) \quad \& \quad (a, 1_2) < (c, d) \quad \& \quad (a, 1_2) < (e, f).$$

We distinguish four cases.

Case I: $c = e$. Then in view of Lemma 4.1 we have $(a, 1_2) = (c, d \wedge f)$ and, consequently, $c = a$, $d = 1_2$ and $f = 1_2$. Therefore, $(c, d) = (a, 1_2)$, a contradiction of (4.1).

Case II: $c \parallel e$. By Lemma 4.1, $(a, 1_2) = (c \wedge e, 1_2)$ so that $a = c \wedge e$. From $c \parallel e$ conclude that $a < c$ and $a < e$. But this contradicts $a \in M(L_1)$.

Case III: $c < e$. Then by Lemma 4.1 $(a, 1_2) = (c, d) \wedge (e, f) = (c, d)$, and we again have a contradiction of (4.1).

Case IV: $e < c$. This case can be treated by similar methods as the third case.

(ii) Let $(a, 0_2) \in M(L_3)$. Then, by (3.1), $0_2 \neq 1_2$. Suppose

$$0_2 = b' \wedge b'' \quad \& \quad 0_2 < b' \quad \& \quad 0_2 < b''.$$

Applying Lemma 4.1 we have that

$$(a, 0_2) = (a, b') \wedge (a, b'') \quad \& \quad (a, 0_2) < (a, b') \quad \& \quad (a, 0_2) < (a, b''),$$

a contradiction. Hence $0_2 \in M(L_2)$.

Conversely, let $0_2 \in M(L_2)$. Because of (3.1), $0_2 \neq 1_2$, and, consequently, $(a, 0_2) \neq (1_1, 1_2)$. Suppose to the contrary that

$$(4.2) \quad (a, 0_2) = (a', b') \wedge (c', d') \quad \& \quad (a, 0_2) < (a', b') \quad \& \quad (a, 0_2) < (c', d').$$

Let us distinguish four cases.

Case I: $a' = c'$. By Lemma 4.1, $(a, 0_2) = (a', b' \wedge d')$. Then $a' = a$ and $0_2 = b' \wedge d'$. Using the fact that $0_2 \in M(L_2)$, we get that either $0_2 = b'$ or $0_2 = d'$. In the former case we have $(a', b') = (a, 0_2)$, a contradiction. In the latter case we obtain $(c', d') = (a, 0_2)$, a contradiction.

Case II: $a' \parallel c'$. Again, by Lemma 4.1, $(a, 0_2) = (a' \wedge c', 1_2)$ and it follows that $0_2 = 1_2$ which contradicts (3.1).

Case III: $a' < c'$. Then, by Lemma 4.1, $(a, 0_2) = (a', b')$ which contradicts (4.2).

The fourth case $c' < a'$ is analogous.

Thus $(a, 0_2) \in M(L_3)$.

The proofs of (iii) and (iv) are dual.

(v) Let $(a, b) \in M(L_3)$ where $b \notin \{0_2, 1_2\}$. Assume to the contrary that

$$b = b' \wedge b'' \quad \& \quad b < b' \quad \& \quad b < b''.$$

This, together with Lemma 4.1, implies that

$$(a, b) = (a, b') \wedge (a, b'') \quad \& \quad (a, b) < (a, b') \quad \& \quad (a, b) < (a, b'')$$

which contradicts $(a, b) \in M(L_3)$. Hence $b \in M(L_2)$.

Conversely, let $b \in M(L_2)$ be such that $b \notin \{0_2, 1_2\}$. Then $(a, b) \neq (1_1, 1_2)$.

Assume

$$(4.3) \qquad (a, b) = (c, d) \wedge (e, f) \quad \& \quad (a, b) < (c, d) \quad \& \quad (a, b) < (e, f).$$

We distinguish four cases.

Case I: $c = e$. Then $(a, b) = (c, d \wedge f)$ and, consequently, $a = c$ and $b = d \wedge f$. Since $1 \neq b \in M(L_2)$, we have either $b = d$ or $b = f$. But the both possibilities contradict (4.3).

Case II: $c \parallel e$. Then $(a, b) = (c \wedge e, 1_2)$ which gives $b = 1_2$, a contradiction.

Case III: $c < e$. Here $(a, b) = (c, d)$ and we contradict (4.3). A similar argument can be used for the fourth case $e < c$.

The proof of the remaining assertion (vi) is essentially the same as the one given for the assertion (v). □

Now suppose that the elements of $L_3 = L_1 \odot L_2$ are ardered by

$$(4.4) \qquad (1_1, 1_2) = (a_1, b_1) \to (a_1, b_2) \to \ldots \to (a_m, b_1) \to (a_m, b_2) \to \ldots$$
$$\ldots \to (a_m, b_n) = (0_1, 0_2)$$

where "$\to$" denotes either "$>$" or "$\parallel$". By a J/M-context of the lattice $L_1 \odot L_2$ *regular* with respect to (3.2) and (3.3) we mean the J/M-contexyt regular with respect to (4.4).

Theorem 4.3. *Let $A = (a_{ij}) \in \mathbb{Z}_2^{m \times m}$ be the order matrix of the lattice L_1, i.e., $a_{ij} = 1$ if and only if $a_i \leq a_j$ and $a_{ij} = 0$ otherwise. Then the J/M-context of the ordinal product $L_3 = L_1 \odot L_2$ regular with respect to (3.2) and (3.3) is a matrix $\mathbb{A}$ partitioned into blocks A_{ij} $(i, j = 1, 2, \ldots, m)$ having their elements in $\mathbb{Z}_2$.*

For any $1 \leq s \neq t \leq m$ the matrix A_{st} is a zero matrix provided $a_{st} = 0$; the matrix A_{st} is a 1-matrix provided $a_{st} = 1$.

For any $s = 1, 2, \ldots, m$ the diagonal block A_{ss} of $\mathbb{A}$ is equal to a matrix having one of the following forms:

$$(1) \qquad D := \begin{pmatrix} d_{11} & \ldots & d_{1q} \\ \cdots & \cdots & \cdots \\ d_{p1} & \ldots & d_{pq} \end{pmatrix};$$

$$(2) \qquad \begin{pmatrix} 1 & d_{11} & \ldots & d_{1q} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & d_{p1} & \ldots & d_{pq} \\ 1 & 1 & \ldots & 1 \end{pmatrix};$$

(3)
$$\begin{pmatrix} 1 & d_{11} & \dots & d_{1q} \\ \dots & \dots & \dots & \dots \\ 1 & d_{p1} & \dots & d_{pq} \end{pmatrix};$$

(4)
$$\begin{pmatrix} d_{11} & \dots & d_{1q} \\ \dots & \dots & \dots \\ d_{p1} & \dots & d_{pq} \\ 1 & \dots & 1 \end{pmatrix};$$

where D denotes the J/M-context of L_2 regular with respect to (3.3).

Moreover, (1) *occurs if and only if*

$$a_s \notin M(L_1) \quad \& \quad a_s \notin J(L_1);$$

(2) *occurs if and only if*

$$a_s \in M(L_1) \quad \& \quad a_s \in J(L_1);$$

(3) *occurs if and only if*

$$a_s \in M(L_1) \quad \& \quad a_s \notin J(L_1);$$

(4) *occurs if and only if*

$$a_s \notin M(L_1) \quad \& \quad a_s \in J(L_1).$$

For all $1 \leq i, j \leq m$ the block A_{ij} of $\mathbb{A}$ has the type (r_{ij}, s_{ij}) where r_{ij} is the length of the J-tract $(a_i, *)^J$ and s_{ij} is the length of the M-tract $(a_j, *)^M$.

Proof. 1. Let $s \neq t$ and let $a_{st} = 0$. Then either $a_s > a_t$ or $a_s \parallel a_t$. By the definition of the ordering in L_3, $(a_s, *) \leq (a_t, \dots)$ is not true for any choice of the second component "$*$" of the elements in the J-tract $(a_s, *)^J$ and for any choice of the second component "..." of the elements in the M-tract $(a_t, *)^M$. Thus the block A_{st} is a zero matrix.

2. Let $s \neq t$ and let $a_{st} = 1$. Then $a_s < a_t$. For any element $(a, *)$ of the J-tract $(a_s, *)^J$ and for any element $(a_t, \dots)$ of the M-tract $(a_t, *)^M$ we therefore have $(a_s, *) \leq (a_t, \dots)$. Hence the block A_{st} is a 1-matrix.

Notice that the type of A_{st} has the indicated form in the both cases.

3. Since $a_s \leq a_s$, $a_{ss} = 1$. The block A_{ss} is determined by the M-tract $(a_s, *)^M$ and by the J-tract $(a_s, *)^J$.

We distinguish four cases.

Case I: $a_s \notin M(L_1)$ and $a_s \notin J(L_1)$. Then Lemma 4.2 yields

$$(a_s, 1_2) \notin M(L_3) \quad \& \quad (a_s, 0_2) \notin J(L_3).$$

By the same Lemma, an element (a_s, b_i) belongs to the M-tract $(a_s, *)^M$ if and only if $b_i \in M(L_2)$. Similarly, (a_s, b_j) belongs to the J-tract $(a_s, *)^J$ if and only if $b_j \in J(L_2)$. Now

$$(a_s, b_j) \leq (a_s, b_i) \Leftrightarrow b_j \leq b_i,$$

so that $A_{ss} = D$.

Case II: $a_s \in M(L_1)$ *and* $a_s \in J(L_1)$. Then

$$(a_s, 1_2) \in M(L_3) \quad \& \quad (a_s, 0_2) \in J(L_3).$$

By Lemma 4.2,

$$(a_s, *)^M = (a_s, 1_2), \ldots, (a_s, b_i), \ldots$$

and

$$(a_s, *)^J = \ldots, (a_s, b_j), \ldots, (a_s, 0_2)$$

where b_i runs over the elements of $M(L_2)$ and b_j runs over the elements of $J(L_2)$.
The block A_{ss} can be deduced from Figure 5.

	$(a_s, 1_2)$	...	(a_s, b_i)	...
...	...	...	...	...
(a_s, b_j)	1	...	d_{ij}	...
...	...	...	...	...
$(a_s, 0_2)$	1	...	1	...

Fig. 5

Thus A_{ss} has the form described in (2).
Case III: $a_s \in M(L_1)$ *and* $a_s \notin J(L_1)$. Then

$$(a_s, 1_2) \in M(L_3) \quad \& \quad (a_s, 0_2) \notin J(L_3).$$

The block A_{ss} can now be obtained from Figure 6.

	$(a_s, 1_2)$	...	(a_s, b_i)	...
...	...	...	...	...
(a_s, b_j)	1	...	d_{ij}	...
...	...	...	...	...

Fig. 6

Therefore, A_{ss} has the form of (3).
Case IV: $a_s \notin M(L_1)$ *and* $a_s \in J(L_1)$. Then

$$(a_s, 1_2) \notin M(L_3) \quad \& \quad (a_3, 0_2) \in J(L_3).$$

The block A_{ss} can be deduced from Figure 7.

	...	(a_s, b_i)	...
...	...	...	...
(a_s, b_j)	...	d_{ij}	...
...	...	...	...
$(a_s, 0_2)$	...	1	...

Fig. 7

Notice that the type of A_{st} is of the indicated form in all the cases considered here above. □

Remark 4.4. The ordinal product of two five-element lattices ("the flying kite" multiplied by "the falling kite"[1]) shows that any form (1), (2), (3) and (4) of Theorem 4.3 can occur.

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[1] The terms were proposed by L. Beran.