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Selfdistributive Groupoids Part A1: Non-Idempotent Left Distributive Groupoids

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In this paper, the essentials of the algebraic theory of (generally non-idempotent) left distributive groupoids are presented.

0. Introduction

The (left and right) equations (or identities, laws, etc.) of selfdistributivity for a binary operation (say multiplication) are expressed as

$$\mathbf{x}(\mathbf{y}\mathbf{z}) \doteq (\mathbf{x}\mathbf{y})(\mathbf{x}\mathbf{z}) \qquad \text{and} \qquad (\mathbf{z}\mathbf{y})\,\mathbf{x} \doteq (\mathbf{z}\mathbf{x})(\mathbf{y}\mathbf{x})\,.$$

Inasmuch, for instance, the operation of arithmetic mean satisfies both of them, they were implicitly present from ancient times. On the other hand, the first explicit allusion to selfdistributivity seems to appear in [Pei, 1880]. Looking at the pages 33 and 34 of that article, we can read the following comment on self-distributivity:

"These are other cases of the distributive principle. ... These formulae, which have hitherto escaped notice, are not without interest."

Another early work which is worth mentioning is [Sch, 1887]. We can already see there (p. 249) a particular example of a non-associative distributive groupoid G:

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G	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Of course, G is idempotent and commutative and, in fact, it is the smallest non-trivial Kirkman-Steiner triple system.

The first article fully devoted to selfdistributivity is (perhaps) [BurM, 29]. The paper deals with (two-sided) distributive quasigroups and a portion of the results may be found in (now rarely seen) book [Suš, 37] (pp. 154 − 157).

One-sided selfdistributive structures (namely left distributive quasigroups) appeared a bit later in [Tak, 43]. Two-sided (generally non-idempotent) distributive groupoids were studied in [Rue, 66] and, finally, non-idempotent left distributive groupoids in [Kep, 81].

Idempotent (either one-sided or two-sided) selfdistributive groupoids are known to appear in many algebraic, geometrical, topological and combinatorial contexts and the theory of (two-sided) distributive groupoids is easily transferred to the idempotent case.

On the other hand, the theory of non-idempotent left distributive groupoids (even of those possessing no idempotent elements) has its own flavour and some of these groupoids are of special interest because of their connections to more popular and fashionable objects like opulent cardinal numbers and braid groups. The rôle of selfdistributivity in the Set Theory was more or less known for a long time (first results in this direction are due to P. Dehornoy) and the most important theorems were proved by R. Laver. The relations to the braid groups were studied mainly by P. Dehornoy. Anyway, all this goes beyond the scope of the present treatment which is fully devoted to the essentials of the algebraic theory of (generally non-idempotent) left distributive groupoids. As concerns the applications mentioned above (and more), the kind reader is referred to the excellent monograph [Deh, 2000].

I. Groupoids

I.1 Groupoids – first concepts

1.1 Let G be a groupoid. For every $a \in G$, we define transformations $L_{a,G}$ and $R_{a,G}$ of G by $L_{a,G}(x) = ax$ and $R_{a,G}(x) = xa$ for every $x \in G$. The transformation $L_{a,G}$ ($R_{a,G}$), often denoted only by L_a (R_a), is called the *left (right) translation by a*.

The transformation semigroup $\mathcal{M}_l(G)$ ($\mathcal{M}_r(G)$) generated by all L_a (R_a) is called the *left (right) multiplication semigroup of* G. The transformation semigroup $\mathcal{M}(G)$ generated by all L_a, R_a is called the *multiplication semigroup of* G. Moreover, we define $\mathcal{M}_l^1(G) = \mathcal{M}_l(G) \cup \{\mathrm{id}_G\}$, $\mathcal{M}_r^1(G) = \mathcal{M}_r(G) \cup \{\mathrm{id}_G\}$, $\mathcal{M}^1(G) = \mathcal{M}(G) \cup \{\mathrm{id}_G\}$ (the *left, right, two-sided multiplication monoids of* G).

1.2 Let G be a groupoid. We denote by $\mathrm{End}(G)$ the endomorphism semigroup (in fact, monoid) of G and by $\mathrm{Aut}(G)$ the automorphism group of G.

A subset A of G is said to be *characteristic (fully characteristic)* if $f(A) \subseteq A$ for every $f \in \mathrm{Aut}(G)$ ($f \in \mathrm{End}(G)$).

1.3 Let G be a groupoid. For every $n \geq 1$, define transformations $o_{n,l,G}$ and $o_{n,r,G}$ of G by $o_{1,l,G}(x) = x = o_{1,r,G}(x)$ and $o_{n+1,l,G} = xo_{n,l,G}(x)$, $o_{n,r,G}(x) = o_{n,r,G}(x)\,x$. We put also $o_G = o_{2,l,G}$, $r_G = o_{3,l,G}$ and $s_G = o_{3,r,G}$ ($o_G(x) = xx = x^2$, $r_G(x) = x \cdot xx$, $s_G(x) = xx \cdot x$).

The grupoid G is said to be *uniquely 2-divisible* if o_G is a permutation of G. The inverse permutation is o_G^{-1} and we shall also write $o_G^{-1}(x) = x^{\frac{1}{2}}$ ($o_G(x) = 2x$ and $o_G^{-1}(x) = x/2$ if the operation is denoted additively).

1.4 If A, B are subsets of a groupoid G then $AB = \{ab \mid a \in A,\ b \in B\} \subseteq G$. Further, $\langle A \rangle_G$ will denote the subgroupoid generated by A.

If the intersection of all subgroupoids of G is non-empty, denote it by S, then S is the smallest subgroupoid of G and we put $\langle \emptyset \rangle_G = S$.

1.5 Let G be a groupoid. Then $\sigma(G)$ means the smallest cardinal number $\mathrm{card}(M)$ for a generator set M of G.

The groupoid G is said to be *cyclic* if $\sigma(G) \leq 1$. The groupoid G is said to be *pseudocyclic* if either $\sigma(G) \leq 1$ or G is idempotent and $\sigma(G) = 2$.

It is easy to see that $\sigma(G) = 0$ iff G contains no proper subgroupoid; $\sigma(G) = 1$ iff there is an element $a \in G$ such that a is contained in no proper subgroupoid but G contains at least one proper subgroupoid. Finally, G is pseudocyclic and $\sigma(G) = 2$ iff G is idempotent, non-trivial and every proper subgroupoid of G is one-element.

1.6 If G is a groupoid then $\mathrm{Id}(G) = \{a \in G \mid a = aa\} = \{a \in G \mid o_G(a) = a\}$ denotes the set of idempotent elements.

1.7 Let G be a groupoid. An element $e \in G$ is said to be *left (right) neutral* if $ex = x$ ($xe = x$) for every $x \in G$, i.e., if $L_e = \mathrm{id}_G$ ($R_e = \mathrm{id}_G$). An element e is said to be *neutral* if it is both left and right neutral.

Clearly, G possesses at most one neutral element, usually denoted by 1 or 1_G (0 or 0_G) if the operation is denoted multiplicatively (additively).

An element $e \in G$ is said to be *left (right) constant* if L_e (R_e) is a constant transformation, i.e., if $ex = ey$ ($xe = ye$) for all $x, y \in G$. An element $e \in G$ is said to be *constant* if it is both left and right constant.

An element $e \in G$ is said to be *(left, right) absorbing* (or *annihilating, dominating)* if it is (left, right) constant and $e = ee$.

Clearly, G possesses at most one absorbing element, usually denoted by 0 or 0_G if the operation is denoted multiplicatively.

1.8 Let G be a groupoid and $e \notin G$. We shall define four groupoids $G[e]$, $G[e\}$, $G\{e]$ and $G\{e\}$ as follows: In all the four cases, the underlying set is $G \cup \{e\}$ and G

is a subgroupoid; e is absorbing in $G[e]$; e is left (right) absorbing and right (left) neutral in $G[e\}$ $(G\{e])$; e is neutral in $G\{e\}$.

1.9 Let G be a groupoid. An element $a \in G$ is said to be *left (right) cancellable* if L_a (R_a) is injective. We denote by $\mathscr{C}_l(G)$ $(\mathscr{C}_r(G))$ the set of all left (right) cancellable elements of G and we put $\mathscr{C}(G) = \mathscr{C}_l(G) \cap \mathscr{C}_r(G)$. The elements from $\mathscr{C}(G)$ are called *cancellable.*

The groupoid G is said to be *(left, right) cancellative* if $\mathscr{C}(G) = G$ $(\mathscr{C}_l(G) = G$, $\mathscr{C}_r(G) = G)$.

1.10 Let G be a groupoid. An element $a \in G$ is said to be *left (right) divisible* if L_a (R_a) is projective. We denote by $\mathscr{D}_l(G)$ $(\mathscr{D}_r(G))$ the set of all left (right) divisible elements of G and we put $\mathscr{D}(G) = \mathscr{D}_l(G) \cap \mathscr{D}_r(G)$. The elements from $\mathscr{D}(G)$ are called *divisible.*

The groupoid G is said to be *(left, right) divisible* if $\mathscr{D}(G) = G$ $(\mathscr{D}_l(G) = G$, $\mathscr{D}_r(G) = G)$.

1.11 Let G be a groupoid. We put $\mathscr{P}_l(G) = \mathscr{C}_l(G) \cap \mathscr{D}_l(G)$, $\mathscr{P}_r(G) = \mathscr{C}_r(G) \cap \mathscr{D}_r(G)$ and $\mathscr{P}(\mathscr{G}) = \mathscr{C}(\mathscr{G}) \cap \mathscr{D}(\mathscr{G})$ $(= \mathscr{P}_l(G) \cap \mathscr{P}_r(G))$.

The groupoid G is said to be a *(left, right) quasigroup* if $\mathscr{P}(G) = G$ $(\mathscr{P}_l(G) = G$, $\mathscr{P}_r(G) = G)$.

1.12 Lemma. (i) *The class of (left, right) cancellative groupoids is closed under isomorphic images, subgroupoids and cartesian products.*

(ii) *The class of (left, right) divisible groupoids is closed under homomorphic images and cartesian products.*

(iii) *The class of (left, right) quasigroups is closed under isomorphic images and cartesian product.*

(iv) *A finite groupoid is (left, right) cancellative iff it is (left, right) divisible; if this is so then it is a (left, right) quasigroup.*

(v) *A non-trivial left cancellative (or divisible) groupoid contains no left constant element.*

Proof. Easy. □

1.13 Let G be a left (right) quasigroup. Them $\mathscr{M}_l^*(G)$, $(\mathscr{M}_r^*(G))$ denotes the permutation group generated by all the left (right) translations $L_{a,G}$, $a \in G$ $(R_{a,G}$, $a \in G)$. If G is a quasigroup then $\mathscr{M}^*(G)$ is the permutation group generated by all $L_{a,G}$, $R_{a,G}$, $a \in G$.

I.2 Stable relations and congruences

2.1 Let G be groupoid. A (binary) relation r defined on G is said to be

- *left stable* if $x, a, b \in G$ and $(a, b) \in r$ implies $(xa, xb) \in r$;
- *right stable* if $x, a, b \in G$ and $(a, b) \in r$ implies $(ax, bx) \in r$;
- *stable* if it is both left and right stable;

– *compatible* if $(a, b) \in r$, $(c, d) \in r$ implies $(ac, bd) \in r$;
– *left cancellative* if $x, a, b \in G$ and $(xa, xb) \in r$ implies $(a, b) \in r$;
– *right cancellative* if $x, a, b \in G$ and $(ax, bx) \in r$ implies $(a, b) \in r$;
– *cancellative* if is both left and right cancellative;
– *a congruence* if it is a stable equivalence.

2.2 Lemma. *Let G be a groupoid. Then:*
(i) *Every reflexive and compatible relation is stable.*
(ii) *Every transitive and stable relation is compatible.*
(iii) *A quasiordering is stable iff it is compatible.*
(iv) *If r is a stable quasiordering then* $\ker(r)$ *is a congruence.*
(v) *If G is a finite left quasigroup then a relation is left stable iff it is left cancellative.*
(vi) *If G is a finite (left, right) quasigroup then every congruence is (left, right) cancellative.*
(vii) *A congruence r is (left, right) cancellative iff the factorgroupoid G/r is (left, right) cancellative.*

Proof. Easy. □

2.3 Lemma. *Let r be a relation defined on a groupoid G and let s (t) be the smallest symmetric (transitive) relation containing r. Further, let u be the greatest symmetric relation contained in r and let* $v = r \cup \mathrm{id}_G$.
(i) *If r is (left, right) stable then s, t, u and v are so.*
(ii) *If r is compatible then u is so.*
(iii) *If r is (left, right) cancellative then s and u are so.*
(iv) *If G is (left, right) cancellative (or divisible) and if r is (left, right) cancellative then v (t) is so.*

Proof. Easy. □

2.4 Lemma. *Let r be a relation defined on a groupoid G and let s be the smallest equivalence containing r.*
(i) *If r is stable then s is a congruence.*
(ii) *If G is (left, right) divisible and r is reflexive, stable and (left, right) cancellative then s is a (left, right) cancellative congruence.*
(iii) *If G is a (left, right) quasigroup and r is stable and (left, right) cancellative then s is a (left, right) cancellative congruence.*

Proof. Let $v = r \cup \mathrm{id}_G$ and let u be the smallest symmetric relation containing v. Then s is just the smallest transitive relation containing u and the rest follows from 2.3. □

2.5 Lemma. *Let G be an idempotent groupoid and let r be a non-empty stable and left cancellative relation defined on G. Furthermore, let r satisfy the following condition:*

If $a, b, c \in G$ and $(a, c) \in r$, $(a, bc) \in r$ then $(a, b) \in r$.
Then r is a congruence of G.

Proof. Since $r \neq \emptyset$, $(a, b) \in r$ for some $a, b \in G$. Then $(a, ab) \in r$ (we have $a = aa$), and hence $(a, a) \in r$ by our condition. For every $x \in G$, $(ax, ax) \in r$ and $(x, x) \in r$, since r is right stable and left cancellative. We have proved that r is reflexive.

Now, let $a, b, c \in G$ and $(a, b), (a, c) \in r$. Then $(ab, b), (a, ac) \in r$ and $(ab, ac \cdot b) \in r$ as follows from the stability of r. Using our condition, we get $(ab, ac) \in r$. But r is left cancellative, and henceforth $(b, c) \in r$. Setting $a = c$, we get $(b, a) \in r$, i.e., r is symmetric. Finally, the transitivity easily follows. □

2.6 Lemma. *Let G be a divisible groupoid and let r be a compatible and cancellative relation defined on G. Then r is transitive. In particular, if r is symmetric and reflexive then r is a cancellative congruence on G.*

Proof. Let $a, b, c \in G$ and $(a, b), (b, c) \in r$. Then $a = ad$, $b = ed$ and $c = ef$ for some $d, e, f \in G$ and we have $(a, e), (d, f), (ad, ef) \in r$. This means that $(a, c) \in r$. □

2.7 Lemma. *Let r, s be cancellative congruences of a divisible groupoid G. Then $r \circ s = s \circ r$ is a cancellative congruence.*

Proof. Let $a, b, c \in G$ and $(a, b) \in r$, $(b, c) \in s$. There are $d, e, f \in G$ with $a = ad$, $b = ed$, $c = ef$ and we have $(ad, ed) \in r$, $(ed, ef) \in s$, and hence $(a, e) \in r$, $(d, f) \in s$, $(ad, af) \in s$, $(af, ef) \in r$, $(a, af) \in s$, $(af, c) \in r$. We have proved that $r \circ s \subseteq s \circ r$. Quite similarly $s \circ r \subseteq r \circ s$, and so $r \circ s = s \circ r$ is a congruence of G. On the other hand, $r \circ s$ is just the equivalence generated by $r \cup s$. By 2.4(ii), $r \circ s$ is a cancellative congruence. □

2.8 Lemma. *Let r be a reflexive relation defined on a quasigroup Q and let s be the union of all cancellative congruences contained in r. Then s is a cancellative congruence and it is the greatest cancellative congruence contained in r.*

Proof. Since id_Q is a cancellative congruence, we have $s \neq \emptyset$ and it is easy to see that s is reflexive, stable, cancellative and symmetric. It remains to show that s is transitive. However, if u and v are cancellative congruences of Q and if $(a, b) \in u$ and $(b, c) \in v$ then $(a, c) \in u \circ v$ and $u \circ v$ is a cancellative congruence by 2.7. □

2.9 Lemma. *Let A, B be blocks of a cancellative congruence r of a groupoid G.*

(i) *$(a, b) \in r$ if $a, b \in G$ and $aA \cap bA \neq \emptyset$ ($Aa \cap Ab \neq \emptyset$).*
(ii) *If G is left divisible and $a, b \in G$ then $(a, b) \in r$ iff $aA = bA$.*
(iii) *If G is divisible then $\{xA \mid x \in G\}$ is the set of blocks of r (i.e., the set G/r).*
(iv) *If G is divisible then* $\mathrm{card}(A) = \mathrm{card}(B)$ *and* $\mathrm{card}(G) = \mathrm{card}(A) \cdot \mathrm{card}(G/r)$.

Proof. Easy. □

2.10 Lemma. *Let r be a congruence and s be a (left, right) cancellative congruence of a divisible groupoid G.*

(i) *If* $A \subseteq B$ $(B \subseteq A)$ *for a block* A *of* r *and a block* B *of* s *then* $r \subseteq s$ $(s \subseteq r)$.
(ii) *If* r *and* s *have a common block then* $r = s$.

Proof. (i) First, let $A \subseteq B$ and $(a, b) \in r$. If $c \in A$ then $c = ad$ for suitable $d \in G$ and we have $(ad, bd) \in r$, $bd \in A$, $(ad, bd) \in s$ and $(a, b) \in s$, since s is right cancellative.

Now, let $B \subseteq A$ and $(a, b) \in s$. Again, if $c \in B$ then $a = cd$, $b = ed$ for some $d, e \in G$ and we have $(cd, ed) \in s$, $(c, e) \in s$, $c, e \in B$, $(c, e) \in r$, $(cd, ed) \in r$ and $(a, b) \in r$.
(ii) This follows immediately from (i). □

2.11 Lemma. *Let* r, s *be congruences of a groupoid* G *such that* $r \cap s = \mathrm{id}_G$ *and* $r \circ s = G \times G$. *Then* G *is isomorphic to the cartesian product* $G/r \times G/s$.

Proof. Put $f(x) = (x/r, x/s) \in G/r \times G/s$ for every $x \in G$. Since $r \cap s = \mathrm{id}_G$, f is an injective homomorphism. Let $a, b \in G$. Then $(a, c) \in r$ and $(c, b) \in s$ for some $c \in G$ and we have $f(c) = (a/r, b/s)$. Thus f is an isomorphism. □

2.12 Let G be a groupoid. We denote by ω_G the intersection of all non-identical congruences of G for G non-trivial and we put $\omega_G = \mathrm{id}_G$ for G trivial. The groupoid G is said to be *subdirectly irreducible* if $\omega_G \neq \mathrm{id}_G$; then G is non-trivial and ω_G is the smallest non-identical congruence of G.

The groupoid G is said to be *simple* if it is non-trivial and id_G, $G \times G$ are the only congruences of G (then G is subdirectly irreducible and $\omega_G = G \times G$).

2.13 Let G be a groupoid. If G is non-trivial then $\omega_{c,G}$ $(\omega_{l,c,G}, \omega_{r,c,G})$ will denote the intersection of all non-identical (left, right) cancellative congruences of G ($G \times G$ is always a cancellative congruence). If G is trivial then $\omega_{c,G} = \omega_{l,c,G} = \omega_{r,c,G} = \mathrm{id}_G$.

The groupoid G is said to be *subdirectly c-irreducible (lc-irreducible, rc-irreducible)* if $\omega_{c,G} \neq \mathrm{id}_G$ $(\omega_{l,c,G} \neq \mathrm{id}_G, \omega_{r,c,G} \neq \mathrm{id}_G)$; then G is non-trivial and $\omega_{c,G}$ $(\omega_{l,c,G}, \omega_{r,c,G})$ is the smallest (left, right) cancellative congruence of G.

The groupoid G is said to be *c-simple (lc-simple, rc-simple)* if it is non-trivial and if it possesses no (left, right) cancellative congruence r such that $r \neq \mathrm{id}_G$ and $r \neq G \times G$ (then G is subdirectly c-irreducible (lc-irreducible, rc-irreducible) and $\omega_{c,G} = G \times G$ $(\omega_{l,c,G} = G \times G, \omega_{r,c,G} = G \times G)$).

2.14 Lemma. *Let* G *be a groupoid.*
(i) $\omega_G \subseteq \omega_{l,c,G} \subseteq \omega_{c,G}$, $\omega_G \subseteq \omega_{r,c,G} \subseteq \omega_{c,G}$.
(ii) *If* G *is subdirectly irreducible then it is subdirectly lc-irreducible and rc-irreducible.*
(iii) *If* G *is subdirectly lc-irreducible (rc-irreducible) then it is subdirectly c-irreducible.*
(iv) *If* G *is not (left, right) cancellative then it is subdirectly c-irreducible (lc-irreducible, rc-irreducible).*

(v) *If G is a finite (left, right) quasigroup then* $\omega_G = \omega_{c,G}$ ($\omega_G = \omega_{l,c,G}$, $\omega_G = \omega_{r,c,G}$).

Proof. Obvious. □

2.15 Let G be a groupoid and $a, b \in G$, $a \neq b$. By Zorn's lemma there exists at least one congruence r of G such that r is maximal with respect to $(a, b) \notin r$. Now, the factorgroupoid G/r is subdirectly irreducible ($\omega_{G/r}$ is just the congruence of G/r generated by the pair $(a/r, b/r)$).

Setting $r_{(a,b)} = r$, we get $\mathrm{id}_G = \bigcap r_{(a,b)}$, $(a, b) \in G^{(2)}$. Thus G (if non-trivial) is a subdirect product of subdirectly irreducible groupoids.

2.16 Let G be a non-trivial groupoid and let $a, b \in G$ be such that $(a, b) \notin s$ for a (left, right) cancellative congruence s of G (e.g., if G is (left, right) cancellative and $a \neq b$). By Zorn's lemma there exists at least one (left, right) cancellative congruence r of G such that r is maximal with respect to $s \subseteq r$ and $(a, b) \notin r$. Now, the factorgroupoid G/r is subdirectly c-irreducible (lc-irreducible, rc-irreducible).

2.17 Lemma. *Let G be a left cancellative and right divisible groupoid and let H be a subgroupoid of G such that every (cancellative) congruence of H can be extended to a (cancellative) congruence of G. Suppose further that G is subdirectly (c-)irreducible and that H is a block of a (cancellative) congruence of G. If H is non-trivial then it is subdirectly (c-)irreducibe.*

Proof. H is a block of a cancellative congruence r. Put $s = \omega_G \cap (H \times H)$ ($s = \omega_{c,G} \cap (H \times H)$). It suffices to show that $s \neq \mathrm{id}_H$.

There are elements $a, b, c \in G$ such that $a \neq b$, $(a, b) \in \omega_G$ ($(a, b) \in \omega_{c,G}$) and $ca \in H$. We have $\omega_G \subseteq r$ ($\omega_{c,G} \subseteq r$), $(ca, cb) \in r$, $cb \in H$ and $(ca, cb) \in s$. Since G is left cancellative, $ca \neq cb$. □

2.18 Lemma. *Let G be a groupoid, $e \notin G$ and $H = G[e]$. Then H is subdirectly irreducible iff either G is trivial (then H is simple and $\omega_H = H \times H$) or G is subdirectly irreducible and contains no absorbing element (then $\omega_H = \omega_G \cup \{(e,e)\}$).*

Proof. Easy. □

2.19 Lemma. *Let G be a groupoid, $e \notin G$ and $H = G[e\}$. Then H is subdirectly irreducible iff either G is trivial or G is subdirectly irreducible and contains no left absorbing right neutral element (then $\omega_H = \omega_G \cup \{(e,e)\}$).*

Proof. Easy. □

2.20 Lemma. *Let G be a groupoid, $e \notin G$ and $H = G\{e\}$. Then H is subdirectly irreducible iff either G is trivial or G is subdirectly irreducible and contains no neutral element (then $\omega_H = \omega_G \cup \{(e,e)\}$).*

Proof. Easy. □

2.21 Let G be a groupoid. For every $a \in G$, let $p_{a,G} = \ker(R_a)$ and $q_{a,G} = \ker(L_a)$. Further, let

$$p_G = \bigcap_{a \in G} p_{a,G} \qquad \text{and} \qquad q_G = \bigcap_{a \in G} q_{a,G}.$$

Thus $(x, y) \in p_G$ iff $L_x = L_y$ and $(u, v) \in q_G$ iff $R_u = R_v$.

Finally, put $t_G = p_G \cap q_G$.

2.22 Lemma. *Let G be a groupoid. Then:*

(i) *p_G is a right stable equivalence.*

(ii) *q_G is a left stable equivalence.*

(iii) *If r is an equivalence on G and if $r \subseteq t_G$ then r is a congruence of G.*

(iv) *t_G is a congruence of G.*

(v) *If G is subdirectly irreducible and $t_G \neq \mathrm{id}_G$ then there are two elements $a, b \in G$ such that $a \neq b$ and $\omega_G = t_G = \{(a,b), (b,a)\} \cap \mathrm{id}_G$.*

Proof. Easy. $\square$

2.23 A groupoid G is said to be *left (right) faithful* if $p_G = \mathrm{id}_G$ ($q_G = \mathrm{id}_G$). G is said to be *semifaithful* if $t_G = \mathrm{id}_G$.

2.24 Lemma. *A groupoid G is semifaithful provided at least one of the following conditions is satisfied:*

(1) *$\mathscr{C}_l(G) \cup \mathscr{C}_r(G) \neq \emptyset$.*

(2) *o_G is injective.*

(3) *G is idempotent.*

(4) *G is anticommutative (i.e., $ab \neq ba$ for all $a, b \in G$, $a \neq b$).*

(5) *G is simple and contains at least three elements.*

Proof. Easy (see 2.21). $\square$

2.25 Lemma. *Let G be a commutative idempotent groupoid. Then $p_G = q_G = t_G = \mathrm{id}_G$.*

Proof. Easy. $\square$

2.26 Lemma. *Let r be a left stable equivalence on an idempotent groupoid G. Then every block of r is a subgroupoid of G.*

Proof. Obvious. $\square$

I.3 Ideals

3.1 By a *left (right) ideal* of a groupoid G we mean a non-empty subset I of G such that $GI \subseteq I$ ($IG \subseteq I$). If I is both a left and right ideal then I is called a *(two-sided) ideal.*

Clearly, every left (right) ideal of G is a subgroupoid and the sets G and GG are ideals of G.

We denote by $\mathrm{Int}(G)$ the intersection of all ideals of G; if $\mathrm{Int}(G) \neq \emptyset$ then it is the smallest ideal of G.

The groupoid G is said to be *left-ideal-free (right-ideal-free, ideal-free)* if G is the only left (right, two-sided) ideal of G.

The groupoid G is said to be *ideal-simple* if $\mathrm{card}(I) = 1$ for every ideal I of G, $I \neq G$.

3.2 Lemma. (i) *The intersection of a non-empty set of (left, right) ideals of a groupoid G is either empty or a (left, right) ideal of G.*

(ii) *If I, J are ideals of G then $IJ \subseteq I \cap J$ and $I \cap J$ is an ideal.*

(iii) *The intersection of a finite non-empty set of ideals is an ideal.*

(iv) *The union of a non-empty set of (left, right) ideals is a (left, right) ideal.*

Proof. Easy. $\square$

3.3 A groupoid G is said to be *left (right) uniform* if $I \cap J \neq \emptyset$ whenever I and J are left (right) ideals of G. In this case, the intersection of a finite non-empty set of left (right) ideals is again a left (right) ideal.

3.4 Lemma. *A groupoid G is left uniform iff for all $a, b \in G$ there exist $n, m \geq 1$ and $c_1, \dots, c_n, d_1, \dots, d_m \in G$ such that $c_1(\dots(c_n a)) = d_1(\dots(d_m b))$.*

Proof. Obvious. $\square$

3.5 Lemma. *Let I be an ideal of a groupoid G and $\equiv_I = (I \times I) \cup \mathrm{id}_G$. Then:*

(i) *$\equiv_I$ is a congruence of G.*

(ii) *I is a block of $\equiv_I$ and every other block is a one-element set.*

(iii) *$G/I = G/\equiv_I$ contains an absorbing element.*

(iv) *If $\mathrm{Id}(G) \subseteq I$ then G/I contains just one idempotent element.*

(v) *If is an ideal of G then $\equiv_I \cap \equiv_J = \equiv_{I \cap J}$ and $\equiv_I \circ \equiv_J = \equiv_J \circ \equiv_I = \equiv_{I \cup J}$.*

Proof. Easy. $\square$

3.6 Lemma. (i) *The class of (left, right-) ideal-free groupoids is closed under homomorphic images.*

(ii) *Every left (right) divisible groupoid is right-ideal-free (left-ideal-free).*

(iii) *Every ideal-free groupoid is ideal-simple.*

(iv) *The class of ideal-simple groupoids is closed under homomorphic images.*

(v) *If $e \in G$ then $\{e\}$ is an ideal of G iff e is an absorbing element.*

(vi) *A groupoid G is ideal-simple iff either G is ideal-free or G contains an absorbing element 0 and $\{0\}, G$ are the only ideals of G.*

(vii) *If G is ideal-simple then either $G = GG$ or $\mathrm{card}(GG) = 1$ and $\mathrm{card}(G) = 2$.*

(viii) *If G is ideal-fre then $G = GG$.*

(ix) *Every simple groupoid is ideal-simple.*

Proof. Easy (use 3.5). $\square$

3.7 Lemma. *Let G, H be (left, right-) ideal-free groupoids, G idempotent. Then the cartesian product $G \times H$ is (left, right-) ideal-free.*

Proof. Put $K = G \times H$ and denote by $g : K \to G$, $h : K \to H$ the natural projections. Let I be a (left, right) ideal of G. Then $g(I)$ and $h(I)$ are (left, right) ideals of G and H, resp., and so $g(I) = G$ and $h(I) = H$. Now, let $x \in G$. There is $a \in H$ with $(x, a) \in I$. Then $J = \{y \in H; (x, y) \in I\} \neq \emptyset$ and, for all $y \in J$ and $z \in H$, we have $(x, yz) = (x, y)(x, z) \in I$ and $(x, zy) = (x, z)(x, y) \in I$. Thus J is a (left, right) ideal of H, $J = H$ and $I = K$. □

3.8 Lemma. *The cartesian product of finitely many (left, right-) ideal-free idempotent groupoids is again (left, right-) ideal-free.*

Proof. This follows immediately from 3.7. □

3.9 Lemma. *Let r be a congruence of a groupoid G such that every block of r is either a one-element set or an ideal-free subgroupoid of G. Then every ideal of G is closed under r. Moreover, G is ideal-free iff G/r is so.*

Proor. Let I be an ideal of G, $a \in I$, $b \in G$ and $(a, b) \in r$, $a \neq b$. Then there is an ideal-free subgroupoid H of G such that $a, b \in H$. But $a \in H \cap I$ and $H \cap I$ is an ideal of H. Consequently, $H \subseteq I$ and $b \in I$. The rest is clear. □

3.10 Lemma. *Let I be an ideal of a subdirectly irreducible groupoid G such that every congruence of I can be extended to a congruence of G. Then either G contains an absorbing element 0 and $I = \{0\}$ or I is a subdirectly irreducible groupoid.*

Proof. Let $\mathrm{card}(I) \geq 2$. Then $\omega_G \subseteq \equiv_I$, $\omega_G \cap (I \times I) \neq \mathrm{id}_I$ and the rest is clear. □

3.11 Lemma. *If G is a subdirectly irreducible groupoid then* $\mathrm{Int}(G) \neq \emptyset$.

Proof. If G contains an absorbing element 0 then $\mathrm{Int}(G) = \{0\}$. Now, assume that G contains no absorbing element and let $a, b \in G$ be such that $a \neq b$ and $(a, b) \in \omega_G$. If I is an ideal of G then $\omega_G \subseteq \equiv_I$, and hence $a, b \in I$. This implies $a, b \in \mathrm{Int}(G)$. □

3.12 Let G be a groupoid. We shall define relations u_G, v_G and w_G on G by $(a, b) \in u_G$ (v_G, w_G) iff the elements a, b generate the same left (right, two-sided) ideal of G. Clearly, these relations are equivalences.

3.13 Lemma. *Let G be a groupoid and $a, b \in G$. Then $(a, b) \in u_G$ (v_G, w_G) iff either $a = b$ or $a = f(b)$, $b = g(a)$ for some $f, g \in \mathcal{M}_l(G)$ ($\mathcal{M}_r(G)$, $\mathcal{M}(G)$).*

Proof. Easy. □

3.14 Lemma. *Let G be a groupoid. Then:*

(i) *Every left (right, two-sided) ideal is closed under* u_G (v_G, w_G).
(ii) $u_G = G \times G$ $(v_G = G \times G, w_G = G \times G)$ *iff* G *is left-ideal-free (right-ideal-free, ideal-free).*

3.15 Let G be a groupoid. Define a relation $z_{l,G}$ $(z_{r,G})$ on G by $(a, b) \in z_{l,G}$ $(z_{r,G})$ iff $a = f(b)$ for some $f \in \mathscr{M}_l(G)$ $(\mathscr{M}_r(G))$. Further, put $z_{l,G}^1 = z_{l,G} \cup \mathrm{id}_G$ $(z_{r,G}^1 = z_{r,G} \cup \mathrm{id}_G)$.

3.16 Lemma. *Let* G *be a groupoid. Then:*
(i) $z_{l,G}$ $(z_{r,G})$ *is transitive and* $z_{l,G}^1$ $(z_{r,G}^1)$ *is a quasiordering.*
(ii) $u_G = \ker(z_{l,G}^1)$ $(v_G = \ker(z_{r,G}^1))$.
(iii) *If* $z_{l,G}$ $(z_{r,G})$ *is irreflexive then* $z_{l,G}^1$ $(z_{r,G}^1)$ *is an ordering.*

Proof. Easy. □

3.17 Let G be a groupoid. Define a relation z_G on G by $(a, b) \in z_G$ iff $a = f(b)$ for some $f \in \mathscr{M}(G)$. Further, put $z_G^1 = z_G \cup \mathrm{id}_G$.

3.18 Lemma. *Let* G *be a groupoid. Then:*
(i) z_G *is transitive and* z_G^1 *is a quasiordering.*
(ii) $w_G = \ker(z_G^1)$.
(iii) *If* z_G *is irreflexive then* z_G^1 *is an ordering.*

Proof. Easy. □

3.19 Let G be a groupoid. An ideal I of G is said to be *prime* if $I \cap \{a,b\} \neq \emptyset$ whenever $a, b \in G$ and $ab \in I$.

A left (right) ideal I of G is said to be *left (right) strongly prime* if $b \in I$ whenever $a, b \in G$ and $ab \in I$ $(ba \in I)$.

3.20 Lemma. *Let* G *be a groupoid.*
(i) *An ideal* I *of* G *is prime iff either* $I = G$ *or* $G - I$ *is a subgroupoid of* G.
(ii) *If* I *is a prime ideal of* G *then* $r = I^{(2)} \cup (G - I)^{(2)}$ *is a congruence of* G. *Moreover,* G/r *is a semilattice.*
(iii) *If* $e \in G$ *then* $\{e\}$ *is a prime ideal of* G *iff* e *is an absorbing element of* G, $xy \neq e$ *for all* $x, y \in G$, $x \neq e \neq y$; *in this case,* $G = (G - \{e\})[e]$.
(iv) *The union of a non-empty set of prime ideals of* G *is again a prime ideal.*
(v) *The intersection of a non-empty chain of prime ideals (i.e., a set of prime ideals linearly ordered by inclusion) is either empty or a prime ideal.*
(vi) *If* I *is a prime ideal and* M *is a non-empty generator set of* G *then* $M \cap I \neq \emptyset$ *and* I *is just the ideal generated by* $M \cap I$.

Proof. The first five assertions are easy.
(vi) Let $K = M \cap I$, $N = M - K$ and $L = G - I$. If $L = \emptyset$ then $I = G$ and $K = M \neq \emptyset$. If $L \neq \emptyset$ then L is a subgroupoid of G, $L \neq G$, and hence $M \nsubseteq L$ and $K \neq \emptyset$. Now, denote by J the ideal generated by K. Then $J \subseteq I$ and we can assume that $N \neq \emptyset$.

Let, on the contrary, $a \in I - J$. If $a \in \langle N \rangle_G$, then $N \cap I \neq \emptyset$, a contradiction. Hence $a \notin \langle N \rangle_G$ and this implies $a \in J$, again a contradiction. □

3.21 Lemma. *Let G be a groupoid.*
(i) *If G is cyclic then G contains no proper prime ideal.*
(ii) *If G is finitely generated and $\sigma(G) \geq 1$ then G contains at most $2^{\sigma(G)} - 2$ proper prime ideals.*

Proof. Use 3.20(vi). □

3.22 Lemma. *Let I be a proper prime ideal of a finitely generated groupoid G. Then:*
(i) $\sigma(G - I) \leq \sigma(G) - 1$.
(ii) *If G is pseudocyclic then* $\operatorname{card}(G - I) = 1$.

Proof. (i) We have $\sigma(G) \geq 2$ (see 3.21(i)). Let M be a generator set of G such that $\operatorname{card}(M) = \sigma(G)$. Then $M \nsubseteq H = G - I$ and H is generated by $M \cap H$.
(ii) G is not cyclic, and hence G is idempotent and $\sigma(G) = 2$. By (i), $\sigma(G - I) \leq 1$, and so $G - I$ is a one-element groupoid. □

3.23 Lemma. *A subdirectly irreducible semilattice contains just two element.*

Proof. Let G be a subdirectly irreducible semilattice, i.e., G is a commutative idempotent semigroup and there are $a, b \in G$ such that $a \neq b$ and $(a, b) \in \omega_G$. Furthermore, we can assume that b is not an absorbing element of G. Then $\operatorname{card}(Gb) \geq 2$ and, since Gb is an ideal, we have $a \in Gb$ and $a = ab$. Similarly, if $\operatorname{card}(Ga) \geq 2$ then $b \in Ga$ and $ba = b$, a contradiction. Hence $\operatorname{card}(Ga) = 1$ and a is an absorbing element of G. On the other hand, $a = ab \neq bb = b$, and therefore $(a, b) \notin p_{a,G}$. But p_a is a congruence of G, hence $p_a = \mathrm{id}_G$ and this implies that b is a neutral element of G. Finally, if $x \in G$, $x \neq b$ then $p_x \neq \mathrm{id}_G$, $a = xa = xb = x$. □

3.24 Lemma. *A groupoid G contains no proper prime ideal iff no non-trivial homomorphic image (i.e., no non-trivial factorgroupoid) of G is a semilattice.*

Proof. If no non-trivial image of G is a semilattice then G possesses no proper prime ideal by 3.20(ii). Conversely, if some non-trivial images of G are semilattices then there is a congruence r of G such that G/r is a two-element semilattice (this follows from 3.23). Now, G/r contains an absorbing element and the inverse image of this element is a proper prime ideal of G. □

3.25 Let G be a groupoid. We shall define a relation u_G^c (v_G^c) on G by $(a, b) \in u_G^c$ (v_G^c) iff the elements a and b are contained in the same left (right) strongly prime left (right) ideals, i.e., iff a and b generate the same left (right) strongly prime left (right) ideal.

Clearly, both u_G^c and v_G^c are equivalences on G.

3.26 Lemma. *Let G be a groupoid.*

(i) *A left ideal I of G is left strongly prime iff either $I = G$ or $G - I$ is again a left ideal (then $G - I$ is also left strongly prime).*

(ii) *If I is a left strongly prime left ideal of G then $r = I^{(2)} \cup (G - I)^{(2)}$ is a congruence of G. Moreover, G/r is an RZ-semigroup (see 6.1).*

(iii) *The union of a non-empty set of left strongly prime left ideals is again a left strongly prime left ideal.*

(iv) *The intersection of a non-empty set of left strongly prime left ideals is either empty or a left strongly prime left ideal.*

Proof. Easy. $\square$

3.27 Lemma. *The following conditions are equivalent for a groupoid G:*

(i) $u_G^c = G \times G$.

(ii) *G pocesses no proper left strongly prime left ideal.*

(iii) *No non-trivial homomorphic image of G is an RZ-semigroup.*

Proof. Easy (use 3.26). $\square$

3.28 For a groupoid G, let $\mathscr{I}_l(G)$ ($\mathscr{I}_r(G)$, $\mathscr{I}(G)$) denote the set of left (right, two-sided) ideals of G.

I.4 Closed subgroupoids

4.1 For a groupoid G and $S \subseteq G$, let $\alpha_G(S) = \{x \in G \mid ax \in S \text{ for some } a \in S\}$, $\gamma_G(S) = \{x \in G \mid xa \in S \text{ for some } a \in S\}$ and $\varphi_G(S) = \alpha_G(S) \cup \gamma_G(S)$.

The subset S is said to be *(left, right) closed in G* if $\varphi(S) \subseteq S$ ($\alpha(S) \subseteq S$, $\gamma(S) \subseteq S$). Clearly, S is closed iff it is both left and right closed.

The intersection of a non-empty set of (left, right) closed subsets is again (left, right) closed. Hence, given a subset R of G, we denote by $[R]_G$ ($[R]_G^l$, $[R]_G^r$) the smallest (left, right) closed subset containing R. Clearly, $[R]_G^l \cup [R]_G^r \subseteq [R]_G$.

4.2. **Let** S a subset of a groupoid G.

(i) Put $S_0 = S$ and $S_{i+1} = \varphi(S_i) \cup S_i$ ($\alpha(S_i) \cup S_i$, $\gamma(S_i) \cup S_i$) for every $i \geq 0$. Then $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_i \subseteq S_{i+1} \subseteq \ldots$ and $\bigcup_{i \geq 0} S_i = [S]_G$ ($[S]_G^l$, $[S]_G^r$).

(ii) Put $R_0 = S$, $R_i = \alpha(R_{i-1}) \cup R_{i-1}$ for $i \geq 1$ odd and $R_i = \gamma(R_{i-1}) \cup R_{i-1}$ for $i \geq 2$ even. Again, $R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_i \subseteq R_{i+1} \subseteq \ldots$ and $\bigcup_{i \geq 0} R_i = [S]_G$.

4.3 Lemma. *Let H be a subgroupoid and S be a subset of a groupoid G. Then:*

(i) $H \subseteq \alpha_G(H) \cap \gamma_G(H) \cap \varphi_G(H)$.

(ii) $\alpha_G^i(H) \subseteq \alpha_G^{i+1}(H)$, $\gamma_G^i(H) \subseteq \gamma_G^{i+1}(H)$ *and* $\varphi_G^i(H) \subseteq \varphi_G^{i+1}(H)$ *for every* $i \geq 1$.

(iii) $[H]_G^l = \bigcup_{i \geq 1} \alpha_G^i(H)$, $[H]_G^r = \bigcup_{i \geq 1} \gamma_G^i(H)$, $[H]_G = \bigcup_{i \geq 1} \varphi_G^i(H)$.

(iv) $\varphi_G(H) \subseteq \alpha_G\gamma_G(H) \cap \gamma_G\alpha_G(H)$.

(v) $[H]_G = \bigcup_{i \geq 1} (\alpha_G\gamma_G)^i(H) = \bigcup_{i \geq 1} (\gamma_G\alpha_G)^i(H)$.

(vi) *If S is (left, right) closed in G then $S \cap H$ is (left, right) closed in H.*

(vii) *If $S \subseteq H$, S is (left, right) closed in H and H is (left, right) closed in G* then *S is (left, right) closed in G.*

(viii) *If $S \subseteq H$ and S is (left, right) closed in G then S is (left, right) closed in H.*

(ix) *If $S \subseteq H$ then $[S]^l_H \subseteq [S]^l_G$, $[S]^r_H \subseteq [S]^r_G$ and $[S]_H \subseteq [S]_G$.*

(x) *If f is a projective homomorphism of G onto a groupoid K and if $H = f^{-1}(L)$ is the inverse image of a subgroupoid L* of *K then H is (left, right) closed in G iff L is (left, right) closed in K.*

Proof. Easy observations. □

4.4. Let G be a groupoid. The intersection of a non-empty set of (left, right) closed subgroupoids is either empty or a (left, right) closed subgroupoid. Hence, given a non-empty subset S of G, $\langle S\rangle^c_G$ ($\langle S\rangle^{lc}_G$, $\langle S\rangle^{rc}_G$) will denote the smallest (left, right) closed subgroupoid containing S. Clearly, $[S]_G \subseteq \langle S\rangle^c_G$ ($[S]^l_G \subseteq \langle S\rangle^{lc}_G$, $[S]^r_G \subseteq \langle S\rangle^{rc}_G$).

(i) Put $S_0 = S$, $S_i = \{xy \mid x, y \in S_{i-1}\} \cup S_{i-1}$ for every odd $i \geq 1$ and $S_i = \varphi_G(S_{i-1}) \cup S_{i-1}$ for every odd $i \geq 1$ and $S_i = \varphi_G(S_{i-1}) \cup S_{i-1}$ ($\alpha_G(S_{i-1}) \cup S_{i-1}$, $\gamma_G(S_{i-1}) \cup S_{i-1}$) for every even $i \geq 2$. Then $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots \subseteq S_i \subseteq S_{i+1} \subseteq \ldots$ and $\bigcup_{i \geq 0} S_i = \langle S\rangle^c_G$ ($\langle S\rangle^{lc}_G$, $\langle S\rangle^{rc}_G$).

(ii) If the intersection A of all (left, right) closed subgroupoids is non-empty then A is the smallest (left, right) closed subgroupoid of G and we put $\langle\emptyset\rangle^{lc}_G = A$, ($\langle\emptyset\rangle^{rc}_G = A$).

We denote by $\sigma_c(G)$ ($\sigma_{lc}(G)$, $\sigma_{rc}(G)$) the smallest cardinal number $\operatorname{card}(M)$ for a set M of c-generators (lc-generators, rc-generators) of G. Clearly, $0 \leq \sigma_c(G) \leq \sigma_{lc}(G) \leq \sigma(G)$ and $0 \leq \sigma_c(G) \leq \sigma_{rc}(G) \leq \sigma(G)$.

4.5 A subset S of a groupoid G is said to be *left (rigth) strongly dense in G* if S lc-generates (rc-generates) G, i.e., if $G = \langle S\rangle^{lc}_G$ ($G = \langle S\rangle^{rc}_G$) (see 4.4).

A subset S of G is said to be *dense in G*, if S c-generates G, i.e., if $G = \langle S\rangle^c_G$ (see 4.4). Clearly, if S is left (right) strongly dense in G then S is dense in G.

A subset S of G is said to be *strongly dense in G* if it is both left and right strongly dense in G.

4.6 Lemma. *Let G be a groupoid and S a subset of G. Then:*

(i) *S is left (right) strongly dense in $\langle S\rangle^{lc}_G$ ($\langle S\rangle^{rc}_G$)* (see 4.4(ii) if $S = \emptyset$).

(ii) *S is dense in $\langle S\rangle^c_G$* (see 4.4(ii) if $S = \emptyset$).

(iii) *If H is a (left, right) strongly dense subgroupoid of G, $S \subseteq H$ and $S \neq \emptyset$ is (left, right) strongly dense in H then S is (left, right) strongly dense in G.*

(iv) *If H is a dense subgroupoid of G, $S \subseteq H$ and $S \neq \emptyset$ is dense in H then S is dense in G.*

Proof. (i) Put $K = \langle S\rangle^{lc}_G$. Then K is left closed in G. If L is a left closed subgroupoid of K with $S \subseteq L$ then L is left closed in G by 4.3(vii) and $L = K$.

(ii) Similar to (i).

(iii) We have $H = \langle S\rangle_H^{lc} \subseteq \langle S\rangle_G^{lc} = K$ (by 4.3(vi)). But K is left closed in G and H is left strongly dense in G. Consequently $K = G$.
(iv) Similar to (iii). □

4.7 Lemma. (i) *Every (left, right) closed subgroupoid of a (left, right) divisible groupoid is (left, right) divisible.*
(ii) *A subgroupoid H of a (left, right) quasigroup G is (left, right) closed iff H is also a (left, right) quasigroup.*
(iii) *Let f, g be homomorphisms of a groupoid G into a (left, right) cancellative groupoid K. Then the set $\{x \in G \mid f(x) = g(x)\}$ is either empty or a (left, right) closed subgroupoid of G.*

Proof. Easy. □

4.8 Lemma. *Let H be a subgroupoid of a groupoid G and let f be a homomorphism of H into a groupoid K.*
(i) *If H is left (right) strongly dense in G and K is left (right) cancellative then f can be extended to at most one homomorphism of G into K.*
(ii) *If H is dense in G and K is cancellative then f can be extended to at most one homomorphism of G into K.*

Proof. This is an immediate consequence of 4.7(iii). □

4.9 Lemma. *Let a subgroupoid H be a block of a (left, right) cancellative congruence of a groupoid G. Then H is a (left, right) closed subgroupoid of G.*

Proof. Easy. □

4.10 Lemma. *Let r, s be cancellative congruences of a divisible groupoid G and let A and B be blocks of r and s, resp., such that $A \cap B \neq \emptyset$ and A is a subgroupoid of G. Then $\langle A \cup B\rangle_G^c$ is a block of $r \circ s$* (see 2.7).

Proof. By 2.7, $t = r \circ s = s \circ r$ is a cancellative congruence of G. Let C be the block of t such that $A \subseteq C$. Since $A \cap B \neq \emptyset$, $A \cup B \subseteq C$ and $\langle A \cup B\rangle_G^c \subseteq C$ by 4.9 (clearly, C is a subgroupoid). Now, let $c \in C$ and $a \in A$. Then $(a, b) \in r$ and $(b, c) \in s$ for some $b \in G$. We have $b \in A$ and $db \in B$ for an element $d \in G$. Then $(db, dc) \in s$ implies $dc \in B$. Thus $b, d, b, dc \in \langle A \cup B\rangle_G^c$, and hence $c \in \langle A \cup B\rangle_G^c$. □

4.11 Lemma. *Let G be a left divisible groupoid, r a congruence of G and H a subgroupoid of G such that H contains a block A of r. Then H is closed under r, provided that at least one of the following two conditions is satisfied:*
(1) *H is right divisible and r is left cancellative.*
(2) *H is closed in G.*

Proof. Let $(x, y) \in r$, $x \in H$. If (1) is true then $x = ba$, $y = bc$ for some $a \in A$, $b \in H$, $c \in G$, $(ba, bc) \in r$ and $(a, c) \in r$, since r is left cancellative. Then $c \in A$ and $y = bc \in H$. If (2) is true then $xa \in A$ for some $a \in G$, $(xa, ya) \in r$, $ya \in A$, $x, xa, ya \in H$, and hence $y \in H$, since H is closed. □

4.12 Lemma. *Let a subgroupoid H be a block of a congruence r of a left divisible groupoid G. Put $K = G/r$. Then:*

(i) $\sigma_c(G) \leq \sigma_c(H) + \sigma_c(K)$, *provided that* $\sigma_c(H) \geq 1$.

(ii) $\sigma_c(G) \leq 1 + \sigma_c(K)$, *provided that* $\sigma_c(H) = 0$.

(iii) $\sigma_c(G) \leq \sigma_c(H) + \sigma_c(K) - 1$ *(*$\sigma_c(G) \leq \sigma_c(H)$ *or* $\sigma_c(G) \leq \sigma_c(K)$ *or* $\sigma_c(G) \leq 1$*), provided that* $\sigma_c(H) \geq 1$ *and* $\sigma_c(K) \geq 1$ *(*$\sigma_c(H) \geq 1$ *and* $\sigma_c(K) = 0$ *or* $\sigma_c(H) = 0$ *and* $\sigma_c(K) \geq 1$ *or* $\sigma_c(H) = 0 = \sigma_c(K)$*) and that for all* $x, y \in G$ *there exists a projective endomorphism* f *of* G *such that* $f(x) = y$ *and* r *is invariant under* f.

(iv) *If both H and K are finitely c-generated then G is finitely c-generated.*

Proof. Denote by $g : G \to K$ the natural projection. There are subsets $A \subseteq H$ and $B \subseteq G$ such that $H = \langle A \rangle_H^c$, $\mathrm{card}(A) = \max(1, \sigma_c(H))$, $K = \langle g(B) \rangle_K^c$ and $\mathrm{card}(B) = \sigma_c(K)$. Put $F = \langle A \cup B \rangle_G^c$. Then $F \cap H \neq \emptyset$, $A \subseteq F \cap H$ and $F \cap H$ is a closed subgroupoid of H. Hence $F \cap H = H$, $H \subseteq F$ and F is closed under r by 4.11(2). This implies easily that $g(F)$ is closed in K. However, $g(B) \subseteq g(F)$, hence $g(F) = K$ and $F = G$.

Now, suppose that $B \neq \emptyset$ ($\mathrm{card}(B) = \max(1, \sigma_c(K))$) and there exists a projective endomorphism f of G such that $f(B) \cap H \neq \emptyset$ and r is invariant under f. Then f induces a projective endomorphism k of K such that $gf = kg$. Put $E = \langle A \cup (f(B) = \{a\}) \rangle_G^c$, where $a \in f(B) \cap H$ is arbitrary. Again, $H \subseteq E$, E is closed under r, $f(B) \subseteq E$, $kg(B) = gf(B) \subseteq g(E)$, $g(E) \subseteq L$, where $L = k^{-1}g(E)$ is the inverse image of $g(E)$ under k, L is closed in K, $L = K$ and $g(E) = K$. Consequently $E = G$. $\square$

4.13 Lemma. *Let H be a dense (left strongly dense, right strongly dense) subgroupoid of a groupoid G. Then* $\sigma_c(G) \leq \max(\sigma_c(H), 1)$ *(*$\sigma_{lc}(G) \leq \max(\sigma_{lc}(H), 1)$, $\sigma_{rc}(G) \leq \max(\sigma_{rc}(H), 1)$*).*

Proof. Let A be a subset of H such that $\mathrm{card}(A) = \max(\sigma_c(H), 1)$ and $H = \langle A \rangle_H^c$. Put $K = \langle A \rangle_G^c$. Then $K \cap H$ is closed in H, and so $H \subseteq K$. Since H is dense in G, $K = G$ and $\sigma_c(G) \leq \mathrm{card}(A)$. $\square$

4.14. Lemma. *Let A be a non-empty subset of a (left, right) cancellative groupoid G and let* $\alpha = \max(\aleph_0, \mathrm{card}(A))$. *Then* $\mathrm{card}(\langle A \rangle_G^c) \leq \alpha$ *(*$\mathrm{card}(\langle A \rangle_G^{lc}) \leq \alpha$, $\mathrm{card}(\langle A \rangle_G^{rc}) \leq \alpha$*).*

Proof. The result is clear from 4.4. $\square$

4.15 Lemma. *Let H be a (left, right strongly) dense subgroupoid of a (left, right) cancellative groupoid G. Then* $\mathrm{card}(H) = \mathrm{card}(G)$.

Proof. If H is infinite then the result follows easily from 4.14. If H is finite then H is a (left, right) quasigroup, and consequently H is (left, right) closed in G (see 4.7(ii)) and $H = G$. $\square$

4.16 Let H be a subgroupoid of a groupoid G. We denote by $\mathcal{M}_l(G, H)$ the transformation semigroup generated by all $L_{a,G}$, $a \in H$ in the left multiplication semigroup of G. Thus $\mathcal{M}_l(G, H)$ is a subsemigroup of $\mathcal{M}_l(G)$. Similarly we define $\mathcal{M}_r(G, H)$ and $\mathcal{M}(G, H)$, and we put $\mathcal{M}_l^1(G, H) = \mathcal{M}_l(G, H) \cup \{\mathrm{id}_G\}$, $\mathcal{M}_r^1(G, H) = \mathcal{M}(G, H) \cup \{\mathrm{id}_G\}$.

4.17 Let S be a subset of a groupoid G. Put $\beta_{0,G}(S) = S$ ($\delta_{0,G}(S) = S$). Further, for $n \geq 1$, let $\beta_{n,G}(S)$ ($\delta_{n,G}(S)$) be the set of $x \in G$ such that $a_1(a_2(\dots(a_n x))) \in S$ ($(((xa_1)\, a_2)\dots)\, a_n \in S$) for some $a_1, \dots, a_n \in S$. Clearly, $\beta_{1,G}(S) = \alpha_G(S)$ ($\delta_{1,G}(S) = \gamma_G(S)$) and $\beta_G(S) \subseteq [S]_G^l$ ($\delta_G(S) \subseteq [S]_G^r$), where $\beta_G(S) = \bigcup_{i \geq 0} \beta_{i,G}(S)$ ($\delta_G(S) = \bigcup_{i \geq 0} \delta_{i,G}(S)$).

4.18 Lemma. *Let H be a subgroupoid of a groupoid G. Then:*
(i) $\beta_G(H) = \{x \in G \mid f(x) \in H$ *for some* $f \in \mathcal{M}_l(G, H)\}$.
(ii) $H = \beta_{0,G}(H) \subseteq \beta_{1,G}(H) \subseteq \dots \subseteq \beta_{i,G}(H) \subseteq \beta_{i+1,G}(H) \subseteq \dots$.
(iii) $\beta_{i,G}(H) \subseteq \alpha_G^i(H)$ *for every* $i \geq 0$.

Proof. Easy. □

4.19 Let S be a subset of a groupoid G. Put $\psi_{0,G}(S) = S$ and, for $n \geq 1$, let $\psi_{n,G}(S)$ be the set of $x \in G$ such that ${}_1T_{a_1} \dots {}_nT_{a_n}(x) \in S$ for some ${}_iT \in \{L, R\}$ and $a_i \in S$. Clearly, $\psi_{1,G}(S) = \alpha_G(S) \cup \gamma_G(S) = \varphi_G(S)$ and $\psi_G(S)$ $[S]_G$, where $\psi_G(S) = \bigcup_{i \geq 0} \psi_{i,G}(S)$.

4.20 Lemma. *Let H be a subgroupoid of a groupoid G. Then:*
(i) $\psi_G(H) = \{x \in G \mid f(x) \in H$ *for some* $f \in \mathcal{M}(G, H)\}$.
(ii) $H = \psi_{0,G}(H) \subseteq \psi_{1,G}(H) \subseteq \dots \subseteq \psi_{i,G}(H) \subseteq \psi_{i+1,G}(H) \subseteq \dots$.
(iii) $\psi_{i,G}(H) \subseteq \varphi_G^i(H)$ *for every* $i \geq 0$.

Proof. Easy. □

4.21 Let G be a groupoid. For $a \in G$ and a subset S of G, let $\mu_{a,G}(S) = u \in G \mid au \in S\}$ and $\nu_{a,G}(S) = \{u \in G \mid ua \in S\}$. Clearly, $\alpha_G(S) = \bigcup_{a \in S} \mu_{a,G}(S)$ and $\gamma_G(S) = \bigcup_{a \in S} \nu_{a,G}(S)$.

A subset S of G is said to be *α-stable (γ-stable)* if $S \subseteq \alpha_G(S)$ ($S \subseteq \gamma_G(S)$). Clearly, S is α-stable (γ-stable) iff for every $b \in S$ there exists $a \in S$ with $ab \in S$ ($ba \in S$). If this is true then $\alpha_G(S)$ ($\gamma_G(S)$) is also α-stable (γ-stable).

I.5 Regular groupoids

5.1 A groupoid G is said to be *left (right) regular* if, for all $a, b, c \in G$, $ca = cb$ ($ac = bc$) implies $xa = xb$ ($ax = bx$) for every $x \in G$. The groupoid G is said to be *regular* if it is both left and right regular.

5.2 Lemma. (i) *Every (left, right) cancellative groupoid is (left, right) regular.*
(ii) *The class of (left, right) regular groupoids is closed under isomorphic images, subgroupoids and cartesian products.*

Proof. Obvious. □

5.3 Lemma. *The following conditions are equivalent for a groupoid G:*

(i) *Every element of G is left (right) absorbing.*

(ii) *Every element of G is right (left) neutral.*

(iii) *G satisfies the identity* $\mathbf{x} \doteq \mathbf{xy}$ $(\mathbf{x} \doteq \mathbf{yx})$, *i.e., G is an LZ-semigroup (RZ-semigroup).*

(iv) *Every non-empty subset of G is a right (left) ideal of G.*

(v) *G is idempotent, left (right) regular and contains at least one left (right) absorbing element.*

(vi) *G is idempotent and* $q_G = G \times G$ $(p_G = G \times G)$.

Proof. Easy. □

5.4 Lemma. *The following conditions are equivalent for a groupoid G:*

(i) *Every element of G is left (right) constant.*

(ii) $R_x = R_y$ $(L_x = L_y)$ *for all* $x, y = G$.

(iii) $q_G = G \times G$ $(p_G = G \times G)$.

(iv) *G satisfies the identity* $\mathbf{xy} \doteq \mathbf{xz}$ $(\mathbf{yx} \doteq \mathbf{zx})$, *i.e., G is a left constant groupoid (right constant groupoid)* (see 6.1).

(v) *G is left (right) regular and contains at least one left (right) constant element.*

Proof. Easy. □

5.5 Lemma. *The following conditions are equivalent for a groupoid G:*

(i) *G satisfies the identity* $\mathbf{xy} \doteq \mathbf{zx}$.

(ii) *G satisfies the identity* $\mathbf{xy} \doteq \mathbf{yz}$.

(iii) *G satisfies the identity* $\mathbf{xy} \doteq \mathbf{uv}$, *i.e., G is an Z-semigroup.*

(iv) *G is both a left and right constant groupoid* (see 6.1).

(v) *G is (left, right) regular and contains an absorbing element.*

(vi) $t_G = G \times G$.

Proof. Easy. □

5.6 Lemma. *Let G be a groupoid.*

(i) *If* q_G (p_G) *is left (right) cancellative then G is left (right) regular.*

(ii) *If G is left (right) regular then G is left (right) cancellative iff G is right (left) faithful.*

(iii) *If G is regular then G is cancellative iff G is both left and right faithful.*

(iv) *If G contains a (left, right) neutral element then G is (left, right) regular iff it is (left, right) cancellative.*

(v) *If G is (left, right) regular, idempotent ankd every subgroupoid of G is (left, right) closed in G then G is (left, right) cancellative.*

Proof. Only (v) needs a proof. Let $a, b, c \in G$ and $ab = ac$. Then $b = bb = bc$, and so $b, bc \in H = \langle b \rangle_G$. But H is left closed in G and $H = \{b\}$. This implies $b = c$. □

5.7 Lemma. *Let G be a regular commutative groupoid. Then G is cancellative, provided that at least one of the following three conditions is satisfied:*
(1) *G is idempotent.*
(2) *G is simple and contains at least three elements.*
(3) *G is subdirectly irreducible and G is a semimedial divisible groupoid.*

Proof. (i) If $ab = ac$ for some $a, b, c \in G$ then $b = bb = bc = cb = cc = c$.
(ii) Since $\mathrm{card}(G) \geq 3$, $t_G = \mathrm{id}_G$ and this implies that G is cancellative.
(iii) It suffices to show that $t_G = \mathrm{id}_G$. Assume, on the contrary, that $t_G \neq \mathrm{id}_G$. Since every equivalence contained in t_G is a congruence of G, we have $t_G = \omega_G = \{(a,b),(b,a)\} \cup \mathrm{id}_G$ for some $a, b \in G$, $a \neq b$. Now, G is divisible and not cancellative, and hence G is infinite. There exist elements $x, y, u, v \in G$ with $a = yx$, $b = ux$, $y \notin \{a,b\}$ and $yv \notin \{a,b\}$. We have $yv \cdot xx = yx \cdot vx = a \cdot vx = b \cdot vx = ux \cdot vx = uv \cdot xx$, and so either $uv = yv$ or $\{uv, yv\} \subseteq \{a,b\}$. The latter possibility is excluded, so that $uv = yv$, $(y, u) \in t_G$ and $y = u$. Then $a = yx = ux = b$, a contradiction. □

5.8 Lemma. *Let r be a congruence of a groupoid G such that the factor $H = G/r$ is regular. Then r is cancellative (or, equivalently, H is cancellative) provided that at least one of the following three conditions is satisfied:*
(1) *Every block of r is a closed subgroupoid of G.*
(2) *H is a semifaithful idempotent divisible groupoid, both p_H and q_H are congruences of H, at least one of the blocks of r is left closed in G and at least one is right closed in G.*
(3) *H is a faithful divisible groupoid, both p_H and q_H are congruences of H and at least one of the blocks of r is a closed subset of G.*

Proof. Denote by f the natural projection of G onto H.
(i) If $a, b, c \in G$ and $(ab, ac) \in r$ then $(xb, xc) \in r$ for every $x \in G$, since H is left regular. In particular, $(bb, bc) \in r$. On the other hand, H is idempotent, and so $(bb, b) \in r$ and $(b, bc) \in r$. Thus $b \cdot bc \in A$ for a block A of r and $c \in A$, since A is left closed. This shows that $(b, c) \in r$, i.e., r is left cancellative. Similarly, r is right cancellative.
(ii) Let $(x, y) \in q_H$ and let A be a block of r such that A is a left closed subgroupoid of G. If $a \in A$ then $xz = f(a)$ and $yz = f(b)$ for some $z \in H$ and $b \in G$. Since q_H is a congruence, we have $(xz, yz) \in q_H$ and $xz = xz \cdot xz = xz \cdot yz$, $f(a) = xz = xz \cdot yz = f(ab)$, $(a, ab) \in r$, $a, ab \in A$, $b \in A$ and $xz = f(a) = f(b) = yz$. Since H is regular, $(x, y) \in p_H$, and so $(x, y) \in p_H \cap q_H = t_H = \mathrm{id}_G$. Thus $x = y$, $q_H = \mathrm{id}_H$ and H is left cancellative. Similarly, H is right cancellative.
(iii) Let $(x, y) \in q_H$, let A be a block of r such that A is a closed subset of G and let $a \in A$. Since H is divisible, $xz = f(a)$, $yz = f(b)$ and $xz = f(c) \cdot xz$ for some $z \in H$ and $b, c \in G$. Then $f(a) = xz = f(c) \cdot xz = f(ca)$, $(a, ca) \in r$, $a, ca \in A$ and $c \in A$, since A is right closed. Further, q_H is a congruence of H, hence

$(xz, yz) \in q_H$ and $f(a) = f(c) \cdot xz = f(c) \cdot yz = f(cb)$, $(a, cb) \in r$, $c, cb \in A$ and $b \in A$, since A is left closed in G. Now, $xz = f(a) = f(b) = yz$ and $(x, y) \in p_H$, since H is right regular. We have proved that $q_H \subseteq p_H$, which implies that $q_H = t_H$. But H is semifaithful, i.e., $t_H = \mathrm{id}_H$. Since H is left regular and $q_H = \mathrm{id}_H$, H is left cancellative. Quite similarly, H is right cancellative. □

5.9 Lemma. *Let r be a congruence of a groupoid G such that $H = G/r$ is a right regular divisible groupoid, q_H is a congruence of H and a block A of r is a closed subset of G. Then A is a subgroupoid of G and $q_H = t_H$.*

Proof. We can proceed in the same way as in the proof of 5.8(iii) to show that $q_H = t_H$. Now, let $a, b \in A$. Since H is left divisible, $ac \in A$ for some $c \in G$. However, A is left closed, $c \in A$ and $(c, b) \in r$, $(ac, ab) \in r$, $ab \in A$. □

I.6 Some varieties of groupoids

6.1 A groupoid is said to be
- *idempotent* if it satisfies the identity $\mathbf{x} \doteq \mathbf{xx}$;
- *unipotent* if it satisfies the identity $\mathbf{xx} \doteq \mathbf{yy}$;
- *zeropotent* if it satisfies the identities $\mathbf{xx} \cdot \mathbf{y} \doteq \mathbf{y} \cdot \mathbf{xx} \doteq \mathbf{xx}$;
- *commutative* if it satisfies the identity $\mathbf{xy} \doteq \mathbf{yx}$;
- *elastic* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yx} \doteq \mathbf{xy} \cdot \mathbf{x}$;
- *left alternative* if it satisfies the identity $\mathbf{x} \cdot \mathbf{xy} \doteq \mathbf{xx} \cdot \mathbf{y}$;
- *right alternative* if it satisfies the identity $\mathbf{y} \cdot \mathbf{xx} \doteq \mathbf{y} \cdot \mathbf{xx}$;
- *left symmetric* if it satisfies the identity $\mathbf{x} \cdot \mathbf{xy} \doteq \mathbf{y}$;
- *right symmetric* if it satisfies the identity $\mathbf{yx} \cdot \mathbf{x} \doteq \mathbf{y}$;
- *semisymmetric* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yx} \doteq \mathbf{y}$;
- *LZ-semigroup* if it satisfies the identity $\mathbf{x} \doteq \mathbf{xy}$;
- *RZ-semigroup* if it satisfies the identity $\mathbf{x} \doteq \mathbf{yx}$;
- *left constant* if it satisfies the identity $\mathbf{xy} \doteq \mathbf{xz}$;
- *right constant* if it satisfies the identity $\mathbf{yx} \doteq \mathbf{zx}$;
- *Z-semigroup* if it satisfies the identity $\mathbf{xy} \doteq \mathbf{uv}$;
- *associative* (or *semigroup)* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{xy} \cdot \mathbf{z}$;
- *left permutable* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{y} \cdot \mathbf{xz}$;
- *right permutable* if it satisfies the identity $\mathbf{xy} \cdot \mathbf{z} \doteq \mathbf{xz} \cdot \mathbf{y}$;
- *left modular* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{z} \cdot \mathbf{yx}$;
- *right modular* if it satisfies the identity $\mathbf{xy} \cdot \mathbf{z} \doteq \mathbf{zy} \cdot \mathbf{x}$;
- *A-semigroup* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{uv} \cdot \mathbf{w}$;
- *left semimedial* if it satisfies the identity $\mathbf{xx} \cdot \mathbf{yz} \doteq \mathbf{xy} \cdot \mathbf{xz}$;
- *right semimedial* if it satisfies the identity $\mathbf{yz} \cdot \mathbf{xx} \doteq \mathbf{yx} \cdot \mathbf{zx}$;
- *middle semimedial* if it satisfies the identity $\mathbf{xy} \cdot \mathbf{zx} \doteq \mathbf{xz} \cdot \mathbf{yx}$;
- *left distributive* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{xy} \cdot \mathbf{xz}$;
- *right distributive* if it satisfies the identity $\mathbf{yz} \cdot \mathbf{x} \doteq \mathbf{yx} \cdot \mathbf{zx}$;
- *medial* if it satisfies the identity $\mathbf{xy} \cdot \mathbf{uv} \doteq \mathbf{xu} \cdot \mathbf{yv}$;

6.2 A groupoid is said to be
- *alternative* if it is both left and right alternative;
- *strongly alternative* if it is alternative and elastic;
- *symmetric* if it is both left and right symmetric;
- *semimedial* if it both left and right semimedial;
- *strongly semimedial* if it is semimedial and middle semimedial;
- *distributive* if it is both left and right distributive;
- *semilattice* if it is associative, commutative and idempotent.

6.3 A groupoid G is said to be
- *monoassociative (diassociative)* if every subgroupoid of G generated by at most one (two) elements is associative;
- *monomedial (dimedial, trimedial)* if every subgroupoid of G generated by at most one (two, three) elements is medial;
- *strongly trimedial* if $\langle a, b, c, d\rangle_G$ is a medial subgroupoid of G, whenever $a, b, c, d \in G$ and $ab \cdot cd = ac \cdot bd$.

6.4 Lemma. *A groupoid G is semisymmetric iff it satisfies the identity* $\mathbf{xy} \cdot \mathbf{x} \doteq \mathbf{y}$. *In this case, G is a quasigroup.*

Proof. Let G be semisymmetric. Then $x = (yx)(x \cdot yx) = yx \cdot y$ for all $x, y \in G$. The rest is clear. □

6.5 Lemma. *The following conditions are equivalent for a groupoid G:*
(i) *G is symmetric.*
(ii) *G is left (right) symmetric and semisymmetric.*
(iii) *G is left (right) symmetric and commutative.*
(iv) *G is commutative and semisymmetric.*

Proof. (i) ⇒ (ii). For all $x, y \in G$, $x = (x \cdot xy)(xy) = y \cdot xy$.
(ii) ⇒ (iii). For all $x, y \in G$, $xy = x(x \cdot yx) = yx$.
The remaining implications are similar. □

6.6 Lemma. (i) *Every medial groupoid is strongly trimedial.*
(ii) *Every strongly trimedial groupoid is trimedial.*
(iii) *Every trimedial groupoid is strongly semimedial.*
(iv) *Every commutative groupoid is middle semimedial.*
(v) *An idempotent groupoid is (left, right) semimedial iff it is (left, right) distributive.*
(vi) *Every left (right) modular groupoid is medial.*
(vii) *Every commutative semigroup is medial.*

Proof. (ii) If G is a groupoid and $a, b, c \in G$ then $ab \cdot bc = ab \cdot bc$.
(vi) Let $a, b, c, d \in G$, where G is left modular. Then $ab \cdot cd = d(c \cdot ab) = d(b \cdot ac) = ac \cdot bd$. □

6.7 A semigroup S is said to be *nilpotent of class at most* $n \geq 1$ if it contains an absorbing element 0 and $S^n = 0$ (i.e., $a_1 \dots a_n = 0$ for all $a_1, \dots, a_n \in S$).

6.8 Lemma. (i) *Z-semigroups are just semigroups nilpotent of class at most 2.* (ii) *A-semigroups are just semigroups nilpotent of class at most 3.*

Proof. Easy. $\square$

6.9 For every $n = 1, 2, \dots,$ let us define a left (right) constant groupoid $\mathrm{Cyc}_l(n)$ ($\mathrm{Cyc}_r(n)$) by $\mathrm{Cyc}_l(n) = \{0, 1, \dots, n-1\}$ ($\mathrm{Cyc}_r(n) = \{0, 1, \dots, n-1\}$), $i * j = i + 1$ for $i \neq n - 1$ and $(n-1) * j = 0$ ($i * j = j + 1$ for $j \neq n - 1$ and $i * (n-1) = 0$).

Further, we shall define a left (right) constant groupoid $\mathrm{Cyc}_l(\infty)$ ($\mathrm{Cyc}_r(\infty)$) by $\mathrm{cyc}_l(\infty) = \{0, 1, 2, \dots\}$ ($\mathrm{Cyc}_r(\infty) = \{0, 1, 2, \dots\}$) and $i * j = i + 1$ ($i * j = j + 1$).

6.10 Lemma. *Let G be a simple left constant groupoid. Then just one of the following three cases takes place:*

(i) *There is a prime $p \geq 2$ such that $G \cong \mathrm{Cyc}_l(p)$.*

(ii) *G is a two-element LZ-semigroup.*

(iii) *G is a two-element Z-semigroup.*

Proof. Easy. $\square$

6.11 Lemma. *Let G be a left constant groupoid. Then every cyclic left constant subgroupoid of G is isomorphic to G iff $G \cong \mathrm{Cyc}_l(\alpha)$ for some $1 \leq \alpha \leq \infty$.*

Proof. Easy. $\square$

6.12 Lemma. *Let G, H be left constant groupoids. Then they are isomorphic, provided that G is cyclic, H is a homomorphic image of G and G is a homomorphic image of H.*

Proof. Easy. $\square$

II. General theory of left distributive groupoids

II.1 Basic properties of left distributive groupoids

1.1 Recall that a groupoid is said to be *left* (resp. *right*) *distributive* if it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{xy} \cdot \mathbf{xz}$ (resp. $\mathbf{zy} \cdot \mathbf{x} \doteq \mathbf{zx} \cdot \mathbf{yx}$). A groupoid is said to be *distributive* if it is both left and right distributive. In the sequel, for short, left distributive (right distributive, distributive) groupoids will be also called *LD*-groupoids (*RD*-groupoids, *D*-groupoids). Similarly, idempotent left distributive groupoids will be called *LDI*-groupoids, etc.

1.2 Proposition. *The following conditions are equivalent for a groupoid G:*

(i) *G is left distributive.*

(ii) *Every left translation is an endomorphism of G.*

(iii) $\mathscr{M}_l(G) \subseteq \mathrm{End}(G)$.

Proof. Obvious. $\square$

1.3 Lemma. *Let G be an LD-groupoid.*
(i) *If $a \in \mathrm{Id}(G)$ then $L_a R_a = R_a L_a$.*
(ii) *If $a \in G$ and R_{aa} is injective then $a \in \mathrm{Id}(G)$.*

Proof. (i) $a \cdot xa = ax \cdot aa = ax \cdot a$ for every $x \in G$.
(ii) The equality $a \cdot aa = aa \cdot aa$ implies $a = aa$. □

1.4 Proposition. (i) *Every LD-groupoid satisfies the identity $\mathbf{x} \cdot \mathbf{xx} \doteq \mathbf{xx} \cdot \mathbf{xx}$ (i.e., $r_G = o_G^2$).*
(ii) *Every LDI-groupoid satisfies the identity $\mathbf{x} \cdot \mathbf{yx} \doteq \mathbf{xy} \cdot \mathbf{x}$, i.e., the elasticity.*

Proof. Obvious. □

1.5 Proposition. *Let G be an LD-groupoid. Then:*
(i) $\mathrm{Id}(G)$ *is either empty or a left ideal of G.*
(ii) *If G is right cancellative then G is idempotent.*
(iii) *If G is left-ideal-free then either G is idempotent or* $\mathrm{Id}(G) = \emptyset$.
(iv) *If G is right divisible then either G is idempotent or* $\mathrm{Id}(G) = \emptyset$.

Proof. (i) For $a \in \mathrm{Id}(G)$ and $x \in G$, $xa \cdot xa = x \cdot aa = xa$.
(ii) If follows immediately from 1.3(ii).
(iii) This is a consequence of (i).
(iv) Every right divisible groupoid is left-ideal-free. □

1.6 Proposition. *The following conditions are equivalent for a groupoid G:*
(i) *G is left distributive and left semimedial.*
(ii) *G is left distributive and it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{xx} \cdot \mathbf{yz}$.*
(iii) *G is left semimedial and it satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \doteq \mathbf{xx} \cdot \mathbf{yz}$.*
Moreover, if $G = GG$ then these conditions are equivalent to the following two additional conditions:
(iv) *G is left distributive and it satisfies the identity $\mathbf{xy} \doteq \mathbf{xx} \cdot \mathbf{y}$.*
(v) *G is left semimedial and it satisfies the identity $\mathbf{xy} \doteq \mathbf{xx} \cdot \mathbf{y}$.*

Proof. Easy. □

1.7 Proposition. *An idempotent groupoid is left distributive iff it is left semimedial.*

Proof. Easy. □

1.8 Proposition. *Let G be an LD-groupoid. Then:*
(i) *q_G is a congruence of G.*
(ii) *If G is left cancellative then $q_G = \mathrm{id}_G$ is left cancellative.*
(iii) *If G is right cancellative then q_G is right cancellative.*
(iv) *G/q_G is an idempotent groupoid (i.e., $(x, xx) \in q_G$ for every $x \in G$) iff $GG \subseteq \mathrm{Id}(G)$.*

Proof. (i) We have $q_G = \bigcap \ker(L_x)$, $x \in G$, and all L_x are endomorphisms of G. Hence $\ker(L_x)$ are congruences and their intersection q_G is also a congruence.

(ii) This is clear.

(iii) Let $(ba, ca) \in q_G$ for some $a, b, c \in G$. Then $xb \cdot xa = x \cdot ba = x \cdot ca = cx \cdot xa$, and hence $xb = xc$ for every $x \in G$.

(iv) Clearly, $ax = a \cdot aa$ for all $a, x \in G$ iff $ax = ax \cdot ax$, i.e., iff $ax \in \mathrm{Id}(G)$. $\square$

1.9 Lemma. *Let G be an LD-groupoid.*

(i) *If $a \in G$ is such that L_a is projective then $(a, aa) \in p_G$.*

(ii) *If $a \in G$ is such that L_{aa} is injective then $(a, aa) \in p_G$ iff $aa = aa \cdot a$.*

(iii) *If $(x, xx) \in p_G$ for every $x \in G$ then G is left semimedial and the transformation o_G is an endomorphism of G.*

(iv) *If o_G is injective then $o_G = s_G$ (i.e., $xx = xx \cdot x$ for every $x \in G$).*

Proof. (i) We have $aa \cdot ax = a \cdot ax$ for every $x \in G$ and, since L_a is projective, $aG = G$.

(ii) If $(a, aa) \in p_G$ then obviously $aa = aa \cdot a$. Conversely, if $aa = aa \cdot a$ then $aa \cdot ax = (aa \cdot a)(aa \cdot x) = (aa)(aa \cdot x)$, and so $ax = aa \cdot x$ for every $x \in G$.

(iii) For all $x, y \in G$, $xx \cdot yz = x \cdot yz = xy \cdot xz$.

(iv) First, $o_G(xx) = xx \cdot xx = (xx \cdot x)(xx \cdot x) = o_G(xx \cdot x)$ for every $x \in G$. Since o_G is injective, $o_G(x) = xx = xx \cdot x = s_G(x)$. $\square$

1.10 Proposition. *Let G be an LD-groupoid. Then p_G is a congruence of G, provided that at least one of the following six conditions is satisfied:*

(1) *G is left cancellative and $xx = xx \cdot x$ for every $x \in G$ (i.e., $o_G = s_G$).*

(2) *G is left cancellative and idempotent.*

(3) *G is right regular.*

(4) *G is left divisible.*

(5) *G is medial and $G = GG$.*

(6) *G is right distributive.*

Proof. First, let (1) be satisfied and let $a, b, x, y \in G$, $(a, b) \in p_G$. By 1.9(ii), $xy = xx \cdot y$ and we have $(x \cdot ax)(xa \cdot y) = (xa \cdot xx)(xa \cdot y) = (xa)(xx \cdot y) = xa \cdot xy = x \cdot ay = x \cdot by = xb \cdot xy = (xb)(xx \cdot y) = (xb \cdot xx)(xb \cdot y) = (x \cdot bx)(xb \cdot y) = (x \cdot ax)(xb \cdot y)$. Since G is left cancellative, $xa \cdot y = sb \cdot y$. We have proved that $(xa, xb) \in p_G$.

The condition (2) implies (1). If (6) is satisfied then our assertion is just the dual of 1.8(i). Now, assume that (3) or (4) is satisfied. Let $a, b, x, y \in G$, $(a, b) \in p_G$. Then $xa \cdot xy = x \cdot ay = x \cdot by = xb \cdot xy$. In both cases, we see that $xa \cdot z = xb \cdot z$ for every $z \in G$, i.e., that $(xa, xb) \in p_G$.

Finally, assume that (5) is true. If $a, b, x, y \in G$, (a, b) $\in p_G$ then $xa \cdot yz = xy \cdot az = xy \cdot bz = xb \cdot yz$, and so $(xa, xb) \in p_G$. $\square$

1.11 Proposition. *Let G be an LD-groupoid. Then $(x, xx) \in p_G$ for every $x \in G$ (i.e., G satisfies the identity* $\mathbf{xy} \simeq \mathbf{xx} \cdot \mathbf{y}$*), provided that at least one of the following seven conditions is satisfied:*

(1) *G is left cancellative and $xx = xx \cdot x$ for every $x \in G$.*
(2) *G is idempotent.*
(3) *G is right regular.*
(4) *G is left divisible.*
(5) *G is left semimedial and $G = GG$.*
(6) *o_G is an injective endomorphism of G.*
(7) *o_G is a projective endomorphism of G.*

Proof. If (1) (resp. (3), (4)) is satisfied then the result follows from 1.9(ii) (resp. 1.4(i), 1.9(i)). If (2) is satisfied then the result is trivial, and if (5) is true then $a \cdot xy = ax \cdot ay = aa \cdot xy$ for all $a, x, y \in G$.

Finally, suppose that o_G is an endomorphism of G. Then $ao_G(x) = a \cdot xx = ax \cdot ax = o_G(ax) = o_G(a)\, o_G(x) = aa \cdot x$ for all $a, x \in G$ and the result is clear for o_G projective. If o_G is injective then $o_G(ax) = o_G(a)\, o_G(x) = o_G(a) \cdot xx = o_G(a)\, x \cdot o_G(a)\, x = o_G(o_G(a)\, x)$ implies $ax = o_G(a)\, x = aa \cdot x$. □

1.12 Theorem. *Let G be an LD-groupoid satisfying at least one of the conditions* (1), (2), (3), (4), (5) *from 1.10. Then:*

(i) *p_G is a congruence of G and G/p_G is an LDI-groupoid.*
(ii) *Every block of p_G is a right constant subgroupoid of G.*
(iii) *Every one-generated subgroupoid of G is a right constant groupoid.*
(iv) *G is left semimedial.*
(v) *$o_G = s_G$ and $r_G = o_G^2$ are endomorphisms of G.*

Proof. (i) See 1.10 and 1.11.
(ii) Since G/p_G is idempotent, every block of p_G is a subgroupoid, and hence right constant.
(iii) This is an immediate consequence of (ii).
(iv) We have $xx \cdot yz = x \cdot yz = xy \cdot xz$.
(v) By 1.9(iii), o_G is an endomorphism, and hence $r_G = o_G^2$ is also an endomorphism. Further, $xx = xx \cdot x$, and so $o_G = s_G$. □

1.13 Proposition. *Let G be a right divisible LD-groupoid such that p_G is a congruence of G and G/p_G is idempotent* (see 1.12). *Then there exists $\alpha \in \{1, 2, \dots, \infty\}$ such that every one-generated subgroupoid of G is isomorphic to* $\mathrm{Cyc}_r(\alpha)$.

Proof. Let $a, b \in G$, $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$. There are $c, d \in G$ with $ca = b$ and $db = a$. Then $L_c(A) = B$ and $L_d(B) = A$. According to our assumptions, both A and B are right constant and the rest is clear from I.6.11, I.6.12. □

1.14 Proposition. *Let G be an LD-groupoid.*

(i) *If G is left cancellative then p_G is left cancellative.*
(ii) *If G is right cancellative then $p_G = \mathrm{id}_G$ is right cancellative.*

Proof. (i) Let $a, b, c, x \in G$ and $(ca, cb) \in p_G$. Then $c \cdot ax = ca \cdot cx = cb \cdot cx = c \cdot bx$ and $ax = bx$.
(ii) Obvious. □

1.15 Lemma. *Let G be an LD-groupoid. Then:*

(i) $o_G(G) \subseteq \mathrm{Id}(G)$ *iff G satisfies the identity* $\mathbf{xx} \doteq \mathbf{x} \cdot \mathbf{xx}$ *(i.e., iff* $o_G = r_G = o_G^2$*).*

(ii) $r_G(G) \subseteq \mathrm{Id}(G)$ *iff G satisfies the identity* $\mathbf{x} \cdot \mathbf{xx} \doteq \mathbf{x}(\mathbf{x} \cdot \mathbf{xx})$ *(i.e., iff* $o_G = o_G^3$*).*

(iii) $s_G(G) \subseteq \mathrm{Id}(G)$ *iff G satisfies the identity* $\mathbf{x} \cdot \mathbf{xx} \doteq \mathbf{xx} \cdot \mathbf{x}$ *(i.e., iff* $r_G = o_G^2 = s_G$*).*

Proof. We have $x \cdot xx = xx \cdot xx$, $x(x \cdot xx) = x(xx \cdot xx) = (x \cdot xx)(x \cdot xx)$ and $x \cdot xx = xx \cdot xx = (xx \cdot x)(xx \cdot x)$. □

1.16 Lemma. *Let G be an LD-groupoid. Then:*

(i) *For all* $f, g \in \mathscr{M}_l(G)$ *there exists* $h \in \mathscr{M}_l(G)$ *such that* $fg = hf$.

(ii) $\mathscr{M}_l(G)$ *and* $\mathscr{M}_l^1(G)$ *are left uniform.*

Proof. (i) There are $n \geq 1$ and $a_1, \ldots, a_n \in G$ with $g = L_{a_1} \ldots L_{a_n}$. Since f is an endomorphism of G, we can put $h = L_{f(a_1)} \ldots L_{f(a_n)}$.

(ii) This follows immediately from (i). □

1.17 Lemma. *Let G be an LD-groupoid. Define a relation r on G by* $(a, b) \in r$ *iff* $f(a) = f(b)$ *for some* $f \in \mathscr{M}_l(G)$. *Then r is the smallest left cancellative congruence of G. Moreover:*

(i) *If* $(u, uu) \in r$ *for some* $u \in G$ *then* $\mathrm{Id}(g) \neq \emptyset$.

(ii) *If* $(vv, vv \cdot v) \in r$ *for some* $v \in G$ *then* $zz = zz \cdot z$ *for at least one* $z \in G$.

Proof. Clearly, r is reflexive, symmetric and left cancellative. Further, from 1.16(i) it follows easily that r is transitive and the inclusion $\mathscr{M}_l(G) \subseteq \mathrm{End}(G)$ implies the fact that r is stable. Thus r is a left cancellative congruence of G.

Now, let s be a left cancellative and reflexive relation on G, let $f \in \mathscr{M}_l(G)$, $a, b \in G$ and $f(a) = f(b)$. We have $f = L_{a_1} \ldots L_{a_n}$, and so $a_1(\ldots(a_n a)) = a_1(\ldots(a_n b))$, which implies $(a, b) \in s$. We have proved that $r \subseteq s$.

Finally, if $(u, uu) \in r$ $((vv, vv \cdot v) \in r)$ then $f(u) = f(u) f(u)$ $(f(v) f(v) = (f(v) f(v)) f(v))$ for some $f \in \mathscr{M}_l(G)$. □

1.18 Theorem. *Let G be an LD-groupoid,* $A = \{a \in G \mid aa = aa \cdot a\}$ *and* $B = G - A$. *Then:*

(i) $G = A \cup B$ *and* $A \cap B = \emptyset$.

(ii) *A is either empty or a left ideal.*

(iii) *If G is left cancellative then B is either empty or a left ideal.*

(iv) *If G is left cancellative then* $s = (A \times A) \cup (B \times B)$ *is a left cancellative congruence of G and either* $s = G \times G$ *or* G/s *is a two-element RZ-semigroup.*

(v) *If* $a, b \in G$ *and* $ab = a$ *then* $a \in A$.

(vi) *If G is finite then* $A \neq \emptyset$.

(vii) *If G is finite and left-ideal-free then* $A = G$.

Proof. The assertions (i), (ii), (iii), (iv) are easy and (vii) follows from (vi).

(v) We have $aa = a \cdot ab = aa \cdot ab = aa \cdot a$.

(vi) Consider the left cancellative congruence r defined in 1.17 and put $H = G/r$. Then H is a left cancellative finite groupoid, and hence it is a left quasigroup. By 1.11, $xx = xx \cdot x$ for every $x \in H$. This means that $(vv, vv \cdot v) \in r$ for every $v \in G$ and we can use 1.17(ii). □

1.19 Lemma. *Let G be an L D-groupoid. Then:*

(i) *$(a, b) \in z_{l,G}$ iff $a = f(b)$ for some $f \in \mathcal{M}_l(G)$ (i.e., iff $a = a_1(\ldots(a_n b))$ for some $n \geq 1$ and $a_1, \ldots, a_n \in G$).*

(ii) *$z_{l,G}$ is transitive and left stable.*

(iii) *$(a, b) \in z^1_{l,G}$ iff $a = f(b)$ for some $f \in \mathcal{M}^1_L(G)$.*

(iv) *$z^1_{l,G}$ is a left stable quasiordering.*

(v) *If $z_{l,G}$ is irreflexive then $z^1_{l,G}$ is a left stable ordering.*

(vi) *If G is idempotent then $z_{l,G} = z^1_{l,G}$.*

Proof. Obvious (see I.3.16). □

1.20 Lemma. *Let G be an LD-groupoid. Then:*

(i) *$u_G = \ker(z^1_{l,G})$ is a left stable equivalence.*

(ii) *$(a, b) \in u_G$ iff $a = f(b)$ and $b = g(a)$ for some $f, g \in \mathcal{M}^1_l(G)$.*

(iii) *If G is idempotent then $(a, b) \in u_G$ iff $a = f(b)$ and $b = g(a)$ for some $f, g \in \mathcal{M}_l(G)$.*

(iv) *If G is idempotent then every block of u_G is a subgroupoid of G.*

Proof. Obvious (see 1.19 and I.2.26). □

1.21 Lemma. *Let G be an LD-groupoid. Then:*

(i) *$(a, b) \in u^c_G$ iff $f(a) = g(b)$ for some $f, g \in \mathcal{M}_l(G)$.*

(ii) *u^c_G is a congruence of G, G/u^c_G is an RZ-semigroup and every block of u^c_G is a left ideal.*

(iii) *$u_G \subseteq u^c_G$ and u^c_G is the smallest congruence of G such that the corresponding factor is an RZ-semigroup.*

Proof. (i) If $f(a) = g(b)$ then $(a, b) \in u^c_G$ follows easily from the definition of u^c_G. Now, let $(a, b) \in u^c_G$ and let I be the set of $x \in G$ such that $h(a) = k(x)$ for some $h, k \in \mathcal{M}_l(G)$. Then $a \in I$ and, for every $y \in G$, $k(yx) = k(y)\,k(x) = k(y)\,h(a) = l(a)$, $l = L_{k(y)}h$, and so $yx \in I$ and we have proved that I is a left ideal. On the other hand, if $x = yz$ then $h(a) = j(z)$, $j = kL_y$, and we see that I is left strongly prime. Since $(a, b) \in u^c_G$ and $a \in I$, we must have $b \in I$.

(ii) Clearly, u^c_G is an equivalence and it follows easily from (i) and the left distributivity, that u^c_G is left stable.

Let $a, b \in G$. Then $a \cdot ab = aa \cdot ab$, $L^2_a(b) = L_{aa}(ab)$, and therefore $(ab, b) \in u^c_G$. This implies that $(yx, zx) \in u^c_G$ for all $x, y, z \in G$, and hence u^c_G is right stable, thus being a congruence of G. The rest is clear.

(iii) It follows from 1.20(ii) that $u_G \subseteq u^c_G$ and the rest is clear. □

1.22 Lemma. *Let G be an LD-groupoid and a left quasigroup. Then:*

(i) *$(a, b) \in u^c_G$ iff $b = f(a)$ for some $f \in \mathcal{M}^*_l(G)$.*

(ii) *If the order of* L_a *in the permutation group* $\mathcal{M}_l^*(G)$ *is finite for every* $a \in G$ *(e.g., if* G *is finite), then* $u_G^c = u_G$.

Proof. Easy (use 1.21). □

1.23 Lemma. *Let* G *be an LD-groupoid. Then:*

(i) $(a, b) \in z_{r,G}$ *iff* $a = f(b)$ *for some* $f \in \mathcal{M}_r(G)$ *(i.e., iff* $a = ((ba_1)\dots)a_n$ *for some* $n \geq 1$ *and* $a_1, \dots, a_n \in G$.

(ii) $z_{r,G}$ *is transitive and left stable.*

(iii) $(a, b) \in z_{r,G}^1$ *iff* $a = f(b)$ *for some* $f \in \mathcal{M}_r^1(G)$.

(iv) $z_{r,G}^1$ *is a left stable quasiordering.*

(v) *If* $z_{r,G}$ *is irreflexive then* $z_{r,G}^1$ *is a left stable ordering.*

(vi) *If* G *is idempotent then* $z_{r,G} = z_{r,G}^1$.

Proof. Obvious. (see I.3.16). □

1.24 Lemma. *Let* G *be an LD-groupoid. Then:*

(i) $v_G = \ker(z_{r,G}^1)$ *is a left stable equivalence.*

(ii) $(a, b) \in v_G$ *iff* $a = f(b)$ *and* $b = g(a)$ *for some* $f, g \in \mathcal{M}_r^1(G)$.

(iii) *If* G *is idempotent then* $(a, b) \in v_G$ *iff* $a = f(b)$ *and* $b = g(a)$ *for some* $f, g \in \mathcal{M}_r(G)$.

(iv) *If* G *is idempotent then every block of* v_G *is a subgroupoid of* G.

Proof. Obvious. □

1.25 Lemma. *Let* G *be an LD-groupoid and* $f \in \mathcal{M}(G)$. *Then there are* $g \in \mathcal{M}_l^1(G)$ *and* $h \in \mathcal{M}_r^1(G)$ *such that* $f = hg$ *and either* $g \in \mathcal{M}_l(G)$ *or* $h \in \mathcal{M}_r(G)$.

Proof. We have $a \cdot xb = ax \cdot ab$ for all $a, b, x \in G$, and hence $L_aR_b = R_{ab}L_a$. The rest is clear. □

1.26 Lemma. *Let* G *be an LD-groupoid. Then* $\mathcal{M}(G) = \mathcal{M}_l(G) \cup \mathcal{M}_r(G) \cup \mathcal{M}_r(G)\,\mathcal{M}_l(G)$ *and* $\mathcal{M}^1(G) = \mathcal{M}_r^1(G) \cup \mathcal{M}_l^1(G)$.

Proof. This follows immediately from 1.25. □

1.27 Lemma. *Let* G *be an LD-groupoid. Then:*

(i) $(a, b) \in z_G$ *iff* $a = hg(b)$, *where* $h \in \mathcal{M}_r^1(G)$, $g \in \mathcal{M}_l^1(G)$ *and either* $h \in \mathcal{M}_r(G)$ *or* $g \in \mathcal{M}_l(G)$.

(ii) $(a, b) \in z_G$ *iff there are* $n \geq 0$, $m \geq 0$, $a_1, \dots, a_n, b_1, \dots, b_m \in G$ *such that* $n + m \geq 1$ *and* $a = (((a_1(\dots(a_nb)))\,b_1)\dots)\,b_m$.

(iii) z_G *is transitive and left stable.*

(iv) $(a, b) \in z_G^1$ *iff* $a = hg(b)$ *for some* $h \in \mathcal{M}_r^1(G)$ *and* $g \in \mathcal{M}_l^1(G)$.

(v) z_G^1 *is a left stable quasiordering.*

(vi) *If* z_G *is irreflexive then* z_G^1 *is a left stable ordering.*

(vii) *If* G *is idempotent then* $z_G = z_G^1$.

Proof. Easy (see 1.25, 1.26 and I.3.18). □

1.28 Lemma. *Let G be an LD-groupoid. Then:*

(i) $w_G = \ker(z_G^1)$ *is a left stable equivalence.*

(ii) $(a, b) \in w_G$ *iff* $a = h_1 g_1(b)$ *and* $b = h_2 g_2(a)$ *for some* $h_1, h_2 \in \mathcal{M}_r^1(G)$ *and* $g_1, g_2 \in \mathcal{M}_l^1(G)$.

(iii) *If G is idempotent then* $(a, b) \in w_G$ *iff* $a = h_1 g_1(B)$ *and* $b = h_2 g_2(a)$ *for some* $h_1, h_2 \in \mathcal{M}_r(G)$ *and* $g_1, g_2 \in \mathcal{M}_l(G)$.

(iv) *If G is idempotent then every block of w_G is a subgroupoid.*

Proof. Obvious. □

1.29 Proposition. *Let G be an LD-groupoid.*

(i) *If G possesses a right neutral element then G is an idempotent groupoid satisfying the identity* $\mathbf{xy} \simeq \mathbf{xy} \cdot \mathbf{x}$.

(ii) *If G possesses a neutral element then G is an idempotent semigroup satisfying the identity* $\mathbf{xy} \simeq \mathbf{xyx}$.

Proof. (i) Let $e \in G$ be right neutral. Then $x = xe = x \cdot ee = xe \cdot xe = xx$ and $xy = x \cdot ye = xy \cdot xe = xy \cdot x$ for all $x, y \in G$.

(ii) Let $e \in G$ be neutral. Then $xy = x \cdot ey = xe \cdot xy = x \cdot xy$ and $x \cdot yz = xy \cdot xz = (xy \cdot x)(xy \cdot z) = (xy)(xy \cdot z) = xy \cdot z$ for all $x, y, z \in G$. □

1.30 Lemma. *Every right permutable LD-groupoid is medial.*

Proof. We have $xa \cdot by = (x \cdot by)\, a = (xb \cdot xy)\, a = ((x \cdot xy)\, b)\, a = ((x \cdot xy)\, a)\, b = (xa \cdot xy)\, b = (x \cdot ay)\, b = xb \cdot ay$ for all $a, b, x, y \in G$. □

1.31 Example. Consider the following two-element groupoid G ($= \mathrm{Cyc}_r(2)$):

G	0	1
0	1	0
1	1	0

Then G is an LD-groupoid, G is not right distributive and $\mathrm{Id}(G) = \emptyset$.

1.32 Example. Consider the following two-element groupoid G:

G	0	1
0	0	1
1	0	0

Then G is an LD-groupoid, $p_G = \mathrm{id}_G$ (and hence G/p_G is not idempotent) and $\mathrm{Id}(G) = \{0\}$ is an ideal of G.

1.33 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	1	2
1	0	1	1
2	0	1	2

Then G is an LDI-groupoid and $p_G = \mathrm{id}_G \cup \{(0,2),(2,0)\}$ is not a congruence of G. On the other hand, G is idempotent and hence $(x, xx) \in p_G$ for every $x \in G$ and $o_G = \mathrm{id}_G$ is an automorphism of G. Furthermore, G is left and middle semimedial but G is not right semimedial.

1.34 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	0	1	2
2	0	1	1

Then G is an LD-groupoid and $p_G = \mathrm{id}_G$. On the other hand, G is not idempotent and not left semimedial.

1.35 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	0	0	1
2	0	0	0

Then G is a medial LD-groupoid but p_G is not a congruence of G and $(0, 1) \notin p_G$, $1 = 0 \cdot 0$. Moreover, o_G is an endomorphism of G.

1.36 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	0	1	2
2	1	1	1

Then G is an LD-groupoid and $p_G = \mathrm{id}_G$ is a congruence of G. On the other hand, o_G is not an endomorphism of G.

1.37 Lemma. *Let G be an LD-groupoid.*

(i) *If $a \in G$ is left constant then $aa \in \mathrm{Id}(G)$.*

(ii) *The set of right constant elements is either empty or a left ideal of G.*

(iii) *If $a \in G$ is constant then aa is right absorbing.*

Proof. (i) We have $aa \cdot aa = a \cdot aa = aa$.

(ii) If $a \in G$ is right constant then $y \cdot xa = y \cdot aa = ya \cdot ya = aa \cdot aa = a \cdot aa$ for all $x, y \in G$, and hence $xa = aa$ is also right constant.

(iii) By (i) and (ii), $aa \in \mathrm{Id}(G)$ and aa is right constant. Hence $x \cdot aa = aa \cdot aa = aa$. □

1.38 Lemma. *Let G be an LD-groupoid and let $a, b \in G$ be right constant elements such that $aa = bb$. Then:*
(i) *$ax = bx$ for every $x \in GG$.*
(ii) *If $G = GG$ then $(a, b) \in t_G$.*

Proof. (i) We have $a \cdot uv = au \cdot av = (au \cdot a)(au \cdot v) = (aa)(au \cdot v) = (aa \cdot au)(aa \cdot v) = ((aa \cdot a)(aa \cdot u))(aa \cdot v) = ((aa)(aa \cdot u))(aa \cdot v) = ((bb)(bb \cdot u))(bb \cdot v) = b \cdot uv$ for all $u, v \in G$.
(ii) By (i), $(a, b) \in p_G$. On the other hand, $xa = aa = bb = xb$ for every $x \in G$, and so $(a, b) \in q_G$. Thus $(a, b) \in p_G \cap q_G = t_G$. □

II.2 Ideals of left distributive groupoids

2.1 Lemma. *Let I, J, K be left ideals of an LD-groupoid G. Then:*
(i) *IJ is a left ideal and $IJ \subseteq J$.*
(ii) *$I \cdot JK = IJ \cdot IK$.*
(iii) *$I(J \cup K) = IJ \cup IK$ and $(J \cup K)I = JI \cup KI$.*
(iv) *If $J \subseteq K$ then $IJ \subseteq IK$ and $JI \subseteq KI$.*

Proof. (i) If $a \in I$, $b \in J$ and $x \in G$ then $x \cdot ab = xa \cdot xb \in IJ$.
(ii) If $a \in I$, $b \in J$ and $c \in K$ then $a \cdot bc = ab \cdot ac$, and hence $I \cdot JK \subseteq IJ \cdot IK$. Conversely, if $a, b \in I$, $c \in J$ and $d \in K$ then $ac \cdot bd = (ac \cdot b)(ac \cdot d)$, $ac \cdot b \in I$, $ac \in J$ and $ac \cdot bd \in I \cdot JK$.
(iii) and (iv) This is obvious. □

2.2 Lemma. *Let G be an LD-groupoid such that $G = G^2$.*
(i) *If I is a right ideal and J is an ideal of G then IJ is a right ideal and $IJ \subseteq I \cap J$.*
(ii) *If I, J are ideals of G then IJ is an ideal and $IJ \subseteq I \cap J$.*

Proof. (i) If $a \in I$, $b \in J$ and $x \in G$ then $x = yz$ for some $y, z \in G$ and $ab \cdot x = ab \cdot yz = (ab \cdot y)(ab \cdot z)$. Of course, $ab \cdot y \in I$ and $ab \cdot z \in J$.
(ii) This follows from (i) and 2.1(i). □

2.3 Let G be a groupoid and let $\mathfrak{P}(G)$ denote the set of all subsets of G. Then we have a binary operation defined on $\mathfrak{P}(G)$, namely $AB = \{ab \mid a \in A, b \in B\}$ for all $A, B \in \mathfrak{P}(G)$. In this way, $\mathfrak{P}(G)$ becomes a groupoid. Clearly, $\emptyset$ is an absorbing element of $\mathfrak{P}(G)$ and $\{\emptyset\}$ is a prime ideal of $\mathfrak{P}(G)$. Further, we denote by $\mathscr{R}(G)$ the subgroupoid of $\mathfrak{P}(G)$ generated by G. Then $\mathscr{R}(G)$ is a trivial groupoid iff $G^2 = G$.

2.4 Let G an LD-groupoid. Then the set $\mathscr{I}_l(G)$ of left ideals of G is a subgroupoid of $\mathfrak{P}(G)$ and $\mathscr{I}_l(G)$ is again an LD-groupoid (see 2.1(i), (ii)). Since $G \in \mathscr{I}_l(G)$, $\mathscr{R}(G)$ is a subgroupoid of $\mathscr{I}_l(G)$; in particular, $\mathscr{R}(G)$ is also an LD-groupoid. If G is idempotent then both $\mathscr{I}_l(G)$ and $\mathscr{R}(G)$ are idempotent.

2.5 Let G be an LD-groupoid such that $G = G^2$. By 2.2(ii), $\mathscr{I}(G)$ is a subgroupoid of $\mathscr{I}_l(G)$. Again, since $G \in \mathscr{I}(G)$, we have $\mathscr{R}(G) \subseteq \mathscr{I}(G)$. Further, if

$I, J, K, L \in \mathscr{I}(G)$ and $a \in I$, $b \in J$, $c \in K$, $d \in L$ then $ab \cdot cd = (ab \cdot c)(ab \cdot d) \in IK \cdot JL$, and so $IJ \cdot KL \subseteq IK \cdot JL$. Similarly the converse and we have proved that $\mathscr{I}(G)$ is a medial groupoid.

2.6 Let G be an LDI-groupoid. If I, J, $K \in \mathscr{I}(G)$ and $a \in I$, $b \in J$, $c \in K$ then $a \cdot bc = ab \cdot ac \in IJ \cdot K$ and $ab \cdot c = ab \cdot cc = (ab \cdot c)(ac \cdot c) \in I \cdot JK$. This shows that $\mathscr{I}(G)$ is an idempotent semigroup. By 2.5, $\mathscr{I}(G)$ is medial, and so $\mathscr{I}(G)$ is a D-groupoid. Moreover, for $a \in I$, $b \in J$, $ab = ab \cdot ab = (ab \cdot a)(ab \cdot b) \in JI$. Thus $IJ = JI$ and $\mathscr{I}(G)$ is a semilattice.

2.7 Let G be a groupoid. Then we put $G^{\langle 1 \rangle} = G$ and $G^{\langle n+1 \rangle} = G \cdot G^{\langle n \rangle}$ for every $n \geq 1$. Let $\mathscr{Q}(G) = \{G^{\langle n \rangle} \mid n \geq 1\} \subseteq \mathscr{R}(G)$.

2.8 Lemma. *Let G be an LD-groupoid and $A \in \mathscr{R}(G)$. Then:*
(i) $GA \subseteq A$.
(ii) *If $A \neq G$ and $n \geq 1$ then $G^{\langle n \rangle} \cdot A = GA$.*
(iii) *There exists $m \geq 1$ such that $G^{\langle m \rangle} \subseteq A$.*

Proof. (i) A is a left ideal (see 2.4).
(ii) Let F be an absolutely free groupoid with a one-element free basis $\{x\}$ and let f denote the uniquely determined homomorphism of F onto $\mathscr{R}(G)$ such that $f(x) = G$. Since $A \neq G$, we have $G \neq G^2$ and $A = f(r)$ for some $r \in F$, $l(r) \geq 2$ ($l(r)$ means the length of r). Now, we shall proceed by induction on $l(r) + n$.

First, let $l(r) = 2$. Then $A = G^2$ and $G^{\langle 3 \rangle} = G^{\langle n \rangle} \cdot G^2 = (G^{\langle n \rangle}G)(G^{\langle n \rangle}G) = ((G^{\langle n \rangle}G)((G^{\langle n \rangle}G)G) \subseteq G^{\langle n+1 \rangle} \cdot G^2 = G^{\langle 3 \rangle}$.

Next, let $r = sx$, $l(s) \geq 2$, $B = f(s)$. Then $GA = G^{\langle n \rangle} \cdot BG = (G^{\langle n \rangle} \cdot B)(G^{\langle n \rangle} \cdot G) = ((G^{\langle n \rangle} \cdot B)(G^{\langle n \rangle}) \cdot ((G^{\langle n \rangle} \cdot B)G) \subseteq G^{\langle n+1 \rangle} \cdot BG = G^{\langle n+1 \rangle} \cdot A$, and so $GA = G^{\langle n+1 \rangle} \cdot A$. Similarly, if $r = xs$, $l(s) \geq 2$ then $GA = G^{\langle n \rangle} \cdot GB = (G^{\langle n \rangle} \cdot G)(G^{\langle n \rangle} \cdot B) = ((G^{\langle n \rangle} \cdot G)G^{\langle n \rangle})((G^{\langle n \rangle} \cdot G)B) \subseteq G^{\langle n+1 \rangle} \cdot A$.

Finally, let $r = st$, $l(s) \geq 2$, $l(t) \geq 2$, $B = f(s)$, $C = f(t)$. Then $G^{\langle n \rangle} \cdot A = (G^{\langle n \rangle} \cdot B)(G^{\langle n \rangle} \cdot C) = GB \cdot GC = G \cdot BC = GA$.
(ii) We can assume that $A = BC$ and that $G^{\langle n \rangle} \subseteq B \cap C$ for some $n \geq 2$. Then $G^{\langle n \rangle} \cdot G^{\langle n \rangle} \subseteq A$. However, by (ii), $G^{\langle n \rangle} \cdot G^{\langle n \rangle} = G^{\langle n+1 \rangle}$. $\square$

2.9 Let G be a groupoid and $n \geq 1$. Then we put $G^{\langle n,0 \rangle} = G^{\langle n \rangle}$ and $G^{\langle n,m+1 \rangle} = G^{\langle n,m \rangle} \cdot G$ for every $m \geq 0$.

2.10 Lemma. *Let G be an LD-groupoid. Then $G^{\langle n,m \rangle} \cdot G^{\langle k \rangle} = G^{\langle k+1 \rangle}$ for all $n \geq 1$, $m \geq 0$ and $k \geq 2$.*

Proof. If $G = G^2$ then the result is clear. Hence, assume that $G \neq G^2$. Now, for $m = 0$, our equality follows from 2.8(ii).

Let $k = 2$. We shall proceed by induction on m. We have $G^{\langle 3 \rangle} = G^{\langle n,m \rangle} \cdot G^2 = (G^{\langle n,m \rangle} \cdot G)(G^{\langle n,m \rangle} \cdot G) \subseteq G^{\langle n,m+1 \rangle} \cdot G^2 \subseteq G^{\langle 3 \rangle}$, and so $G^{\langle 3 \rangle} = G^{\langle n,m+1 \rangle} \cdot G^2$.

Let $k \geq 3$. Again, we shall proceed by induction on m. We have $G^{\langle k+1 \rangle} = G^{\langle n,m \rangle} \cdot G^{\langle k \rangle} = G^{\langle n,m \rangle} \cdot (G \cdot G^{\langle k-1 \rangle}) = (G^{\langle n,m \rangle} \cdot G)(G^{\langle n,m \rangle} \cdot G^{\langle k-1 \rangle}) = G^{\langle n,m+1 \rangle} \cdot G^{\langle k \rangle}$. $\square$

2.11 Lemma. *Let G be an LD-groupoid, $n \geq 1, m \geq 1$. Then $G \cdot G^{\langle n,m \rangle} = G^{\langle 3 \rangle}$.*

Proof. Again, we assume that $G \neq G^2$. We shall proceed by induction on m. Now, $G \cdot G^{\langle n,m \rangle} = (G \cdot G^{\langle n,m-1 \rangle}) \cdot G^2$. If $m \geq 2$ then $G \cdot G^{\langle n,m-1 \rangle} = G^{\langle 3 \rangle}$ by induction and $G^{\langle 3 \rangle} \cdot G^2 = G^{\langle 3 \rangle}$ by 2.10. If $m = 1$ then $G \cdot G^{\langle n,m-1 \rangle} = G^{\langle n+1 \rangle}$ and $G^{\langle n+1 \rangle} \cdot G^2 = G^{\langle 3 \rangle}$ again by 2.10. $\square$

2.12 Lemma. *Let G be an LD-groupoid, $n \geq 1, m \geq 0, k \geq 1$ and $l \geq 1$. Then $G^{\langle n,m \rangle} \cdot G^{\langle k,l \rangle} = G^{\langle 3 \rangle}$.*

Proof. Let $G \neq G^2$. If $k = 1 = l$ then the result follows from 2.10. If $k \geq 2$ and $l = 1$ then $G^{\langle n,m \rangle} \cdot G^{\langle k,1 \rangle} = (G^{\langle n,m \rangle} \cdot G^{\langle k \rangle}) \cdot (G^{\langle n,m \rangle} \cdot G) = G^{\langle k+1 \rangle} \cdot G^{\langle n,m+1 \rangle} = G \cdot G^{\langle n,m+1 \rangle} = G^{\langle 3 \rangle}$ by 2.10, 2.8(ii) and 2.11.

Now, let $l \geq 2$. We shall proceed by induction on l. We have $G^{\langle n,m \rangle} \cdot G^{\langle k,l \rangle} = (G^{\langle n,m \rangle} G^{\langle k,l-1 \rangle})(G^{\langle n,m \rangle} \cdot G) = G^{\langle 3 \rangle} G^{\langle n,m+1 \rangle} = G \cdot G^{\langle n,m+1 \rangle} = G^{\langle 3 \rangle}$ by induction, 2.8(ii) and 2.11. $\square$

2.13. Proposition. *Let G be an LD-groupoid. Then:*
(i) $G^{\langle n,m \rangle} \cdot G^{\langle k,l \rangle} = G^{\langle 3 \rangle}$ *for all* $n \geq 1, m \geq 1, k \geq 1, l \geq 1$.
(ii) $G^{\langle n,m \rangle} \cdot G^{\langle k,0 \rangle} = G^{\langle k+1,0 \rangle}$ *for all* $n \geq 1, m \geq 0, k \geq 2$.
(iii) $G^{\langle n,m \rangle} \cdot G^{\langle 1,0 \rangle} = G^{\langle n,m+1 \rangle}$ *for all* $n \geq 1, m \geq 0$.

Proof. See 2.10 and 2.12. $\square$

2.14 Corollary. *Let G be an LD-groupoid. Then:*
(i) $\mathcal{R}(G) = \{G^{\langle n,m \rangle} \mid n \geq 1, m \geq 0\}$.
(ii) *If* $G \neq G^2$ *then* $\mathcal{Q}(G) - \{G\} = \{G^{\langle k \rangle} \mid k \geq 2\}$ *is a left ideal of* $\mathcal{R}(G)$. $\square$

2.15 Construction. Denote by D_0 the set of all ordered pairs (n, m), where n, m are integers, $n \geq 1$, $n \neq 2$ and $m \geq 0$. We shall define a multiplication on D_0 as follows: $(n,m)(k,l) = (3,0)$ if $l \geq 1$; $(n,m)(k,0) = (k+1,0)$ if $k \geq 3$; $(n,m)(1,0) = (n, m+1)$. Now, D_0 becomes a groupoid and it is easy to check that D_0 is an LD-groupoid. Namely, for $u = (n, m)$, $v = (k, l)$ and $z = (p, q)$ from D_0, we have $u \cdot vz = uv \cdot uz = (4, 0)$ if $q \geq 1$, $u \cdot vz = uv \cdot uz = (p+2, 0)$ if $q = 0$, $p \geq 3$, and $u \cdot vz = uv \cdot uz = (3, 0)$ if $q = 0$, $p = 1$. Proceeding similarly, we can show that D_0 is medial and $uv \cdot z \neq uz \cdot vz$ for all $u, v, z \in D_0$. In particular, D_0 is not right distributive. Furthermore, $\mathrm{Id}(D_0) = \emptyset$, $p_{D_0} = \mathrm{id}_{D_0}$, D_0/q_{D_0} is a right constant groupoid and $((n, m), (k, l)) \in q_{D_0}$ iff either $(n, m) = (k, l)$ or $m \geq 1, l \geq 1$ (D_0/q_{D_0} is isomorphic to the right constant groupoid $*$ defined on the set of positive integers by $i * j = j + 1$: $(n, m) \to 2$ if $m \geq 1$ and $(n, 0) \to n$, and so $D_0/q \cong \mathrm{Cyc}_r(\infty)$).

Define a relation $\leq_0$ on D_0 by $(n, m) \leq_0 (k, l)$ iff at least one of the following four cases takes place: $k \leq n, m = l$; $3 \leq m < l$; $3 \leq n, k = 1$; $k = 1, 0 \leq l < m$. It is easy to check that $\leq_0$ is a linear ordering of D_0 and that $\leq_0$ is stable (with respect to the operation of the groupoid D_0).

Finally, notice that the groupoid D_0 is generated by the element $(1, 0)$, and hence D_0 is cyclic and $\sigma(D_0) = 1$.

2.16 Theorem. *Let G be an LD-groupoid. Define a mapping $f : D_0 \to \mathscr{R}(G)$ by $f(n,m) = G^{\langle n,m\rangle}$. Then:*

(i) *f is a projective homomorphism of the groupoid D_0 onto the groupoid $\mathscr{R}(G)$.*

(ii) *If $(n,m),(k,l) \in D_0$ and $(n,m) \leq_0 (k,l)$ then $f(n,m) = G^{\langle n,m\rangle} \subseteq G^{\langle k,l\rangle} = f(k,l)$.*

Proof. (i) This follows from 2.13, the definition of the operation of D_0 and the fact that $f(1,0) = G$.

(ii) First, let $k \leq n, m = l$. We have $G^{\langle n\rangle} = G(\ldots(G \cdot G^{\langle k\rangle}))$, where G appears $(n-k)$-times, and hence $G^{\langle n\rangle} \subseteq G^{\langle k\rangle}$, since $G^{\langle k\rangle}$ is a left ideal. This also implies $G^{\langle n,m\rangle} \subseteq G^{\langle k,l\rangle}$.

Next, let $3 \leq n$ and $0 \leq m < l$. If $m = 0$ then $G^{\langle n,0\rangle} \subseteq G^{\langle 3\rangle} = G \cdot G^{\langle k,l\rangle} \subseteq G^{\langle k,l\rangle}$. If $m \geq 1$ then $G^{\langle n,0\rangle} \subseteq G^{\langle k,l-m\rangle}$, and therefore $G^{\langle n,m\rangle} = ((G^{\langle n,0\rangle} \cdot G)\ldots)\, G \subseteq ((G^{\langle k,l-m\rangle} \cdot G)\ldots)\, G = G^{\langle k,l\rangle}$.

Now, let $3 \leq n$ and $k = 1$. With respect to the preceding case, we can assume that $l \leq m$. Now $G^{\langle n,m\rangle} = ((G^{\langle n,m-l\rangle} \cdot G)\ldots)\, G \subseteq (GG)\ldots)\, G = G^{\langle 1,l\rangle}$. Finally, if $k = 1$ and $0 \leq l < m$ then we can proceed similarly. □

2.17 Corollary. *Let G be an LD-groupoid. Then $\mathscr{R}(G)$ is a medial LD-groupoid which is linearly ordered by inclusion (this ordering is stable).* □

2.18 Example. Consider the following three-element groupoid G:

G	0	1	2
0	1	2	2
1	1	2	2
2	1	2	2

Then G is an LD-groupoid, it is right constant and $\mathscr{R}(G) = \mathscr{I}_l(G) = \{G^{\langle 1\rangle}, G^{\langle 2\rangle}, G^{\langle 3\rangle}\}$; we have $G^{\langle 2\rangle} = \{1,2\}, G^{\langle 3\rangle} = \{2\}$ and $G^{\langle 3\rangle} = G^{\langle 1\rangle} \cdot G^{\langle 2\rangle}$ is not a right ideal. Moreover, the groupoids G and $\mathscr{R}(G)$ are isomorphic.

2.19 Example. Consider the following four-element groupoid G:

G	0	1	2	3
0	0	0	0	0
1	0	0	3	0
2	0	0	1	0
3	0	0	3	0

Then G is an LD-groupoid, $\mathscr{R}(G) = \{G^{\langle 1,0\rangle}, G^{\langle 1,1\rangle}, G^{\langle 1,2\rangle}, G^{\langle 3,0\rangle}\}$, $G^{\langle 1,1\rangle} = \{0,1,3\}$, $G^{\langle 1,2\rangle} = \{0,3\}$, $G^{\langle 3,0\rangle} = \{0\}$, every element of $\mathscr{R}(G)$ is an ideal, $\mathscr{R}(G) = \mathscr{I}(G) = \mathscr{I}_r(G) \neq \mathscr{I}_l(G) = \mathscr{R}(G) \cup \{A\}$, where $A = \{0,1\}$ is a left ideal but not a right ideal ($\mathscr{I}_l(G)$ is not linearly ordered by inclusion).

2.20 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

Then G is an LD-groupoid and G is commutative (in fact, G is a semigroup), $\mathscr{R}(G) = \{G^{\langle 1\rangle}, G^{\langle 2\rangle}\} \neq \mathscr{I}(G)$ and $\mathscr{I}(G)$ is not linearly ordered by inclusion.

2.21 Lemma. *Let G be an LD-groupoid and $a \in G$. Then the set of all $x \in G$ such that $f(x) = g(a)$ for some $f, g \in \mathscr{M}_l(G)$ is just the left strongly prime left ideal generated by a.*

Proof. See the proof of 1.21(i). □

II.3 Dense subgroupoids of left distributive groupoids

3.1 Lemma. *Let H be a subgroupoid of an LD-groupoid G. Then:*
(i) *For all $f, g \in \mathscr{M}_l(G, H)$ there exists $h \in \mathscr{M}_l(G, H)$ such that $fh = hf$.*
(ii) *$\mathscr{M}_l(G, H)$ and $\mathscr{M}_l^1(G, H)$ are left uniform.*

Proof. We can proceed in the same way as in the proof of 1.16. □

3.2 Lemma. *Let H be a subgroupoid of an LD-groupoid G. Then $\langle H\rangle_G^{lc} = [H]_G^l = \beta_G(H) = \{x \in G \mid f(x) \in H$ for some $f \in \mathscr{M}_l(G, H)\} = \bigcup_{i\geq 1} \alpha_G^i(H)$.*

Proof. We have $\beta_G(H) \subseteq [H]_G^l \subseteq \langle H\rangle_G^{lc}$ (see I.4.17 and I.4.4). On the other hand, if $f(x), g(y) \in H$ then $fg(y) \in H$ and $fg = hf$ for some $h \in \mathscr{M}_l(G, H)$ (see 3.1). Now, $fg(xy) = fg(x)\, fg(y) = hf(x)\, fg(y) \in H$, i.e., $\beta_G(H)$ is a subgroupoid of G. Similarly, if $f(x), g(xy) \in H$ then $hf(x)\, fg(y) = fg(x)\, fg(y) = fg(xy) \in H$, and so $k(y) \in H$, where $k = L_{hf(x)} fg \in \mathscr{M}_l(G, H)$. We have proved that $\beta_G(H)$ is a left closed subgroupoid of G. Consequently, $\langle H\rangle_G^{lc} \subseteq \beta_G(H)$. Finally, $[H]_G^l = \bigcup \alpha_G^i(H)$ by I.4.3(iii). □

3.3 Lemma. *Let H be a subgroupoid of an LD-groupoid G. Then for all $n \geq 1$ and $x_1, \dots, x_n \in \langle H\rangle_G^{lc}$ there exists $f \in \mathscr{M}_l(G, H)$ with $f(x_1), \dots, f(x_n) \in H$.*

Proof. By 3.2, $\langle H\rangle_G^{lc} = \beta_G(H)$, and hence $f_1(x_1) \in H$ for some $f_1 \in \mathscr{M}_l(G, H)$. Since $\beta_G(H)$ is a subgroupoid, we have $f_1(x_2) \in \beta_G(H)$, and so $f_2 f_1(x_2) \in H$ for an $f_2 \in \mathscr{M}_l(G, H)$. Clearly, $f_2 f_1(x_1) \in H$ and the rest is clear by induction. □

3.4 Theorem. *Let H be a left strongly dense subgroupoid of an LD-groupoid G. Then:*
(i) *For all $n \geq 1$ and $x_1, \dots, x_n \in G$ there exists $f \in \mathscr{M}_l(G, H)$ with $f(x_1), \dots, f(x_n) \in H$.*
(ii) *Every left cancellative congruence r of H can be uniquely extended to a left cancellative conguence s of G.*

(iii) *If s is a left cancellative congruence of G and $r = s \cap (H \times H)$ then s is a cancellative congruence of G iff r is cancellative congruence of H.*
(iv) *If G is left cancellative and H is cancellative then G is cancellative.*
(v) *If G is a left quasigroup and H is right divisible then G is right divisible.*
(vi) *If G is left cancellative then the groupoids H and G satisfy the same groupoid identities (i.e., they generate the same groupoid variety).*

Proof. (i) See 3.3.
(ii) Define s by $(x, y) \in s$ iff $(f(x), f(y)) \in r$ for some $f \in \mathcal{M}_l(G, H)$. Then s is clearly symmetric, s is reflexive by (i) and the transitivity of s follows easily from 3.1(i). Thus s is an equivalence on G. Moreover, $s \cap (H \times H) = r$, since r is left cancellative.

If $(x, y) \in s$, $(f(x), f(y)) \in r$, $f \in \mathcal{M}_l(G, H)$ and $z \in G$ then $gf(x)$, $gf(y)$, $gf(z) \in H$ for some $g \in \mathcal{M}_l(G, H)$ (by (i)) and $gf(zx) = gf(z)\,gf(x)$, $gf(zy) = gf(z)\,gf(y)$, $(gf(x), gf(y)) \in r$, $(gf(zx), gf(zy)) \in r$ and $(zx, zy) \in s$. Quite similarly $(xz, yz) \in s$ and we have proved that s is a congruence of G.

Now, let $x, y, z \in G$, $f \in \mathcal{M}_l(G, H)$ and $(f(zx), f(zy)) \in r$. Again, we have $gf(x), gf(y), gf(z) \in H$ for some $g \in \mathcal{M}_l(G, H)$, $(gf(z)\,gf(x), gf(z)\,gf(y)) \in r$ and $(gf(x), gf(y)) \in r$, since is left cancellative. Thus $(x, y) \in s$ and we have proved that s is left cancellative. If r right cancellative then, proceeding similarly, we can show that s is right cancellative.

Finally, let t be a congruence of G such that $t \cap (H \times H) = r$. If $(x, y) \in t$ and $f \in \mathcal{M}_l(G, H)$ is such that $f(x), f(y) \in H$ then $(f(x), f(y)) \in t$ implies $(f(x), f(y)) \in r$ and $(x, y) \in s$. Thus $t \subseteq s$. Now, assume that t is left cancellative, $x, y \in G$, $f \in \mathcal{M}_l(G, H)$ and $(f(x), f(y)) \in r$. Then $(f(x), f(y)) \in t$, and so $(x, y) \in t$ due to the left cancellativity of t. Consequently, $s \subseteq t$, and so $s = t$.
(iii) See the preceding part of the proof.
(iv) The identity relations id_H and id_G are left cancellative congruences of H and G, respectively, and id_G extends id_H. Since H is cancellative, id_H is so, and hence id_G is cancellative by (iii). However, this means that G is cancellative.
(v) Let $x, y \in G$. Then $f(x), f(y) \in H$ for some $f \in \mathcal{M}_l(G, H)$ and, since H is right divisible, there is $a \in H$ such that $af(x) = f(y)$. Now, G is a left quasigroup, hence f is a permutation and $f(y) = af(x) = f(x)$, $b = f^{-1}(a)$, $y = bx$.
(vi) Let $u, v \in \mathbf{W}$ be such that $u \doteq v$ holds in H and let $h : \mathbf{W} \to G$ be a homomorphism. Then there is $f \in \mathcal{M}_l(G, H)$ such that $fh(x) \in H$ for each variable x occurring in u and v. Further, there is a homomorphism $k : \mathbf{W} \to H$ such that $k(x) = fh(x)$. Now, $fh(u) = k(u) = k(v) = fh(v)$ and, since G is left cancellative, $h(u) = h(v)$. $\square$

3.5 Proposition. *Let H be a left strongly dense subgroupoid of an LD-groupoid G and let φ be a homomorphism of H into an LD-groupoid K such that K is a left quasigroup. Then φ can be extended in a unique way to a homomorphism of G into K.*

Proof. Let A be a subgroupoid of G such that $H \subseteq A$, φ can be extended to a homomorphism $\psi : A \to K$ and A is maximal with respect to these properties. We are going to show that A is left closed in G.

For, let $a \in A$ and $B = \mu_G(A)$. Then B is a subgroupoid of G and $A \subseteq B$. Now, if $x \in B$ then $\psi(ax) = \psi(a)\,\xi(x)$ for just one element $\xi(x) \in K$ and it is easy to check that $\xi : B \to K$ is a homomorphism such that $\xi \,|\, A = \psi$. Then $B = A$ due to the maximality of A and we have proved that A is left closed in G. Since $H \subseteq A$ and H is left strongly dense in G, we must have $A = G$ and φ is extended to $\psi : G \to K$. The unicity of ψ follows from I.4.8(i). □

3.6 Lemma. *Let H be a subgroupoid of an LD-groupoid G and let $a \in H$ and $K = \mu_{a,G}(H)$. Then K is a subgroupoid of G, $H \subseteq K$ and $\varphi = L_{a,K}$ is a homomorphism of K into H. This homomorphism is injective (projective), provided that G is left cancellative (left divisible).*

Proof. Obvious. □

3.7 Lemma. *Let H be a subgroupoid of an LD-groupoid G, $n \geq 0$ and $m = 2^n - 1$. Then $\alpha_G^n(H) \subseteq \beta_{m,G}(H) \subseteq \alpha_G^m(H)$.*

Proof. By induction on n. The result is clear for $n = 0$. Now, let $x \in \alpha_G^{n+1}(H)$. Then $ax = b$ for some $a, b \in \alpha_G^n(H)$ and there are $a_1, \dots, a_m, b_1, \dots, b_m \in H$ such that $c = a_1(\dots(a_m a)) \in H$ and $b_1(\dots(b_m b)) \in H$. From this we immeadiately obtain $b_1(\dots(b_m(c(a_1(\dots(a_m x)))))) \in H$. The rest is clear. □

3.8 Let H be a left strongly dense subgroupoid of an LD-groupoid G and suppose that $\sigma_{lc}(H) \leq \aleph_0$. Then there is a countable non-empty subset S of H such that $H = \langle S \rangle_H^{lc}$. The subgroupoid A generated by S is also countable and $H = \langle A \rangle_A^{lc}$.

Now, consider a bijective mapping $f : A \times \mathbb{N} \to \mathbb{N}$, $\mathbb{N}$ being the set of positive integers, $f^{-1}(i) = (g(i), h(i))$, $g(i) \in A$, $h(i) \in \mathbb{N}$. Put $K_0 = H$ and $K_i = \mu_{g(i),G}(K_{i-1})$ for each $i \geq 1$. Then $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_i \subseteq K_{i+1} \subseteq \dots$ and all K_i are subgroupoids of G. Hence $K = \bigcup_{i \geq 0} K_i$ is a subgroupoid of G and $H \subseteq K$.

(i) By induction on $n \geq 0$ we show that $\beta_{n,G} \subseteq K$. This is clear for $n = 0$. Now, let $n \geq 1$, $a_1, \dots, a_n \in A$, $a \in G$, $a_1(\dots(a_n a)) \in A$. By the induction hypothesis, $a_n a \in K$, and so $a_n a \in K_m$ for some $m \geq 0$. Clearly, there is $i > m$ such that $g(i) = a_n$. Then $a_n a \in K_{i-1}$, and hence $a \in K_i \subseteq K$.

(ii) By (i) and 3.2, $\langle A \rangle_G^{lc} = \beta_G(A) \subseteq K$. However, $H \subseteq \langle A \rangle_G^{lc}$ and H is left strongly dense in G. Consequently, $\langle A \rangle_G^{lc} = K = G$.

(iii) Put $\varrho_i = L_{g(i),K_i}$ for each $i \geq 1$. Then ϱ_i is a homomorphism of K_i into K_{i-1}, and so $\eta_i = \varrho_1 \dots \varrho_{i-1}\varrho_i$ is a homomorphism of K_i into H.

If G is left cancellative then all ϱ_i and η_i are injective, and hence all K_i are isomorphic to subgroupoids of H.

If G is left divisible then $\varrho_i(K_i) = K_{i-1}$ and $\eta_i(K_i) = H$.

If G is a left quasigroup then all ϱ_i and η_i are isomorphisms, and hence all K_i are isomorphic to H.

3.9 Lemma. *Let H be a subgroupoid of an LD-groupoid G. Then:*
(i) $\mathscr{M}(G, H) = \mathscr{M}_l(G, H) \cup \mathscr{M}_r(G, H) \cup \mathscr{M}_r(G, H)\, \mathscr{M}_l(G, H)$.
(ii) $\mathscr{M}^1(G, H) = \mathscr{M}_r^1(G, H)\, \mathscr{M}_l^1(G, H)$.
(iii) *If H is left divisible then*

$$\mathscr{M}(G, H) = \mathscr{M}_l(G, H) \cup \mathscr{M}_r(G, H) \cup \mathscr{M}_l(G, H)\, \mathscr{M}_r(G, H)$$

and $\mathscr{M}^1(G, H) = \mathscr{M}_l^1(G, H)\, \mathscr{M}_r^1(G, H)$.

Proof. We have $L_a R_b = R_{ab} L_a$ for all $a, b \in H$. If H is left divisible then $a = bc$ for some $c \in H$ and $R_a L_b = L_b R_c$. □

3.10 Lemma. *Let H be a subgroupoid of an LD-groupoid G. Then $\psi_G(H) \subseteq \beta_G \delta_G(H)$. Moreover, if H is left divisible then $\psi_G(H) \subseteq \delta_G \beta_G(H)$.*

Proof. See I.4.19, I.4.20 and the preceding lemma. □

II.4 Cancellable and divisible elements of left distributive groupoids

4.1 Proposition. *Let G be an LD-groupoid. Then:*
(i) *$\mathscr{C}_l(G)$ is either empty or a left closed subgroupoid of G.*
(ii) *$\mathscr{D}_l(G)$ is either empty or a subgroupoid of G.*
(iii) *$\mathscr{P}_l(G)$ is either empty or a left closed subgroupoid of G.*
(iv) *$\mathscr{D}_l(G)\, \mathscr{D}_r(G) \subseteq \mathscr{D}_r(G)$ and $\mathscr{P}_l(G)\, \mathscr{P}_r(G) \subseteq \mathscr{P}_r(G)$.*
(v) *If both $\mathscr{C}_l(G)$ and $\mathscr{D}_l(G)$ are non-empty then $\mathscr{D}_l(G)$ is an idempotent groupoid. If, moreover, $\mathscr{D}(G) \neq \emptyset$ then G is idempotent.*
(vi) *If $\mathscr{P}(G) \neq \emptyset$ then G is idempotent.*

Proof. First, $L_x L_y = L_{xy} L_x = L_{xy \cdot x} L_{xy}$ for all $x, y \in G$ and (i), (ii), (iii) are easily seen. Further, $L_x R_y = R_{xy} L_x$ and (iv) is clear. Now, let $a \in \mathscr{D}_l(G)$ and $b \in \mathscr{C}_r(G)$. Then $b = ac$ for some $c \in G$ and we have $ab = a \cdot ac = aa \cdot ac = aa \cdot b$, which implies $a = aa$. If, moreover, $\mathscr{D}(G) \neq \emptyset$ then $\mathrm{Id}(G) = G$, since $\mathscr{D}(G) \subseteq \mathrm{Id}(G)$ and $\mathrm{Id}(G)$ is a left ideal. □

4.2 Proposition. *Let G be an LD-groupoid. Put $\mathscr{C}_l^*(G) = \{a \in \mathscr{C}_l(G) \mid aa = aa \cdot a\}$. Then:*
(i) *$\mathscr{C}_l^*(G)$ is either empty or a left closed subgroupoid of G.*
(ii) *If $\mathscr{C}_l^*(G) \neq \emptyset$ then $\mathscr{C}_l^*(G)$ is a left strongly prime left ideal of the groupoid $\mathscr{C}_l(G)$.*
(iii) *$\mathscr{P}_l(G) \subseteq \mathscr{C}_l^*(G)$.*
(iv) *$(a, aa) \in p_G$ for every $a \in \mathscr{C}_l^*(G)$.*

Proof. Put $A = \{a \in G \mid aa = aa \cdot a\}$ (see 1.18). If $a \in \mathscr{C}_l(G)$ and $ab \in A$ then $a(bb \cdot b) = (ab \cdot ab)(ab) = (ab)(ab) = a \cdot bb$ implies $b \in A$. Now (i) and (ii) are clear from 4.1(i) and 1.18(ii), (iii). Finally, (iii) and (iv) follow from 1.9(i), (ii). □

4.3 Proposition. *Let G be an LD-groupoid and $a \in \mathscr{C}_l^*(G)$. Then there exists an LD-groupoid K and an element $b \in K$ with the following properties:*

(i) *G is a left strongly dense subgroupoid of K, $b \in \mathscr{C}_l^*(K)$ and $a = bb = ab$, $(a, b) \in p_K$.*

(ii) $K = \mu_{a,K}(G) = \beta_{1,K}(G)$.

(iii) *$G = aK = bK$ and the translation $L_{a,K} = L_{b,K}$ is an isomorphism of K onto G.*

(iv) $\mathscr{C}_l(K) = \mu_{a,K}(\mathscr{C}_l(G)), \mathscr{C}_l(G) = a\mathscr{C}_l(K) \subseteq \mathscr{C}_l(K)$.

(v) $\mathscr{C}_l^*(K) = \mu_{a,K}(\mathscr{C}_l^*(G)), \mathscr{C}_l^*(G) = a\mathscr{C}_l^*(K) \subseteq \mathscr{C}_l^*(K)$.

(vi) *If 0 is an absorbing element of G then 0 is also absorbing in K.*

Proof. Put $H = aG$ and $\varphi = L_{a,G}$. Then φ is an isomorphism of G onto H and $\varphi(a) = aa$. Now, it is clear that there exists an LD-groupoid K such that G is a subgroupoid of $K, G = bK$ for an element $b \in \mathscr{C}_l^*(K)$ and $\psi = L_{b,K}$ is an isomorphism of K onto G, $\psi \mid G = \varphi$. We have $\psi(b) = bb = a$, $(a, b) \in p_K$ (by 4.2(iv)) and $G = aK = bK$. The rest is obvious. $\square$

4.4 Proposition. *Let G be a LD-groupoid and $a \in \mathscr{C}_l^*(G)$. Then there exists an LD-groupoid K with the following properties:*

(i) *G is a left strongly dense subgroupoid of K and $a \in \mathscr{P}_l(K)$.*

(ii) *K is the union of a chain $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_i \subseteq K_{i+1} \subseteq \ldots$ of subgroupoids such that $K_0 = G$, $K_i = aK_{i+1}$ for each $i \geq 0$ (thus all K_i are isomorphic to G).*

(iii) *For every $x \in K$ there is $n \geq 0$ with $L^n_{a,K}(x) = a(\ldots(ax)) \in G$ (thus $K = \beta_K(G)$).*

(iv) *$\mathscr{C}_l(G) \subseteq \mathscr{C}_l(K)$ and $\mathscr{C}_l^*(G) \subseteq \mathscr{C}_l^*(K)$.*

(v) *K is (left, right) cancellative (regular) iff G is so.*

(vi) *K is (left, right) divisible, provided that G is so.*

(vii) *$\omega_G \subseteq \omega_K$; K is subdirectly irreducible, provided that G is so.*

(viii) *K is simple, provided that G is so.*

(ix) *$p_K = \mathrm{id}_K$, provided that $p_G = \mathrm{id}_G$.*

(x) *K contains an absorbing element iff G does; in the positive case, the absorbing elements coincide.*

(xi) *The groupoids G and K satisfy the same groupoid identities.*

Proof. The chain $K_0 = G \subseteq K_1 \subseteq K_2 \subseteq \ldots$ is constructed by means of 4.3, $K_i = aK_{i+1}$, and $K = \bigcup_{i \geq 0} K_i$. The assertions of the proposition are easy consequences of 4.3 and the fact that all the links of the chain $\ldots \subseteq K_i \subseteq K_{i+1} \subseteq \ldots$ are isomorphic to G. For instance, if G is subdirectly irreducible, $r \neq \mathrm{id}_K$ is a congruence of K and $(u, v) \in r$, $u \neq v$ then $L^n_a(u), L^n_a(v) \in G$ for some $n \geq 0$, $L^n_a(u) \neq L^n_a(v)$, and so $r \cap (G \times G) \neq \mathrm{id}_G$ and $\omega_G \subseteq r$. $\square$

4.5 Theorem. *Let G be an LD-groupoid. Then there exists an LD-groupoid Q with the following properties:*

(i) *G is a left strongly dense subgroupoid of Q and* $\mathrm{card}(Q) = \mathrm{card}(G)$.

(ii) *$\mathscr{C}_l(G) \subseteq \mathscr{C}_l(Q)$ and $\mathscr{C}_l^*(G) \subseteq \mathscr{C}_l^*(Q) = \mathscr{P}_l(Q)$.*

(iii) *If $x \in Q$ then there exist $n \geq 1$ and $a_1, \ldots, a_n \in \mathscr{C}_l^*(G)$ such that $a_1(\ldots(a_nx)) \in G$.*

(iv) *Q is (left, right) cancellative (divisible, regular), provided that G is so.*
(v) $\omega_G \subseteq \omega_Q$; *$Q$ is subdirectly irreducible, provided that G is so.*
(vi) *Q is simple, provided that G is so.*
(vii) $p_Q = \mathrm{id}_Q$, *provided that* $p_G = \mathrm{id}_G$.
(viii) *Q contains an absorbing element iff G does; in the positive case, the absorbing elements coincide.*
(ix) *The groupoids Q and G satisfy the same groupoid identities.*

Proof. We can assume that $\mathscr{C} = \mathscr{C}_L^*(G) \neq \emptyset$. The rest of the proof is divided into several parts:

(i) Let $\alpha > 1$ be an ordinal number such that $\mathscr{C} = \{a_\beta \mid 1 \le \beta < \alpha\}$. Now, we shall construct a chain G_β, $0 \le \beta < \alpha$, of groupoids as follows: $G_0 = G$; if $1 \le \beta < \alpha$ and β is not limit then G_β is (by 4.3) such that $a_\beta G_\beta = G_{\beta-1}$; if $1 < \beta < \alpha$ and β is limit then G_β is such that $a_\beta G_\beta = \bigcup_{0 \le \gamma < \beta} G_\gamma$ (again, by 4.3). Put $K = \bigcup_{0 \le \beta < \alpha} G_\beta$. By transfinite induction we can show that K satisfies the properties (i), (iii), ..., (ix) and that $\mathscr{C}_l(G) \subseteq \mathscr{C}_l(K)$, $\mathscr{C}_l^*(G) \subseteq \mathscr{C}_l^*(K)$. Moreover, for every $a \in \mathscr{C}_l^*(G)$, $G \subseteq aK$.

(ii) Define a chain $Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots$ of groupoids in such a way that $Q_0 = G$ and, for $i \ge 0$, Q_{i+1} is constructed by means of (i) (starting from Q_i). Put $Q = \bigcup_{i \ge 0} Q_i$. If $x \in \mathscr{C}_l^*(Q)$, $y \in Q$ then $x \in \mathscr{C}_L^*(Q_i)$, $y \in Q_i$ for some $i \ge 0$, and hence $y = xz$ for some $z \in Q$. Thus $\mathscr{C}_l^*(Q) - \mathscr{P}_l(Q)$ (use 4.2(iii)). In the rest, we can use (i) and proceed similarly as in the proof of 4.4 (to prove (iii), put $H = \{x \in Q \mid f(x) \in G$ for some $f \in \mathscr{M}_l(Q, \mathscr{C})\}$ and show that H is left closed in Q). □

4.6 Lemma. *Let G be an LD-groupoid such that $\mathscr{C} = \mathscr{C}_l^*(G) \neq \emptyset$. Then the transformation semigroups $\mathscr{M}_l(G, \mathscr{C})$ and $\mathscr{M}_l^1(G, \mathscr{C})$ are cancellative.*

Proof. Every transformation from $\mathscr{M}_l(G, \mathscr{C})$ is injective, and this implies that the semigroup is left cancellative. Now, let $f, g, h \in \mathscr{M}_l(G, \mathscr{C})$ be such that $fh = gh$. There are $n \ge 1$ and $a_1, \dots, a_n \in \mathscr{C}$ such that $h = L_{a_1} \dots L_{a_n}$ and we shall proceed by induction on n. Put $k = L_{a_1} \dots L_{a_{n-1}}$ ($k = \mathrm{id}_G$ if $n = 1$) and $a = a_n$. We have $fk(ax) = gk(ax)$ for every $x \in G$. Consequently, $b = fk(aa) = gk(aa)$ and $bfk(x) = fk(aa \cdot x) = fk(ax) = gk(ax) = gk(aa \cdot x) = bgk(x)$. But $b \in \mathscr{C}$, and hence $fk(x) = gk(x)$ and $fk = gk$. Then $f = g$ by the induction hypothesis. □

4.7 Remark. Let G be an LD-groupoid and $a \in \mathscr{D}_l(G)$. Put $\varphi = L_a$. Then φ is a projective homomorphism of G onto G, $(a, \varphi(a)) \in p_G$ and there is $b \in G$ such that $\varphi(b) = ab = a$. Moreover, $\ker(\varphi) = q_{a,G}$ and if r is a left cancellative congruence of G such that $r \subseteq \ker(\varphi)$ then $r = \mathrm{id}_G$ and G is a left quasigroup (if $(x, y) \in r$ then $x = au$, $y = av$, $(u, v) \in r \subseteq \ker(\varphi)$ and $x = au = av = y$).

4.8 Example. Consider the following three-elem,ent groupoid G:

G	0	1	2
0	0	0	0
1	0	1	2
2	0	1	1

Then G is an LD-groupoid, $\mathscr{C}_l(G) = \mathscr{D}_l(G) = \mathscr{P}_l(G) = \{1\}$ is a left closed subgroupoid which is not right closed and $\mathscr{C}_r(G) = \mathscr{D}_r(G) = \mathscr{P}_r(G) = \{2\}$ is not a subgroupoid of G. Moreover, G is not idempotent.

4.9 Proposition. *Let G be a subdirectly irreducible LD-groupoid. Then either $q_G \neq \mathrm{id}_G$ or $\mathscr{C}_l(G) \neq \emptyset$.*

Proof. Suppose that $\mathscr{C}_l(G) \neq \emptyset$. Then, for every $x \in G$, L_x is not injective, $q_{x,G} = \ker(L_x) \neq \mathrm{id}_G$ is a congruence of G and $\omega_G \subseteq q_{x,G}$. If $(a, b) \in \omega_G$, $a \neq b$ then $xa = xb$, and so $(a, b) \in q_G$. □

II.5 Left cancellative left distributive groupoinds – first observations

5.1 Proposition. *Let G be a left cancellative LD-groupoid. Then:*

(i) $\mathscr{C} = \mathscr{C}_l^*(G) = \{a \in G \mid aa = aa \cdot a\}$ *is either empty or a left strongly prime left ideal of G.*

(ii) $\mathscr{C} = \{a \in G \mid (a, aa) \in p_G\}$.

(iii) *p_G is a left cancellative and right stable equivalence.*

(iv) *$\mathscr{M}_l(G)$ and $\mathscr{M}_l^1(G)$ are left cancellative left uniform semigroups.*

(v) *If $\mathscr{C} \neq \emptyset$ then $\mathscr{M}_l(G, \mathscr{C})$ and $\mathscr{M}_l^1(G, \mathscr{C})$ are cancellative left uniform semigroups.*

(vi) *Either $\mathrm{Id}(G) = \emptyset$ or $\mathrm{Id}(G)$ is a left strongly prime left ideal of G.*

Proof. See 1.18, 1.9(ii), 1.14(i), 1.16 and 4.6 (if $ab \in \mathrm{Id}(G)$ then $ab = ab \cdot ab = a \cdot bb$, and so $b = bb$). □

5.2 Proposition. *Let G be a left cancellative LD-groupoid. Then $G = \mathscr{C}_l^*(G)$ (i.e., G satisfies the identity $\mathbf{xx} \simeq \mathbf{xx} \cdot \mathbf{x}$) iff $(x, xx) \in p_G$ for every $x \in G$ (i.e., iff G satisfies the identity $\mathbf{xy} \simeq \mathbf{xx} \cdot \mathbf{y}$). Moreover, if these equivalent conditions are satisfied then:*

(i) *p_G is a congruence of G and G/p_G is a left cancellative LDI-groupoid.*

(ii) *G is left semimedial and o_G is an endomorphism of G.*

(iii) *$\mathscr{M}_l(G)$ and $\mathscr{M}_l^1(G)$ are cancellative left uniform semigroups.*

Proof. See 5.1 and 1.12. □

5.3 Theorem. *The following conditions are equivalent for a left cancellative LD-groupoid G:*

(i) *G satisfies the identity $\mathbf{xx} \simeq \mathbf{xx} \cdot \mathbf{x}$.*

(ii) *G satisfies the identity $\mathbf{xy} \simeq \mathbf{xx} \cdot \mathbf{y}$.*

(iii) *G satisfies the identity $\mathbf{x} \cdot \mathbf{yz} \simeq \mathbf{xx} \cdot \mathbf{yz}$.*

(iv) *G satisfies the identity* $\mathbf{xx}\cdot\mathbf{yz} \doteq \mathbf{xy}\cdot\mathbf{xz}$ *(i.e., G is left semimedial).*
(v) *G satisfies the identity* $\mathbf{xx}\cdot\mathbf{yy} \doteq \mathbf{xy}\cdot\mathbf{xy}$ *(i.e., o_G is an endomorphism of G).*
(vi) *G satisfies the identity* $\mathbf{x}\cdot\mathbf{yy} \doteq \mathbf{xx}\cdot\mathbf{yy}$.
(vii) *G can be embedded into a left distributive left quasigroup.*

Proof. (i) implies (ii) by 5.2, (ii) implies (iii) trivially, (iii) implies (iv) by the left distributivity, (iv) implies (v) trivially and (v) implies (vi) by the left distributivity.

Let (vi) be satisfied and let $x \in G$. Then $x(xx\cdot x) = (xx\cdot xx)(xx) = xx\cdot xx = x\cdot xx$, and hence $xx\cdot x = xx$, i.e., (i) is satisfied.

The condition (vii) implies (i) by 1.11(4). Now, let (i) be satisfied and consider the LD-groupoid Q constructed in 4.5. Then G is a subgroupoid of Q, Q is left cancellative, Q satisfies $\mathbf{xx} \doteq \mathbf{xx}\cdot\mathbf{x}$ and $Q = \mathscr{C}_l^*(Q) = \mathscr{P}_l(Q)$. Thus Q is a left quasigroup. $\square$

5.4 A left cancellative LD-groupoid satisfying the equivalent conditions of 5.3 will be called *pseudoidempotent* (clearly, every left cancellative LDI-groupoid is pseudoidempotent).

5.5 Remark. Let G be a pseudoidempotent left cancellative LD-groupoid. We shall exhibit here two alternative proofs of the fact that G can be embedded into a left distributive left quasigroup.

(i) We can assume without loss of generality that G is infinite. Let S be a set such that $G \subseteq S$ and $\mathrm{card}(S) > \mathrm{card}(G)$. Denote by $\mathfrak{M}$ the set of pseudoidempotent left cancellative LD-groupoids K such that G is a left strongly dense subgroupoid of K and the underlying set of K is a subset of S. The set $\mathfrak{M}$ is non-empty (we have $G \in \mathfrak{M}$) and it is ordered by $K \leq L$ if K is a subgroupoid of L (then K is left strongly dense in L). By Zorn's lemma, let Q be a maximal element of $\mathfrak{M}$. We are going to show that Q is a left quasigroup. For, let $a, b \in Q$. By I.4.15, $\mathrm{card}(Q) = \mathrm{card}(G) < \mathrm{card}(S)$, and hence there exists a groupoid $P \in \mathfrak{M}$ such that $Q \leq P$ and $Q = aP$ (use 4.3). Since Q is maximal, we must have $Q = P$, and so $b = ac$ for some $c \in Q$.

(ii) First, let G be finitely generated, $G = \langle A\rangle_G$ for a non-empty finite set $A \subseteq G$. Let $f : A \times \mathbb{N} \to \mathbb{N}$ be a bijection (see 3.8). Put $Q_0 = G$ and, for $i \geq 1$, let Q_i be such that Q_{i-1} is a subgroupoid of Q_i and $Q_{i-1} = g(i)\,Q_i$ (by 4.3), $f^{-1}(i) = (g(i), h(i))$. We are going to show that $Q = \bigcup_{i\geq 0} Q_i$ is a left quasigroup. It is easy to see that $A \subseteq \mathscr{P} = \mathscr{P}_l(Q)$. Since $\mathscr{P}$ is a subgroupoid of Q, we also have $G \subseteq \mathscr{P}$. However, $\mathscr{P}$ is a left closed subgroupoid (see 4.1(iii)) and G is left strongly dense in Q. Consequently, $\mathscr{P} = Q$.

In the general case, G can be embedded into a filtered product of its finitely generated subgroupoids. Every such subgroupoid can be embedded (by the first part of this proof) into a left distributive left quasigroup, and then G can be embedded into the corresponding filtered product of these left quasigroups which is again a left distributive left quasigroup.

(iii) By 5.2(iii), $\mathscr{M}_l(G)$ is a cancellative left uniform semigroup. Then $\mathscr{M}_l(G)$ is a subsemigroup of its group $\mathscr{N}$ of left fractions. Define an operation $*$ on $\mathscr{N}$ by $u * v = uvu^{-1}$. Then $\mathscr{N}(*)$ is an LDI-groupoid and a left quasigroup. The mapping $\varphi : a \to L_a \in \mathscr{N}$ is a homomorphism of G into $\mathscr{N}(*)$ and $\ker(\varphi) = p_G$. Thus G/p_G can be embedded into $\mathscr{N}(*)$.

5.6 Example. Let $\mathscr{A}$ be the set of non-projective injective transformations of an infinite set A. Define an operation $*$ on $\mathscr{A}$ by $(f * g)(f(a)) = fg(a)$ and $(f * g)(b) = b$ for all $f, g \in \mathscr{A}$, $a \in A$ and $b \in A - f(A)$. Then $\mathscr{A}(*)$ is a left cancellative LD-groupoid and $\mathscr{C}_l^*(\mathscr{A}(*)) = \emptyset$. In particular, $\mathscr{A}(*)$ is not pseudoidempotent, and hence it cannot be embedded into a left distributive left quasigroup.

5.7 Theorem. *Let H be a left strongly dense subgroupoid of an LD-groupoid G such that $H \subseteq \mathscr{C}_l(G)$. Then:*

(i) $G = \mathscr{C}_l(G)$ *is left cancellative,* $\operatorname{card}(G) = \operatorname{card}(H)$ *and for every $x \in G$ there exist $n \geq 1$ and $a_1, \dots, a_n \in H$ such that $a_1(\dots(a_n x)) \in H$.*

(ii) If *K is a finitely generated subgroupoid of G then K is isomorphic to a subgroupoid of H.*

(iii) *The groupoids G and H satisfy the same groupoid identities.*

(iv) $\omega_H \subseteq \omega_G$; *$G$ is subdirectly irreducible, provided that H is so.*

(v) $\omega_{l,c,H} = \omega_{l,c,G} | H$; *$G$ is subdirectly lc-irreducible iff H is so.*

(vi) *G is lc-simple iff H is so.*

(vii) *G is cancellative iff H is so.*

(viii) $p_H = p_G | H$.

Proof. (i) By 4.1(i), $\mathscr{C}_l(G)$ is left closed in G, and hence $G = \mathscr{C}_l(G)$. The other assertions follow from I.4.15 and 3.2.

(ii) Let A be a non-empty finite set such that $K = \langle A \rangle_G$. By 3.4(i), $f(A) \subseteq H$ for some $f \in \mathscr{M}_l(G, H)$. Then $f(K) \subseteq H$ and the groupoids $H, f(K)$ are isomorphic, since f is an injective endomorphism of G.

(iii) Use (ii) or 3.4(vi).

(iv) Let $r \neq \mathrm{id}_G$ be a congruence of G, $s = r | H$, $v \in G$, $u \neq v$, $(u, v) \in r$. Then $f(u), f(v) \neq H$ for some $f \in \mathscr{M}_l(G, H)$, $(f(u), f(v)) \in s$, $f(u) \neq f(v)$ and $s \neq \mathrm{id}_H$. Consequently, $\omega_H \subseteq s \subseteq r$, and hence $\omega_H \subseteq \omega_G$.

(v) and (vi). See 3.4(ii), (iii).

(vii) See 3.4(iv).

(viii) Let $(a, b) \in p_H$ and $x \in G$. There are $n \geq 1$ and $a_1, \dots, a_n \in H$ such that $a_1(\dots(a_n x)) \in H$. Now, $aa_i = ba_i = c_i$, $c_1(\dots(c_n \cdot ax)) = a(a_1(\dots(a_n x))) = b(a_1(\dots(a_n x))) = c_1(\dots(c_n \cdot bx))$, $ax = bx$ and $(a, b) \in p_G$. $\square$

5.8 Let G be a left strongly dense subgroupoid of a left distributive left quasigroup Q. Then we shall say that Q is a *left quasigroup-envelope* of G and we shall write $Q = Q_l(G)$.

With respect to 5.3, an LD-groupoid G possesses a left quasigroup-envelope iff G is left cancellative and pseudoidempotent.

5.9 Theorem. *Let G be a pseudoidempotent left cancellative LD-groupoid.*

(i) *If Q and P are left quasigroup-envelopes of G then there exists just one isomorphism $f : Q \to P$ such that $f \mid G = \mathrm{id}_G$ (i.e., a G-isomorphism).*

(ii) *If $g : G \to H$ is a homomorphism, where H is a pseudoidempotent left cancellative LD-groupoid, and if Q and P are left quasigroup-envelopes of G and H, respectively, then there exists just one homomorphism $f : Q \to P$ such that $f \mid G = g$. Moreover, f is injective (projective), provided that g is so.*

(iii) *If G is a subgroupoid of a left distributive left quasigroup P then $\langle G \rangle_P^{lc}$ is a left quasigroup-envelope of G.*

Proof. Clearly, (i) follows from (ii) and (iii) is evident. Now, we shall prove (i). By 3.5, g can be extended in a unique way to a homomorphism $f : Q \to P$. If g is injective then $\ker(g) = \mathrm{id}_G$. However, $\ker(f)$ extends $\ker(g)$, and so $\ker(f) = \mathrm{id}_Q$ by 3.4(ii). If $g(G) = H$ then $H \subseteq f(Q) \subseteq P$. But, $f(Q)$ is a left quasigroup, and hence it is left closed in P. On the other hand, H is left strongly dense in P, and therefore $f(Q) = P$. □

5.10. Theorem. *Let G be a pseudoidempotent left cancellative LD-groupoid and $Q = Q_l(G)$. Then:*

(i) $\mathrm{card}(Q) = \mathrm{card}(G)$ *and Q, G satisfy the same groupoid identities.*

(ii) *Q is right cancellative (regular) iff G is so.*

(iii) *Q is right divisible, provided that G is so.*

(iv) *$\omega_G \subseteq \omega_Q$; Q is subdirectly irreducible, provided that G is so.*

(v) *$\omega_{l,c,G} = \omega_{l,c,Q} \mid G$; Q is subdirectly lc-irreducible iff G is so.*

(vi) *Q is simple, provided that G is so.*

(vii) *Q is lc-simple iff G is so.*

(viii) *$p_G = p_Q \mid G$ and $p_G = \mathrm{id}_Q$ iff $p_G = \mathrm{id}_G$.*

Proof. (i) is proved in 5.7(i), (iii); (ii) and (iii) follow from 3.4(iv) and (v), respectively; (iv), (v) and (vii) are proved in 5.7(iv), (v) and (vi), respectively.

(vi) This follows from 4.5(vi), however we shall present a direct proof here.

Let K be a subgroupoid of Q maximal with respect to the properties that $G \subseteq K$ and K is simple. We show that K is left closed in Q (then $K = Q$). Indeed, if $a \in K$ and $L = \mu_{a,Q}(K)$ then $K \subseteq L$ and $aL = K$ (since Q is a left quasigroup). Hence K and L are isomorphic, L is simple and $L = K$.

(viii) By 5.7(viii), $p_G = p_Q \mid G$, and so $p_Q = \mathrm{id}_Q$ implies $p_G = \mathrm{id}_G$. If $p_G = \mathrm{id}_G$ and $(u, v) \in p_Q$ then $f(u), f(v) \in G$ for some $f \in \mathscr{M}_l(Q, G)$, $(f(u), f(v)) \in p_G$ (since p_Q is a congruence), $f(u) = f(v)$ and $u = v$. □

5.11 Remark. Let G be a pseudoidempotent left cancellative LD-groupoid such that G is infinite countable and G is not a left quasigroup. Put $Q = Q_l(G)$. Then there exists a chain $G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_i \subseteq G_{i+1} \subseteq \ldots$ of subgroupoids of Q and elements $a_i \in G$ such that $G_0 = G$, $\bigcup_{i \geq 0} G_i = Q$ and $G_i \neq G_{i-1} = a_i G_i$ for each $i \geq 1$ (all the subgroupoids G_i are isomorphic to G). The existence of such a chain follows from 3.8 (see also the first part of 5.5(ii), where we could take A to be also infinite countable).

5.12 Remark. Let G be an LD-groupoid. If $a \in \mathscr{C}_l(G)$ then there exists an LD-groupoid K such that G is a subgroupoid of K, $\mathscr{C}_l(G) = \mathscr{C}_l(K)$ and $G = bK$, $a = bb$ for some $b \in \mathscr{C}_l(K)$ (to show this, we proceed similarly as in 4.3).

If G is left cancellative then G is a left strongly dense subgroupoid of a left cancellative LD-groupoid P such that $o_P(P) = P$ and G, P satisfy the same groupoid identities.

5.13 Remark. Let G be a right cancellative LD-groupoid. Then G is idempotent (see 1.5(ii)) and $p_G = \mathrm{id}_G$. Consequently, p_G is a congruence and G/p_G is idempotent.

II.6 Left divisible left distributive groupoids – first observations

6.1 Proposition. *Let G be a left divisible LD-groupoid. Then:*

(i) *p_G is a congruence of G and G/p_G is idempotent.*

(ii) *The semigroups $\mathscr{M}_l(G)$ and $\mathscr{M}_l^1(G)$ are right cancellative.*

(iii) $\mathscr{M}(G) = \mathscr{M}_l(G) \cup \mathscr{M}_r(G) \cup \mathscr{M}_r(G)\mathscr{M}_l(G) = \mathscr{M}_l(G) \cup \mathscr{M}_r(G) \cup \mathscr{M}_l(G)\mathscr{M}_r(G)$ and $\mathscr{M}^1(G) = \mathscr{M}_l^1(G)\mathscr{M}_r^1(G) = \mathscr{M}_r^1(G)\mathscr{M}_l^1(G)$.

(iv) *$\mathscr{D}_r(G)$ is either empty or a left ideal of G.*

(v) *If $\mathscr{D}_r(G) \neq \emptyset$ and G is left-ideal-free then G is divisible.*

(vi) *If $\mathscr{C}_r(G) \neq \emptyset$ then G is idempotent.*

Proof. See 1.12, 3.9, 4.1(iv), (v). □

6.2 Proposition. *Let G be an LD-groupoid and a left quasigroup and let $\mathscr{L}(G)$ denote the subgroup in $\mathscr{M}_l^*(G)$ generated by all $L_xL_y^{-1}$, $x, y \in G$. Then:*

(i) *$\mathscr{L}(G)$ is a normal subgroup of $\mathscr{M}_l^*(G)$ and the corresponding factorgroup is cyclic.*

(ii) *If $a, b \in G$ and $(a, b) \in u_G^c$ then L_a, L_b are conjugate in $\mathscr{M}_l^*(G)$.*

(iii) *G is medial iff $L_yL_x^{-1}L_z = L_zL_x^{-1}L_y$ for all $x, y, z \in G$ and iff $\mathscr{L}(G)$ is abelian.*

(iv) *$\mathscr{P}_r(G)$ is either empty or a left ideal of G.*

(v) *If $\mathscr{P}_r(G) \neq \emptyset$ and G is left-ideal-free then G is a quasigroup.*

Proof. (i) We have $L_zL_xL_y^{-1}L_z^{-1} = L_{zx}L_zL_y^{-1}L_z^{-1} = L_{zx}L_y^{-1}L_{yz}L_z^{-1} \in \mathscr{L}(G)$ for all $x, y, z \in G$. The rest is clear.

(ii) Let $(a, b) \in u_G^c$. Then $b = f(a)$ for some $f \in \mathscr{M}_l^*(G)$ (see 1.22(i)) and we have $L_b = fL_af^{-1}$, since f is an automorphism of G.

(iii) G is medial iff $L_{xy}L_z = L_{xz}L_y$ for all $x, y, z \in G$. But $L_{xy} = L_xL_yL_x^{-1}$, $L_{xz} = L_xL_zL_x^{-1}$, and so G is medial iff $L_yL_x^{-1}L_z = L_zL_x^{-1}L_z = L_zL_x^{-1}L_y$. If this is so then $L_yL_x^{-1}L_zL_u^{-1} = L_zL_x^{-1}L_yL_u^{-1} = L_zL_u^{-1}L_yL_z^{-1}$ for all $x, y, z, u \in G$, and hence $\mathscr{L}(G)$ is abelian. Conversely, if $\mathscr{L}(G)$ is abelian then $L_yL_x^{-1}L_zL_x^{-1} = L_zL_x^{-1}L_yL_x^{-1}$ and $L_yL_x^{-1}L_z = L_zL_x^{-1}L_y$.

(iv) See 4.1(iv).
(v) This follows immediately from (iv). □

6.3. Example. Let G be a non-trivial group such that all non-unit elements of G are conjugate. Define a binary operation $*$ on $H = G - \{1\}$ by $x * y = xyz^{-1}$. Then $H(*)$ is a divisible LDI-groupoid and a left quasigroup. Moreover, $p_{H(*)} = \mathrm{id}_H = q_{H(*)}$ and $H(*)$ is not right regular.

6.4 Example. Let $G(+) = \mathbb{Z}_{2^\infty}$ and define a multiplication on G by $xy = -x + 2y$. Then G becomes a divisible IM-groupoid (hence a DI-groupoid) and a right quasigroup. If $a \in G$ is such that $a \neq 0$ and $2a = 0$ then $L_0 \neq L_a$ and $L_0L_0 = L_0L_a$ in $\mathscr{M}_l(G)$. Consequently, $\mathscr{M}_l(G)$ is not left cancellative.

6.5 Example. Let $G(+) = \mathbb{Z}_{2^\infty}$, $a \in G$, $a \neq 0$, $2a = 0$ and $xy = 2x - y + a$ for all $x, y \in G$. Then G is a divisible medial LD-groupoid and a left quasigroup. Moreover, $\mathrm{Id}(G) = \emptyset$ and G is not right distributive. Since every right cancellative LD-groupoid is idempotent, G is not a homomorphic image of an LD-groupoid which is also a right quasigroup. If $x \in G$ then $\langle x \rangle_G \cong \mathrm{Cyc}_r(2)$.

6.6 Remark. Let G be a right divisible LD-groupoid. Then either $\mathrm{Id}(G) = \emptyset$ or G is idempotent (see 1.5(iv)). Similarly, either G satisfies $\mathbf{xx} \doteq \mathbf{xx} \cdot \mathbf{x}$ or $xx \neq xx \cdot x$ for every $x \in G$ (see 1.18). If p_G is a congruence of G and G/p_G is idempotent then there exists $1 \leq \alpha \leq \infty$ such that every cyclic subgroupoid of G is isomorphic to $\mathrm{Cyc}_r(\alpha)$ (see 1.13).

6.7 Example. Let $G(+) = \mathbb{Z}_{2^\infty}$ and let $H = G \cup G^{(2)}$. Define an operation $*$ on H by $a * x = a + 2x$, $a * (x, y) = (-a + 2x, -a + 2y)$, $(a, b) * x = -a - 2b + 4x$ and $(a, b) * (x, y) = (-a - 2b + 4x, -a - 2b + 4y)$ for all $a, b, x, y \in G$. It is not difficult to check that $H(*)$ is a left divisible LD-groupoid.

Now, let $e \in G$ be an element such that $4e = 0 \neq 2e$. Then $(0, 0) * (e, 0) = (0, 0) * (0, 0)$ and $0 * (e, 0) = (2e, 0) \neq (0, 0) = 0 * (0, 0)$. This shows that $H(*)$ is not left regular. Further, $p_{H(*)} = \mathrm{id}_H \cup \{((a, b), (c, d)) \mid a + 2b = c + 2d\}$, and we have $((2e, 0), (0, 0)) \notin p_{H(*)}$. This also shows that the left divisible LDI-groupoid $H(*)/p_{H(*)}$ is not left regular.

II.7 Simple left distributive groupoids – first observations

7.1 Lemma. *Let G be a simple LD-groupoid. Put $A = \{a \in G \mid ax = aa \text{ for each } x \in G\}$ (i.e., A is the set of left constant elements of G), $B = \{b \in A \mid bb \in \mathscr{C}_l(G)\}$ and $C = \{c \in A \mid cc \in A\}$. Then:*
(i) *$G = A \cup \mathscr{C}_l(G)$ and $A \cap \mathscr{C}_l(G) \neq \emptyset$.*
(ii) *For each $a \in A$ there exists an idempotent $e(a) \in \mathrm{Id}(G)$ such that $aa = e(a) = ax$ for every $x \in G$; if $a \in C$ then $e(a) \in C$ and $e(a)$ is a left absorbing element of G.*
(iii) *$A = B \cup C$ and $B \cap C = \emptyset$.*
(iv) *C is either empty or a right ideal of G.*

(v) $\mathscr{C}_l(G)\,A \subseteq A$, $\mathscr{C}_l(G)\,B \subseteq B$ *and* $\mathscr{C}_l(G)\,C \subseteq C$.

(vi) *If* G *contains at least three elements then either* $\mathscr{C}_l(G) = G$ *or* $C = G$ *or* $\mathrm{card}(\mathscr{C}_l(G)) = \mathrm{card}(B) = \mathrm{card}(C) = 1$.

Proof. (i) Let $a \in G$. Then $r = q_{a,G} = \ker(L_a)$ is a congruence of G, and hence either $r = G \times G$ and $a \in A$ or $r = \mathrm{id}_G$ and $a \in \mathscr{C}_l(G)$. Thus $G = A \cup \mathscr{C}_l(G)$. On the other hand, $A \cap \mathscr{C}_l(G) = \emptyset$, since G is non-trivial.

(ii) For $a \in A$, $aa \cdot aa = a \cdot aa = aa = e(a)$ and the rest is clear.

(iii) This follows from (i).

(iv) If $a \in C$ and $x \in G$ then $ax = aa = e(a) \in C$.

(v) Let $a \in \mathscr{C}_l(G)$, $b \in B$, $c \in C$ and $x \in G$. Then $ab \cdot ax = a \cdot bx = ae(b)$, and hence L_{ab} is not injective, $ab \notin \mathscr{C}_l(G)$ and $ab \in A$. But $ab \cdot ab = ae(b) \in \mathscr{C}_l(G)$, since $\mathscr{C}_l(G)$ is a subgroupoid of G (see 4.1(i)), and so $ab \in B$. Similarly, $ac \in A$, $ac \cdot ac = ae(c)$ and $ae(c) \cdot ax = a \cdot e(c)\,x = ae(c)$, so that $ac \in C$.

(vi) Put $r = (\mathscr{C}_l(G) \times \mathscr{C}_l(G)) \cup (B \times B) \cup (C \times C)$. Then r is an equivalence and we are going to show that r is a congruence of G.

For, let $a, b, c \in G$, $(a, b) \in r$. If $c \in \mathscr{C}_l(G)$ then $(ca, cb) \in r$ by (v). If $c \notin \mathscr{C}_l(G)$ then $c \in A$, $ca = cb$ and again $(ca, cb) \in r$. We have proved that r is left stable. Further, if $a, b, c \in \mathscr{C}_l(G)$ then $ac, bc \in \mathscr{C}_l(G)$ and $(ac, bc) \in r$. If $a, b \in B$ then $ac = e(a)$, $bc = e(b)$, $e(a), e(b) \in \mathscr{C}_l(G)$ and again $(ac, bc) \in r$. If $a, b \in C$ then $ac, bc \in C$ by (iv) and we have $(ac, bc) \in r$. Finally, if $a, b \in \mathscr{C}_l(G)$ and $c \in B$ (resp. $c \in C$) *then* $ac, bc \in B$ (resp. $ac, bc \in C$) and $(ac, bc) \in r$.

We have proved that r is a congruence of G. If $r = G \times G$ then either $\mathscr{C}_l(G) = G$ or $B = G$ or $C = G$. However, if $B = G$ then $\mathscr{C}_l(G) = \emptyset$ and this is not possible. Finally, if $r \neq G \times G$ then $r = \mathrm{id}_G$ and all the three sets are one-element. □

7.2 It is easy to check that every two-element LD-groupoid is isomorphic to one of the following six pair-wise non-isomorphic two-element LD-groupoids:

$D(1)$	0	1
0	0	0
1	0	1

$D(2)$	0	1
0	0	1
1	0	1

$D(3)$	0	1
0	0	0
1	1	1

$D(4)$	0	1
0	0	0
1	0	0

$D(5)$	0	1
0	0	1
1	0	0

$D(6)$	0	1
0	1	0
1	1	0

7.3 Consider the following three-element groupoid $D(30)$ (see V.6.1):

$D(30)$	0	1	2
0	0	0	0
1	0	1	2
2	1	1	1

Then $D(30)$ is a simple LD-groupoid and $o_{D(30)}$ is not an endomorphism of $D(30)$.

7.4 Theorem. *Let G be a simple LD-groupoid. Then exactly one of the following three cases takes place:*

(i) *G is a two-element groupoid (and then G is isomorphic to one of the groupoids $D(1), \ldots, D(6)$ from 7.2).*

(ii) *G is isomorphic to the LD-groupoid $D(30)$ from 7.3.*

(iii) *G contains at least three elements and G is left cancellative.*

Proof. Suppose that G contains at least three elements and that G is not left cancellative. Then $\mathscr{C}_l(G) \neq G$.

First, let $C = G$, where C is from 7.1. There is a mapping $e : G \to \mathrm{Id}(G)$ such that $xy = e(x)$ and $e(e(x)) = e(x)$ for all $x, y \in G$. Since $\ker(e)$ is a congruence of G, either $\ker(e) = G \times G$ or $\ker(e) = \mathrm{id}_G$. If $\ker(e) = G \times G$ then G is a Z-semigroup, and then G contains just two elements (every equivalence is a congruence), a contradiction. If $\ker(e) = \mathrm{id}_G$ then $x = e(x)$ for every $x \in G$, G is an LZ-semigroup and, again, G is a two-element groupoid.

We have proved that $C \neq G$. By 7.1(vi), each of the sets $\mathscr{C}_l(G)$, B, C contains only one element, say $\mathscr{C}_l(G) = \{a\}$, $B = \{b\}$, $C = \{c\}$. Now, $aa = a$, $ab = b$, $ac = c$, $ba = bb = bc = a$, $ca = cb = cc = c$ (see 7.1), and hence G is isomorphic to $D(30)$ $(a \to 1, b \to 2, c \to 0)$. □

7.5 Theorem. (i) *The groupoids $D(1), \ldots, D(6)$, $D(30)$ and $\mathrm{Cyc}_r(p)$, $p \geq 3$ a prime number, are pair-wise non-isomorphic finite simple LD-groupoids.*

(ii) *If G is a finite simple LD-groupoid then either G is isomorphic to one of the groupoids from (i) or G is an idempotent left quasigroup with $p_G = \mathrm{id}_G$.*

Proof. (i) See 7.2, 7.3 and I.6.9.

(ii) In view of 7.4, we can assume that G is left cancellative. Then G is a left quasigroup, and hence p_G is a congruence of G and G/p_G is idempotent (see 1.12). If $p_G = \mathrm{id}_G$ then G is idempotent. If $p_G = G \times G$ then G is a right constant groupoid (see I.6.10). □

7.6 Theorem. *Let G be a simple left cancellative LD-groupoid. Then:*

(i) *Either G is pseudoidempotent or $xx \neq xx \cdot x$ for every $x \in G$.*

(ii) *If G is pseudoidempotent then either G is isomorphic to $D(2)$ or to $\mathrm{Cyc}_r(p)$ for a prime $p \geq 2$ or G is idempotent and $p_G = \mathrm{id}_G$.*

(iii) *If G is idempotent and $p_G = \mathrm{id}_G$ then there exists a simple LDI-groupoid Q such that Q is a left quasigroup, $p_Q = \mathrm{id}_Q$ and G is a left strongly dense subgroupoid of Q and $\mathrm{card}(Q) = \mathrm{card}(G)$.*

Proof. (i) This follows easily from 1.18(iv).

(ii) This follows from the fact that p_G is a congruence of G and G/p_G is idempotent (see 5.2, 5.3 and 5.4).

(iii) We can put $Q = Q_l(G)$ (see 5.10). □

7.7 Proposition. *Let G be a simple LD-groupoid such that o_G is an endomorphism of G (i.e., G satisfies the identity* $\mathbf{x} \cdot \mathbf{yy} \doteq \mathbf{xx} \cdot \mathbf{yy}$*). Then either G is isomorphic to one of the groupoids $D(1), \ldots, D(6)$, $\mathrm{Cyc}_r(p)$, $p \geq 3$ being a prime number, or G is left cancellative, idempotent and contains at least three elements.*

Proof. Put $r = \ker(o_G)$. If $r = G \times G$ then G is unipotent and $xx = xx \cdot xx = x \cdot xx$ for every $x \in G$, and hence G is not left cancellative and we can use 7.4 to show that G is isomorphic to one of $D(4)$, $D(5)$.

Now, assume $r = \mathrm{id}_G$. With respect to 7.4, either G is isomorphic to one of $D(1)$, $D(2)$, $D(3)$, $D(6)$ or G is a left cancellative groupoid containing at least three elements. By 5.3, G is pseudoidempotent and the rest is clear. □

II.8 Comments and open problems

This chapter is based essentially on [Kep,81] and [Kep,94b] (see also [KepP,91], …, [KepP,95b] and [BashJK,?]). The ideal theory of left distributive groupoids (see II.2) was initiated by [Bir,86] and the important example 5.6 is taken from [Deh,89b].

The following problems remain open:

Do there exist non-pseudoidempotent simple left cancelative LD-groupoids?

Is p_G a congruence of G for every right divisible LD-groupoid G?

Is every (right) divisible LD-groupoid left regular?

Is every left divisible LD-groupoid a homomorphic image of an LD-groupoid which is also a left quasigroup?

Which LD-groupoids can be embedded into (left, right) divisible LD-groupoids?

III. Subdirect decompositions of some non-idempotent left distributive groupoids

III.1 Introduction

1.1 Let G be an LD-groupoid. We shall say that G is

- *delightful* if satisfies the identity $\mathbf{xx} \cdot \mathbf{y} \doteq \mathbf{x} \cdot \mathbf{yy}$;
- *strongly delightful* if it is delightful and satisfies the identity $(\mathbf{xx} \cdot \mathbf{y})\,\mathbf{z} \doteq \mathbf{xy} \cdot \mathbf{z}$;
- an *LDA-groupoid* if is delightful and satisfies the identity $\mathbf{x} \cdot \mathbf{xx} \doteq \mathbf{y} \cdot \mathbf{yy}$.

1.2 Lemma. *Let G be an LD-groupoid. Then:*

(i) $x \cdot yz = (xy \cdot x)(xy \cdot z)$ *for all* $x, y, z \in G$.

(ii) *If G is elastic then* $x \cdot yx = xy \cdot x = (xy)(x \cdot xx) \in \mathrm{Id}(G)$ *for all* $x, y \in G$.

Proof. (i) $x \cdot yz = xy \cdot xz = (xy \cdot x)(xy \cdot z)$.

(ii) $xy \cdot x = x \cdot yx = xy \cdot xx = (xy \cdot x)(xy \cdot x) = (x \cdot yx)(x \cdot yx) = x(yx \cdot yx) = x(y \cdot xx) = (xy)(x \cdot xx)$. □

1.3 Theorem. *Let G be a delightful LD-groupoid. Then:*

(i) *G satisfies the identity $\mathbf{x} \cdot \mathbf{xx} \doteq \mathbf{xx} \cdot \mathbf{x}$ (i.e., $r_G = s_G$).*

(ii) *$\mathrm{Id}(G)$ is an ideal of G and $x \cdot xx \in \mathrm{Id}(G)$ for every $x \in G$.*

(iii) *r_G is an endomorphism of G, $r_G(G) = \mathrm{Id}(G)$ and $r_G \mid \mathrm{Id}(G)$ is the identity mapping.*

(iv) *$H = G/\mathrm{Id}(G)$ is an LDA-groupoid.*

(v) $\ker(r_G) \cap \equiv_{\mathrm{Id}(G)} = \mathrm{id}_G$.

(vi) *G is the subdirect product of $\mathrm{Id}(G)$ and H.*

(vii) *Every block of $\ker(r_G)$ is a subgroupoid and an LDA-groupoid.*

Proof. (i) Obvious.

(ii) First, $(x \cdot xx)(x \cdot xx) = x(xx \cdot xx) = xx \cdot xx = x \cdot xx$, $x \cdot xx \in \mathrm{Id}(G)$ and $Id(G)$ is a left ideal by 1.5(i). Moreover, if $a \in \mathrm{Id}(G)$ and $y \in G$ then $ay \cdot ay = a \cdot yy = aa \cdot y = ay$ and we see that $\mathrm{Id}(G)$ is an ideal.

(iii) $(x \cdot xx)(y \cdot yy) = (xx \cdot xx)(y \cdot yy) = (xx)(y \cdot yy)^2 = (xx)(y \cdot yy) = x(y \cdot yy)^2 = x(y \cdot yy) = (xy)(xy \cdot xy)$ by (ii) and the rest is clear.

(iv) This follows from (ii).

(v) and (vi). If $(a, b) \in \ker(r_G)$ and $a, b \in \mathrm{Id}(G)$ then $a = a \cdot aa = b \cdot bb = b$.

(vii) This is obvious. □

1.4 Proposition. *Let G be an LDA-groupoid. Then G contains just one idempotent element 0. Moreover, 0 is an absorbing element of G and $x \cdot xx = 0 = xx \cdot x$ for every $x \in G$.*

Proof. By 1.3(ii), $\mathrm{Id}(G)$ is an ideal, and hence $\mathrm{Id}(G) = \{0\}$ is a one-element set. □

1.5 Remark. Let G be a delightful LD-groupoid and $x, y, z \in G$. Then:
$x \cdot xy = xx \cdot xy = (xx \cdot x)(xx \cdot y) = (x \cdot xx)(x \cdot yy) = x(xx \cdot yy) = x(x(yy \cdot yy)) = x(x(y \cdot yy)) \in \mathrm{Id}(G)$,
$xx \cdot y = x \cdot yy = xy \cdot xy = (xy \cdot x)(xy \cdot y)$,
$(xx \cdot y)(xx \cdot y) = xx \cdot yy = x(yy \cdot yy) = x(y \cdot yy) = (xx \cdot xx)y = (x \cdot xx)y \in \mathrm{Id}(G)$,
$x \cdot yx = xy \cdot xx = (xy \cdot xy)x = (x \cdot yy)x = (xx \cdot y)x$,
$(x \cdot yx)(x \cdot yx) = x(y \cdot xx) = x(yy \cdot x) = (x \cdot yy)(xx) = (xy \cdot xy)(xx) = (xy)(xx \cdot xx) = (xy)(x \cdot xx) \in \mathrm{Id}(G)$,
$(xy \cdot x)(xy \cdot x) = xy \cdot xx = x \cdot yx$,
$xx \cdot yz = (xx \cdot y)(xx \cdot z) = (x \cdot yy)(x \cdot zz) = x(yy \cdot zz) = x(y(zz \cdot zz)) = x(y(z \cdot zz)) \in \mathrm{Id}(G)$,
$xx \cdot yz = x(yz \cdot yz) = x(y \cdot zz)$,
$xy \cdot zz = (xy \cdot xy)z = (x \cdot yy)z = (xx \cdot y)z$.

Moreover, if G is an LDA-groupoid then $x \cdot xy = (x \cdot yx)(x \cdot yx) = (xx \cdot y)(xx \cdot y) = xx \cdot yz = 0$.

1.6 Proposition. *Let G be an elastic delightful LD-groupoid. Then:*

(i) *$x \cdot yz \in \mathrm{Id}(G)$ for all $x, y, z \in G$.*

(ii) *$(xy \cdot z)(xy \cdot z) = (x \cdot yy)z = (xx \cdot y)z \in \mathrm{Id}(G)$ for all $x, y, z \in G$.*

(iii) *G is left semimedial.*

Proof. (i) By 1.2(ii), $x \cdot yx = xy \cdot x \in \mathrm{Id}(G)$ for all $x, y \in G$. However, $\mathrm{Id}(G)$ is an ideal by 1.3(ii), and so $x \cdot yz \in \mathrm{Id}(G)$ by 1.2(i).

(ii) $(xy \cdot z)(xy \cdot z) = (x \cdot yy)z = (xx \cdot y)z \in \mathrm{Id}(G)$ by (i) (since $x \cdot yy \in \mathrm{Id}(G)$).

(iii) The assertion is clear if G is idempotent, and hence, with respect to 1.3(vi), we can assume that G is an LDA-groupoid. Then $x \cdot yz = xy \cdot xz = 0$ by (i) and $xx \cdot yz = x(yz \cdot yz) = 0$ also by (i). □

1.7 Proposition. (i) *A delightful LD-groupoid G is strongly delightful iff $G/\mathrm{Id}(G)$ is a semigroup. If this is so, then G is elastic and $G/\mathrm{Id}(G)$ is an A-semigroup.*

(ii) *A groupoid G is strongly delightful LDA-groupoid iff G is an A-semigroup.*

Proof. Put $H = G/\mathrm{Id}(G)$. First, let G be strongly delightful. Then $xy \cdot x = (xx \cdot y)x = (x \cdot yy)x = (xy \cdot xy)x = xy \cdot xx = x \cdot yx$, G is elastic, $x \cdot yz \in \mathrm{Id}(G)$ by 1.6(i) and $(xy \cdot z)(xy \cdot z) = xy \cdot zz = (xy \cdot xy)z = (x \cdot yy)z = (xx \cdot y)z = xy \cdot z$, so that $xy \cdot z \in \mathrm{Id}(G)$ as well. This implies that H is a semigroup.

Conversely, if H is a semigroup then both H and $\mathrm{Id}(G)$ are strongly delightful, and hence G is so by 1.2(vi). □

1.8 Proposition. *If G is a strongly delightful LD-groupoid then every block of* $\ker(r_G)$ *is an A-semigroup. Moreover, G is a D-groupoid iff* $\mathrm{Id}(G)$ *is so.*

Proof. Use 1.3(vii) and 1.7(ii). □

1.9 Theorem. *Let G be a D-groupoid. Then G is strongly delightful, elastic and semimedial.*

Proof. First, $xx \cdot y = xy \cdot xy = x \cdot yy$ by the left and right distributivity and we have proved that G is delightful. Now, by 1.3(vi), G is the subdirect product of $\mathrm{Id}(G)$ and $H = G/\mathrm{Id}(G)$, where $\mathrm{Id}(G)$ is idempotent (and hence strongly delightful) and H is an LDA-groupoid. Now, it suffices to show that H is strongly delightful. But H is a delightful D-groupoid, H contains an absorbing element 0 and, for $u, u, w \in H$, $uv \cdot w = uw \cdot vw = (uw \cdot v)(uw \cdot w) = (uw \cdot v)(uw \cdot ww) = (uw \cdot v)((u \cdot ww)(w \cdot ww)) = 0$, since $w \cdot ww = 0$.

We have proved that G is strongly delightful. By 1.7(i), G is elastic and, by 1.6(iii), G is left semimedial. Since G is right distributive, G is right semimedial by the left-right symmetry. □

1.10 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	0	1	2
2	0	0	0

Then G is an elastic LD-groupoid and $\mathrm{Id}(G) = \{0,1\}$ is not an ideal. Consequently, G is not delightful. Furthermore, p_G is a congruence of G, G/p_G is idempotent and o_G is an endomorphism of G.

1.11 Example. Consider the following five-element groupoid G.

G	0	1	2	3	4
0	0	0	0	0	0
1	0	0	3	0	0
2	0	0	0	0	0
3	0	0	4	0	0
4	0	0	0	0	0

It is easy to check that G is an elastic LDA-groupoid and that G is not strongly delightful (in this case, it means that G is not a semigroup).

1.12 (i) Let G, H be delightful LD-groupoids, $I = \mathrm{Id}(G)$, $J = \mathrm{Id}(H)$, $A = G/I$ and $B = H/J$. Let $f : G \to H$ be a homomorphism. Then $f(I) \subseteq J$, and so $g = f \mid I$ is a homomorphism of I into J. If f is injective then, trivially, g is injective. If f is projective and $u \in J$ then $u = f(x)$ for some $x \in I$ and $f(x \cdot xx) = u \cdot uu = u$, $x \cdot xx \in I$; consequently, g is projective. Further, f induces a homomorphism $h : A \to B$, $h(x/I) = f(x)/J$. Again, h is injective (projective), provided that f is so.

(ii) Let G_i be a non-empty family of delightful LDI-groupoids and $G = \prod G_i$. Then $\mathrm{Id}(G) = \prod \mathrm{Id}(G_i)$ and $\prod G_i/\mathrm{Id}(G_i)$ is isomorphic to a subgroupoid of $G/\mathrm{Id}(G)$. Moreover, if all the LDA-groupoids $G_i/\mathrm{Id}(G_i)$ are unipotent (or Z-semigroups) then $G/\mathrm{Id}(G)$ is unipotent (or a Z-semigroup).

III.2 Construction of strongly delightful left distributive groupoids

2.1 (i) Let G be a strongly delightful LD-groupoid, $I = \mathrm{Id}(G)$, $r = r_G$ and, for every $i \in I$, let $A(i)$ be the block of r such that $i \in A(i)$. Then I is an ideal of G, I is an LDI-groupoid, $A(i)$ is an A-semigroup and $i = 0_i$ is an absorbing element of $A(i)$ (see 1.3 and 1.8). Further, $G = \bigcup_{i \in I} A(i)$ is the disjoint union.

Let $i, j \in I$. If $a \in A(i)$, $b \in A(j)$ then $r(ab) = r(a)\, r(b) = ij \in I$ and $ab \in A(ij)$. We get a mapping $g_{i,j} : A(i) \times A(j) \to A(ij)$, $g_{i,j}(a, b) = ab$.

Let $i \in I$, $A(i, 2) = A(i)\, A(i) = \{xy \mid x, y \in A(i)\}$ and $A(i, 1) = A(i) - A(i, 2)$. If $j \in I$, $a \in A(i, 2)$ and $b \in A(j)$ then $ab \in I \cap A(ij) = \{ij\}, ab = ij$. Similarly, $ba = ji$ and the following condition is satisfied:

(1) $$g_{i,j}(A(i, 2) \times A(j)) = \{0_{ij}\} = g_{i,j}(A(i) \times A(j, 2)) \text{ for all } i, j \in I,\ i \neq j\,.$$

For $i \in I$, let $B(i) = \{x \in A(i) \mid xA(i) = 0_i = A(i)\, x\}$. Clearly, $A(i, 2) \subseteq B(i)$. If $j \in I$, $a \in A(i, 1)$ and $b \in A(j, 1)$ then $ab \in B(ij)$. Hence:

(2) $$g_{i,j}(A(i, 1) \times A(j, 1) \subseteq B(ij) \text{ for all } i, j \in I, i \neq j\,.$$

Finally, for $i, j \in I$, let $C(i,j) = g_{i,j}(A(i,1) \times A(j,1)) - \{0_{ij}\}$. If $k \in I$, $a \in C(i,j)$ and $b \in A(k)$, then $ab = 0_{ij \cdot k}$ and $ba = 0_{k \cdot ij}$. Now, we can formulate our last condition:

(3) If $i, j \in I, i \neq j, C(i,j) \neq \emptyset$ and if $k \in I, k \neq ij$ then
$$g_{ij,k}(C(i,j) \times A(k)) = \{0_{ij \cdot k}\} \text{ and } g_{k,ij}(A(k) \times C(i,j)) = \{0_{k \cdot ij}\}.$$

(ii) Now, conversely, let I be an LDI-groupoid and $A(i)$, $i \in I$, be a family of pairwise disjoint A-semigroups (their absorbing elements being denoted by 0_i). For all $i, j \in I$, $i \neq j$, let there be given mappings $g_{i,j} : A(i) \times A(j) \to A(ij)$ such that the conditions (1), (2) and (3) from (i) are satisfied (the sets $A(i,1)$, $A(i,2)$, $B(i)$ and $C(i,j)$ are defined in the same way as in (i)).

Put $G = \bigcup_{i \in I} A(i)$ and define an operation $*$ on g by $x * y = xy$ if $x, y \in A(i)$ for some $i \in I$ and $x * y = g_{i,j}(x,y)$ if $x \in A(i)$, $y \in A(j)$ and $i \neq j$. It requires just a tedious checking to show that $G(*)$ is a strongly delightful LD-groupoid, $\mathrm{Id}(G(*)) = \{0_i \mid i \in I\} \cong I$ and $A(i)$, $i \in I$, are just the blocks of $\ker(r_{G(*)})$. Clearly, $G(*)$ is a D-groupoid iff I is so.

2.2 Theorem. *Every strongly delightful LD-groupoid is constructed from an LDI-groupoid and a family of disjoint A-semigroups in the way described in 2.1.*

Proof. See 2.1. □

2.3 Example. Let I be an LDI-groupoid and A be an A-semigroup such that $A \cap I = \emptyset$. Further, let g be a mapping of $B = A - \{0\}$ (0 being the absorbing element of A) into I such that $g(xy) = g(x)\,g(y)$ whenever $x, y \in B$ and $xy \neq 0$. Put $G = B \cup I$ and define an operation $*$ on G as follows: $x * y = xy$ if $x, x \in B$ and $xy \neq 0$; $x * y = g(x)\,g(y)$ if $x, y \in B$, $xy = 0$; $x * y = xg(y)$ and $y * x = g(y)\,x$ for all $x \in I$, $y \in B$; $x * y = xy$ for all $x, y \in I$.

Clearly, $I = \mathrm{Id}(G(*))$ is an ideal of the groupoid $G(*)$ and $G(*)/I \cong A$. Moreover, $r_{G(*)} \mid B = g$ is a homomorphism of $G(*)$ onto I and $G(*)$ is the subdirect product of I and A. Consequently, $G(*)$ is a strongly delightful LD-groupoid and $G(*)$ is distributive iff I is so.

2.4 Example. Let I be an LDI-groupoid and A be an A-semigroup such that $I \cap A = \{0\}$, where 0 is the absorbing element of A. Put $G = I \cup A$ and define an operation $*$ on G as follows: $x * y = xy$ if either $x, y \in I$ pr $x, y \in A$; $x * y = x0$ and $y * x = 0x$ if $x \in I$ and $y \in A$. Then $G(*)$ is a strongly delighful LD-groupoid, $I = \mathrm{Id}(G(*))$ and $G(*)/I \cong A$.

2.5 Proposition. *The following conditions are equivalent for a delightul LD-groupoid G (and then G is strongly delightful):*

(i) $(x, xx) \in p_G$ *for every* $x \in G$ *(i.e., G satisfies* $\mathbf{xy} \doteq \mathbf{xx} \cdot \mathbf{y}$*),*
(ii) *The factorgroupoid* G/q_G *is idempotent.*
(iii) $GG \subseteq \mathrm{Id}(G)$.
(iv) $G/\mathrm{Id}(G)$ *is a Z-semigroup.*

Proof. (i) implies (ii). We have $xy = xx \cdot y = x \cdot yy$ for all $x, y \in G$, and hence $(y, yy) \in q_G$, which means that G/q_G is idempotent.

Proceeding conversely, we can show that (ii) implies (i) and the rest is clear. □

2.6 Proposition. *Let G be an LD-groupoid. Then the factorgroupoid G/t_G is idempotent iff G is delightful and $G/\mathrm{Id}(G)$ is a Z-semigroup.*

Proof. If G/t_G is idempotent then $xx \cdot y = xy = x \cdot yy$ for all $x, y \in G$. □

III.3 Splitting strongly delightful left distributive groupoids

3.1 Let G be a strongly delightful LD-groupoid. For every $i \in \mathrm{Id}(G)$, let $A_G(i)$ (or only $A(i)$) be the block of $\ker(r_G)$ containing i. Then $A(i)$ is an A-semigroup and i is an absorbing element of $A(i)$.

We shall say that G is *balanced* if all $A(i)$, $i \in \mathrm{Id}(G)$, are isomorphic.

We shall say that G is *splitting* if G is isomorphic to the cartesian product $I \times A$ for an LDI-groupoid I and an A-semigroup A; then $\mathrm{Id}(G) \cong I$, $A_G(i) \cong A$ for every $i \in \mathrm{Id}(G)$ and G is balanced.

3.2 Lemma. *Let G be a strongly delightful LD-groupoid. Then G is splitting iff there exist an A-semigroup A and isomorphisms $g_i : A_G(i) \to A$, $i \in \mathrm{Id}(G)$, such that $g_i(x)\, g_j(y) = g_{ij}(xy)$ for all $i, j \in Id(G)$, $x \in A_G(i)$ and $y \in A_G(j)$.*

Proof. The direct implication is clear and, as concerns the converse one, the mapping $x \to (r_G(x), g_{r_G(x)}) \in \mathrm{Id}(G) \times A$ is an isomorphism of G onto $\mathrm{Id}(G) \times A$. □

3.3 Proposition. *Let G be a strongly delightful LD-groupoid. The following conditions are equivalent:*

(i) *G is splitting and $A_G(i)$ is a Z-semigroup for every $i \in \mathrm{Id}(G)$.*
(ii) *G is splitting and G satisfies the equivalent conditions of 2.5.*
(iii) *G is balanced and $G/\mathrm{Id}(G)$ is a Z-semigroup.*

Proof. It suffices to show that (iii) implies (i). Coose $u \in \mathrm{Id}(G)$ and, for each $i \in \mathrm{Id}(G)$, let g_i be an isomorphism of $A(i)$ onto $A(u)$. Then $g_i(x)\, g_j(y) = g_{ij}(xy) = u$ for all $i, j \in \mathrm{Id}(G)$, $x \in A(i)$, $y \in A(j)$ and we can use 3.2. □

3.4 Proposition. *Let G be a strongly delightful LD-groupoid such that $G/\mathrm{Id}(G)$ is a Z-semigroup and $\mathrm{card}(A_G(i)) = \mathrm{card}(A_G(j))$ for all $i, j \in \mathrm{Id}(G)$. Then G is splitting.*

Proof. This follows from 3.3, since two Z-semigroups are isomorphic iff they have the same cardinality. □

3.5 Proposition. *Let G be a strongly delightful LD-groupoid such that $\mathrm{Id}(G)$ is a quasitrivial groupoid and $\mathrm{card}(A_G(i)) = 2$ for every $i \in \mathrm{Id}(G)$. Then G is splitting.*

Proof. With respect to 3.4, it is enough to show that $G/\mathrm{Id}(G)$ is a Z-semigroup. Suppose, on the contrary, that $ab \notin \mathrm{Id}(G)$ for some $a, b \in G$. Then $a \in A(i)$, $b \in A(j)$, $i, j \in \mathrm{Id}(G)$, $i \neq j$, $ab \in A(ij)$, $an \neq ij$ and $a \neq i$, $b \neq j$. Since $\mathrm{Id}(G)$ is quasitrivial, we can assume that $ij = i$ (the other case, $ij = j$, being similar). Then $a, ab \in A(i) - \{i\}, a = ab$, $a = ab \cdot b \in \mathrm{Id}(G)$ and this is a contradiction with the fact that G is strongly delightful (see 1.7(i)). $\square$

3.6 Theorem. *Let I be an LD-groupoid and A an A-semigroup. Then every balanced strongly delightful LD-groupoid G with* $\mathrm{Id}(G) \cong I$ *and* $A_G(i) \cong A$ *($i \in \mathrm{Id}(G)$) is splitting iff at least one of the following three cases takes place:*

(a) *I is trivial.*

(b) *A is trivial.*

(c) *I is quasitrivial and* $\mathrm{card}(A) = 2$ *(then A is a Z-semigroup).*

Proof. If I is trivial then G is an A-semigroup. If A is trivial then G is idempotent. If (c) is true then G is splitting by 3.5. The rest of the proof is divided into three parts:

(i) Let I be non-trivial and let A be not a Z-semigroup. Consider a family $A(i)$, $i \in I$, of pair-wise disjoint A-semigroups isomorphic to A and denote by 0_i the absorbing element of $A(i)$. Further, put $G = \bigcup_{i \in I} A(i)$ and $g_{ij}(x, y) = 0_{ij}$ for all $i, j \in I$, $i \neq j$, $x \in A(i)$, $y \in A(j)$. It is easy to check that the conditions (1), (2) and (3) from 2.1 are satisfied and we obtain a strongly delightful LD-groupoid $G(*)$ such that $\mathrm{Id}(G(*)) \cong I$ and $A_{G(*)}(i) = A(i) \cong A$. In particular, $G(*)$ is balanced and $G(*)/\mathrm{Id}(G(*))$ is not a Z-semigroup. Furthermore, $x * y \in \mathrm{Id}(G(*))$, whenever $x, y \in G$ and $r_{G(*)}(x) \neq r_{G(*)}(y)$. Now, suppose that G is splitting and that $\varphi : K = I \times A \to G(*)$ is an isomorphism. Since A is not a Z-semigroup, there are $a, b \in A$ such that $ab \neq 0$. Let $i, j \in I$, $i \neq j$, $u = (i, a)$, $v = (j, b)$, $uv \in I \times A$. Then $r_K(u) \neq r_K(v)$ and $uv \notin \mathrm{Id}(K)$, and hence $r_{G(*)}(\varphi(u)) \neq r_{G(*)}(\varphi(v))$ and $\varphi(u)\,\varphi(v) \notin \mathrm{Id}(G(*))$, a contradiction.

(ii) Suppose that I is not quasitrivial and that $\mathrm{card}(A) = 2$. Consider a family $A(i)$ of pair-wise disjoint two-element Z-semigroups with the absorbing elements 0_i and put $G = \bigcup_{i \in I} A(i)$. There are $k, l \in I$ such that $k \neq kl \neq l$, and hence also $k \neq l$. Let $A(k) = \{0_k, a\}$, $A(l) = \{0_l, b\}$ and $A(kl) = \{0_{kl}, c\}$. The elements a, b, c are pair-wise different. Now, define mappings $g_{i,j} : A(i) \times A(j) \to A(ij)$ for all $i, j \in I$, $i \neq j$, by $g_{i,j}(x, y) = 0_{ij}$ in all cases except for the one when $i = k, j = l$, $x = a$, $y = b$. Then $g_{k,l}(a, b) = c$. Obviously, the conditions (1), (2), (3) from 2.1 are satisfied and we get a strongly delightful LD-groupoid $G(*)$ such that $\mathrm{Id}(G(*)) \cong I$ and $\mathrm{card}(A_{G(*)}) = 2$. Further, $a * b = c \notin \mathrm{Id}(G(*))$, $G(*)/\mathrm{Id}(G(*))$ is not a Z-semigroup and $G(*)$ is not splitting by 3.3.

(iii) Let I be non-trivial and let A be a Z-semigroup containing at least three elements. Again, consider a family $A(i)$, $i \in I$, of pair-wise disjoint A-semigroups isomorphic to A and with the absorbing elements 0_i and put $G = \bigcup_{i \in I} A(i)$. There are $k, l \in I$, $k \neq l$, and $a \in A(k) - \{0_k\}, b \in A(l) = \{0_l\}, c \in A(kl) - \{0_{kl}\}$

such that the elements a, b, c are pair-wise different. Now, we can proceed similarly as in the foregoing part. $\square$

3.7 Example. Let I be a non-trivial LDI-groupoid and A be a non-trivial A-semigroup such that $A \cap I = \{0\}$, where 0 is the absorbing element of A. Put $G = I \cup A$ and define $*$ by $x * y = xy$ for all $x, y \in I$, $u * v = uv$ for all $u, v \in A$ and $x * u = 0x$, $u * x = x0$ for all $x \in I$, $u \in A$ (see 2.4). Then $G(*)$ is a strongly delightful LD-groupoid, $\mathrm{Id}(G(*)) \cong I$, $G(*)/\mathrm{Id}(G(*)) \cong A$ and $G(*)$ is not splitting.

3.8 Proposition. *Let G be a regular delightful LD-groupoid. Then G is isomorphic to the cartesian product of a regular LDI-groupoid and a Z-semigroup. Hence G is strongly delightful and balanced.*

Proof. Easy. $\square$

III.4 Varieties of strongly delightful left distributive groupoids – first observations

4.1 Throughout this section, let $\mathscr{I}$ denote the variety of LDI-groupoids and $\mathscr{A}$ that of A-semigroups. Further, let $\mathscr{A}_0$ denote the variety of trivial groupoids, $\mathscr{A}_1$ the variety of Z-semigroups, $\mathscr{A}_2$ the variety of commutative A-semigroups satisfying the identity $\mathbf{xx} \doteq \mathbf{yy}$ ((i.e., the variety of unipotent commutative A-semigroups), $\mathscr{A}_3$ the variety of commutative A-semigroups, $\mathscr{A}_4$ the variety of unipotent A-semigroups and let $\mathscr{A}_5 = \mathscr{A}$.

It is easy to check that $\mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \mathscr{A}_3 \subseteq \mathscr{A}_5$, $\mathscr{A}_2 \subseteq \mathscr{A}_4 \subseteq \mathscr{A}_5$, and that there are no other inclusions except for those which follow by transitivity. Moreover, $\mathscr{A}_0, \ldots, \mathscr{A}_5$ are pair-wise distinct and they are the only subvarieties of $\mathscr{A}$.

4.2 Proposition. *Let $\mathscr{V}$ be a variety of strongly delightful LD-groupoids.*

(i) *$\mathscr{V}$ is generated by $(\mathscr{V} \cap \mathscr{I}) \cup (\mathscr{V} \cap \mathscr{A})$.*

(ii) *If $\mathscr{V} \cap \mathscr{A} \subseteq \mathscr{A}_3$ and $\mathscr{V} \cap \mathscr{A} \nsubseteq \mathscr{A}_1$ then every groupoid from $\mathscr{V}$ is commutative.*

Proof. (i) This result follows immediately from the fact that every strongly delightful LD-groupoid is a subdirect product of the LDI-groupoid $\mathrm{Id}(G)$ and the A-semigroup $G/\mathrm{Id}(G)$.

(ii) Let, on the contrary, $G \in \mathscr{V}$ be not commutative. Since G/I, $I = \mathrm{Id}(G)$, is a commutative A-semigroup, the LDI-groupoid I is non-commutative, i.e. $ab \neq ba$ for some $a, b \in I$. Further, $\mathscr{V} \cap \mathscr{A} \nsubseteq \mathscr{A}_1$ and there exist $H \in \mathscr{V} \cap \mathscr{A}$ and $u, v \in H$ such that $uv \notin \mathrm{Id}(H)$. The groupoid $K = I \times H$ belongs to $\mathscr{V}$, and so $K/\mathrm{Id}(K)$ is commutative. On the other hand, $u, v \notin \mathrm{Id}(H)$, and hence $x = (a, u)$, $y = (b, v) \notin \mathrm{Id}(K)$; furthermore, $xy \neq yx$, $xy \notin \mathrm{Id}(K)$ and this shows that $K/\mathrm{Id}(K)$ is not commutative, a contradiction. $\square$

4.3 Proposition. *Let $\mathscr{W}$ be a variety of LDI-groupoids and $\mathscr{U}$ a variety of A-semigroups such that either every groupoid from $\mathscr{W}$ is commutative or*

$\mathscr{U} \neq \mathscr{A}_2, \mathscr{A}_3$. *Denote by* $\mathscr{V}$ *the class of strongly delightful LD-groupoids G such that* $\mathrm{Id}(G) \in \mathscr{W}$ *and* $G/\mathrm{Id}(G) \in \mathscr{U}$. *Then* $\mathscr{V}$ *is a variety of LD-groupoids and* $\mathscr{V} \cap \mathscr{I} = \mathscr{W}, \mathscr{V} \cap \mathscr{A} = \mathscr{U}$.

Proof. In view of 1.12(i), $\mathscr{V}$ is closed under subgroupoids and homomorphic images. If $\mathscr{U} = \mathscr{A}_0, \mathscr{A}_5$ then $\mathscr{V}$ is clearly closed under cartesian products. If $\mathscr{U} = \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4$ then the result follows from 1.12(ii). □

4.4 Let $\mathscr{V}$ be a variety of strongly delightful LD-groupoids and denote by $\mathscr{L}(\mathscr{V})$ the lattice of subvarieties of $\mathscr{V}$ (more precisely, to be in better accordance with the basic set theory, the dual lattice of fully invariant congruences of a free strongly delightful LD-groupoid of countably infinite rank). For $\mathscr{T} \in \mathscr{V}$, put $\varphi(\mathscr{T}) = (\mathscr{T} \cap \mathscr{I}, \mathscr{T} \cap \mathscr{A}) \in \mathscr{L}(\mathscr{V} \cap \mathscr{I}) \times \mathscr{L}(\mathscr{V} \cap \mathscr{A})$ and let $\mathfrak{M}$ be the collection of ordered couples $(\mathscr{W}, \mathscr{U})$, where $\mathscr{W} = \mathscr{L}(\mathscr{V} \cap \mathscr{I}), \mathscr{U} \in \mathscr{L}(\mathscr{V} \cap \mathscr{A})$, and either every groupoid from $\mathscr{W}$ is commutative or $\mathscr{U} \neq \mathscr{A}_2, \mathscr{A}_3$. Then $\mathfrak{M}$ is a lattice with respect to the induced ordering $((\mathscr{W}_1, \mathscr{U}_1) \leq (\mathscr{W}_2, \mathscr{U}_2)$ iff $\mathscr{W}_1 \subseteq \mathscr{W}_2$ and $\mathscr{U}_1 \subseteq \mathscr{U}_2)$ and φ is an isomorphism of the lattice $\mathscr{L}(\mathscr{V})$ onto $\mathfrak{M}$ (this follows easily from 4.2 and 4.3).

Now, put $\mathscr{W} = \mathscr{V} \cap \mathscr{I}$ and $\mathscr{U} = \mathscr{V} \cap \mathscr{A}$. We have the following six cases:

(i) $\mathscr{U} = \mathscr{A}_0$, and then $\mathscr{V} = \mathscr{W} \subseteq \mathscr{I}$ and $\mathscr{L}(\mathscr{V}) = \mathscr{L}(\mathscr{W})$.

(ii) $\mathscr{U} = \mathscr{A}_1$, and then $\mathscr{L}(\mathscr{V}) \cong \mathscr{L}(\mathscr{W}) \times \mathscr{C}_2$ (where $\mathscr{C}_2$ denotes a two-element chain).

(iii) $\mathscr{U} = \mathscr{A}_2$, and then every groupoid from $\mathscr{V}$ is commutative and $\mathscr{L}(\mathscr{V}) \cong \mathscr{L}(\mathscr{W}) \times \mathscr{C}_3$ (where $\mathscr{C}_3$ is a three-element chain).

(iv) $\mathscr{U} = \mathscr{A}_3$, and then every groupoid from $\mathscr{V}$ is commutative and $\mathscr{L}(\mathscr{V}) \cong \mathscr{L}(\mathscr{W}) \times \mathscr{C}_4$ (where $\mathscr{C}_4$ is a four-element chain).

(v) $\mathscr{U} = \mathscr{A}_4$, and then $\mathscr{L}(\mathscr{V}) \cong \mathfrak{M}$, where $\mathfrak{M} = \{(\mathscr{W}_1, \mathscr{U}_1) \mid \mathscr{W}_1 \in \mathscr{L}(\mathscr{W})$, and either $\mathscr{U}_1 = \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_4$ or every groupoid from $\mathscr{W}_1$ is commutative and $\mathscr{U}_1 = \mathscr{A}_2\}$.

(vi) $\mathscr{U} = \mathscr{A}_5$, and then $\mathscr{L}(\mathscr{V}) \cong \mathfrak{M}$, where $\mathfrak{M} = \{(\mathscr{W}_1, \mathscr{U}_1) \mid \mathscr{W}_1 \in \mathscr{L}(\mathscr{W})$, and either $\mathscr{U}_1 = \mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_4, \mathscr{A}_5$ or every groupoid from $\mathscr{W}_1$ is commutative and $\mathscr{U}_1 = \mathscr{A}_2, \mathscr{A}_3\}$.

4.5 Remark. Let $\mathscr{V}$ be a variety of strongly delightful LD-groupoids, $\mathscr{W} = \mathscr{V} \cap \mathscr{I}$ and $\mathscr{U} = \mathscr{V} \cap \mathscr{A}$. Suppose that $\mathscr{U} \nsubseteq \mathscr{A}_1$ and $\mathscr{W}$ contains some non-commutative groupoids. Then $\mathscr{A}_2 \subseteq \mathscr{U} \in \{\mathscr{A}_4, \mathscr{A}_5\}$ and the varieties $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_4, \mathscr{V}_1, \mathscr{V}_2$ (where $\mathscr{V}_1$ is generated by $\mathscr{W} \cup \mathscr{A}_1$ and $\mathscr{V}_2$ by $\mathscr{W} \cup \mathscr{A}_4$) are subvarieties of $\mathscr{V}$ and form a five-element non-modular sublattice of $\mathscr{L}(\mathscr{V})$. Consequently, the lattice $\mathscr{L}(\mathscr{V})$ of subvarieties of $\mathscr{V}$ is not modular.

4.6 Construction. Let $\mathscr{V}$ be a variety of strongly delightful LD-groupoids such that both $\mathscr{W} = \mathscr{V} \cap \mathscr{I}$ and $\mathscr{U} = \mathscr{V} \cap \mathscr{A}$ are non-trivial varieties. Let X and Y be two disjoint non-empty sets of the same cardinality and let $f : X \to Y$ be a bijection.

Now, let $G(*)$ be a free groupoid in $\mathscr{W}$ having Y as a set of free generators and, similarly $H(\circ)$ be free over X in $\mathscr{U}$; $H(\circ)$ is then an A-semigroup and possesses an absorbing element 0. Put $F = G \cup (H - \{0\})$, and define a mapping $g : K = H - \{0\} \to G$ as follows: $g(x) = f(x)$ for every $x \in X$; if $x, y \in X$ and $x \circ y \neq 0$ then $g(x \circ y) = f(x) * f(y)$. Notice that the mapping g is well defined: If $x \circ y \neq 0$

then necessarily $x, y \in X$ and $x \circ y = x_1 \circ y_1$ implies that either $x = x_1$, $y = y_1$ (and then $f(x) * f(y) = f(x_1) * f(y_1)$) or $x = y_1, y = x_1$. However, in the latter case, $\mathscr{U} \subseteq \mathscr{A}_3$, $\mathscr{U} \nsubseteq \mathscr{A}_1$, every groupoid from $\mathscr{W}$ is commutative by 4.2(ii) and again $f(x) * f(y) = f(x_1) * f(y_1)$.

Now, define a multiplication on $F : uv = u \circ v$ for all $u, v \in K$, $u \circ v \neq 0$; $uv = g(u) * g(v)$ for all $u, v \in K, u \circ v = 0$; $uv = g(u) * v$ and vu $=$ v $*$ g(u) for all $u \in K, v \in G$; $uv = u * v$ for all $u, v \in G$. It is easy to check that F is a free groupoid in $\mathscr{V}$ and that X is a set of free generators of F.

III.5 Left distributive groupoids with just one idempotent element

5.1 Let G be an LD-groupoid such that $\mathrm{card}(\mathrm{Id}(G)) = 1$. By 1.5(i), $\mathrm{Id}(G) = \{z\}$ is a left ideal and this means that z is a right absorbing element. Throughout this section, we shall use the notation $z = 0$ (more precisely, $z = 0_G$).

5.2 Proposition. *Let G be an LD-groupoid such that* $\mathrm{card}(\mathrm{Id}(G)) = 1$. *Then:*

(i) *The set* $A = \{a \in G \mid 0a = 0\}$ *is a left ideal of G.*

(ii) $x \cdot 0y = 0 \cdot xy = 0x \cdot 0y$ *for all* $x, y \in G$.

(iii) *If G is left cancellative then A is left strongly prime.*

(iv) *If either G is right regular or* L_0 *is projective then* $(x, 0x) \in p_G$ *for every* $x \in G$.

(v) *If G is elastic then* 0 *is an absorbing element of G (i.e.,* $A = G$*).*

(vi) *If G is a semigroup then* 0 *is an absorbing element of G.*

(vii) *If G is right distributive then* 0 *is an absorbing element of G.*

Proof. (i) If $a \in A$ and $x \in G$ then $0 \cdot xa = 0x \cdot 0a = 0x \cdot 0 = 0$.

(ii) $x \cdot 0y = x0 \cdot xy = 0 \cdot xy = 0x \cdot 0y$.

(iii) If $ab \in A$ then $a0 = 0 = 0 \cdot ab = a \cdot 0b$ (see (ii)), and hence $0 = 0b$ and $b \in A$.

(vi) This follows easily from (ii).

(v) By (ii), $0x = x0 \cdot x = x \cdot 0x = 0x \cdot 0x$, and hence $0x \in \mathrm{Id}(G)$ and $0x = 0$ for every $x \in G$.

(vi) This follows from (v), since every semigroup is elastic.

(vii) Since G is right distributive, $\mathrm{Id}(G) = \{0\}$ is a right ideal, and so 0 is left absorbing. □

5.3 Proposition. *An LD-groupoid G is an LDA-groupoid iff G is delightful and* $\mathrm{card}(\mathrm{Id}(G)) = 1$.

Proof. See 1.3 and 1.4. □

5.4 Proposition. *Let G be a unipotent LD-groupoid. Then:*

(i) $\mathrm{card}(\mathrm{Id}(G)) = 1$ *and* $xx = 0$ *for every* $x \in G$.

(ii) *G is delightful iff* 0 *is an absorbing element.*

(iii) $(x, xx) \in p_G$ *for every* $x \in G$ *iff G is a Z-semigroup.*

Proof. (i) and (ii) are obvious.

(iii) If $(x, 0) \in p_G$ for every $x \in G$ then $0 = yy = 0y$ and $xy = 0y = 0$. □

5.5 Proposition. *Let G be a groupoid satisfying the identity* $\mathbf{x}\cdot\mathbf{yz}\doteq\mathbf{u}\cdot\mathbf{vw}$. *Then:*

(i) *G is an LD-groupoid and* $\operatorname{card}(\operatorname{Id}(G)) = 1$.

(ii) *$GG \subseteq A = \{a\in G \mid 0a = 0\}$ and A is an ideal of G.*

Proof. (i) Obviously, G is an LD-groupoid and $\operatorname{card}(\operatorname{Id}(G)) \le 1$. Now, if $0 = a\cdot bc$, $a, b, c \in G$ then $0\cdot 0 = 0(a\cdot bc) = 0$.

(ii) Obvious. □

5.6 Let G be a groupoid satisfying the identity $\mathbf{x}\cdot\mathbf{yz}\doteq\mathbf{u}\cdot\mathbf{vw}$. Then G is an LD-groupoid and if G is, moreover, delightful then we shall say that G is an LDB-groupoid.

5.7 Proposition. (i) *Every LDB-groupoid is an LDA-groupoid.*

(ii) *Every unipotent LDA-groupoid is zeropotent.*

(iii) *Every zeropotent LD-groupoid is an LDA-groupoid.*

(iv) *Every finite zeropotent LD-groupoid is an LDB-groupoid.*

Proof. (i), (ii) and (iii). Obvious.

(iv) Let G be a finite zeropotent LD-groupoid and denote by Q the set of ordered triples $(a, b, c) \in G^{(3)}$ such that $a\in bc \neq 0$.

Now, define a mapping $f : G\times G\times \mathbb{N}_0 \to G$ by $f(a, b, 0) = a$, $f(a, b, 1) = ab$ and $f(a, b, n) = f(a, b, n-1)\,f(a, b, n-2)$ for all $a, b \in G$ and $n \ge 2$. The rest of the proof is divided into five parts:

(iv1) Proceeding by induction on $n \ge 0$, we show that $f(ab, b, n) = f(a, b, n+1)$.

Indeed, the equality is clear for $n \le 1$. However, if $n\ge 2$ then $f(ab, b, n) = f(ab, b, n-1)\,f(ab, b, n-2) = f(a, b, n)\,f(a, b, n-1) = f(a, b, n+1)$.

(iv2) By induction on $n \ge 0$, we show that $af(a, b, n) = 0$.

First, we have $af(a,b,0) = aa = 0$ and $af(a,b,1) = a\cdot ab = aa\cdot ab = 0\cdot ab = 0$. For $n\ge 2$, $af(a,b,n) = a\cdot f(a,b,n-1)\,f(a,b,n-2) = af(a,b,n-1)\cdot af(a,b,n-2) = 0\cdot 0 = 0$.

(iv3) By induction on $n\ge 1$, we show that $f(a, b, n)\,(f(a, b, n-1)\,c) = a\cdot bc$ for all $a, b, c\in G$.

For $n = 1$, the equality is just the left distributive law. For $n\ge 2$, we can write $f(a, b, n)\,(f(a, b, n-1)\,c) = (f(a, b, n-1)\,f(a, b, n-2))\,(f(a, b, n-1)\,c) = f(a, b, n-1)\,(f(a, b, n-2)\,c) = a\cdot bc$.

(iv4) Let $(a, b, c)\in Q$. We are going to show by induction on $n\ge 0$ that the elements $f(a, b, 0), \dots, f(a, b, n)$ are pair-wise different.

For $n = 0$, there is nothing to prove. Let $n\ge 1$. Then, by induction, the elements $f(a, b, 0), \dots, f(a, b, n-1)$ are pair-wise different. On the other hand, $ab\cdot ac = a\cdot bc \neq 0$, and hence $(ab, a, c)\in Q$ and $f(ab, a, 0), \dots, f(ab, a, n-1)$ are also pair-wise different. Using (iv1), we see that the elements $f(a, b, 1), \dots, f(a, b, n)$ are pair-wise different and it remains to show that $a = f(a, b, 0) \neq f(a, b, n)$. If this is not true then $a\cdot bc = f(a, b, n)\,(f(a, b, n-1)\,c) = a\cdot f(a, b, n-1)\,c = af(a, b, n-1)\cdot ac = 0\cdot ac = 0$ (by (iv3) and (iv2)), a contradiction.

(iv5) It follows immediately from (iv4) and the finiteness of G that $Q = \emptyset$. □

5.8 Example. Consider the following three-element groupoid G:

G	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

Then G is a commutative LDB-groupoid, but G is not unipotent.

5.9 Remark. There are examples of infinite zeropotent LD-groupoids which are not LDB-groupoids (see [Deh, 98b]).

III.6 Comments and open problems

The material of this chapter is based mainly on [Rue, 66] and [JezKN, 81]. Proposition 5.7 appeared in [Jez, 95].

A general task is to find various classes of LD-groupoids with "nice" subdirect decompositions into idempotent LD-groupoids and LD-groupoids having just one idempotent. Also, more information on the latter groupoids should be of interest.

IV. Constructions and examples of left distributive groupoids

IV.1 Various constructions of left distributive groupoids

1.1 Let be a right constant groupoid and let f be the transformation of G such that $xy = f(y)$ for all $x, y \in G$. Then:

(i) G is a medial LD-groupoid.

(ii) G is right distributive (or delightful, strongly delightful, elastic, associative) iff $f^2 = f$.

(iii) G is idempotent iff $f = \mathrm{id}_G$.

(iv) G is commutative iff f is constant.

(v) G is left symmetric iff $f^2 = \mathrm{id}_G$.

(vi) G is right symetric (or semisymmetric, symmetric) iff G is trivial.

(vii) $\mathrm{Id}(G) = \{a \in G \mid f(a) = a\}$ and $\mathrm{Id}(G)$ is an ideal of G iff $\mathrm{Id}(G) \neq \emptyset$ and $f^2 = f$.

(viii) G is regular.

(ix) G is left cancellative (left divisible) iff f is injective (projective).

(x) Both $o_G = s_G$ and x_G are endomorphisms of G.

(xi) $p_G = G \times G$ (and hence p_G is a congruence of G and G/p_G is idempotent).

(xii) $q_G = t_G = \ker(f)$.

(xiii) $(a, b) \in z_{l,G}$ iff $a = f^n(b)$ for some $n \geq 1$.

(xiv) $(a, b) \in z_{r,G}$ iff $a \in f(G)$.

1.2 Example. Define an operation $*$ on the set $\mathbb{N}$ of positive integers by $x * y = y + 1$. Then $G = \mathbb{N}(*)$ is a right constant groupoid and an LD-groupoid (see 1.1). Furthermore, $G = \langle 1 \rangle_G$, i.e., G is cyclic. By 1.1(xii), $(a, b) \in z_{l,G}$ iff $a > b$, and hence $z_{l,G}$ is irreflexive and $z^1_{l,G} = \geq$ is the dual of the usual ordering of the set $\mathbb{N}$.

1.3 Let G be a non-trivial groupoid such that $G = A \cup B$, where A is the set of left neutral elements of G and $ax = ay \in \mathrm{Id}(G)$ for all $a \in B$ and $x, y \in G$ (in particular, every element from B is left constant and $A \cap B = \emptyset$). Then:

(i) G is an LD-groupoid. (Indeed, if $a, b, c \in G$ then $a \cdot bc = bc = ab \cdot ac$ for $a \in A$ and $a \cdot bc = e = ee = ab \cdot ac$ for $a \in B$ and $e = ax \in \mathrm{Id}(G)$.)

(ii) $\mathrm{Id}(G) = A \cup C$, where $C \subseteq B$ and C is the set of left absorbing elements of G.

(iii) $\mathrm{Id}(G)$ is an ideal of G iff either $C = B$ (i.e., G is idempotent) or $B = G$ (and then G is a left constant groupoid).

(iv) G is idempotent iff $C = B$ (i.e., every element from B is left absorbing).

(v) G is distributive (or delightful, strongly delightful) iff either $B = G$ (and then G is a left constant groupoid) or $C = B$ and $\mathrm{card}(B) \geq 1$ (and then either $A = G$ and G is an RZ-semigroup or $B = \{0\}$ and $G = A[0]$). (Indeed, let G be distributive and $a \in A$. Then $x = ax = aa \cdot x = ax \cdot ax = xx$ for each $x \in G$. If $z \in C$ then $z = zx = az \cdot x = ax \cdot zx = xz$ and z is an absorbing element.)

(vi) G is elastic iff $aa \in B$ for each $a \in B$ (i.e., iff either $B = \emptyset$ or B is a subgroupoid of G).

(vii) p_G is a congruence of G.

(viii) G/p_G is idempotent iff $aa \in B$ for each $a \in B$ (see (vi)).

(ix) o_G is an endomorphism of G iff either $\mathrm{card}(A) = 1$ and $xx \in A$ for each $x \in G$ or $aa \in B$ for every $a \in B$. (Let $e = aa \notin B$ for some $a \in B$. Then, for each $c \in A$, $c = ec = aa \cdot cc = ac \cdot ac = ee = e$. Moreover, for $b \in B$, $bb = e \cdot bb = aa \cdot bb = ab \cdot ab = ee = e$).

1.4 Let G be a groupoid such that $G = A \cup B$, where A is a subgroupoid of G, A is an LD-groupoid, $B \neq \emptyset$ and every element from B is left neutral and right absorbing in G. Then:

(i) G is an LD-groupoid.

(ii) G is distributive iff A is a DI-groupoid satisfying the identities $\mathbf{x} \doteq \mathbf{yx} \cdot \mathbf{x}$ and $\mathbf{xy} \doteq \mathbf{y} \cdot \mathbf{xy}$.

(iii) G is idempotent (or delightful, strongly delightful, elastic) iff A is idempotent.

(iv) p_G is a congruence of G iff p_A is a congruence of A and the set of left neutral elements of A is either empty or a left ideal of A.

(v) $(x, xx) \in p_G$ for every $x \in G$ iff $(a, aa) \in p_A$ for every $a \in A$.

(vi) o_G is an endomorphism of G iff o_A is an endomorphism of A.

1.5 Let G be an LD-groupoid and $n \geq 1$. Put $H = G^{(n)}$ (the set of ordered n-tuples) and define an operation $*$ on H by

$$(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (x_1(x_2(\ldots(x_n y_1))), \ldots, x_1(x_2(\ldots(x_n y_n)))).$$

Then $H(*)$ is an LD-gropoid. Moreover, if G is left cancellative (left divisible) then $H(*)$ is so.

1.6 Let G be an LD-groupoid and $H = \bigcup_{i\geq 1} G^{(i)}$. Define an operation $*$ on H by

$$(x_1, \ldots, x_n) * (y_1, \ldots, y_m) = (x_1(\ldots(x_n y_1)), \ldots, x_1(\ldots(x_n y_m))).$$

Then $H(*)$ is an LD-groupoid. Moreover, if G is left cancellative (left divisible) then $H(*)$ is so.

For $n \geq 1$, let $H_n = \bigcup_{i=1}^{n} G^{(i)}$. Then H_n is a left ideal of $H(*)$ and H_n is left strongly prime.

Similarly, all $G^{(n)}$ are left strongly prime left ideals of $H(*)$.

1.7 Let f be an endomorphism of an LD-groupoid G such that $(f(x), f^2(x)) \in p_G$ for every $x \in G$. Define an operation $*$ on G by $x * y = f(xy)\,(= f(x)\,f(y))$. Then $G(*)$ is again an LD-groupoid. If G is idempotent then $o_{G(*)} = f$, and hence $o_{G(*)}$ is an endomorphism of $G(*)$ and $(x, x * x) \in p_{G(*)}$ for every $x \in G$.

1.8 (i) Let G be an LD-groupoid such that $f = o_G$ is an automorphism of G and $(x, xx) \in p_G$ for every $x \in G$ (i.e., G satisfies $\mathbf{xy} \doteq \mathbf{xx} \cdot \mathbf{y}$). Put $x \circ y = f^{-1}(xy)$ $(= f^{-1}(x)\,f^{-1}(y))$ for all $x, y \in G$. Then $G(\circ)$ is an LDI-groupoid, f is an automorphism of $G(\circ)$, $(f(x), f^2(x)) \in p_{G(\circ)}$ for every $x \in G$ and $xy = f(x \circ y)$ for all $x, y \in G$ (compare with 1.7).

(ii) Let G be an LD-groupoid such that o_G is an injective endomorphism of G. Starting from the imbedding $o(G) \subseteq G$, we can construct a chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_i \subseteq G_{i+1} \subseteq \ldots$ of groupoids isomorphic to G such that $o(G_{i+1}) = G_i$. Then $H = \bigcup_{\geq 0} G_i$ is an LD-groupoid satisfying the same identities as G and o_H is an automorphism of H.

1.9 Proposition. *Let G be an LD-groupoid and $e \notin G$. Then:*

(i) *$G[e]$ is an LD-groupoid.*

(ii) *$G\{e]$ is an LD-groupoid.*

(iii) *$G[e\}$ is an LD-groupoid iff G is an idempotent groupoid satisfying the identities* $\mathbf{xy} \doteq \mathbf{x} \cdot \mathbf{yz}$ *and* $\mathbf{xy} \doteq \mathbf{xy} \cdot \mathbf{x}$.

(iv) *$G\{e\}$ is an LD-groupoid iff G is an idempotent semigroup satisfying* $\mathbf{xy} \doteq \mathbf{xyx}$.

Proof. (i) and (ii) are easy.

(iii) Assume that $G[e\}$ is an LD-groupoid. Then $xy = x \cdot ye = xy \cdot xe = xy \cdot x$ and $x = xe = x \cdot ey = xe \cdot xy = x \cdot xy$ for all $x, y \in G$. From this, $x = x(x \cdot xx) = xx$ and $x \cdot yz = xy \cdot xz = (xy \cdot xz = (xy \cdot x)(xy \cdot z) = (xy)(xy \cdot z) = xy$.

(iv) Use 1.29(ii). □

IV.2 Group constructions of left distributive groupoids

2.1 Let f be an endomorphism of a group G, $g(x) = xf(x)^{-1}$ for every $x \in G$, let $a \in G$ and let $x * y = g(x)\,f(y)\,a$ $(= xf(x)^{-1}f(y)\,a = xf(x^{-1}y)\,a)$ for all $x, y \in G$.

Now, $x * (y * z) = x * (y * z) = x * (g(y) f(z) a) = g(x) fg(y) f^2(z) f(a)$ and $(x * y) * (x * z) = (g(x) f(y) a) * (g(x) f(z) a) = g(g(x) f(y) a) fg(x) f^2(z) f(a)$. Consequently, $x * (y * z) = (x * y) * (x * z)$ iff $g(x) fg(y) = g(g(x) f(y) a) fg(x)$. However, $g(x) fg(y) = xf(x)^{-1} f(y) f^2(y)^{-1}$ and $g(g(x) f(y) a) fg(x) = g(xf(x)^{-1} f(y) a) f(xf(x)^{-1}) = xf(x)^{-1} f(y) af(a)^{-1} f^2(y)^{-1} f^2(x) f(x)^{-1} f(x) f^2(x)^{-1} = xf(x)^{-1} f(y) af(a)^{-1} f^2(y)^{-1}$. Thus we have proved the following assertion (the other assertions are also easy to check):

(i) $G(*)$ is an LD-groupoid iff $f(a) = a$.
(ii) $x * x = xa$ for every $x \in G$.
(iii) $G(*)$ is a D-groupoid iff $a = 1$ and $fg(x) fg(y) = fg(y) fg(x)$ for all $x, y \in G$.
(iv) Either $\mathrm{Id}(G(*)) = \emptyset$ or $a = 1$ and $G(*)$ is idempotent.
(v) $G(*)$ is a regular groupoid.
(vi) $G(*)$ is left (right) cancellative iff f (g) is injective.
(vii) $G(*)$ is left (right) divisible iff f (g) is projective.
(viii) $o_{G(*)}$ is an endomorphism of $G(*)$ iff $g(a) f(x) = f(x) g(a)$ for every $x \in G$.
(ix) $(u, v) \in p_{G(*)}$ iff $f(u^{-1}v) = u^{-1}v$.
(x) If $f(a) = a$ (see (i)) then $p_{G(*)}$ is a congruence of $G(*)$.

2.2 Let f be an endomorphism of a group G, let $a \in G$ and let $x * y = xf(y) af(x)^{-1}$ for all $x, y \in G$.

Now, $x * (y * z) = x * (yf(z) af(y)^{-1}) = xf(y) f^2(z) f(a) f^2(y)^{-1} af(x)^{-1}$ and $(x * y) * (x * z) = (xf(y) af(x)^{-1}) * (xf(z) af(x)^{-1}) =$
$xf(y) af(x)^{-1} f(x) f^2(z) f(a) f^2(x)^{-1} af^2(x) f(x) f(a)^{-1} f^2(y)^{-1} f(x)^{-1} =$
$xf(y) af^2(z) f(a) f^2(x)^{-1} af^2(x) f(a)^{-1} f^2(y)^{-1} f(x)^{-1}$. Consequently, $x * (y * z) = (x * y) * (x * z)$ iff $f^2(z) f(a) f^2(y)^{-1} a = af^2(z) f(a) f^2(x)^{-1} af^2(x) f(a)^{-1} f^2(y)^{-1}$, or equivalently, $f^2(z)^{-1} a^{-1} f^2(z) \cdot f(a) \cdot f^2(y)^{-1} af^2(y) = f(a) \cdot f^2(x)^{-1} af^2(x) \cdot f(a)^{-1}$.

If $af^2(u) = f^2(u) a$ for every $u \in G$ then the equality $x * (y * z) = (x * y) * (x * z)$ is equivalent to $a^{-1} f(a) a = f(a) af(a)^{-1}$, which is the same as $af(a) a = f(a) af(a)$.

Conversely, if $G(*)$ is an LD-groupoid then $1 * (1 * 1) = (1 * 1) * (1 * 1)$ implies $a^{-1} f(a) a = f(a) af(a)^{-1}$ and $1 * (1 * z) = (1 * 1) * (1 * z)$ implies $f^2(z) a^{-1} f^2(z) f(a) a = f(a) af(a)^{-1} = a^{-1} f(a) a, f^2(z) a^{-1} f^2(z) = a^{-1}$ and $a^{-1} f^2(z) = f^2(z) a^{-1}$ for every $z \in G$; of course then also $af^2(u) = f^2(u) a$ for every $u \in G$.

(i) $G(*)$ is an LD-groupoid iff $af(a) a = f(a) af(a)$ and $af^2(u) = f^2(u) a$ for every $u \in G$.
(ii) $x * x = xf(x) af(x)^{-1}$ for every $x \in G$.
(iii) $G(*)$ is idempotent iff $a = 1$.
(iv) $G(*)$ is left regular.
(v) $G(*)$ is left cancellative (left divisible) iff f is injective (projective).

2.3 Let f be an endomorphism of a group G and let $a \in G$.

(i) If $x * y = axf(x)^{-1} f(y)$ $(= axf(x^{-1}y))$ for all $x, y \in G$ then $G(*)$ is an LD-groupoid iff $f(a) = a$ and $auf(u)^{-1} = uf(u)^{-1} a$ for every $u \in G$.
(ii) If $x * y = xaf(x)^{-1} f(y)$ $(= xaf(x^{-1}))$ then $G(*)$ is an LD-groupoid iff $f(a) = a$.

(iii) If $x * y = xf(x)^{-1} af(y)$ then $G(*)$ is an LD-groupoid iff $f(a) = a$ and $af(f(u)u^{-1}) = f(f(u)u^{-1})a$ for every $u \in G$.

(iv) If $x * y = axf(y)f(x)^{-1}$ $(= axf(yx^{-1}))$ then $G(*)$ is an LD-groupoid iff $f(a) = a$ and $a \in Z(G)$ (the centre of G).

(v) If $x * y = xaf(y)f(x)^{-1}$ $(= xaf(yx^{-1}))$ then $G(*)$ is an LD-groupoid iff $f(a) = a$ and $af(u) = f(u)a$ for every $u \in G$.

(vi) If $x * y = xf(y)f(x)^{-1}a$ $(= xf(yx^{-1})a)$ then $G(*)$ is an LD-groupoid iff $f(a) = a$ and $af(u) = f(u)a$ for every $u \in G$.

IV.3 One particular example

3.1 Throughout this section, let F be a free group with an infinite countable basis $\{a_1, a_2, a_3, \dots\}$.

For every $i \geq 1$, define endomorphisms s_i and t_i of F by $s_i(a_i) = a_i a_{i+1} a_i^{-1}$, $t_i(a_i) = a_{i+1}$, $s_i(a_{i+1}) = a_i$, $t_i(a_{i+1}) = a_{i+1}^{-1} a_i a_{i+1}$ and $s_i(a_j) = t_i(a_j) = a_j$ for every $j \geq 1, j \neq i, i+1$.

Clearly, $s_i t_i(a_k) = a_k = t_i s_i(a_k)$ for each $k \geq 1$ and this shows that s_i, t_i are mutually inverse automorphisms of F.

Let S denote the subgroup generated by all s_i in the automorphism group of F and let T be the subgroup generated by s_j, $j \geq 2$.

3.2 Lemma. *$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for every $i \geq 1$.*

Proof. Put $f = s_i s_{i+1} s_i$ and $g = s_{i+1} s_i s_{i+1}$. Then $f(a_j) = a_j = g(a_j)$ for $j \neq i, i+1, i+2$ and $f(a_i) = s_i s_{i+1}(a_i a_{i+1} a_i^{-1}) = s_i(a_i a_{i+1} a_{i+2} a_{i+1}^{-1} a_i^{-1}) = a_i a_{i+1} a_{i+2} a_{i+1}^{-1} a_i^{-1} = s_{i+1}(a_i a_{i+1} a_i^{-1}) = s_{i+1} s_i(a_i) = g(a_i)$, $f(a_{i+1}) = s_i s_{i+1}(a_i) = s_i(a_i) = a_i a_{i+1} a_i^{-1} = s_{i+1}(a_i a_{i+2} a_i^{-1}) = s_{i+1} s_i(a_{i+1} a_{i+2} a_{i+1}^{-1}) = g(a_{i+1})$, $f(a_{i+2}) = s_i s_{i+1}(a_{i+2}) = s_i(a_{i+1}) = a_i = s_{i+1}(a_i) = s_{i+1} s_i(a_{i+1}) = g(a_{i+2})$. □

3.3 Lemma. *$s_i s_j = s_j s_i$ for all $1 \leq i < i + 2 \leq j$.*

Proof. Similar to that of 3.2. □

3.4 Lemma. *Let $i \geq 1$ and let n be an integer. Then $s_i(a_i^n) = a_i a_{i+1}^n a_i^{-1}$, $s_i(a_{i+1}^n) = a_i^n$ and $s_i(a_j^n) = a_j^n$ for every $j \geq 1, j \neq i, i+1$.*

Proof. The equalities follow easily from the definition of s_i. □

3.5 Every element $w \in F$, $w \neq 1$, has a uniquely determined reduced form

$$w = a_{i_1}^{k_1} \dots a_{i_n}^{k_n},$$

where $n \geq 1$, $k_1, \dots, k_n$ are non-zero integers and $i_1 \neq i_2 \neq i_3 \neq \dots \neq i_n$.

Now, let W (V) denote the set of $w \in F$, $w \neq 1$, such that $i_1 \neq 1 \neq i_n$ $(i_1 \neq 1, 2, i_n \neq 1, 2)$.

3.6 Lemma. *If $i \neq 1$ then $s_i(W) = W$.*

Proof. Let E be the subgroup of F generated by $\{a_2, a_3, \dots\}$ and let $E^* = E - \{1\}$. Clearly, $s_i(E^*) = E^* = t_i(E^*)$. Now, let $w \in W$. Then just one of the following cases takes place:

(i) $w \in E^*$ and $s_i(w)$, $t_i(w) \in E^* \subseteq W$.

(ii) $w = u_1 a_1^{n_1} u_2 a_1^{n_2} u_3 \dots a_1^{n_k} u_{k+1}$, where $k \geq 1$, $u_1, \dots, u_{k+1} \in E^*$ and $n_1, \dots, n_k$ are non-zero integers. Then $s_i(w) = s_i(u_1)\, a_1^{n_1}\, s_i(u_2)\, a_1^{n_2} \dots a_1^{n_k}\, s_i(u_{k+1}) \in W$ and, similarly, $t_i(w) \in W$. $\square$

3.7 Lemma. $s_1(a_1 W a_1^{-1}) \subseteq a_1 W a_1^{-1}$.

Proof. Let $w \in W$. We have to distinguish the following cases:

(i) $w \in V$.
Then $s_1(w) \in V$, $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} s_1(w)\, a_1 a_2^{-1} a_1^{-1}$, $a_2 a_1^{-1} s_1(w)\, a_1 a_2^{-1} \in W$.

(ii) $w = a_2^k$.
Then $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} a_1^k a_1 a_2^{-1} a_1^{-1} = a_1 a_2 a_1^k a_2^{-1} a_1^{-1}$ and $a_2 a_1^k a_2^{-1} \in W$.

(iii) $w = a_2^k v$, $v \in V$.
Then $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} a_1^k s_1(v)\, a_1 a_2^{-1} a_1^{-1} = a_1 a_2 a_1^{k-1} s_1(v)\, a_1 a_2^{-1} a_1^{-1}$ and $a_2 a_1^{k-1} s_1(v)\, a_1 a_2^{-1} \in W$.

(iv) $w = v a_2^k$, $v \in V$.
Then $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} s_1(v)\, a_1^k a_1 a_2^{-1} a_1^{-1} = a_1 a_2 a_1^{-1} s_1(v)\, a_1^{k+1} a_2^{-1} a_1^{-1}$ and $a_2 a_1^{-1} s_1(v)\, a_1^{k+1} a_2^{-1} \in W$.

(v) $w = a_2^{k_1} a_1^{l_1} a_2^{k_2} a_1^{l_2} \dots a_2^{k_n} a_1^{l_n} a_2^{k_{n+1}}$, $n \geq 1$.
Then $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} a_1^{k_1} a_1 a_2^{l_1} a_1^{-1} a_1^{k_2} a_1 a_2^{l_2} a_1^{-1} \dots a_1^{k_n} a_1 a_2^{l_n} a_1^{-1} a_1^{k_{n+1}} a_1 a_2^{-1} a_1^{-1} = a_1 a_2 a_1^{k_1} a_2^{l_1} a_1^{k_2} a_2^{l_2} \dots a_1^{k_n} a_2^{l_n} a_1^{k_{n+1}} a_2^{-1} a_1^{-1}$ and $a_2 a_1^{k_1} a_2^{l_1} \dots a_1^{k_n} a_2^{l_n} a_1^{k_{n+1}} a_2^{-1} \in W$.

(vi) $w = a_2^{k_1} a_1^{l_1} \dots a_2^{k_n} a_1^{l_n} v$, $n \geq 1$, $v \in V$.
Then $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} a_1^{k_1} a_2^{l_1} a_1^{-1} s_1(v)\, a_1 a_2^{-1} a_1^{-1} = a_1 a_2 a_1^{k_1} a_2^{l_1} \dots a_1^{k_n} a_2^{l_n} a_1^{-1} s_1(v)\, a_1 a_2^{-1} a_1^{-1}$ and $a_2 a_1^{k_1} a_2^{l_1} \dots a_1^{k_n} a_2^{l_n} a_1^{-1} s_1(v)\, a_1 a_2^{-1} \in W$.

(vii) $w = v a_1^{k_1} a_2^{l_2} \dots a_1^{k_n} a_2^{l_n}$, $n \geq 1$, $v \in V$.
Then $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} s_1(v)\, a_1 a_2^{k_1} a_1^{-1} a_1^{l_1} \dots a_1 a_2^{k_n} a_1^{-1} a_1^{l_n} a_1 a_2^{-1} a_1^{-1} = a_1 a_2 a_1^{-1} s_1(v)\, a_1 a_2^{k_1} a_1^{l_1} \dots a_2^{k_1} a_1^{l_n} a_2^{-1} a_1^{-1}$ and $a_2 a_1^{-1} s_1(v)\, a_1 a_2^{k_1} a_1^{l_1} \dots a_2^{k_n} a_1^{l_n} a_2^{-1} \in W$.

(viii) $w = a_2^{k_1} a_1^{l_1} \dots a_2^{k_n} a_1^{l_n} v a_1^{i_1} a_2^{j_1} \dots a_1^{i_m} a_2^{j_m}$, $n \geq 1$, $m \geq 1$, $v \in V$.
Then $s_1(a_1 w a_1^{-1}) = a_1 a_2 a_1^{-1} a_1^{k_1} a_1 a_2^{l_1} a_1^{-1} \dots a_1^{k_n} a_1 a_2^{l_n} a_1^{-1} s_1(v)\, a_1 a_2^{i_1} a_1^{-1} a_1^{j_1} \dots a_1 a_2^{i_m} a_1^{-1} a_1^{j_m} a_1 a_2^{-1} a_1 = a_1 a_2 a_1^{k_1} a_2^{l_1} \dots a_1^{k_n} a_2^{l_n} a_1^{-1} s_1(v)\, a_1 a_2^{i_1} a_1^{j_1} \dots a_2^{i_m} a_1^{j_m} a_2^{-1} a_1$ and $a_2 a_1^{k_1} a_2^{l_1} \dots a_1^{k_n} a_2^{l_1} a_1^{-1} s(v)\, a_1 a_2^{i_1} a_1^{j_1} \dots a_2^{i_m} a_1^{j_m} a_2^{-1} \in W$. $\square$

3.8 Lemma. *Let* $n \geq 1$, $r_1, \dots, r_n \in T$ *and* $r = r_1 s_1 r_2 s_1 \dots r_n s_1 r_{n+1}$. *Then* $r(a_1) \neq a_1$, *and hence* $r \neq \mathrm{id}_F$.

Proof. By induction on n we show that $r(a_1) \in a_1 W a_1^{-1}$. If $n = 1$ then $r_1 s_1 r_2(a_1) = r_1 s_1(a_1) = r_1(a_1 a_2 a_1^{-1}) = a_1 r_1(a_2)\, a_1^{-1} \in a_1 W a_1^{-1}$, since $r_1(a_2) \in W$. Now, let $n \geq 2$ and $s = r_2 s_1 \dots r_n s_1 r_{n+1}$. Then $s(a_1) = a_1 w a_1^{-1}$ for some $w \in W$ and $r(a_1) = r_1 s_1 s(a_1) = r_1(a_1 u a_1^{-1}) = a_1 r_1(u)\, a_1^{-1}$, where $s_1(a_1 w a_1^{-1}) = a_1 u a_1^{-1}$, $u \in W$ and $r_1(u) \in W$ by 3.6 and 3.7. $\square$

3.9 Lemma. *There exists a uniquely determined endomorphism σ of S with the following properties:*
(i) $\sigma(s_i) = s_{i+1}$ *for every* $i \geq 1$.
(ii) $s_1\sigma(s_1)\, s_1 = \sigma(s_1)\, s_1\sigma(s_1)$.
(iii) $s_1\sigma^2(r) = \sigma^2(r)\, s_1$ *for every* $r \in S$.
(iv) σ *is injective and* $\sigma(S) = T$.

Proof. Define an endomorphism s of F by $s(a_j) = a_{j+1}$ for every $j \geq 1$. One may check easily that $s_1 = s_{i+1}s$ for every $i \geq 1$. Now, let $r \in S$, $r = s_{i_1}^{k_1} \dots s_{i_n}^{k_n} = s_{j_1}^{l_1} \dots s_{j_m}^{l_m}$, where $n \geq 1$, $m \geq 1$, $k_1, \dots, k_n, l_1, \dots, l_m \in \{\pm 1\}$. Then $sr = s_{i_1+1}^{k_1} \dots s_{i_n+1}^{k_1} = s_{j_1+1}^{l_1} \dots s_{j_m+1}^{l_1} s$ and this implies that the endomorphisms $r_1 = s_{i_1+1}^{k_1} \dots s_{i_n+1}^{k_n}$ and $r_2 = s_{j_1+1}^{l_1} \dots s_{j_m+1}^{l_m}$ coincide on E. On the other hand, $r_1(a_1) = a_1 = r_2(a_1)$, and hence $r_1 = r_2$. Now, we can put $\sigma(r) = r_1$ and we get an endomorphism satisfying (i), (ii) and (iii) (see 3.2 and 3.3). Clearly, $\sigma(S) = T$. Finally, if $\sigma(p) = \sigma(q)$ for some $p, q \in S$ then $sp = \sigma(p)\, s = \sigma(q)\, s$ and $p = q$, since s is an injective endomorphism of F. □

3.10 Define a binary operation $*$ on S by $p * q = p\sigma(q)\, s_1\sigma(p^{-1})$ for all $p, q \in S$. With respect to 3.9 and 2.2(i), $S(*)$ is an LD-groupoid. We shall prove that the relation $z_{r,S(*)}$ is irreflexive (see 1.23).

Let $p, q, q_1, \dots, q_n \in S$, $n \geq 1$, be such that $p = (((q * q_1) * q_2) * \dots) * q_n$. Then we have $p = q\sigma(q_1)\, s_1\sigma(q^{-1})\, \sigma(q_2)\, s_1\sigma(q * q_1)^{-1} \dots \sigma(q_n)\, s_1\sigma(((q * q_1) * \dots) * q_{n-1})^{-1} = q\sigma(r_1)\, s_1\sigma(r_2)\, s_1 \dots \sigma(r_n)\, s_1\sigma(r_{n+1})$, where $r_1 = q_1$, $r_2 = q^{-1}q_2$, $r_3 = (q * q_1)^{-1}q_3, \dots, r_n = ((((q * q_1) * q_2) * \dots) * q_{n-2})^{-1}q_n$, $r_{n+1} = ((q * q_1) * \dots) * q_{n-1})^{-1}$. From this, $\mathrm{id}_F = p^{-1}qr$, where $r = \sigma(r_1)\, s_1\sigma(r_2)\, s_1 \dots \sigma(r_n)\, s_1\sigma(r_{n+1})$. Clearly, $\sigma(r_i) \in T$ and $p^{-1}q \neq \mathrm{id}_F$ by 3.8. Thus $p \neq q$.

The endomorphism σ is injective and consequently the groupoid $S(*)$ is left cancellative.

IV.4 Two-element left distributive groupoids

4.1 Consider the following six two-element groupoids:

$D(1)$	0	1
0	0	0
1	0	1

$D(2)$	0	1
0	0	1
1	0	1

$D(3)$	0	1
0	0	0
1	1	1

$D(4)$	0	1
0	0	0
1	0	0

$D(5)$	0	1
0	0	1
1	0	0

$D(6)$	0	1
0	1	0
1	1	0

It is easy to check that these six groupoids are pair-wise non-isomorphic LD-groupoids and that every two-element LD-groupoid is isomorphic to one of them. Some properties of the groupoids are listed in the following table:

	D	LSM	RSM	MSM	M	S	C	I	E	Dl	Pi	Pc	Oe	Id	La	Ra	Ln	Rn	G^{op}
$D(1)$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	1	1	1	1	$D(1)$
$D(2)$	+	+	+	+	+	+	−	+	+	+	+	+	+	+	0	2	2	0	$D(3)$
$D(3)$	+	+	+	+	+	+	−	+	+	+	+	+	+	+	2	0	0	2	$D(2)$
$D(4)$	+	+	+	+	+	+	+	−	+	+	+	+	+	+	1	1	0	0	$D(4)$
$D(5)$	−	−	−	−	−	−	−	−	−	−	−	+	+	−	0	1	1	0	−
$D(6)$	−	+	+	+	+	−	−	−	−	−	+	+	+	−	0	0	0	0	−

Explanation: D … distributive; LSM … left semimedial; RSM … right semimedial; MSM … middle semimedial; M … medial; S … associative; C … commutative; I … idempotent; E … elastic; Dl … delightful; Pi … $(x, xx) \in p_G$ for every $x \in G$ (i.e., $\mathbf{xy} \doteq \mathbf{xx} \cdot \mathbf{y}$); Pc … p_G is a congruence of G; Oe … o_G is an endomorphism of G (i.e., $\mathbf{x} \cdot \mathbf{yy} \doteq \mathbf{xx} \cdot \mathbf{yy}$); Id … $\mathrm{Id}(G)$ is an ideal of G; La (Ra) … the number of left (right) absorbing elements; Ln (Rn) … the number of left (right) neutral elements; G^{op} … the opposite groupoid is isomorphic to … (only in the two-sided distributive case).

IV.5 Three-element left distributive idempotent groupoids

5.1 Consider the following seventeen three-element groupoids:

$D(7)$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$D(8)$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$D(9)$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

$D(10)$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

$D(11)$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

$D(12)$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

$D(13)$	0	1	2
0	0	0	0
1	1	1	1
2	0	0	2

$D(14)$	0	1	2
0	1	0	0
1	0	1	0
2	0	1	2

$D(15)$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

$D(16)$	0	1	2
0	0	2	1
1	0	1	2
2	0	1	2

$D(17)$	0	1	2
0	0	0	0
1	2	1	1
2	1	2	2

$D(18)$	0	1	2
0	0	1	2
1	0	1	0
2	0	1	2

$D(19)$	0	1	2
0	0	0	0
1	1	1	1
2	2	0	2

$D(20)$	0	1	2
0	0	0	0
1	1	1	1
2	0	1	2

$D(21)$	0	1	2
0	0	0	0
1	1	1	1
2	1	0	2

$D(22)$	0	1	2
0	0	1	2
1	0	1	1
2	0	2	2

$D(23)$	0	1	2
0	0	1	2
1	1	1	1
2	0	1	2

(i) $D(7) \cong D(1)[e]$ $(\cong D(1)\{e\})$, $D(9) \cong D(3)[e]$ and $D(10) \cong D(2)[e]$ are LD-groupoids by 1.9(i) ($D(7)$ is a semilattice).
(ii) $D(12) \cong D(2)\{e]$, $D(22) \cong D(3)\{e]$ and $D(23) \cong D(1)\{e]$ are LD-groupoids by 1.9(ii) ($D(12)$ is an RZ-semigroup).
(iii) $D(11) \cong D(3)[e\}$ is an LZ-semigroup.
(iv) $D(20) \cong D(3)\{e\}$ is an LD-groupoid by 1.9(iv).
(v) $D(8)$ is a subdirect product of two copies of $D(1)$, and hence it is a semilattice.
(vi) $D(13)$ is a subdirect product of $D(1)$ and $D(3)$, and hence it is a D-semigroup.
(vii) $D(14)$ is a subdirect product of $D(1)$ and $D(2)$, and hence it is a D-semigroup.
(viii) $D(15)$ is an IM-quasigroup, $D(16)$, $D(17)$, $D(18)$, $D(19)$ are IM-groupoids ($D(17) = D(16)^{\mathrm{op}}$ and $D(19) = D(18)^{\mathrm{op}}$).
(ix) $D(21)$ is an LDI-groupoid (since 0, 1 are left absorbing, it is enough to show that $2 \cdot yz = 2y \cdot 2z$).
(x) All the groupoids $D(7), \ldots, D(23)$ are LDI-groupoids and possess the following properties (see p. 72).

Explanation: See 4.1; $Si \ldots$ G is subdirectly irreducible. Notice that an idempotent groupoid is left (right) semimedial iff it is left (right) distributive.

(xi) Assigning the ordered quadruple (La, Ra, Ln, Rn) to each of the groupoids $D(i)$ (see the foregoing table), we see that these groupoids are pair-wise non-isomorphic with possible exceptions of the pairs $D(13)$, $D(21)$ and $D(18)$, $D(23)$. However $D(13)$, $D(18)$ are right distributive and $D(21)$, $D(23)$ are not. Thus we have shown that the groupoids $D(7), \ldots, D(23)$ are pair-wise non-isomorphic.

In the remaining part of this section, we show that every three-element LDI-groupoid is isomorphic to one of the groupoids $D(7), \ldots, D(23)$.

	D	MSM	M	S	C	Pc	Si	La	Ra	Ln	Rn	G^{op}
$D(7)$	+	+	+	+	+	+	−	1	1	1	1	$D(7)$
$D(8)$	+	+	+	+	+	+	−	1	1	0	0	$D(8)$
$D(9)$	+	+	+	+	−	+	+	1	1	0	2	$D(10)$
$D(10)$	+	+	+	+	−	+	+	1	1	2	0	$D(9)$
$D(11)$	+	+	+	+	−	+	−	3	0	0	3	$D(12)$
$D(12)$	+	+	+	+	−	+	−	0	3	3	0	$D(11)$
$D(13)$	+	+	+	+	−	+	−	2	0	0	1	$D(14)$
$D(14)$	+	+	+	+	−	+	−	0	2	1	0	$D(13)$
$D(15)$	+	+	+	−	+	+	+	0	0	0	0	$D(15)$
$D(16)$	+	+	+	−	−	+	+	0	1	2	0	$D(17)$
$D(17)$	+	+	+	−	−	+	+	1	0	0	2	$D(16)$
$D(18)$	+	+	+	−	−	+	+	0	2	2	0	$D(19)$
$D(19)$	+	+	+	−	−	+	+	2	0	0	2	$D(18)$
$D(20)$	−	−	−	+	−	+	+	2	0	1	1	−
$D(21)$	−	−	−	−	−	+	+	2	0	0	1	−
$D(22)$	−	−	−	−	−	+	+	0	1	1	0	−
$D(23)$	−	+	−	−	−	−	+	0	2	2	0	−

5.2 Lemma. *Let G be a three-element LD-groupoid such that* $\mathrm{Id}(G) \neq \emptyset$ *and G contains no left and no right absorbing elements. Then G is commutative, distributive and idempotent.*

Proof. Let $G = \{a, b, c\}$. Since $\mathrm{Id}(G)$ is a left ideal of G and G contains no right absorbing elements, $\mathrm{Id}(G)$ possesses at least two elements, say $a, b \in \mathrm{Id}(G)$. Now, we have to distringuish the following cases:

(i) Let $\mathrm{Id}(G) = \{a, b\}$. We can further assume that $cc = a$. Since $\mathrm{Id}(G)$ is a left ideal, we have $ab, ba, ca, cb \in \{a, b\}$ and $a = aa = a \cdot cc = ac \cdot ac$ implies $ac \in \{a, c\}$. If $ac = a$ then, since a is not left absorbing, $ab = b$ and $a \cdot cb = ac \cdot ab = a \cdot ab = ab = b$, $cb = b$ and b is right absorbing, a contradiction. Hence $ac = c$ and $ca = c \cdot cc = cc \cdot cc = aa = a$, and so $ba = b$, since a is not right absorbing. On the other hand, $bc = b \cdot ac = ba \cdot bc = b \cdot bc$. Since b is not left absorbing, we have $bc \in \{a, c\}$. If $bc = a$ then $a = bc = b \cdot ba = b$, a contradiction. Hence $bc = c$ and $b = ba = b \cdot cc = bc \cdot cc = a$, again a contradiction.

(ii) Let G be idempotent and $ac = b$. Then $a \cdot ca = ac \cdot a = ba$ and $cb = c \cdot ac = ca \cdot c$. If $ca = a$ then $a = ba$ and a is right absorbing, a contradiction. If $ca = b$ then $ab = a \cdot ca = ac \cdot a = ba$, $cb = c \cdot ac = ca \cdot c = bc$ and G is commutative. Finally, if $ca = c$ then $cb = c \cdot ac = ca \cdot c = cc = c$ and c is left absorbing, a contradiction.

(iii) Let G be idempotent and $ac = c$, $bc = a$. Then $a = a \cdot bc = ab \cdot ac = ab \cdot c$, and so $ab = b$. Further, $a = bc = b \cdot ac = ba \cdot bc = ba \cdot a$, $b = bb = b \cdot ab = ba \cdot b$ and $ba \neq c$, since b is not right absorbing. If $ba = b$ then $a = ba \cdot a = ba = b$, a contradition. Hence $ba = a$. If $cb = c$ then, since a is not right absorbing, we have $ca = b$ and $c = cb = c \cdot ab = ca \cdot cb = bc = a$, a contradiction. If $cb = a$ then $a = ba = b \cdot cb = bc \cdot b = ab = b$, again a contradiction. Finally, if $cb = b$ then b is right absorbing and this is not possible.

(iv) Let G be idempotent and $ac = c$, $bc = b$. If $ba = a$ then $b = bc = b \cdot ac = ba \cdot bc = ab$, $b \cdot ca = bc \cdot ba = ba = a$, and hence $ca = a$ and a is right absorbing, a contradiction. If $ba = b$ then b is left absorbing, a contradiction. Thus $ba = c$ and $b = bc = b \cdot ac = ba \cdot bc = cb$, $ab = a \cdot bc = ab \cdot ac = ab \cdot c$, and therefore $ab = c$. From this, $ca = ab \cdot a = a \cdot ba = ac = c$ and G is commutative.

(v) Let G be commutative. If $ac = c$ then $bc \neq c$ (since c is not right absorbing) and either (iii) or (iv) applies. If $ac = b$ then (ii) applies. Finally, if $ac = a$ then $ab \neq a$ and, replacing c by b, we can proceed in the same way as in (ii), (iii) and (iv). □

5.3 Lemma. *Let G be a three-element LDI-groupoid containing an absorbing element. Then G is distributive.*

Proof. Let $G = \{a, b, c\}$, where a is the absorbinbg element, and let $x, y, z \in G$. If $a \in \{x, y, z\}$ then $xy \cdot z = a = xz \cdot yz$. Hence, assume $\{x, y, z\} \subseteq \{b, c\}$. However, then one of the following cases takes place: $bb \cdot b = bb \cdot bb$, $bb \cdot c = bc \cdot bc$, $bc \cdot b = bb \cdot cb$, $bc \cdot c = bc \cdot cc$, $cb \cdot b = cb \cdot bb$, $cb \cdot c = cc \cdot b = cb \cdot cb$, $cc \cdot c = cc \cdot cc$. □

5.4 Lemma. *Let G be a three-element LDI-groupoid containing at least two left absorbing elements. Then G is isomorphic to one of the groupoids $D(11)$, $D(13)$, $D(19)$, $D(20)$, $D(21)$.*

Proof. Let $G = \{a, b, c\}$, where a, b are left absorbing. If $(ca, cb) = (c, c)$ then $G \cong D(11)$; if $(ca, cb) = (a, a)$ or $(ca, cb) = (b, b)$ then $G \cong D(13)$; if $(ca, cb) = (c, a)$ or $(ca, cb) = (b, c)$ then $G \cong D(19)$; if $(ca, cb) = (a, b)$ then $G \cong D(20)$; if $(ca, cb) = (b, a)$ then $G \cong D(21)$. If $ca = a$ and $cb = c$ then $c = cb = c \cdot ba = cb \cdot ca = ca = a$, a contradiction. If $ca = c$, $cb = b$ then $c = ca = c \cdot ab = ca \cdot cb = cb = b$, again a contradiction. □

5.5 Lemma. *Let G be a three-element LDI-groupoid containing just one left absorbing element and no right absorbing elements. Then $G \cong D(17)$.*

Proof. Let $G = \{a, b, c\}$ and let a be the only left absorbing element. Since a is not right absorbing, we can assume that $ca \neq a$. Now, let us distinguish the following cases:

(i) Let $ca = ba = b$. Then $b = ba \in b \cdot ac = ba \cdot bc = b \cdot bc$, and so $bc = a$, since b is not left absorbing. Further, $b = ba = b \cdot ca = bc \cdot ba = ab = a$, a contradiction.

(ii) Let $ca = b$ and $ba = c$. Then $c = ba = b \cdot ab = ba \cdot b = cb$ and $c \cdot bc = cb \cdot c = c$. Hence $bc \in \{b, c\}$. If $bc = b$ then $G \cong D(17)$. If $bc = c$ then $b = bb = b \cdot ca = bc \cdot ba = cc = c$, a contradiction.

(iii) Let $ca = b$ and $ba = a$. Then $b = bb = b \cdot ca = bc \cdot ba = bc \cdot a$, and hence $bc = c$. Then also $b = ca = c \cdot ac = ca \cdot c = bc = c$, a contradiction.

(iv) Let $ca = cb = c$. Then c is left absorbing, a contradiction.

(v) Let $ca = c$ and $cb = a$. Then $c \cdot ba = cb \cdot ca = ac = a$, and so $ba = b$. Since b is not left absorbing, we have $bc \in \{a, c\}$. If $bc = a$ then $c = ca = c \cdot bc = cb \cdot c = ac = a$, a contradiction. If $bc = c$ then $a = cb = bc \cdot b = b \cdot cb = ba = b$, again a contradiction.

(vi) Let $ca = c$ and $cb = b$. If $bc = b$ then $c \cdot ba = cb \cdot ca = bc = b$, $ba = b$ and b is left absorbing, a contradiction. If $bc = c$ then $c \cdot ab = ca \cdot cb = cb = b$, and hence $b = ab = a$, a contradiction. Finally, if $bc = a$ then $c = ca = c \cdot bc = cb \cdot c = bc = a$, a contradiction. □

5.6 Lemma. *Let G be a three-element LDI-groupoid containing a right absorbing element but no left absorbing elements. Then G is isomorphic to one of the groupoids $D(12)$, $D(14)$, $D(16)$, $D(18)$, $D(22)$, $D(23)$.*

Proof. Let $G = \{a, b, c\}$, a being right absorbing. Since a is not left absorbing, we can assume that $ac \neq a$. Now, consider the following cases:

(i) Let $ac = b$. Then $cb = c \cdot ac = ca \cdot c = ac = b$. If $bc = a$ then $b = bb = b \cdot ac = ba \cdot bc = aa = a$, a contradiction. If $bc = b$ and $ab = a$ then $b = bb = b \cdot ac = ba \cdot bc = ba \cdot b = ab = ba = a$, a contradiction. If $bc = c$ and $ab = c$ then $c = ab = a \cdot bc = ab \cdot ac = cb = b$, a contradiction. If $bc = b$ and $ab = b$ then $G \cong D(14)$. If $bc = c$ and $ab = a$ then $b = ac = a \cdot bc = ab \cdot ac = a \cdot ac = ab = a$, a contradiction. If $bc = c$ and $ab = b$ then $G \cong D(18)$. If $bc = c$ and $ab = c$ then $G \cong D(16)$.

(ii) Let $ac = c$ and $ab = a$. Then $bc = b \cdot ac = ba \cdot bc = a \cdot bc = ab \cdot ac = ac = c$ and $a \cdot cb = ac \cdot ab = ca = a$. From this, it follows that $cb \in \{a, b\}$. If $cb = a$ then $G \cong D(14)$. If $cb = b$ then $G \cong D(23)$.

(iii) Let $ac = c$, $ab = b$ and $cb = a$. Then $c \cdot bc = cb \cdot c = ac = c$, $bc = c$ and $G \cong D(18)$.

(iv) Let $ac = c$, $ab = b$ and $cb = b$. If $bc = a$ then $G \cong D(18)$. If $bc = b$ then $G \cong D(23)$. If $bc = c$ then $G \cong D(12)$.

(v) Let $ac = c$, $ab = b$ and $cb = c$. If $bc = a$ then $a = ca = c \cdot bc = cb \cdot c = cc = c$, a contradiction. If $bc = b$ then $G \cong D(22)$. If $bc = c$ then $G \cong D(23)$.

(vi) Let $ac = ab = c$. Then $bc = b \cdot ab = ba \cdot b = ab = c$ and $a \cdot cb = ac \cdot ab = cc = c$. This implies that $cb \in \{b, c\}$. If $cb = b$, then $G \cong D(18)$. If $cb = c$ then $G \cong D(14)$. □

5.7 Proposition. (i) *The seventeen groupoids $D(7), \ldots, D(23)$ are pair-wise non-isomorphic three-element LDI-groupoids.*

(ii) *Every three-element LDI-groupoid is isomorphic to one of $D(7), \ldots, D(23)$.*

Proof. (i) See 5.1.

(ii) Let $G = \{a, b, c\}$ be a three-element LDI-groupoid. The rest of this point is divided into five parts:

(a) Let G contain an absorbing element. By 5.3, G is a DI-groupoid and we can assume that a is absorbing in G. If $\{b, c\}$ is a subgroupoid of G then G is isomorphic to one of $D(7)$, $D(9)$, $D(10)$. Hence, let $\{b, c\}$ be not a subgroupoid of G and let $bc = a$ (the other case, $cb = a$, being similar). Then $a = ca = c \cdot bc = cb \cdot c$ and $cb \neq c$. If $cb = b$ then $b = b \cdot cb = bc \cdot b = ab = a$, a contradiction. Thus $cb = a$ and $G \cong D(8)$.

(b) Let G contain no left and no right absorbing elements. By 5.2, G is a CDI-groupoid. If G is not subdirectly irreducible then G is a subdirect product of copies of $D(1)$ (since $D(1)$ is up to isomorphism the only two-element CDI-groupoid), and hence G is a semilattice. But every finite semilattice contains an absorbing element and we have proved that G is subdirectly irreducible. If $p_G \neq \mathrm{id}_G$, say $(a, b) \in p_G$ then $a = aa = ba = ab = bb = b$, a contradiction. Thus $p_G = q_G = \mathrm{id}_G$ and $\mathscr{C}_l(G) \neq \emptyset$ by II.4.9; we can assume that $a \in \mathscr{C}_l(G)$. Then $L_a = R_a$ is a permutation. If a is a neutral element of G then $bc = a$ (otherwise either b or c would be absorbing) and $a = bc = b \cdot ac = ba \cdot bc = b \cdot bc = ba = b$, a contradiction. Thus a is not neutral and we have $ab = c = ba$, $ac = b = ca$. Further, $a \cdot bc = ab \cdot ac = cb = bc$, which implies $bc = a = cb$. Now, it is clear that $G \cong D(15)$.

(c) Let G contain at least two left absorbing elements. By 5.4, G is isomorphic to one of $D(11)$, $D(13)$, $D(19)$, $D(20)$, $D(21)$.

(d) Let G contain just one left absorbing element but no right absorbing element. By 5.5, $G \cong D(17)$.

(e) Let G contain at least one right absorbing element but no left absorbing element. By 5.6, G is isomorphic to one of $D(12)$, $D(14)$, $D(16)$, $D(18)$, $D(22)$, $D(23)$. □

IV.6 Three-element left distributive groupoids with two idempotent elements

6.1 Consider the following twelve three-element groupoids:

$D(24)$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

$D(25)$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	1

$D(26)$	0	1	2
0	0	0	0
1	1	1	1
2	0	0	0

$D(27)$	0	1	2
0	0	1	0
1	0	1	0
2	0	1	0

$D(28)$	0	1	2
0	0	0	0
1	0	1	2
2	0	0	0

$D(29)$	0	1	2
0	0	1	2
1	0	1	1
2	0	1	1

$D(30)$	0	1	2
0	0	0	0
1	0	1	2
2	1	1	1

$D(31)$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	1

$D(32)$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	1

$D(33)$	0	1	2
0	0	0	2
1	0	1	2
2	0	0	0

$D(34)$	0	1	2
0	0	1	2
1	0	1	0
2	0	1	0

$D(35)$	0	1	2
0	0	1	2
1	0	1	2
2	0	0	0

(i) $D(24)$ and $D(25)$ are subdirect products of $D(1)$ and $D(4)$, and so $D(24)$, $D(25)$ are CD-semigroups. Moreover, $D(25) \cong D(4)[e]$.

(ii) $D(26)$ ($D(27)$) is a subdirect product of $D(3)$ ($D(2)$) and $D(4)$, and so $D(26)$ ($D(27)$) is a D-semigroup.

(iii) $D(28)$ is a medial LD-semigroup (0,2 are left constant, 0 is right absorbing and 1 is left neutral).

(iv) $D(29) \cong D(4)\{e]$ is a medial LD-groupoid (see 1.9(ii); 0,1 are right absorbing, 0 is left neutral and $(1, 2) \in p_G$).

(v) $D(30)$ is an LD-groupoid (0,2 are left constant and 1 is left neutral).

(vi) $D(31) \cong D(5)[e]$ is an LD-groupoid by 1.9(i).

(vii) $D(32) \cong D(5)\{e]$ is an LD-groupoid by 1.9(ii). Moreover, it is a subdirect product of $D(2)$ and $D(5)$.

(viii) $D(33)$ is subdirect product of $D(1)$ and $D(5)$, and therefore $D(33)$ is an LD-groupoid.

(ix) $D(34)$ is an LD-groupoid (0,1 are right absorbing, 0 is left neutral, $(1, 2) \in p_G$, $2 \cdot y2 = 2y \cdot 22 = 0$ for every $y \in G$).

(x) $D(35)$ is an LD-groupoid (0 is right absorbing, 0,1 are left neutral and 2 is left constant).

(xi) All the groupoids $D(24), \ldots, D(35)$ are LD-groupoids with $\mathrm{card}(\mathrm{Id}(G)) = 2$ and they possess the properties listed in the following table:

	D	LSM	RSM	MSM	M	S	C	E	Dl	Pi	Pc	Ol	Id	Si	La	Ra	Ln	Rn	G^{op}
$D(24)$	+	+	+	+	+	+	+	+	+	+	+	+	+	−	1	1	0	0	$D(24)$
$D(25)$	+	+	+	+	+	+	+	+	+	+	+	+	+	−	1	1	0	0	$D(25)$
$D(26)$	+	+	+	+	+	+	−	+	+	+	+	+	+	−	2	0	0	0	$D(27)$
$D(27)$	+	+	+	+	+	+	+	−	+	+	+	+	+	−	0	2	0	0	$D(26)$
$D(28)$	−	+	+	+	+	+	−	+	−	+	+	+	−	+	1	1	1	0	−
$D(29)$	−	+	+	+	+	−	−	−	−	+	+	+	−	+	0	2	1	0	−
$D(30)$	−	−	−	−	−	−	−	−	−	−	+	−	−	+	1	0	1	0	−
$D(31)$	−	−	−	−	−	−	−	−	−	−	+	+	−	+	1	1	1	0	−
$D(32)$	−	−	−	−	−	−	−	−	−	−	+	+	−	−	0	2	2	0	−
$D(33)$	−	−	−	−	−	−	−	−	−	−	+	+	−	−	0	1	1	0	−
$D(34)$	−	−	−	−	−	−	−	−	−	−	−	+	−	+	0	2	1	0	−
$D(35)$	−	−	−	−	−	−	−	−	−	−	−	−	−	+	0	1	2	0	−

Explanation: See 4.1 and 5.1.

(xii) Considering the items of the foregoing table and taking into account that they are invariant under isomorphisms, we see easily that the groupoids $D(24), \ldots, D(35)$ are pair-wise non-isomorphic with possible exception of $D(24)$, $D(25)$. However, 0 is absorbing in the both groupoids, 0 appears seven times in the table of $D(24)$ and only five times in the table of $D(25)$. Consequently, $D(24)$ and $D(25)$ are not isomorphic.

In the remaining part of this section, we show that $D(24), \ldots, D(35)$ are (up to isomorphism) the only three-element LD-grupoids having just two idempotents.

6.2 Lemma. *Let G be a three-element LD-groupoid such that G contains an absorbing element and* $\operatorname{card}(\operatorname{Id}(G)) = 2$. *Then G is isomorphic to one of* $D(24)$, $D(25)$, $D(28)$, $D(31)$.

Proof. Let $G = \{a, b, c\}$, where a is absorbing, $bb = b$ and $cc \neq c$. Since $\operatorname{Id}(G)$ is a left ideal, we have $cb \in \{a, b\}$. Now, we shall distinguish the following cases:

(i) Let $cc = a$ and $bc = a$. Then $b \cdot cb = bc \cdot b = ab = a$, and hence $cb = a$ and $G \cong D(24)$.

(ii) Let $cc = a$ and $bc = b$. Then $a = ba = b \cdot cc = bc \cdot bc = bb = b$, a contradiction.

(iii) Let $cc = a$ and $bc = c$. Then $a = a \cdot cb = cc \cdot cb = c \cdot cb$, and hence $cb = a$ and $G \cong D(28)$.

(iv) Let $cc = b$. Then $cb = c \cdot cc = cc \cdot cc = bb = b$, $b = b \cdot cb = bc \cdot b$ and $bc \in \{b, c\}$. If $bc = b$ then $G \cong D(25)$. If $bc = c$ then $G \cong D(31)$. $\square$

6.3 Lemma. *Let G be a three-element LD-groupoid such that G contains at least two left absorbing elements and* $\operatorname{card}(\operatorname{Id}(G)) = 2$. *Then* $G \cong D(26)$.

Proof. Let $G = \{a, b, c\}$, where a, b are left absorbing and $cc \neq c$. We can assume that $cc = a$. Further, $ca, cb \in \operatorname{Id}(G) = \{a, b\}$, $ca = c \cdot cc = cc \cdot cc = aa = a$, $c \cdot cb = cc \cdot cb = a \cdot cb = a$, and hence $cb = a$ and $G \cong D(26)$.

6.4 Lemma. *Let G be a three-element LD-groupoid containing just one left absorbing element, no right absorbing element and such that* $\operatorname{card}(\operatorname{Id}(G)) = 2$. *Then* $G \cong D(30)$.

Proof. Let $G = \{a, b, c\}$, where a is left absorbing, $bb = b$ and $cc \neq c$. Again, $ba, ca, cb \in \{a, b\}$.

(i) Let $cc = a$. Then $ca = c \cdot cc = cc \cdot cc = aa = a$, and, since a is not right absorbing, $ba = b$. On the other hand, $b = ba = b \cdot cc = bc \cdot bc$, $bc = b$ is left absorbing, a contradiction.

(ii) Let $cc = b$ and $bc = b$. Then $bc = c \cdot cc = cc \cdot cc = bb = b$, $c \cdot ba = cb \cdot ca = b \cdot ca = bc \cdot ba = b \cdot ba$. If $ba = a$ then $ca = a$ and a is absorbing, which is not true. Hence $ba = b$ and b is left absorbing, again a contradiction.

(iii) Let $cc = b$ and $bc \neq b$. We have $b = bb = b \cdot cc = bc \cdot bc$, and so $bc = c$. If $ba = b$ then $b = ba = b \cdot ac = ba \cdot bc = bc = c$, a contradiction. Hence $ba = a$, and then $ca = b$, since a is not absorbing. We have proved that $G \cong D(30)$. □

6.5 Lemma. *Let G be a three-element LD-groupoid containing at least two right absorbing elements and such that* $\mathrm{card}(\mathrm{Id}(G)) = 2$. *Then G is isomorphic to one of* $D(27)$, $D(29)$, $D(32)$, $D(34)$.

Proof. Let $G = \{a.b, c\}$, where a, b are right absorbing and $cc = a$. Then $a = aa = a \cdot cc = ac \cdot ac$, and hence $ac \in \{a, c\}$. Similarly, $a = ba = b \cdot cc = bc \cdot bc$ and bc $\in \{a, c\}$. The rest is clear. □

6.6 Lemma. *Let G be a three-element LD-groupoid containing just one right absorbing element, no left absorbing element and such that* $\mathrm{card}(\mathrm{Id}(G)) = 2$. *Then G is isomorphic to one of* $D(33)$, $D(35)$.

Proof. Let $G = \{a, b, c\}$, where a is right absorbing, $bb = b$ and $cc \neq c$. Then $\mathrm{Id}(G) = \{a, b\}$, and so $ab, cb \in \{a, b\}$.

(i) Let $cc = a$ and $ab = b$. Then $cb = a$, since b is not right absorbing and $a = ba = b \cdot cb = bc \cdot b$, so that $bc = c$. Finally, $a = aa = a \cdot cb = ac \cdot ab = ac \cdot b$, $ac = c$ and $G \cong D(35)$.

(ii) Let $cc = a$ and $ab \neq b$. Then $ab = a$, $a = aa = a \cdot cc = ac \cdot ac$, $ac \neq a$, since a is not absorbing, and so $ac = c$. Further, $c \cdot cb = cc \cdot cb = a \cdot cb = ac \cdot ab = ca = a$, $cb \neq b$, $cb = a$, and $bc = b \cdot ac = ba \cdot bc = a \cdot bc = ab \cdot ca = ac = c$. Thus $G \cong D(33)$.

(iii) Let $cc = b$. Then $ab = a \cdot cc = ac \cdot ac$, $b = bb = b \cdot cc = bc \cdot bc$, $cb = c \cdot cc = cc \cdot cc = bb = b$. Since b is not right absorbing, $ab = a$, and so $a = ab = ac \cdot ac$ implies that $ac = a$ and a is left absorbing, a contradiction. □

6.7 Proposition. (i) *The twelve groupoids* $D(24), \ldots, D(35)$ *are pair-wise non-isomorphic three-element LD-groupoids containing just two idempotents.*

(ii) *Every three-element LD-groupoid containing just two idempotents is isomorphic to one of* $D(24), \ldots, D(35)$.

Proof. (i) See 6.1.

(ii) Combine 5.2, 6.2, ..., 6.6. □

IV.7 Three-element unipotent left distributive groupoids

7.1 Consider the following ten three-element groupoids:

$D(36)$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

$D(37)$	0	1	2
0	0	0	0
1	0	0	1
2	0	0	0

$D(38)$	0	1	2
0	0	0	1
1	0	0	0
2	0	0	0

$D(39)$	0	1	2
0	0	0	1
1	0	0	1
2	0	0	0

$D(40)$	0	1	2
0	0	1	0
1	0	0	0
2	0	1	0

$D(41)$	0	1	2
0	0	1	1
1	0	0	0
2	0	0	0

$D(42)$	0	1	2
0	0	1	2
1	0	0	0
2	0	0	0

$D(43)$	0	1	2
0	0	2	1
1	0	0	0
2	0	0	0

$D(44)$	0	1	2
0	0	1	2
1	0	0	0
2	0	1	0

$D(45)$	0	1	2
0	0	1	2
1	0	0	2
2	0	1	0

(i) $D(36)$ is a Z-semigroup.

(ii) $D(37)$, $D(38)$ and $D(39)$ are medial LD-groupoids (easy to check directly).

(iii) $D(40)$ and $D(41)$ are subdirect products of $D(4)$ and $D(5)$, and hence $D(40)$, $D(41)$ are LD-groupoids.

(iv) $D(42)$ is an LD-groupoid by 1.3(i).

(v) $D(43)$ is an LD-groupoid (clearly, L_0 is an autoímorphism of $D(43)$).

(vi) $D(44)$ is an LD-groupoid ($2 \cdot yz = 2y \cdot 2z$ for $y \neq 0 \neq z$, $y \neq z$, and the remaining cases are clear).

(vii) $D(45)$ is a subdirect product of two copies of $D(5)$, and so $D(45)$ is an LD-groupoid.

(viii) Taking into account 5.2 and 5.4 and the fact that $D(5)$ is a homomorphic image of $D(39), \dots, D(45)$, we have the following table:

	D	LSM	RSM	MSM	M	S	C	E	Dl	Pi	Pc	Ol	Id	Si	Lc	Rc	Lp	Rp	G^{op}
$D(36)$	+	+	+	+	+	+	+	+	+	+	+	+	+	−	3	3	0	0	$D(36)$
$D(37)$	−	+	+	+	+	−	−	+	+	−	−	+	−	+	2	2	0	0	−
$D(38)$	−	+	+	+	+	−	−	−	−	−	−	+	−	+	2	2	0	0	−
$D(39)$	−	+	+	+	+	−	−	−	−	−	+	+	−	+	1	2	0	0	−
$D(40)$	−	−	−	−	−	−	−	−	−	−	+	+	−	−	1	2	0	0	−
$D(41)$	−	−	−	−	−	−	−	−	−	−	+	+	−	−	2	1	0	0	−
$D(42)$	−	−	−	−	−	−	−	−	−	−	+	+	−	+	2	1	1	0	−
$D(43)$	−	−	−	−	−	−	−	−	−	−	+	+	−	+	2	1	1	0	−
$D(44)$	−	−	−	−	−	−	−	−	−	−	+	+	−	+	1	1	1	0	−
$D(45)$	−	−	−	−	−	−	−	−	−	−	+	+	−	−	0	1	1	0	−

Explanation: See 4.1 and 5.1; Lc $(Rc)\ldots$ the number of left (right) constant elements; $Lp = \operatorname{card}(\mathscr{P}_l(G))$, $Rp = \operatorname{card}(\mathscr{P}_r(G))$.

(ix) Considering the foregoing table, we see easily that the groupouds $D(36),\ldots,$ $D(45)$ are pair-wise non-isomorphic with possible exception of $D(42)$, $D(43)$. But $D(42)$ possesses a left neutral element and $D(43)$ does not.

7.2 Lemma. *Let G be a three-element unipotent LD-groupoid such that $xy = 0$ (0 being the only idempotent of G) for all $x, y \in G$, $x \neq 0 \neq y$, $x \neq y$. Then G is isomorphic to one of $D(36)$, $D(38)$, $D(41)$, $D(42)$, $D(43)$.*

Proof. Let $G = \{a, b, c\}$; we have $bc = 0 = cb$. If $0b = 0$ and $0c = c$ then $0 = 00 = 0 \cdot bc = 0b \cdot 0c = 0c = c$, a contradiction. If $0b = b$ and $0c = 0$ then $0 = 00 = 0 \cdot cb = 0c \cdot 0b = 0b = b$, a contradiction. The remaining cases are clear from the following table:

$0b$	0	0	b	b	c	c	c
$0c$	0	b	b	c	0	b	c
$G \cong$	$D(36)$	$D(38)$	$D(41)$	$D(42)$	$D(38)$	$D(43)$	$D(41)$

□

7.3 Lemma. *Let G be a three-element unipotent LD-groupoid such that $xy = x$ for some $x, y \in G$, $x \neq 0 \neq y$. Then G is isomorphic to one of $D(37)$, $D(39)$.*

Proof. We can assume that $G = \{0, b, c\}$ and $bc = b$. Then $0 = bb = b \cdot bc = bb \cdot bc = 0 \cdot bc = 0b$, $c \cdot cb = cc \cdot cb = 0 \cdot cb = 0c \cdot 0b = 0c \cdot 0 = 0$, and hence $cb \in \{0, c\}$. Further, $0b = 0 \cdot bc = 0b \cdot 0c = 0 \cdot 0c$, and therefore $0c \in \{0, b\}$.

If $cb = 0$ and $0c = 0$ then $G \cong D(37)$. If $cb = 0$ and $0c = b$ then $G \cong D(39)$. If $cb = c$ then $b = bc = b \cdot cb = bc \cdot bb = bc \cdot 0 = 0$, a contradiction. □

7.4 Lemma. *Let G be a three-element unipotent LD-groupoid such that $xy = y$ for some $x, y \in G$, $x \neq 0 \neq y$. Then G is isomorphic to one of $D(40)$, $D(44)$, $D(45)$.*

Proof. We can assume that $G = \{0, b, c\}$ and $bc = c$. Then $c = b \cdot bc = bb \cdot bc = 0 \cdot bc = 0c$, $c = 0 \cdot bc = 0b \cdot 0c = 0b \cdot c$, and so $0b \in \{0, b\}$. If $0b = 0$ and $cb = 0$ then $G \cong D(40)$. If $ob = 0$ and $cb = b$ then $b = cb = c \cdot cb = cc \cdot cb = 0 \cdot cb = 0c \cdot 0b = c0 = 0$, a contradiction. If $0b = 0$ and $cb = c$ then $0 = c0 = c \cdot 0b = c0 \cdot cb = 0c = c$, a contradiction. If $0b = b$ and $cb = 0$ then $G \cong D(44)$. If $0b = b$ and $cb = b$ then $G \cong D(45)$. If $0b = b$ and $cb = c$ then $0 = cc = c \cdot cb = cc \cdot cb = 0 \cdot cb = 0c \cdot 0b = cb = c$, a contradiction. □

7.5 Proposition. (i) *The ten groupoids $D(36), \ldots, D(45)$ are pair-wise non-isomorphic three-element unipotent LD-groupoids.*

(ii) *Every three-element unipotent LD-groupoid is isomorphic to one of $D(36), \ldots,$ $D(45)$.*

Proof. (i) See 7.1.

(ii) Combine 7.2, 7.3 and 7.4. □

8.1 Consider the following seven three-element groupoids:

$D(46)$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

$D(47)$	0	1	2
0	0	0	0
1	0	0	1
2	0	0	1

$D(48)$	0	1	2
0	0	0	1
1	0	0	0
2	0	0	1

$D(49)$	0	1	2
0	0	0	1
1	0	0	1
2	0	0	1

$D(50)$	0	1	2
0	0	0	0
1	0	2	1
2	0	2	1

$D(51)$	0	1	2
0	0	1	2
1	0	2	1
2	0	2	1

$D(52)$	0	1	2
0	0	2	1
1	0	2	1
2	0	2	1

(i) $D(46)$ is a commutative A-semigroup, and hence an LD-groupoid.
(ii) $D(47)$ and $D(48)$ are medial LD-groupoids (easy to check directly).
(iii) $D(49)$ and $D(52)$ are right constant groupoids, and hence they are LD-groupoids.
(iv) $D(50) \cong D(6)[e]$ and $D(51) \cong D(6)\{e\}$ and so $D(50)$ and $D(51)$ are LD-groupoids by 1.9(i), (ii).
(v) Taking into account 5.2 and the fact that $D(6)$ is isomorphic to a subgroupoid of $D(50)$, $D(51)$ and $D(52)$, we have the following table:

	D	LSM	RSM	MSM	M	S	C	E	Dl	Pi	Pc	Ol	Id	Si	La	Ra	Ln	Rn	G^{op}
$D(46)$	+	+	+	+	+	+	+	+	+	−	+	+	+	+	1	1	0	0	$D(46)$
$D(47)$	−	+	+	+	+	−	−	−	−	−	−	+	+	+	1	1	0	0	−
$D(48)$	−	+	+	+	+	−	−	−	−	−	−	+	−	+	0	1	0	0	−
$D(49)$	−	+	+	+	+	−	−	−	−	+	+	+	−	+	0	1	0	0	−
$D(50)$	−	+	+	+	+	−	−	−	−	+	+	+	+	+	1	1	0	0	−
$D(51)$	−	+	+	+	+	−	−	−	−	+	+	+	−	+	0	1	1	0	−
$D(52)$	−	+	+	+	+	−	−	−	−	+	+	+	−	−	0	1	0	0	−

Explanation: See 4.1 and 5.1.

(vi) The foregoing table shows that $D(46), \dots, D(52)$ are pair-wise non-isomorphic (with possible exception of $D(49)$, $D(52)$, but these are evidently non-isomorphic).

8.2 Proposition. (i) *The seventeen groupoids $D(36), \dots, D(52)$ are pair-wise non-isomorphic three-element LD-groupoids with just one idempotent element.*

(ii) *Every three-element LD-groupoid with just one idempotent element is isomorphic to one of* $D(36), \dots, D(52)$.

Proof. (i) The groupoids $D(36), \dots, D(45)$ are unipotent and $D(46), \dots, D(52)$ are not. The statement now follows from 7.5(i) and 8.1.

(ii) With respect to 7.5(ii), we can assume that $G = \{a, b, c\}$ is a non-unipotent three-element LD-groupoid and that a is the only idempotent of G. Since $\mathrm{Id}(G)$ is a left ideal, a is a right absorbing element. Further, since G is not unipotent, we can assume that $cc \neq a$. Then $cc = b$ and either $bb = a$ or $bb = c$.

(ii1) Let $bb = a$. Then $cb = c \cdot cc = cc \cdot cc = bb = a$, $a = bb = b \cdot cc = bc \cdot bc$, and so $bc \in \{a, b\}$. Further, $ab = cb \cdot cc = c \cdot bc \in \{ca, cb\} = \{a\}$, i.e., $ab = a$. If $ac = a = bc$ then $G \cong D(46)$. If $ac = a$ and $bc = b$ then $G \cong D(47)$. If $ac = b$ and $bc = a$ then $G \cong D(48)$. If $ac = b = bc$ then $G \cong D(49)$. If $ac = c$ and $bc = a$ then $a = aa = a \cdot bc = ab \cdot ac\alpha \cdot ac = ac = c$, a contradiction. If $ac = c$ and $bc = b$ then $b = bc = b \cdot ac = ba \cdot bc = ab = a$, a contradiction.

(ii2) Let $bb = c$. Then $bc = b \cdot bb = bb \cdot bb = cc\beta$ and $cb = c \cdot cc = cc \cdot cc = bb = c$. Further, $ab = a \cdot cc = ac \cdot ac$ and $ac = a \cdot bb = ab \cdot ab$. Now, it is clear that $ab = a$ iff $ac = a$, $ab = b$ iff $ac = b$. If $ab = a$ then $G \cong D(50)$. If $ab = b$ then $G \cong D(51)$. If $ab = c$ then $G \cong D(52)$. □

IV.9 Three-element left distributive groupoids without idempotent elements

9.1 Consider the following two three-element groupoids:

D(53)	0	1	2
0	1	0	0
1	1	0	0
2	1	0	0

D(54)	0	1	2
0	1	2	0
1	1	2	0
2	1	2	0

Both $D(53)$ and $D(54)$ are right constant groupoids, and hence they are medial LD-groupoids (see 1.1). Clearly, they are not isomorphic and we have the following table (see 1.1 again):

	D	M	S	C	E	Dl	Pi	Pc	Ol	Si
$D(53)$	−	+	−	−	−	−	+	+	+	−
$D(54)$	−	+	−	−	−	−	+	+	+	+

Explanation: See 4.1 and 5.1.

9.2 Proposition. (i) *The two groupoids* $D(53)$ *and* $D(54)$ *are non-isomorphic three-element LD-groupoids without idempotent elements.*

(ii) *Every three-element LD-groupoid without idempotent elements is isomorphic to one of* $D(53)$, $D(54)$.

Proof. (i) See 9.1.

(ii) Let $G = \{a, b, c\}$, $\mathrm{Id}(G) = \emptyset$. It is easy to see that we can restrict ourselves to the following two cases:

(ii1) Let $aa = b$, $bb = a$ and $cc = a$. Then $ab = a \cdot aa = aa \cdot aa = bb\alpha$, $ba = b \cdot bb = bb \cdot bb = aa = b$, $b = aa = a \cdot cc = aa \cdot ac$, and hence $ac = a$. Similarly, $b = ba = b \cdot cc = bc \cdot bc$ and $bc = a$. Finally, $b = aa = cc \cdot cc = ca$ and $cb = c \cdot aa = ca \cdot ca = bb = a$. We have proved that $G \cong D(53)$.

(ii2) Let $aa = b$, $bb = c$ and $cc = a$. Then $ab = a \cdot aa = aa \cdot aa = bb = c$, $bc = b \cdot bb = bb \cdot bb = cc = a$, $ca = c \cdot cc = cc \cdot cc = aa = b$. Moreover, $b = aa = b \cdot aa = a \cdot cc = ac \cdot ac$, and so $ac = a$; $c = bb = b \cdot aa = ba \cdot ba$, and so $ba = b$; $a = cc = c \cdot bb = cb \cdot cb$, and so $cb = c$. Thus $G \cong D(54)$. $\square$

IV.10 Three-element left distributive groupouds – the concluding table

10.1 By 5.1(x), 6.1(xi), 7.1(viii), 8.1(v) and 9.1, we have the following table (see p. 84).

Explanation: See 4.1 and 5.1; *Sm*... the groupoid G is simple (it is easy to check that $D(15)$, $D(30)$, $D(54)$ are the only simple groupoids among $D(7), \ldots, D(54)$).

IV.11 Number of isomorphism types of at most six element left distributive groupoids

11.1 The following table shows the number of all LD-groupoids and the number of their isomorphism types on a given set of at most 6 elements:

Elements	1	2	3	4	5	6
Groupoids	1	9	224	14067	3717524	?
Iso types	1	6	48	720	33425	35527485

11.2 The following table specifies the numbers of isomorphism types of LD-groupoids (from 1 up to 5 elements) according to the number of idempotent elements:

Idempotents / Elements	0	1	2	3	4	5
1	0	1	0	0	0	0
2	1	2	3	0	0	0
3	2	17	12	17	0	0
4	25	233	179	142	141	0
5	704	21699	3936	3115	2267	1704

	D	LSM	RSM	MSM	M	S	C	I	E	Dl	Pi	Pc	Ol	Id	Si	Sm	La	Ra	Ln	Rn	G^{op}
$D(7)$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	–	–	1	1	1	1	$D(7)$
$D(8)$	+	+	+	+	+	+	+	+	+	+	+	+	+	+	–	–	1	1	0	0	$D(8)$
$D(9)$	+	+	+	+	+	+	–	+	+	+	+	+	+	+	+	–	1	1	0	2	$D(10)$
$D(10)$	+	+	+	+	+	+	–	+	+	+	+	+	+	+	+	–	1	1	2	0	$D(9)$
$D(11)$	+	+	+	+	+	+	–	+	+	+	+	+	+	+	–	–	3	0	0	3	$D(12)$
$D(12)$	+	+	+	+	+	+	–	+	+	+	+	+	+	+	–	–	0	3	3	0	$D(11)$
$D(13)$	+	+	+	+	+	+	–	+	+	+	+	+	+	+	–	–	2	0	0	1	$D(14)$
$D(14)$	+	+	+	+	+	+	–	+	+	+	+	+	+	+	–	–	0	2	1	0	$D(13)$
$D(15)$	+	+	+	+	+	–	+	+	+	+	+	+	+	+	+	+	0	0	0	0	$D(15)$
$D(16)$	+	+	+	+	+	–	–	+	+	+	+	+	+	+	+	–	0	1	2	0	$D(17)$
$D(17)$	+	+	+	+	+	–	–	+	+	+	+	+	+	+	+	–	1	0	0	2	$D(16)$
$D(18)$	+	+	+	+	+	–	–	+	+	+	+	+	+	+	+	–	0	2	2	0	$D(19)$
$D(19)$	+	+	+	+	+	–	–	+	+	+	+	+	+	+	+	–	2	0	0	2	$D(18)$
$D(20)$	–	+	–	–	–	+	–	+	+	+	+	+	+	+	+	–	2	0	1	1	–
$D(21)$	–	+	–	–	–	–	–	+	+	+	+	+	+	+	+	–	2	0	0	1	–
$D(22)$	–	+	–	–	–	–	–	+	+	+	+	+	+	+	+	–	0	1	1	0	–
$D(23)$	–	+	–	+	–	–	–	+	+	+	+	–	+	+	+	–	0	2	2	0	–
$D(24)$	+	+	+	+	+	+	+	–	+	+	+	+	+	+	–	–	1	1	0	0	$D(24)$
$D(25)$	+	+	+	+	+	+	+	–	+	+	+	+	+	+	–	–	1	1	0	0	$D(25)$
$D(26)$	+	+	+	+	+	+	–	–	+	+	+	+	+	+	–	–	2	0	0	0	$D(27)$
$D(27)$	+	+	+	+	+	+	–	–	+	+	+	+	+	+	–	–	0	2	0	0	$D(26)$
$D(28)$	–	+	+	+	+	+	–	–	+	–	+	+	+	–	+	–	1	1	1	0	–
$D(29)$	–	+	+	+	+	–	–	–	–	–	+	+	+	–	+	–	0	2	1	0	–
$D(30)$	–	–	–	–	–	–	–	–	–	–	–	+	–	–	+	+	1	0	1	0	–
$D(31)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	+	–	1	1	1	0	–
$D(32)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	–	–	0	2	2	0	–
$D(33)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	–	–	0	1	1	0	–
$D(34)$	–	–	–	–	–	–	–	–	–	–	–	–	+	–	+	–	0	2	1	0	–
$D(35)$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	+	–	0	1	2	0	–
$D(36)$	+	+	+	+	+	+	+	–	+	+	+	+	+	+	–	–	3	3	0	0	$D(36)$
$D(37)$	–	+	+	+	+	–	–	–	+	+	–	–	+	–	+	–	2	2	0	0	–
$D(38)$	–	+	+	+	+	–	–	–	–	–	–	–	+	–	+	+	2	2	0	0	–
$D(39)$	–	+	+	+	+	–	–	–	–	–	–	+	+	–	+	–	1	2	0	0	–
$D(40)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	–	–	1	2	0	0	–
$D(41)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	–	–	2	1	0	0	–
$D(42)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	+	–	0	1	1	0	–
$D(43)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	+	–	2	1	1	0	–
$D(44)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	+	–	1	1	1	0	–
$D(45)$	–	–	–	–	–	–	–	–	–	–	–	+	+	–	–	–	0	1	1	0	–
$D(46)$	+	+	+	+	+	+	+	–	+	+	–	+	+	+	+	–	1	1	0	0	$D(46)$
$D(47)$	–	+	+	+	+	–	–	–	–	–	–	–	+	+	+	–	1	1	0	0	–
$D(48)$	–	+	+	+	+	–	–	–	–	–	–	–	+	–	+	–	0	1	0	0	–
$D(49)$	–	+	+	+	+	–	–	–	–	–	+	+	+	–	+	–	0	1	0	0	–
$D(50)$	–	+	+	+	+	–	–	–	–	–	+	+	+	+	+	–	1	1	0	0	–
$D(51)$	–	+	+	+	+	–	–	–	–	–	+	+	+	–	+	–	0	1	1	0	–
$D(52)$	–	+	+	+	+	–	–	–	–	–	+	+	+	–	+	–	0	1	0	0	–
$D(53)$	–	+	+	+	+	–	–	–	–	–	+	+	+	–	–	–	0	0	0	0	–
$D(54)$	–	+	+	+	+	–	–	–	–	–	+	+	+	–	+	+	0	0	0	0	–

11.3 The following table contains the numbers of isomorphism types of at most five-element LD-groupoids satisfying some basic identities:

Identity / Elements	LD	D	M	S	C	I	IM	IS	CI
1	1	1	1	1	1	1	1	1	1
2	6	4	5	4	2	3	3	3	1
3	48	19	32	16	7	17	13	9	3
4	720	120	405	93	24	141	71	38	7
5	33425	921	25185	682	103	1704	449	179	22

IV.12 Comments and open problems

Group constructions of idempotent self-distributive groupoids are quite common (e.g., the operation of mean, $(x, y) \to (x + y)/2$, in a uniquely 2-divisible Abelian group or the operation of conjugation, $(x, y) \to y^x$, in any group). Group constructions of non-idempotent (left) distributive groupoids were introduced in [Kep, 81]. A substantial progress was made by P. Dehornoy, who (using indirect methods) came to the constructions 2.2 and 2.3 which are the most sophisticated up to now. These constructions then yield the very important example IV.3 which is essentially due to P. Dehornoy again (the present formulation comes from D. Larue).

Three-element LDI-groupoids were classified in [Kep, 81] and the enumerating tables 11.1, 11.2, 11.3 are due to [Jez, 95].

The following open problem might be of interest: For $n \geq 1$, let $\alpha(n)$ denote the number of iso-types of LD-groupoids having n elements and, for $m \geq 1$, let $\alpha(n, m)$ be the number of iso-types of those n-element LD-groupoids which have just m idempotent elements. Find

$$\lim_{n \to \infty} \frac{\alpha(n, m)}{\alpha(n)}$$

for every $m \geq 1$.

List of symbols

$A \cdot B$ ... the set of all products ab, $a \in A$, $b \in B$ (denoted also by AB)

$\langle A \rangle_G$ ... subgroupoid of G generated by a subset $A \subseteq G$

$A_G(i)$ ... block of $\ker(r_G)$ containing an element $i \in \mathrm{Id}(G)$ in a strongly delightful groupoid G

$\alpha_G(S)$ ... the set of all $x \in G$ such that $ax \in S$ for some $a \in S$, S being a subset of a groupoid G

$\mathrm{Aut}(G)$ ... the automorphism group of a groupoid G

$\beta_{n,G}(S)$ ... the set of all $x \in G$ such that $a_1(a_2(\dots(a_n x))) \in S$ for some $a_1, \dots, a_n \in S$ ($\beta_{0,G}(S) = S$), S being a subset of G

$\beta_G(S)$ ... the set $\bigcup_{i \geq 0} \beta_{i,G}(S)$

$\mathrm{card}(M)$ ... the cardinality of a set M

$\mathscr{C}_l(G)$ ... the set of all left cancellable elements of a groupoid G

$\mathscr{C}_r(G)$ ... the set of all right cancellable elements of a groupoid G

$\mathscr{C}(G)$ ... the set of all cancellable elements of a groupoid G

$\mathscr{C}_l^*(G)$ ... the set of all $a \in \mathscr{C}_l(G)$ such that $aa = aa \cdot a$

$\mathrm{Cyc}_l(n)$ ... groupoid defined on $\{0, 1, \dots, n-1\}$ by $i * j = i + 1$ for $i \neq n - 1$ and $(n-1) * j = 0$

$\mathrm{Cyc}_r(n)$ ... groupoid defined on $\{0, 1, \dots, n-1\}$ by $i * j = j + 1$ for $j \neq n - 1$ and $i * (n-1) = 0$

$\mathrm{Cyc}_l(\infty)$ ... groupoid defined on $0, 1, \dots$ by $i * j = i + 1$

$\mathrm{Cyc}_r(\infty)$ ... groupoid defined on $0, 1, \dots$ by $i * j = j + 1$

$\gamma_G(S)$ ... the set of all $x \in G$ such that $xa \in S$ for some $a \in S$

$\mathscr{D}_l(G)$ ... the set of all left divisible elements of a groupoid G

$\mathscr{D}_r(G)$ ... the set of all right divisible elements of a groupoid G

$\mathscr{D}(G)$ ... the set of all divisible elements of a groupoid G

$\delta_{n,G}(S)$ ... the set of all $x \in G$ such that $(((xa_1)a_2)\dots)a_n \in S$ for some $a_1, \dots, a_n \in S$ ($\delta_{0,G}(S) = S$), S being a subset of a groupoid G

$\delta_G(S)$ ... the set $\bigcup_{i \geq 0} \delta_{i,G}(S)$

$\mathrm{End}(G)$ ... the endomorphism monoid of a groupoid G

$\varphi_G(S)$ ... the set $\alpha_G(S) \cup \gamma_G(S)$

$G[e]$ ... groupoid defined on the set $G \cup \{e\}$ ($e \notin G$) such that G is a subgroupoid of $G[e]$ and e is an absorbing element of $G[e]$

$G[e\}$ ... groupoid defined on the set $G \cup \{e\}$ ($e \notin G$) such that G is a subgroupoid of $G[e\}$ and e is left absorbing and right neutral

$G\{e]$ ... groupoid defined on the set $G \cup \{e\}$ ($e \notin G$) such that G is a subgroupoid of $G\{e\}$ and e is right absorbing and left neutral

$G\{e\}$ ... groupoid defined on the set $G \cup \{e\}$ ($e \notin G$) such that G is a subgroupoid of $G\{e\}$ and e is a neutral element of $G\{e\}$

$G[e, f]$ ... groupoid defined on the set $G \cup \{e\}$ ($e \notin G$) such that G is a subgroupoid of $G[e, f]$, $xe = e$ and $ey = f(y)$ for all $x \in G \cup \{e\}$ and $y \in G$, where G is an $LSLD$-groupoid and f is an automorphism of G such that $f^2 = \mathrm{id}_G$ and $(x, f(x)) \in p_G$ for every $x \in G$

$G^{(n)}$ ... the set of ordered n-tuples of elements of G

$G^{\{n\}}$ ... subset of G defined inductively by $G^{(1)} = G$ and $G^{\{n+1\}} = G \cdot G^{\{n\}}$

$G^{\{n,m\}}$ ... subset of G defined inductively by $G^{\{n,0\}} = G^{\{n\}}$ and $G^{\{n,m+1\}} = G^{\{n,m\}} \cdot G$

G^{op} ... the opposite groupoid of a groupoid G

id_G ... the identical mapping (relation) on a set G

$\mathrm{Id}(G)$ ... the set of all idempotent elements of a groupoid G

$\mathscr{I}(G)$ ... the set of all two-sided ideals of a groupoid G

$\mathscr{I}_l(G)$ ... the set of all left ideals of a groupoid G

$\mathscr{I}_r(G)$ ... the set of all right ideals of a groupoid G

$\mathrm{Int}(G)$ ... intersection of all ideals of a groupoid G

ip_G ... relation defined on an $LSLD$-groupoid G by $(a, b) \in ip_G$ iff either $a = b$ or $a = bb$

$\ker(f)$ ... the kernel equivalence of a mapping f ($(a, b) \in \ker(f)$ iff $f(a) = f(b)$)

$L_{a,G}$ ... left translation by an element a in a groupoid G, $L_{a,G}(x) = ax$ for all $x \in G$ (denoted also by L_a)

$\mathscr{L}(G)$ ... subgroup in $\mathscr{M}_l^*(G)$, G being a quasigroup, generated by all mappings $L_x L_y^{-1}$, $x, y \in G$

$\mathscr{L}(\mathscr{V})$ ... the lattice of subvarieties of a variety $\mathscr{V}$

$\mathscr{M}(G)$ ... the multiplication semigroup of a groupoid G, i.e., the subsemigroup of the transformation monoid of the set G generated by all left and right translations

$\mathscr{M}^1(G)$ ... the multiplication monoid $\mathscr{M}(G) \cup \{\mathrm{id}_G\}$

$\mathscr{M}_l(G)$ ... the left multiplication semigroup of a groupoid G (generated by all left translations)

$\mathscr{M}_l^1(G)$ ... the left multiplication monoid of a groupoid G ($\mathscr{M}_l^1(G) = \mathscr{M}_l(G) \cup \{\mathrm{id}_G\}$)

$\mathscr{M}_r(G)$ ... the right multiplication semigroup of a groupoid G (generated by all right translations)

$\mathscr{M}_r^1(G)$ ... the right multiplication monoid of a groupoid G ($\mathscr{M}_r^1(G) = \mathscr{M}_r(G) \cup \{\mathrm{id}_G\}$)

$\mathscr{M}^*(G)$ ... permutation group generated by all (left and right) translations in a quasigroup G

$\mathscr{M}_l^*(G)$ ... permutation group generated by all left translations in a left quasigroup G

$\mathscr{M}_r^*(G)$ ... permutation group generated by all right translations in a right quasigoup G

$\mathscr{M}(G, H)$ ... the transformation semigroup generated in $\mathscr{M}(G)$ by all left and right translations $L_{a,G}$, $R_{a,G}$, $a = H$, H being a subgroupoid of G

$\mathscr{M}^1(G, H)$ ... the transformation monoid $\mathscr{M}(G, H) \cup \{\mathrm{id}_G\}$

$\mathscr{M}_l(G, H)$ ... the transformation semigroup generated in $\mathscr{M}_l(G)$ by all left translations $L_{a,G}$, $a \in H$, H being a sugroupoid of G

$\mathscr{M}_l^1(G, H)$ ... the transformation monoid $\mathscr{M}_l(G, H) \cup \{\mathrm{id}_G\}$

$\mathscr{M}_r(G, H)$ ... the transformation semigroup generated in $\mathscr{M}_r(G)$ by all right translations $R_{a,G}$, $a \in H$, H being a sugroupoid of G

$\mathscr{M}_r^1(G, H)$ ... the transformation monoid $\mathscr{M}_r(G, H) \cup \{\mathrm{id}_G\}$

$\mu_{a,G}(S)$ ... the set of all $u \in G$ such that $au \in S$, S being a subset of a groupoid G

$\nu_{a,G}(S)$ ... the set of all $u \in G$ such that $ua \in S$, S being a subset of a groupoid G

o_G ... transformation of a groupoid G defined by $o_G(x) = xx$ for all $x \in G$

p_G ... relation on a groupoid G defined by $p_G = \bigcap_{a \in G} \ker(R_{a,G})$

$\mathfrak{P}(G)$ ... groupoid of all subsets of a groupoid G with multiplication defined by $A \cdot B = AB = \{ab \mid a \in A, b \in B\}$ for all $A, B \subseteq G$

$\mathscr{P}(G)$ ... the set of all elements of a groupoid G which are both cancellable and divisible

$\mathscr{P}_l(G)$ ... the set of all elements of a groupoid G which are both left cancellable and left divisible

$\mathscr{P}_r(G)$ ... the set of all elements of a groupoid G which are both right cancellable and right divisible

$\psi_{n,G}(S)$ ... the set of all $x \in G$ such that ${}_1T_{a_1} \dots {}_nT_{a_n}(x) \in S$ for some $n \geq 1$, ${}_iT \in \{L, R\}$ and $a_i \in S$, $i = 1, \dots, n$, S being a subset of a groupoid G ($\psi_{0,G}(S) = S$)

$\psi_G(S)$ ... the set $\bigcup_{i \geq 0} \psi_{i,G}(S)$

q_G ... relation on a groupoid G defined by $q_G = \bigcap_{a \in G} \ker(L_{a,G})$

$\mathscr{Q}(G)$ ... the set of all $G^{\{n\}}$, $n \geq 1$

$Q_l(G)$ ... the left-quasigroup-envelope of a groupoid G

$R_{a,G}$ ... right translation by an element a a groupoid G, $R_{a,G}(x) = xa$ for all $x \in G$

$\mathscr{R}(G)$ ... the subgroupoid of $\mathfrak{P}(G)$ generated by G (obviously, $\mathscr{Q}(G) \subseteq \mathscr{R}(G)$)

$[R]_G$ ... the smallest closed subset of a groupoid G containing a set $R \subseteq G$

$[R]_G^l$ ... the smallest left closed subset of a groupoid G containing a set $R \subseteq G$

$[R]_G^r$ ... the smallest right closed subset of a groupoid G containing a set $R \subseteq G$

r_G ... transformation of a groupoid G defined by $r_G(x) = x \cdot xx$ for all $x \in G$

s_G ... transformation of a groupoid G defined by $s_G(x) = xx \cdot x$ for all $x \in G$

$\langle S \rangle_G^c$ ... the smallest closed subgroupoid of a groupoid G containing a set $S \subseteq G$

$\langle S \rangle_G^{lc}$ ... the smallest left closed subgroupoid of a groupoid G containing a set $S \subseteq G$

$\langle S \rangle_G^{rc}$ ... the smallest right closed subgroupoid of a groupoid G containing a set $S \subseteq G$

$\sigma(G)$ ... minimal cardinality of a generating set of a groupoid G

$\sigma_c(G)$ ... minimal cardinality of a set M of c-generators of a groupoid G (G is the least closed subgroupoid containing M)

$\sigma_{lc}(G)$ ... minimal cardinality of a set M of lc-generators of a groupoid G (G is the least left closed subgroupoid containing M)

$\sigma_{rc}(G)$ ... minimal cardinality of a set M of rc-generators of a groupoid G (G is the least right closed subgroupoid containing M)

t_G ... relation defined on a groupoid G by $t_G = p_G \cap q_G$ (i.e., $(x, y) \in t_G$ iff $L_x = L_y$ and $R_x = R_y$)

u_G ... relation defined on a groupoid G by $(a, b) \in u_G$ iff the elements a and b generate the same left ideal

u_G^c ... relation defined on a groupoid G by $(a, b) \in u_G^c$ iff the elements a and b generate the same left strongly prime left ideal

v_G ... relation defined on a groupoid G by $(a, b) \in v_G$ iff the elements a and b generate the same right ideal

v_G^c ... relation defined on a groupoid G by $(a, b) \in v_G^c$ iff the elements a and b generate the same right strongly prime right ideal

w_G ... relation defined on a groupoid G by $(a, b) \in w_G$ iff the elements a and b generate the same two-sided ideal of G

z_G ... relation defined on a groupoid G by $(a, b) \in z_G$ iff $a = f(b)$ for some $f \in \mathscr{M}(G)$

z_G^1 ... relation defined on a groupoid G by $z_G^1 = z_G \cup \{\mathrm{id}_G\}$

$z_{l,G}$ ... relation defined on a groupoid G by $(a, b) \in z_{l,G}$ iff $a = f(b)$ for some $f \in \mathscr{M}_l(G)$

$z_{l,G}^1$ ... relation defined on a groupoid G by $z_{l,G}^1 = z_{l,G} \cup \{\mathrm{id}_G\}$

$z_{r,G}$ ... relation defined on a groupoid G by $(a, b) \in z_{r,G}$ iff $a = f(b)$ for some $f \in \mathscr{M}_r(G)$

$z_{r,G}^1$ ... relation defined on a groupoid G by $z_{r,G}^1 = z_{r,G} \cup \{\mathrm{id}_G\}$

ω_G ... the intersection of all non-identical congruences of a groupoid G ($\omega_G = \mathrm{id}_G$ if G is a trivial groupoid)

$\omega_{c,G}$ ... the intersection of all non-identical cancellative congruences of a groupoid G

$\omega_{l,c,G}$ ... the intersection of all non-identical left cancellative congruences of a groupoid G

$\omega_{r,c,G}$ ... the intersection of all non-identical right cancellative congruences of a groupoid G

Abbreviations of groupoid varieties

A- A-semigroup (satisfying $x \cdot yz = uv \cdot w$)

CD- commutative distributive groupoid (satisfying $x \cdot yz = xy \cdot xz$ and $xy = yx$)

CDI- commutative distributive idempotent groupoid (satisfying $x \cdot yz = xy \cdot xz$, $xx = x$ and $xy = yx$)

DI- distributive idempotent groupoid (satisfying $x \cdot yz = xy \cdot xz$, $xy \cdot z = xz \cdot yz$ and $xx = x$)

LD- left distributive groupoid (satisfying $x \cdot yz = xy \cdot xz$)

LDA- groupoid satisfying $x \cdot yz = xy \cdot xz$, $xx \cdot y = x \cdot yy$ (i.e., left distributive delightful groupoid) and $x \cdot xx = y \cdot yy$

LDB- groupoid satisfying $xx \cdot y = x \cdot yy$ and $x \cdot yz = u \cdot vw$

LDI- left distributive idempotent groupoid (satisfying $x \cdot yz = xy \cdot xz$ and $xx = x$)

$LSLD$- left symmetric left distributive groupoid (satisfying $x \cdot yz = xy \cdot xz$ and $x \cdot xy = y$)

LSLDI- left symmetric left distributive idempotent groupoid (satisfying $x \cdot yz = xy \cdot xz$, $xx = x$ and $x \cdot xy = y$)

LZ- semigroup of left zeros (satisfying $x = xy$)

IM- idempotent medial groupoid (satisfying $xx = x$ and $xy \cdot uv = xu \cdot yv$)

RD- right distributive groupoid (satisfying $xy \cdot z = xz \cdot yz$)

RDI- right distributive idempotent groupoid (satisfying $xy \cdot z = xz \cdot yz$ and $xx = x$)

RZ- semigroup of right zeros (satisfying $x = yx$)

Z- *Z*-semigroup (satisfying $xy = uv$)

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