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1 Introduction

Basic properties of random coefficient autoregressive models (RCA) are studied in [7], where processes of random parameters and errors are supposed to be independent and consist of independent and identically distributed random variables. There exist extended literature that generalize these basic models. The assumptions of an error process are weakened for example in papers [1] and [6]. While in the former one homoscedastic martingale differences are supposed, in the latter one independent but heteroscedastic errors are considered. A generalized RCA model in which processes of random coefficients and disturbances are permitted to be correlated is studied in [3] under homoscedastic assumption.

In [5] we have extended the results of [6]. Firstly, we have proved strong consistency and asymptotic normality of an OLS estimator of β in a heteroscedastic RCA(1) model under weaker conditions than in [6]. Further, we have extended these results for a WLS estimator and moreover for the process $\{V_t\}$ with unknown mean μ. Paper [5] is a reduced form of [4] where full versions of all proofs and all important lemmas are given.

In this paper results of [5] are generalized. We are dealing here with two heteroscedastic RCA(1) models where the assumption of independence is weakened. In the first one the error process is supposed to be martingale differences, in the second one the same behaviour is assumed for centered random coefficients. In the

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paper we will show that all asymptotic results (strong consistency and asymptotic normality) derived in [5] hold unchanged under these generalized conditions. Since all proofs of essential theorems are substantially based on several auxiliary lemmas we will concentrate our attention to proofs of such lemmas. Because the proofs of the theorems can be then made analogously as in [4], we will present here main steps of the proofs only and the rest is left to the reader.

At the end of the paper a process of martingale differences for which all derived asymptotic properties hold is constructed.

2 Model definition

Let us suppose that the behaviour of the process $\{X_t\}$ is described by the RCA(1) model

$$X_t = b_t X_{t-1} + Y_t, \qquad t = 1, \dots, n \tag{1}$$

together with the following basic assumptions: $EX_0 = 0$, $0 < EX_0^2 = \sigma_0^2 < \infty$, Y_t, $t = 1, \dots, n$ are random variables with $EY_t = 0\ \forall t$, $0 < EY_t^2 = \sigma_t^2 < \infty$ which are independent of X_0 and b_t, $t = 1, \dots, n$ are random variables with $Eb_t = \beta$, $0 < Eb_t^2 = \sigma_b^2 < \infty\ \forall t$ which are independent of X_0 and of $\{Y_t\}$.

We will frequently use the two following representations of the model (1). The first one has the form of an autoregression with a constant coefficient:

$$X_t = \beta X_{t-1} + B_t X_{t-1} + Y_t = \beta X_{t-1} + u_t, \tag{2}$$

where $u_t = B_t X_{t-1} + Y_t$ and $B_t = b_t - \beta$. To keep unified notation let us denote $\sigma_B^2 := EB_t^2$, so the equation $\sigma_B^2 = \sigma_b^2 - \beta^2$ holds.

The second one is expressed in terms $\{Y_t\}$ and $\{b_t\}$ only. This representation is given by formula (3) (for convenience let us denote $Y_0 := X_0$):

$$X_t = \sum_{j=0}^{t} c_{t,j-1} Y_{t-j}, \tag{3}$$

where $c_{t,j} := \prod_{i=0}^{j} b_{t-i}$ and $c_{t,-1} := 1$.

Let us define the system of σ-fields $\mathscr{F}_t$ for $t = 0, 1, \dots$ in the following way: $\mathscr{F}_0 = \sigma(X_0)$, $\mathscr{F}_t = \sigma(X_0, Y_1, B_1, \dots, Y_t, B_t)$, $t \geq 1$. Let us denote $\mathscr{F} = (\mathscr{F}_t, t \geq 0)$.

We will study asymptotic behaviour of OLS and WLS estimators of β in model (2) given by the following formulas:

$$\hat{\beta} = \frac{\sum_{t=1}^{n} X_t X_{t-1}}{\sum_{t=1}^{n} X_{t-1}^2}, \tag{4}$$

$$\hat{\beta}_W = \frac{\sum_{t=1}^{n} \frac{1}{\sigma_t^2} X_t X_{t-1}}{\sum_{t=1}^{n} \frac{1}{\sigma_t^2} X_{t-1}^2}. \tag{5}$$

Further, we will concentrate on the process $\{V_t\}$ for which $EV_t = \mu$ is unknown and such that $X_t = V_t - \mu$ satisfies model (1). In this model we will study asymptotic properties of the following estimators:

$$\hat{\mu} = \frac{1}{n} \sum_{t=1}^{n} V_t. \tag{6}$$

$$\hat{\hat{\beta}} = \frac{\sum_{t=1}^{n} (V_t - \hat{\mu})(V_{t-1} - \hat{\mu})}{\sum_{t=1}^{n} (V_{t-1} - \hat{\mu})^2}, \tag{7}$$

$$\hat{\hat{\beta}}_W = \frac{\sum_{t=1}^{n} \frac{1}{\sigma_t^2} (V_t - \hat{\mu})(V_{t-1} - \hat{\mu})}{\sum_{t=1}^{n} \frac{1}{\sigma_t^2} (V_{t-1} - \hat{\mu})^2}. \tag{8}$$

In [5] strong consistency and asymptotic normality are proved under a crucial assumption

A0: both $\{Y_t\}$ and $\{b_t\}$ are processes of independent random variables

and several mainly technical assumptions.

In the next two sections we will generalize these results into the cases where Assumption A0 is released and $\{Y_t\}$ or $\{b_t\}$ are permitted to consist of dependent random variables. The first one is concerned with the model where $\{Y_t\}$ is an $\mathscr{F}_t$-martingale difference sequence ($\mathscr{F}_t$-m.d.s.), the second is dealing with the model where $\{B_t\}$ is an $\mathscr{F}_t$-m.d.s.

3 Model with martingale difference errors

Let us suppose that behaviour of the stochastic process $\{X_t\}$ is described by the RCA(1) model (1) defined in Section 2. In this case let us assume that the process $\{Y_t\}$ is an $\mathscr{F}_t$-m.d.s.

3.1 Strong consistency

In accordance with weakened assumption about $\{Y_t\}$ we have to slightly change the set of assumptions given in [5] under which strong consistency of considered estimators is still valid. In the sequel let us assume:

A0′: $\{Y_t\}$ is an $\mathscr{F}_t$-m.d.s., $\{b_t\}$ is a process of independent random variables,

A1: $E|X_0|^{2+\delta} < \infty$ and $\omega_t := E|Y_t|^{2+\delta} \le K < \infty\ \forall t$ and for some $\delta > 0$,

A2: $\omega_b := \sup_t E|b_t|^{2+\delta} < 1$ for some $\delta > 0$,

A3: $E\left(Y_t^2 \mid \mathscr{F}_{t-1}\right) = \sigma_t^2$ a.s.,

A4: $\frac{1}{n}\sum_{t=1}^{n}\sigma_t^2 \xrightarrow[n\to\infty]{} \sigma^2 > 0.$

One can see that opposed to [5] here we have to assume one additional Assumption A3.

Auxiliary lemmas.

All lemmas given in Paragraph 1.2.1 in [4] hold analogously under Assumptions A0′, A1 − A4 but some proofs have to be slightly modified. In this part we will summarized all lemmas, but we will present only such proofs that have to be modified due to different assumptions.

For $0 \le j < k$ let us define $\mathscr{G}_{t-k}^{t-j} := \sigma\left(\mathscr{F}_{t-k} \cup \sigma\left(Y_{t-k+1}, \ldots, Y_{t-j}\right)\right)$.

Remark 3.1. *In the sequel, C will denote a general positive constant, the value of which may change in different formulas or even in different places in the same formula.*

Lemma 3.1. *Under A0′, the second moment of the process $\{X_t\}$ is given by formula*

$$EX_t^2 = \sum_{j=0}^{t}\sigma_{t-j}^2\sigma_b^{2j}. \tag{9}$$

Proof. Using (3) we can write

$$EX_t^2 = \sum_{j=0}^{t}EY_{t-j}^2Ec_{t,j-1}^2 + 2\sum\sum_{0\le j<k\le t}E\left(Y_{t-j}Y_{t-k}\right)E\left(c_{t,j-1}c_{t,k-1}\right) = \sum_{j=0}^{t}\sigma_{t-j}^2\sigma_b^{2j}$$

since $E\left(Y_{t-j}Y_{t-k}\right) = E\left[Y_{t-k}E\left(Y_{t-j}|\mathscr{F}_{t-j-1}\right)\right] = 0$ for $j < k$. □

Lemma 3.2. *Assumptions A0′, A1 and A2 imply that there exists a constant $C > 0$ such that*

$$E|X_t|^{2+\delta} \le C < \infty \quad \forall t \text{ and some } \delta > 0. \tag{10}$$

Proof. Directly using Minkowski inequality. □

Lemma 3.3. *Under Assumptions A0′, A1 − A3, the following processes are $L_{1+\varepsilon}$-uniformly bounded $\mathscr{F}_t$-m.d.s. for $\varepsilon = \frac{\delta}{2}$.*

a) $Z_t^{(1)} = u_t$,
b) $Z_t^{(2)} = X_{t-1}u_t$,
c) $Z_t^{(3)} = X_{t-1}b_tY_t$,
d) $Z_t^{(4)} = X_{t-1}^2\left(b_t^2 - \sigma_b^2\right)$,
e) $Z_t^{(5)} = Y_t^2 - \sigma_t^2$.

Proof. Martingale difference property in cases a), b) and d) is a trivial consequence of definition of $\mathcal{F}_t$-m.d.s., in case e) it leads directly from Assumption A3. The only non-trivial modification must be done in case c):

$$E(Z_t^{(3)}|\mathcal{F}_{t-1}) = X_{t-1}E[Y_tE(b_t|\mathcal{G}_{t-1}^t)|\mathcal{F}_{t-1}] = X_{t-1}\beta E(Y_t|\mathcal{F}_{t-1}) = 0 \text{ a.s.}$$

$L_{1+\varepsilon}$-uniform boundedness is a trivial consequence of Assumptions A0′, A1 and A2, Minkowski inequality and Lemma 3.2 in all but two cases b) and c), where an additional argument of Hölder inequality for $p = q = 2$ must be used:

b) $E|Z_t^{(2)}|^{1+\varepsilon} \leq (E|X_{t-1}|^{2+\delta})^{\frac{1}{2}}(E|u_t|^{2+\delta})^{\frac{1}{2}} \leq$

$C\left[(E|X_{t-1}B_t|^{2+\delta})^{\frac{1}{2+\delta}} + (E|Y_t|^{2+\delta})^{\frac{1}{2+\delta}}\right]^{\frac{2+\delta}{2}} \leq C,$

c) $E|Z_t^{(3)}|^{1+\varepsilon} \leq (E|X_{t-1}b_t|^{2+\delta})^{\frac{1}{2}}(E|Y_t|^{2+\delta})^{\frac{1}{2}} \leq C.$ □

Let us recall the definition of mixingales:

Definition 3.1. *Let $\{X_t\}$ be integrable random variables on a probability space $(\Omega, \mathcal{A}, P)$ equipped by the filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$. The sequence of pairs $\{X_t, \mathcal{F}_t\}$ is called* ***L_p-mixingale*** *if, for $p \geq 1$, there exist sequences of non-negative constants $\{c_t\}$ and $\{\xi_s\}$, $s = 0, 1, \ldots,$ such that $\xi_s \xrightarrow[s\to\infty]{} 0$ and*

i) $\|E(X_t|\mathcal{F}_{t-s})\|_p \leq c_t\xi_s,$

ii) $\|X_t - E(X_t|\mathcal{F}_{t+s})\|_p \leq c_t\xi_{s+1}$

hold for all $t, s \geq 0$, where $\|X\|_p$ denotes the norm in $L_p(\Omega, \mathcal{A}, P)$ defined as $\|X\|_p = (E|X|^p)^{\frac{1}{p}}$.

Furher, L_p-mixingale $\{X_t, \mathcal{F}_t\}$ is of a **size** $-\varphi_0$ *for $\varphi_0 > 0$ if $\xi_s = O(s^{-\varphi})$ for $\varphi > \varphi_0$.*

Lemma 3.4. *Under Assumptions A0′, A1 and A2, the following properties hold for some $\delta > 0$:*

a) the sequence $\{X_t, \mathcal{F}_t\}$ is $L_{2+\delta}$-mixingale of an arbitrary size.

If moreover Assumption A3 holds, then

b) the sequence $\{X_t^2 - EX_t^2, \mathcal{F}_t\}$ is $L_{1+\varepsilon}$-mixingale of an arbitrary size for $\varepsilon = \frac{\delta}{2}$.

Proof. Since $E(X_t|\mathcal{F}_{t+s}) = X_t$ a.s. and $E(X_t^2|\mathcal{F}_{t+s}) = X_t^2$ a.s. for $s \geq 0$, it remains to verify only condition i) of Definition 3.1.

Case a):

Firstly, for $s > 0$ we have

$$E(X_t|\mathcal{F}_{t-s}) = \sum_{j=0}^{s-1} E(c_{t,j-1}Y_{t-j}|\mathcal{F}_{t-s}) + \sum_{j=s}^{t} Y_{t-j}E(c_{t,j-1}|\mathcal{F}_{t-s}) =$$

$$= \sum_{j=0}^{s-1} E\left[Y_{t-j}E(c_{t,j-1}|\mathcal{F}_{t-j})|\mathcal{F}_{t-s}\right] + \sum_{j=s}^{t} Y_{t-j}\prod_{i=s}^{j-1}(b_{t-i})\, Ec_{t,s-1} = \beta^s X_{t-s} \quad \text{a.s.}$$

Hence Lemma 3.2 directly implies that $\|E(X_t|\mathcal{F}_{t-s})\|_{2+\delta} \leq |\beta|^s C$. Since $|\beta| < 1$, the statement of case a) is proved.

Case b):
Firstly, let us show that for $s > 0$

$$E\left(\left(X_t^2 - EX_t^2\right) \mid \mathscr{F}_{t-s}\right) = \sigma_b^{2s} \sum_{j=s}^{t} \sum_{k=s}^{t} Y_{t-j} Y_{t-k} \prod_{i=s}^{j-1}(b_{t-i}) \prod_{i=s}^{k-1}(b_{t-i}) - \sum_{j=s}^{t} \sigma_{t-j}^2 \sigma_b^{2j} =$$
$$= \sigma_b^{2s}\left(X_{t-s}^2 - EX_{t-s}^2\right) \quad \text{a.s.} \tag{11}$$

This expression is the same as in Lemma 1.4 in [4], but in this case arguments are slightly different:

i) For $j, k < s$ both $c_{t,j-1}$ and $c_{t,k-1}$ are independent of $\mathscr{F}_{t-s}$, but Y_{t-j} and Y_{t-k} are dependent on it, so we have to use conditioning in the following way:

$$E\left(\sum_{j=0}^{s-1} \sum_{k=0}^{s-1} c_{t,j-1} c_{t,k-1} Y_{t-j} Y_{t-k} \middle| \mathscr{F}_{t-s} \right) = \sum_{j=0}^{s-1} E\left[Y_{t-j}^2 E\left(c_{t,j-1}^2 \mid \mathscr{G}_{t-s}^{t-j}\right) \middle| \mathscr{F}_{t-s} \right] +$$
$$+ 2 \sum_{0 \le j < k \le s-1} \sum E\left[Y_{t-j} Y_{t-k} E\left(c_{t,j-1} c_{t,k-1} \mid \mathscr{G}_{t-s}^{t-j}\right) \middle| \mathscr{F}_{t-s} \right] = \sum_{j=0}^{s-1} \sigma_{t-j}^2 \sigma_b^{2j} \quad \text{a.s.}$$

ii) For $j < s \le k$, using analogous arguments we have:

$$E\left(c_{t,j-1} c_{t,k-1} Y_{t-j} Y_{t-k} \mid \mathscr{F}_{t-s}\right) = Y_{t-k} \prod_{i=s}^{k-1}(b_{t-i})\, E\left(c_{t,j-1} c_{t,s-1} Y_{t-j} \mid \mathscr{F}_{t-s}\right) =$$
$$= Y_{t-k} \prod_{i=s}^{k-1}(b_{t-i})\, E\left[Y_{t-j} E\left(c_{t,j-1} c_{t,s-1} \mid \mathscr{G}_{t-s}^{t-j}\right) \middle| \mathscr{F}_{t-s} \right] = 0 \quad \text{a.s.}$$

iii) The case for $k < s \le j$ is analogous.

iv) For $j, k \ge s$, since both Y_{t-j} and Y_{t-k} are $\mathscr{F}_{t-s}$-measurable, derivation is unchanged, thus:

$$E\left(c_{t,j-1} c_{t,k-1} Y_{t-j} Y_{t-k} \mid \mathscr{F}_{t-s}\right) = Y_{t-j} Y_{t-k} \prod_{i=s}^{j-1}(b_{t-i}) \prod_{i=s}^{k-1}(b_{t-i})\, Ec_{t,s-1}^2 \quad \text{a.s.}$$

These results together imply (11). The rest of the proof is a direct consequence of Lemma 3.2 and Assumption A2. □

Lemma 3.5. *Under Assumptions A0′, A1 and A2, the following relations hold for any deterministic function g such that $|g(t)| \le C\,\forall t$:*

a) $\displaystyle \frac{1}{n} \sum_{t=1}^{n} g(t)\, X_t \xrightarrow[n \to \infty]{a.s.} 0,$

b) $\displaystyle \frac{1}{n} \sum_{t=1}^{n} g(t)\left(X_t^2 - EX_t^2\right) \xrightarrow[n \to \infty]{a.s.} 0.$

Proof. It is a direct consequence of Lemma 3.4 and Theorem 20.16 in [2] since for any deterministic function g such that $|g(t)| \le C\,\forall t$ both $\{g(t)X_t, \mathscr{F}_t\}$ and $\{g(t)(X_t^2 - EX_t^2), \mathscr{F}_t\}$ remain to be $L_{1+\varepsilon}$-mixingales with respect to constants $c_t = C$. □

Lemma 3.6. *Let $\{X_n\}$ be a stochastic sequence such that $X_n \xrightarrow[n\to\infty]{a.s.} 0$. Then $c_nX_n \xrightarrow[n\to\infty]{a.s.} 0$ for any sequence of constants $\{c_n\}$ such that $|c_n| \leq C\,\forall n$.*

Proof. It is directly seen from the fact that

$$X_n \xrightarrow[n\to\infty]{a.s.} 0 \Leftrightarrow \lim_{k\to\infty} P\left(\bigcup_{n=k}^{\infty}[|X_n| > \varepsilon]\right) = 0 \quad \text{for any } \varepsilon > 0. \qquad \square$$

Theorems

Since all auxiliary lemmas remain valid under assumptions of this generalized model, we can directly formulate the following theorems for both zero and non-zero mean case. They can be proved analogously as Theorems 1.1., 1.2., 2.1., 2.2. and 2.3. in [4].

Theorem 3.1. *Under Assumptions A0′, A1–A4, $\hat{\beta} \xrightarrow[n\to\infty]{a.s.} \beta$ holds. If moreover*
A5: $0 < N \leq \sigma_t^2\ \forall t$
holds, then also $\hat{\beta}_W \xrightarrow[n\to\infty]{a.s.} \beta$.

Proof. Combining (2) and (4) we get $\hat{\beta} - \beta = \left(\frac{1}{n}\sum_{t=1}^{n} X_{t-1}u_t\right)\left(\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\right)^{-1}$. In the first step it is shown that $\frac{1}{n}\sum_{t=1}^{n} X_{t-1}u_t \xrightarrow[n\to\infty]{a.s.} 0$. This arises from Lemma 3.3 b) and from Theorem 20.11 in [2]. Further, using Lemma 3.5 b) one can show that $\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2 \xrightarrow[n\to\infty]{a.s.} \frac{\sigma^2}{1 - (\beta^2 + \sigma_B^2)} > 0$, which concludes the first part of the proof.

In case of $\hat{\beta}_W$ the difference $\hat{\beta}_W - \beta$ can be rewritten in the way $\hat{\beta}_W - \beta = \left(\frac{1}{nc_n}\sum_{t=1}^{n}\frac{1}{\sigma_t^2}X_{t-1}u_t\right)\left(\frac{1}{nc_n}\sum_{t=1}^{n}\frac{1}{\sigma_t^2}(X_{t-1}^2 - EX_{t-1}^2) + 1\right)^{-1}$, where $c_n = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma_t^2}EX_{t-1}^2$. Then the proof is done analogously as in the previous case, additional arguments for existence of all a.s.-limits are ensured by Lemmas 3.5. and 3.6. $\square$

Theorem 3.2. *Under Assumptions A0′, A1 and A2, $\hat{\mu} \xrightarrow[n\to\infty]{a.s.} \mu$ holds. Under Assumptions A0′, A1–A4, $\hat{\hat{\beta}} \xrightarrow[n\to\infty]{a.s.} \beta$ and if moreover A5 holds, then $\hat{\hat{\beta}}_W \xrightarrow[n\to\infty]{a.s.} \beta$.*

Proof. Since $\hat{\mu} - \mu = \frac{1}{n}\sum_{t=1}^{n}(V_t - \mu)$, the first statement is a direct consequence of Lemma 3.5 a).

Further, denoting $X_t := V_t - \mu$ and $\bar{X} := \frac{1}{n}\sum_{t=1}^{n} X_t = \hat{\mu} - \mu$ we can rewrite the model $(V_t - \mu) = b_t(V_{t-1} - \mu) + Y_t$ into the form

$$X_t - \bar{X} = \beta(X_{t-1} - \bar{X}) - (1 - \beta)\bar{X} + u_t, \tag{12}$$

where $u_t = B_tX_{t-1} + Y_t$. Multiplying previous equation (12) by $(X_{t-1} - \bar{X})$ and summing it over $t = 1, \ldots, n$ we can get after some algebra

$$\hat{\hat{\beta}} - \beta = \frac{\frac{1}{n}\sum_{t=1}^{n} X_{t-1}u_t - \bar{X}\frac{1}{n}\sum_{t=1}^{n} u_t - (1-\beta)\bar{X}\frac{1}{n}\sum_{t=1}^{n}(X_{t-1} - \bar{X})}{\frac{1}{n}\sum_{t=1}^{n}(X_{t-1} - \bar{X})^2}. \tag{13}$$

Firstly, it can be shown that $\frac{1}{n}\sum_{t=1}^{n}(X_{t-1} - \bar{X})^2 \xrightarrow[n\to\infty]{a.s.} \frac{\sigma^2}{1 - (\beta^2 + \sigma_B^2)} > 0$. Further, due to Theorem 20.11 in [2] both $\frac{1}{n}\sum_{t=1}^{n} X_{t-1}u_t$ and $\frac{1}{n}\sum_{t=1}^{n} u_t$ converge a.s. to 0. Finally, the fact that $\hat{\mu} \xrightarrow[n\to\infty]{a.s.} \mu$ concludes that $\hat{\hat{\beta}} \xrightarrow[n\to\infty]{a.s.} \beta$.

In case of $\hat{\hat{\beta}}_W$ we proceed similarly, Lemmas 3.5 and 3.6 have to be moreover used. □

3.2 Asymptotic normality

Similarly as in the basic model in [5] it is necessary to extend the set of assumptions in order to derive asymptotic distribution of studied estimators. Let us modify Assumptions A1 and A2 in the following way:

A1′: $E|X_0|^{4+\delta} < \infty$ and $\eta_t := E|Y_t|^{4+\delta} \leq K < \infty\ \forall t$ and for some $\delta > 0$,

A2′: $\eta_b := \sup_t E|b_t|^{4+\delta} < 1$ for some $\delta > 0$, moreover $Eb_t^4 = \gamma_b\ \forall t$.

Moreover let us assume:

A6: $E(Y_t^3|\mathscr{F}_{t-1}) = \alpha_t$ a.s., where $\alpha_t = EY_t^3$,

A7: $E(Y_t^4|\mathscr{F}_{t-1}) = \gamma_t$ a.s., where $\gamma_t = EY_t^4$,

A8: $\frac{1}{n}\sum_{t=1}^{n}\gamma_t \xrightarrow[n\to\infty]{} \gamma$,

A9: $\frac{1}{n}\sum_{t=1}^{n}\sigma_t^2 EX_{t-1}^2 \xrightarrow[n\to\infty]{} \bar{\sigma}^2$.

On can check that in addition to the basic model in [5] we have to suppose moreover Assumptions A3, A6 and A7.

Auxiliary lemmas

Again, let us summarize all important lemmas and modified proofs.

Lemma 3.7. *Assumptions A0′, A1′ and A2′ imply that there exists a constant $C > 0$ such that*

$$E|X_t|^{4+\delta} \leq C < \infty \qquad \forall t \text{ and some } \delta > 0. \tag{14}$$

Proof. Analogously as for Lemma 3.2. □

Lemma 3.8. *Under Assumptions A0′, A1′, A2′, A3, A6 and A7, the following processes are all $L_{1+\varepsilon}$-uniformly bounded $\mathscr{F}_t$-m.d.s. for $\varepsilon = \frac{\delta}{4}$:*

a) $Z_t^{(1)} = X_{t-1}(Y_t^3 - \alpha_t)$,
b) $Z_t^{(2)} = X_{t-1}Y_t^3(b_t - \beta)$,
c) $Z_t^{(3)} = X_{t-1}^2(Y_t^2 - \sigma_t^2)$,
d) $Z_t^{(4)} = X_{t-1}^2 Y_t^2(b_t^2 - \sigma_b^2)$,
e) $Z_t^{(5)} = X_{t-1}^3 b_t^3 Y_t$,
f) $Z_t^{(6)} = X_{t-1}^4(b_t^4 - \gamma_b)$,
g) $Z_t^{(7)} = Y_t^4 - \gamma_t$.

Proof. Martingale difference property is directly seen from definition of $\mathscr{F}_t$-m.d.s. and Assumptions A3, A6 and A7 in cases a), c), f) and g). In remaining cases we have to use conditioning in the following way:

b) $E(Z_t^{(2)}|\mathscr{F}_{t-1}) = X_{t-1}E\left[Y_t^3 E(b_t - \beta|\mathscr{G}_{t-1}^t)|\mathscr{F}_{t-1}\right] = 0$ a.s.,

d) $E(Z_t^{(4)}|\mathscr{F}_{t-1}) = X_{t-1}^2 E\left[Y_t^2 E(b_t^2 - \sigma_b^2|\mathscr{G}_{t-1}^t)|\mathscr{F}_{t-1}\right] = 0$ a.s.,

e) $E(Z_t^{(5)}|\mathscr{F}_{t-1}) = X_{t-1}^3 E\left[Y_t E(b_t^3|\mathscr{G}_{t-1}^t)|\mathscr{F}_{t-1}\right] = 0$ a.s.,

$L_{1+\varepsilon}$-uniform boundedness is trivial in cases f) and g). Hölder inequality for $p = q = 2$ in cases c) and d) and for $p = 4$, $q = \frac{4}{3}$ in cases a), b) and e) analogously as in Lemma 3.3 can be used. □

Theorems

Theorems 1.3., 1.4., 2.4., 2.5. and 2.6. from [4] about the asymptotic distribution of $\hat{\beta}$, $\hat{\beta}_W$, $\hat{\mu}$, $\hat{\hat{\beta}}$ and $\hat{\hat{\beta}}_W$ can be now formulated for this generalized model. Since again all important lemmas hold unchanged all proofs can be done similarly as those in [4].

Theorem 3.3. *Under Assumptions A0′, A1–A4, the asymptotic distribution of* $\sqrt{n}(\hat{\mu} - \mu)$ *is* $N\left(0, \dfrac{\sigma^2(1+\beta)}{(1-\beta)(1-(\beta^2+\sigma_B^2))}\right)$.

Proof. After some algebra it can be derived that

$$\sqrt{n}(\hat{\mu} - \mu) = U_n + \left(\frac{1}{s_n}\sum_{t=1}^{n}\rho_{n,t}u_t\right)\left(\sqrt{\frac{n}{s_n^2}}\right)^{-1},$$

where $\rho_{n,t} := \dfrac{1-\beta^{n-t+1}}{1-\beta}$, $s_n^2 := \sum_{t=1}^{n}\rho_{n,t}^2 E u_t^2$ and U_n is a random variable for which $U_n \xrightarrow[n\to\infty]{p} 0$ holds. Firstly, it is shown that

$$\frac{1}{n}s_n^2 \xrightarrow[n\to\infty]{} \frac{\sigma^2}{1-\beta}\left[\frac{1+\beta}{1-(\beta^2+\sigma_B^2)}\right] > 0. \tag{15}$$

Further, it has to be proved that $\dfrac{1}{s_n}\sum_{t=1}^{n}\rho_{n,t}u_t$ has the limiting distribution $N(0,1)$. Since $\{u_t\}$ in an $\mathscr{F}_t$-m.d.s. it is sufficient to check conditions of the central limit theorem that are of the form:

$$\text{i)}\ \frac{\sigma_B^2 \sum_{t=1}^{n} \rho_{n,t}^2 (V_{t-1} - \mu)^2 + \sum_{t=1}^{n} \rho_{n,t}^2 \sigma_t^2}{\sigma_B^2 \sum_{t=1}^{n} \rho_{n,t}^2 E(V_{t-1} - \mu)^2 + \sum_{t=1}^{n} \rho_{n,t}^2 \sigma_t^2} \xrightarrow[n\to\infty]{p} 1,$$

$$\text{ii)}\ \frac{1}{s_n^2} \sum_{t=1}^{n} E(\rho_{n,t}^2 E(V_{t-1} - \mu)^2 I_{[|\rho_{n,t}(V_{t-1} - \mu)| \geq \varepsilon s_n]}) \xrightarrow[n\to\infty]{} 0 \qquad \text{for all } \varepsilon > 0.$$

The first condition follows from Lemma 3.5b) since $|\rho_{n,t}^2| \leq \frac{4}{(1-\beta)^2}$. The second one is a consequence of (15) and the fact that $\sum_{t=1}^{n} \left|\frac{1-\beta^{n-t+1}}{1-\beta}\right|^{2+\delta} E|V_{t-1} - \mu|^{2+\delta} \leq nC$. □

Theorem 3.4. *Under Assumptions A0′, A1′, A2′, A3, A4, A6–A9, the asymptotic distribution of* $\sqrt{n}(\hat{\beta} - \beta)$ *and* $\sqrt{n}(\hat{\hat{\beta}} - \beta)$ *is* $N\left(0, \Delta\left(\frac{1 - (\beta^2 + \sigma_B^2)}{\sigma^2}\right)^2\right)$, *where*

$$\Delta = \sigma_B^2 \frac{6(\beta^2 + \sigma_B^2)\,\bar{\sigma}^2 + \gamma}{1 - \gamma_b} + \bar{\sigma}^2. \tag{15}$$

Proof. We can write $\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{s_n} \sum_{t=1}^{n} X_{t-1} u_t\right)\left(\sqrt{\frac{n}{s_n^2}} \frac{1}{n} \sum_{t=1}^{n} X_{t-1}^2\right)^{-1}$. Firstly, it can be derived that $\frac{1}{n} s_n^2 \xrightarrow[n\to\infty]{} \Delta$ holds for $s_n^2 := \sum_{t=1}^{n} E(X_{t-1}^2 u_t^2)$. Thus, it is sufficient to show that $\frac{1}{s_n} \sum_{t=1}^{n} X_{t-1} u_t$ has the asymptotic distribution $N(0, 1)$. Due to Lemma 3.3 b), the central limit theorem for martingale differences can be used. Hence, it remains to verify the following conditions:

$$\text{i)}\ \frac{\sigma_B^2 \sum_{t=1}^{n} X_{t-1}^4 + \sum_{t=1}^{n} \sigma_t^2 X_{t-1}^2}{\sigma_B^2 \sum_{t=1}^{n} EX_{t-1}^4 + \sum_{t=1}^{n} \sigma_t^2 EX_{t-1}^2} \xrightarrow[n\to\infty]{p} 1,$$

$$\text{ii)}\ \frac{1}{s_n^2} \sum_{t=1}^{n} E(X_{t-1}^2 u_t^2 I_{[|X_{t-1} u_t| \geq \varepsilon s_n]}) \xrightarrow[n\to\infty]{} 0 \qquad \text{for all } \varepsilon > 0.$$

Firstly, combining $X_t^4 = (b_t X_{t-1} + Y_t)^4$ and $EX_t^4 = E(b_t X_{t-1} + Y_t)^4$ one can show that $\frac{1}{n} \sum_{t=1}^{n} (X_{t-1}^4 - EX_{t-1}^4) \xrightarrow[n\to\infty]{a.s.} 0$. This result together with Lemma 3.5b) imply condition i). Condition ii) follows from $\frac{1}{n} s_n^2 \xrightarrow[n\to\infty]{} \Delta$ and Assumptions A1′ and A2′.

In case of $\hat{\hat{\beta}}$ we can write

$$\sqrt{n}(\hat{\hat{\beta}} - \beta) = \frac{\frac{1}{s_n} \sum_{t=1}^{n} X_{t-1} u_t}{\sqrt{\frac{n}{s_n^2} \frac{1}{n} \sum_{t=1}^{n} (X_{t-1} - \bar{X})^2}} - \frac{\sqrt{n}\bar{X}\left[\frac{1}{n} \sum_{t=1}^{n} u_t + (1-\beta)\frac{1}{n} \sum_{t=1}^{n} (X_{t-1} - \bar{X})\right]}{\frac{1}{n} \sum_{t=1}^{n} (X_{t-1} - \bar{X})^2}.$$

It can be shown that the latter term converges in probability to 0. Convergence in distribution of the first term can be proved similarly as in case of $\hat{\beta}$. □

In order to formulate the theorem concerning with asymptotic distribution of $\hat{\beta}_W$ and $\hat{\hat{\beta}}_W$ we have to assume a stronger version of Assumption A4 in the form:

A4′: $\sigma_n^2 \xrightarrow[n\to\infty]{} \sigma^2 > 0.$

Theorem 3.5. *Under Assumptions A0′, A1′, A2′, A3, A4′, A5—A8, the asymptotic distribution of* $\sqrt{n}(\hat{\beta}_W - \beta)$ *and* $\sqrt{n}(\hat{\hat{\beta}}_W - \beta)$ *is the same as that given in Theorem 3.4.*

Proof. Analogously as the proof of Theorem 3.4. Existence of all corresponding limits is ensured by Assumption A4′. □

4 Model with martingale difference coefficients

In this section let us deal with the RCA(1) model (1) for which all basic assumptions are satisfied. Let us suppose now that the process $\{B_t\}$ is an $\mathscr{F}_t$-m.d.s. while the process $\{Y_t\}$ remains to consist of independent random variables.

4.1 Strong consistency

To preserve strong consistency we have to strengthen Assumption A2 and change Assumption A3 in comparison with the previous model. In the sequel let us assume:

A0*: $\{B_t\}$ is an $\mathscr{F}_t$-m.d.s., $\{Y_t\}$ is a process of independent random variables,

A2*: $\sup_t E(|b_t|^{2+\delta} | \mathscr{F}_{t-1}) \leq K_2 < 1$ a.s. for some $\delta > 0$,

A3*: $E(b_t^2 | \mathscr{F}_{t-1}) = \sigma_b^2$ a.s.

Auxiliary lemmas

Analogously as in Section 3 let us define $\mathscr{H}_{t-k}^{t-j} := \sigma(\mathscr{F}_{t-k} \cup \sigma(B_{t-k+1}, \ldots, B_{t-j}))$ for $0 \leq j < k$.

Remark 4.1. *Let us denote* $\omega_b := \sup_t E(|b_t|^{2+\delta})$, *then Assumption A2* implies that* $\omega_b < 1$.

Lemma 4.1. *Assumptions A0* and A3* imply that* $E(B_t^2 | \mathscr{F}_{t-1}) = \sigma_B^2$ *a.s.*

Proof. Obvious. □

Lemma 4.2. *Under Assumptions A0* and A3*, for* $j \leq s$ *the following equalities hold:*

$$Ec_{t,j-1} = \beta^j, \qquad E(c_{t,j-1} | \mathscr{F}_{t-s}) = \beta^j \ a.s.,$$
$$Ec_{t,j-1}^2 = \sigma_b^{2j}, \qquad E(c_{t,j-1}^2 | \mathscr{F}_{t-s}) = \sigma_b^{2j} \ a.s.$$

Proof. Straightforward. □

Lemma 4.3. *Under Assumptions A0* and A3*, the second moment of the process $\{X_t\}$ is given by formula (9).*

Proof. Directly based on results of Lemma 4.2 using (3). □

Lemma 4.4. *Under Assumptions A0*, A1, A2*, inequality (10) holds.*

Proof. Applying Minkowski inequality for $p = 2 + \delta$ to expression (3) and using subsequent conditioning we get:

$$\left(E|X_t|^{2+\delta}\right)^{\frac{1}{2+\delta}} \le \sum_{j=0}^{t}\left(E|c_{t,j-1}|^{2+\delta}\, E|Y_{t-j}|^{2+\delta}\right)^{\frac{1}{2+\delta}} \le$$

$$\le \sum_{j=0}^{t}\left(\omega_{t-j}E\left(\prod_{i=1}^{j-1}|b_{t-i}|^{2+\delta}\, E\left(|b_t|^{2+\delta}\,|\,\mathscr{F}_{t-1}\right)\right)\right)^{\frac{1}{2+\delta}} \le$$

$$\le \sum_{j=0}^{t}\left(\omega_{t-j}K_2E\left(\prod_{i=2}^{j-1}|b_{t-i}|^{2+\delta}\, E\left(|b_{t-1}|^{2+\delta}\,|\,\mathscr{F}_{t-2}\right)\right)\right)^{\frac{1}{2+\delta}} \le \sum_{j=0}^{t}(\omega_{t-j}K_2^j)^{\frac{1}{2+\delta}} \le C. \quad \square$$

Lemma 4.5. *Under Assumptions A0*, A1, A2* and A3*, the following processes are $L_{1+\varepsilon}$-uniformly bounded $\mathscr{F}_t$-m.d.s. with respect to the filtration $\mathscr{F}$ for $\varepsilon = \frac{\delta}{2}$.*

a) $Z_t^{(1)} = u_t$, *d)* $Z_t^{(4)} = X_{t-1}^2(b_t^2 - \sigma_b^2)$,
b) $Z_t^{(2)} = X_{t-1}u_t$, *e)* $Z_t^{(5)} = b_t^2 - \sigma_b^2$.
c) $Z_t^{(3)} = X_{t-1}b_tY_t$,

Proof. Martingale difference property in cases a), b), d) and e) is trivial, in case c) it is seen from the following derivations:

$$E(Z_t^{(3)}\,|\,\mathscr{F}_{t-1}) = X_{t-1}E(b_tY_t\,|\,\mathscr{F}_{t-1}) = X_{t-1}E\left[b_tE(Y_t\,|\,\mathscr{H}_{t-1}^t)\,|\,\mathscr{F}_{t-1}\right] = 0 \text{ a.s.}$$

$L_{1+\varepsilon}$-uniform boundedness for e) is trivial, in remaining cases Minkowski and Hölder inequalities together with conditioning are useful:

a) $$\left(E|Z_t^{(1)}|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \le \left(E|X_{t-1}B_t|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} + \left(E|Y_t|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \le$$
$$\le \left(E|X_{t-1}|^{2+\delta}\right)^{\frac{1}{2+\delta}}\left(E|B_t|^{2+\delta}\right)^{\frac{1}{2+\delta}} + C \le C,$$

b) $$E|Z_t^{(2)}|^{1+\varepsilon} \le \left(E|X_{t-1}|^{2+\delta}\right)^{\frac{1}{2}}\left(E|u_t|^{2+\delta}\right)^{\frac{1}{2}} \le C\left[E|(B_t+\beta)X_{t-1} - \beta X_{t-1} + Y_t|^{2+\delta}\right]^{\frac{1}{2}} \le$$
$$\le C\left[\left(E|X_{t-1}b_t|^{2+\delta}\right)^{\frac{1}{2+\delta}} + \beta\left(E|X_{t-1}|^{2+\delta}\right)^{\frac{1}{2+\delta}} + \left(E|Y_t|^{2+\delta}\right)^{\frac{1}{2+\delta}}\right]^{\frac{2+\delta}{2}} \le$$
$$\le C\left[\left(E\left[|X_{t-1}|^{2+\delta}\, E\left(|b_t|^{2+\delta}\,|\,\mathscr{F}_{t-1}\right)\right]\right)^{\frac{1}{2+\delta}} + C\right]^{\frac{2+\delta}{2}} \le C,$$

c) $$E|Z_t^{(2)}|^{1+\varepsilon} \le \left(E|X_{t-1}Y_t|^{2+\delta}\right)^{\frac{1}{2}}\left(E|b_t|^{2+\delta}\right)^{\frac{1}{2}} \le C,$$

d) $$\left(E|Z_t^{(4)}|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} = \left(E|X_{t-1}^2b_t^2 - X_{t-1}^2\sigma_b^2|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \le \left(E|X_{t-1}b_t|^{2+\delta}\right)^{\frac{1}{1+\varepsilon}} +$$
$$+ \sigma_b^2\left(E|X_{t-1}|^{2+\delta}\right)^{\frac{1}{1+\varepsilon}} \le \left(E\left[|X_{t-1}|^{2+\delta}\, E\left(|b_t|^{2+\delta}\,|\,\mathscr{F}_{t-1}\right)\right]\right)^{\frac{1}{1+\varepsilon}} + C \le C. \quad \square$$

Lemma 4.6. *Under Assumptions A0*, A1, A2*, Lemma 3.4, case a) holds. If moreover Assumption A3* holds, then Lemma 3.4 case b) holds.*

Proof. As in the previous case it is sufficient to verify condition i) of Definition 3.1 only. Derivation has to be slightly modified:

Case a):

Firstly, for $s > 0$ using Lemma 4.2 we have

$$E(X_t|\mathscr{F}_{t-s}) = \sum_{j=0}^{s-1} E\left[Y_{t-j}E(c_{t,j-1}|\mathscr{F}_{t-j})|\mathscr{F}_{t-s}\right] + \sum_{j=s}^{t} Y_{t-j}\prod_{i=s}^{j-1}(b_{t-i})\,E(c_{t,s-1}|\mathscr{F}_{t-s}) = \beta^s X_{t-s} \text{ a.s.}$$

Then Lemma 4.4 directly implies a).

Case b):

Firstly, we can again derive that for $s > 0$ expression (11) holds, but arguments differ from the previous case:

i) For $j, k < s$ both Y_{t-j} and Y_{t-k} are independent on $\mathscr{F}_{t-s}$, but $b_{t,j-1}$ and $b_{t,k-1}$ are dependent on it, so we have to use conditioning in the following way:

$$E\left(\sum_{j=0}^{s-1}\sum_{k=0}^{s-1} c_{t,j-1}c_{t,k-1}Y_{t-j}Y_{t-k}\middle|\mathscr{F}_{t-s}\right) = \sum_{j=0}^{s-1} E\left[Y_{t-j}^2 E(c_{t,j-1}^2|\mathscr{F}_{t-j})|\mathscr{F}_{t-s}\right] +$$

$$+ 2\sum\sum_{0\le j<k\le s-1} E\left[Y_{t-j}Y_{t-k}\prod_{i=j}^{k-1}(b_{t-i})\,E(c_{t,j-1}^2|\mathscr{F}_{t-j})\middle|\mathscr{F}_{t-s}\right] = \sum_{j=0}^{s-1}\sigma_{t-j}^2\sigma_b^{2j} +$$

$$+ 2\sum\sum_{0\le j<k\le s-1} \sigma_b^{2j}E\left[Y_{t-k}\prod_{i=j}^{k-1}(b_{t-i})\,E(Y_{t-j}|\mathscr{H}_{t-k}^{t-j})\middle|\mathscr{F}_{t-s}\right] = \sum_{j=0}^{s-1}\sigma_{t-j}^2\sigma_b^{2j} \text{ a.s.}$$

ii) For $j < s \le k$, using analogous arguments we have:

$$E(c_{t,j-1}c_{t,k-1}Y_{t-j}Y_{t-k}|\mathscr{F}_{t-s}) = Y_{t-k}\prod_{i=s}^{k-1}(b_{t-i})\,E(c_{t,j-1}c_{t,s-1}Y_{t-j}|\mathscr{F}_{t-s}) =$$

$$= Y_{t-k}\prod_{i=s}^{k-1}(b_{t-i})\,E\left[\prod_{i=j}^{s-1}(b_{t-i})\,Y_{t-j}E(c_{t,j-1}^2|\mathscr{F}_{t-j}\middle|\mathscr{F}_{t-s}\right] =$$

$$= \sigma_b^{2j}Y_{t-k}\prod_{i=s}^{k-1}(b_{t-i})\,E\left[\prod_{i=j}^{s-1}(b_{t-i})\,E(Y_{t-j}|\mathscr{H}_{t-s}^{t-j}\middle|\mathscr{F}_{t-s}\right] = 0 \text{ a.s.}$$

iii) The case for $k < s \le j$ is analogous.

iv) The result for $j, k \ge s$ is, due to Lemma 4.2, the same as in the proof of Lemma 3.4.

Further we can proceed analogously as in the proof of Lemma 3.4 using Remark 4.1 and Lemma 4.4. □

Theorems

Having proved the previous lemmas we can now summarize results about strong consistency of $\hat{\beta}$, $\hat{\beta}_W$, $\hat{\mu}$, $\hat{\hat{\beta}}$ and $\hat{\hat{\beta}}_W$ analogously as in Theorems 3.1 and 3.2. In their proofs we can follow exactly the same steps, in this case their verification is based on previous lemmas.

Theorem 4.1. *Under Assumptions A0*, A1, A2*, A3* and A4,* $\hat{\beta} \xrightarrow[n\to\infty]{a.s.} \beta$ *holds. If moreover A5 holds, then also* $\hat{\beta}_W \xrightarrow[n\to\infty]{a.s.} \beta$.

Theorem 4.2. *Under Assumptions A0*, A1, A2*,* $\hat{\mu} \xrightarrow[n\to\infty]{a.s.} \mu$ *holds. Under Assumptions A0*, A1, A2*, A3* and A4,* $\hat{\hat{\beta}} \xrightarrow[n\to\infty]{a.s.} \beta$ *and if moreover A5 holds, then* $\hat{\hat{\beta}}_W \xrightarrow[n\to\infty]{a.s.} \beta$.

4.2 Asymptotic normality

To preserve the same results about asymptotic distribution of the estimators of β and μ we have to change the previous set of assumptions. Let us assume:
A2:** $\sup_t E(|b_t|^{4+\delta} | \mathscr{F}_{t-1}) \le K_2 < 1$ a.s. for some $\delta > 0$, moreover $Eb_t^4 = \gamma_b \; \forall t$,
A7*: $E(b_t^4 | \mathscr{F}_{t-1}) = \gamma_b$ a.s.

Auxiliary lemmas

Lemma 4.7. *Under Assumptions A0*, A1' and A2**, inequality (14) holds.*

Proof. Analogously as for Lemma 4.4. □

Lemma 4.8. *Under Assumptions A0*, A1', A2**, A3* and A7*, the following processes are all* $L_{1+\varepsilon}$*-uniformly bounded* $\mathscr{F}_t$*-m.d.s. for* $\varepsilon = \frac{\delta}{4}$:

a) $Z_t^{(1)} = X_{t-1}(Y_t^3 - \alpha_t)$, *d)* $Z_t^{(4)} = X_{t-1}^2 Y_t^2 (b_t^2 - \sigma_b^2)$,
b) $Z_t^{(2)} = X_{t-1} Y_t^3 (b_t - \beta)$, *e)* $Z_t^{(5)} = X_{t-1}^3 b_t^3 Y_t$,
c) $Z_t^{(3)} = X_{t-1}^2 (Y_t^2 - \sigma_t^2)$, *f)* $Z_t^{(6)} = X_{t-1}^4 (b_t^4 - \gamma_b)$.

Proof. Martingale difference property immediately follows from the definition of $\mathscr{F}_t$-m.d.s. in cases a) and c) and from Assumption A7* in case f). In remaining cases we can verify this property by conditioning in the following way:

b) $E(Z_t^{(2)} | \mathscr{F}_{t-1}) = X_{t-1} E\big[(b_t - \beta)\, E(Y_t^3 | \mathscr{H}_{t-1}^t) \big| \mathscr{F}_{t-1}\big] = 0$ a.s.,
d) $E(Z_t^{(4)} | \mathscr{F}_{t-1}) = X_{t-1}^2 E\big[(b_t^2 - \sigma_b^2)\, E(Y_t^2 | \mathscr{H}_{t-1}^t) \big| \mathscr{F}_{t-1}\big] = 0$ a.s.,
e) $E(Z_t^{(5)} | \mathscr{F}_{t-1}) = X_{t-1}^3 E\big[b_t^3 E(Y_t | \mathscr{H}_{t-1}^t) \big| \mathscr{F}_{t-1}\big] = 0$ a.s.

$L_{1+\varepsilon}$-uniform boundedness is trivial in cases a) and c), in case f) it is directly seen from Assumption A2** and Lemma 4.7 using Minkowski inequality and conditioning analogously as in Lemma 4.5 d). Hölder inequality for $p = q = 2$ in cases b) and d) and for $p = 4$, $q = \frac{4}{3}$ together with conditioning and Assumption A2** in case e) can be used. □

Theorems

Since all lemmas from Paragraph 3.2 remain valid also for this generalized model, Theorems 3.3, 3.4 and 3.5 can be reformulated in the following way. One can easily check that proofs of the following theorems can be done analogously without any significant changes.

Theorem 4.3. *Under Assumptions A0*, A1, A2*, A3* and A4, the asymptotic distribution of* $\sqrt{n}(\bar{\hat{\mu}} - \mu)$ *is the same as that given in Theorem 3.3.*

Theorem 4.4. *Under Assumptions A0*, A1', A2**, A3*, A4, A7*, A8 and A9, the asymptotic distribution of* $\sqrt{n}(\bar{\hat{\beta}} - \beta)$ *and* $\sqrt{n}(\bar{\hat{\hat{\beta}}} - \beta)$ *is the same as that given in Theorem 3.4.*

Theorem 4.5. *Under Assumptions A0*, A1', A2**, A3*, A4', A5, A7* and A8, the asymptotic distribution of* $\sqrt{n}(\bar{\hat{\beta}}_W - \beta)$ *and* $\sqrt{n}(\bar{\hat{\hat{\beta}}}_W - \beta)$ *is the same as that given in Theorem 3.4.*

5 Construction of a martingale difference sequence

In the previous sections we have studied asymptotic properties of some estimators in heteroscedastic RCA(1) models that contain processes of martingale differences. To prove strong consistency and asymptotic normality we have to require quite strong assumptions about conditioned and unconditioned moments up to order four for martingale differences. These requirements are expressed by Assumptions A3, A6, A7 and A3*, A7*. One can put a question how restrictive these assumptions are and what form can have such a sequence. Some of the possible constructions of a martingale difference sequence that satisfies given moment restrictions are presented in the sequel.

Let us define a stochastic process $\{Z_t\}$ for $t = 1, \dots, n$ by general formula (17)

$$Z_t = \begin{cases} U_t & \text{for } Z_{t-1} \geq 0, \\ V_t & \text{for } Z_{t-1} < 0. \end{cases} \tag{17}$$

Put $Z_0 := X_0$, where X_0 is a random variable with $EX_0 = 0$, further let $\{U_t\}$ and $\{V_t\}$ be independent sequences of independent random variables, independent of X_0 such that EU_t^4 and EV_t^4 exist. Let us define a system of σ-fields $\mathcal{F}_t := \sigma(Z_0, Z_1, \dots, Z_t)$.

For such process the following equalities hold for $i = 1, \dots, 4$:

$$E(Z_t^i | \mathcal{F}_{t-1}) = I_{[Z_{t-1} \geq 0]} EU_t^i + I_{[Z_{t-1} < 0]} EV_t^i \quad \text{a.s.,}$$
$$EZ_t^i = P(Z_{t-1} \geq 0)\, EU_t^i + P(Z_{t-1} < 0)\, EV_t^i = r_{t-1} EU_t^i + (1 - r_{t-1})\, EV_t^i,$$

where $r_{t-1} = P(Z_{t-1} \geq 0)$.

From the previous equations it is clearly seen that conditions

$$\begin{aligned} &E(Z_t | \mathcal{F}_{t-1}) = 0 \quad \text{a.s.,} \\ &E(Z_t^i | \mathcal{F}_{t-1}) = EZ_t^i \quad \text{as. for } i = 2, 3, 4 \end{aligned} \tag{18}$$

are satisfied, if the following equalities hold:

$$\begin{aligned} EU_t &= EV_t = 0, \quad & EU_t^3 &= EV_t^3, \\ EU_t^2 &= EV_t^2, & EU_t^4 &= EV_t^4. \end{aligned} \tag{19}$$

This construction requires independence of $\{U_t\}$ and $\{V_t\}$. In the following we will give an example where both processes can be dependent.

We can for example suppose that the processes $\{U_t\}$ and $\{V_t\}$ behave according to the following relations:

$$U_t = A_t W_t,$$
$$V_t = B_t W_t,$$

where $\{A_t\}, \{B_t\}$ and $\{W_t\}$ are independent processes of independent random variables. In this case given moment conditions (18) are satisfied under the following restrictions:

$$\begin{aligned} EW_t &= 0, \quad \text{or} \quad EA_t = EB_t = 0, \\ EW_t^3 &= 0, \quad \text{or} \quad EA_t^3 = EB_t^3, \\ EA_t^2 &= EB_t^2, \\ EA_t^4 &= EB_t^4, \end{aligned} \tag{20}$$

More specially we can suppose that random variables A_t, B_t reduce to non-zero constants a_t and b_t. This assumption leads to very simple conditions in the form:

$$\begin{aligned} EW_t &= 0, \\ EW_t^3 &= 0, \\ a_t &= -b_t. \end{aligned} \tag{21}$$

The last case looks very simply but the process $\{Z_t\}$ has in this case some interesting properties. Let us suppose that $P(W_t \geq 0) = p \in (0, 1)\ \forall t$ and $a_t > 0$. Then it is easy to derive that

$$r_t = \frac{1}{2} - \left(\frac{1}{2} - p\right)(2p - 1)^t \xrightarrow[t \to \infty]{} \frac{1}{2}.$$

Hence probability r_t depends on a parameter p in an exponential way and converges to $\frac{1}{2}$. If $p > \frac{1}{2}$, then convergence is monotone and $r_t > \frac{1}{2}$. In case of $p < \frac{1}{2}$ values r_t oscillate around $\frac{1}{2}$ and for $p = \frac{1}{2}$ also $r_t = \frac{1}{2}\ \forall t$ holds. In case of $p = 0$ they periodically reaches values 0 and 1 while if $p = 1$, then $r_t = 1\ \forall t$. Changing the parameter p and the sign of a_t we can influence behaviour of r_t and hence properties of $\{Z_t\}$.

These constructions of the process $\{Z_t\}$ can be analogously used for martingale differences $\{Y_t\}$ or $\{B_t\}$ (with respect to the filtration $\mathscr{F}$) in previously discussed generalized RCA(1) models.

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