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A Note on Weak Convergence On Martingale Measures

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It is investigated the topology of the set of all distributions of pairs (X, Y) such that Y is a real continuous local martingale on the canonical filtration of the process X and X is a stochastic process on some separable metric space T.

It is offered another idea how to prove and generalize the result of weak relative closedness stated in ŠTĚPÁN, ŠEVČÍK (2000) and another approach to the convergence of continuous local martingales given by Remark 2.

V článku se vyšetřuje topologie množiny všech rozdělení dvojic (X, Y) takových, že Y je spojitý reálný lokální martingal vzhledem ke kanonické filtraci procesu X a X je stochastický proces na nějakém separabilním metrickém prostoru T.

Dále se nabízí jiná myšlenka, jak dokázat a zobecnit výsledek slabé relativní uzavřenosti ze článku ŠPĚPÁN, ŠEVČÍK (2000), a jiný přístup ke konvergenci spojitých lokálních martingalů daný poznámkou 2.

Se investiga la topología de todas las distribuciones de las parejas (X, Y), dónde que Y sea un continuo local martingalo con respecto a la filtración canónica de un proceso X y Y sea un proceso estocástico con los valores en un espacio métrico separable T.

Se ofrece una otra idea de comprobar y generalizar el resultado de ŠTĚPÁN, ŠEVČÍK (2000) y el otro enfoque a la convergencia de los martingales locales continuos dado por el comentario 2.

1. Notations and results

Fix a metric space T and denote its Borel σ-algebra by $\mathscr{B}(T)$. Write $\mathbb{C}(T) = \mathbb{C}(\mathbb{R}^+, T)$ for the space of all continuous functions from $\mathbb{R}^+ = [0, \infty)$ to $\mathbb{R}$ and

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endow the space by the metric topology of uniform convergence on compact intervals in $\mathbb{R}^+$ that names $\mathbb{C}(T)$ to be a separable space if T is separable one and to be a Polish space if T is Polish one. In what follows, we complement and generalize the result of [4] related to the convergence in distribution of random elements (X, Y), where $Y = (Y_t, t \geq 0)$ is a continuous[1] $\mathscr{F}_t^X$ or $\mathscr{F}_t^{X,P}$-local martingale with values in $(\mathbb{R}. \mathscr{B}(\mathbb{R}))$ and $X = (X_t, t \geq 0)$ a continuous stochastic process with values in $(T, \mathscr{B}(T))$.

Assuming without loss of generality that the underlying probability space $(\Omega, \mathscr{F}, P)$ is complete, we employed the notation

$$(1) \qquad \mathscr{F}_t^X = \sigma\{[X_s \in B], s \leq t, B \in \mathscr{B}(T)\} \text{ and } \mathscr{F}_t^{X,P} = \sigma(\mathscr{F}_t^X \cup \mathscr{N}_P),$$

where $\mathscr{N}_P = \{N \in \mathscr{F}, P(N) = 0\}$. We refer to [2] for the definition of local martingale and elements of stochastic analysis, generally.

Having two sets of functions $\mathscr{K} \subseteq \mathbb{R}^K$ and $\mathscr{L} \subseteq \mathbb{R}^L$, we shall agree to denote by $\mathscr{K} \otimes \mathscr{L}$ the set of all functions $h : K \times L \to \mathbb{R}$ of the form

$$(2) \qquad h(k, l) = f(k) \cdot f(l), \text{ where } k \in K, l \in L$$

and $f \in \mathscr{K}$, $g \in \mathscr{L}$.

Similarly, $\otimes_S \mathscr{S}$ denotes the set of all functions $F : T^{[0,\infty)} \to \mathbb{R}$ of the form

$$(3) \qquad F(u) = f_1(u(s_1)) \dots f_k(u(s_k)), \; u = (u(t), t \geq 0) \in T^{[0,\infty)},$$

where $\{s_1, \dots, s_k\}$ and $\{f_1, \dots, f_k\}$ go through all finite subsets of $S \subseteq [0, \infty)$ and $\mathscr{S} \subseteq \mathbb{R}^T$, respectively.

Our main result reads as follows.

Proposition. *Let X_n, X be stochastic processes with values in separable metric space T and Y_n, Y be real-valued continuous processes such that*

(i) Y is an $\mathscr{F}_t^X$-adapted process

(ii) Y_n is an $\mathscr{F}_t^{X_n}$-local martingale for every $n \in \mathbb{N}$

(iii) $Ef(X_n, Y_n) \to Ef(X, Y)$ as $n \to \infty$ for every $f \in \otimes_S C_b(T) \otimes C_b(\mathbb{C})$ such that

(a) $S = \mathbb{R}^+$ or

(b) S is a dense subset of $\mathbb{R}^+$ and X is a right-continuous process.

Then Y is an $\mathscr{F}_t^X$-local martingale. The assertion will stay to be valid if we replace the filtrations $\mathscr{F}_t^X$ and $\mathscr{F}_t^{X_n}$ by the completed ones $\mathscr{F}_t^{X,P}$ and $\mathscr{F}_t^{X_n,P}$, respectively.

Assuming that X and X_n are continuous T-valued processes, we may interpret the pairs (X_n, Y_n) and (X, Y) in Proposition as continuous stochastic processes with values in $T \times \mathbb{R}$, i.e. as $\mathbb{C}(\mathbb{R}^+, T \times \mathbb{R})$-valued random variables, where $\mathbb{C}(\mathbb{R}^+, T \times \mathbb{R})$ is provided by the separable metric topology of uniform convergence on compact sets in $\mathbb{R}^+$. Recall that $\mathbb{C}(Z_1, Z_2)$ denotes the space of all continuous mappings

[1] especially, Y is a random variable with values in $\mathbb{C} = \mathbb{C}(\mathbb{R})$

from topological space Z_1 to topological space Z_2. Using the above interpretation, we may look at condition (iii) in Proposition as a weakened condition of

$$(4) \qquad (X_n, Y_n) \to (X, Y) \text{ in distribution as } n \to \infty \text{ in } C(\mathbb{R}^+, T \times \mathbb{R}).$$

We denote by $\mathbb{A}$ the family of all pairs of continuous processes (X, Y) on some probability space $(\Omega, \mathscr{F}, P)$, which may be different for different pairs (X, Y), such that Y is a real-valued $\mathscr{F}_t^X$-adapted process and X is a T-valued process and by $\mathbb{L}$ the family of all $(X, Y) \in \mathbb{A}$ such that Y is an $\mathscr{F}_t^X$-local martingale. The families $\mathbb{A}$, $\mathbb{L}$ are not sets, of course, but the following families of distributions

$$(5) \qquad \mathscr{L}(\mathbb{A}) = \{\mathscr{L}(X, Y), (X, Y) \in \mathbb{A}\}, \quad \mathscr{L}(\mathbb{L}) = \{\mathscr{L}(X, Y), (X, Y) \in \mathbb{L}\}$$

are subsets of $\mathscr{P}(\mathbb{C}(T \times \mathbb{R}))$, where $\mathscr{P}(\mathbb{C}(T \times \mathbb{R}))$ denotes the set of all Borel probability measures on metric space $\mathbb{C}(T \times \mathbb{R})$. Recall that $\mathscr{L}(Z)$ denotes the probability distribution of a r.v. Z.

It this notation, we get by Proposition.

Corollary 1. *Let T be a separable metric space, then $\mathscr{L}(\mathbb{L})$ is a relatively weakly closed set in $\mathscr{L}(\mathbb{A})$.*

Remark 1. Similar assertion holds if we consider processes Y that may be $\mathscr{F}_t^{X,P}$-adapted in the definition of $\mathbb{A}$ and that are $\mathscr{F}_t^{X,P}$-local martingales in the definition of $\mathbb{L}$ by Proposition. But in fact, the sets $\mathscr{L}(\mathbb{A})$, $\mathscr{L}(\mathbb{L})$ would not change if we worked with the completed filtrations instead of with the canonical ones.

Note only that the values $\mathbb{E} f(X, Y)$ in (iii) (a) of Proposition determine distribution of (X, Y) as a random variable with values in $\mathbb{C}(\mathbb{R}^+, T \times \mathbb{R})$ or, more generally.

Lemma 1. *Let P, Q be probability measures on σ-algebra $\mathscr{B} := \otimes^{[0,\infty)}\mathscr{B}(T) \otimes \mathscr{B}(\mathbb{C})$ such that for every $f \in \otimes_{[0,\infty)} C_b(T) \otimes C_b(\mathbb{C})$ $\int f \, \mathrm{d}P = \int f \, \mathrm{d}Q$. Then $P = Q$.*

Proof. The following set of functions $\otimes_{[0,\infty)} C_b(T) \otimes C_b(\mathbb{C})$, generating $\mathscr{B}$, is obviously closed under products and it is a subset of a linear set of bounded functions

$$(6) \quad \mathscr{H} = \left\{ f : T^{[0,\infty)} \times \mathbb{C} \to \mathbb{R} \text{ bounded } \mathscr{B} \text{ measurable, } \int f \, \mathrm{d}P = \int f \, \mathrm{d}Q \right\},$$

which is closed under bounded pointwise limits and contains constants. Applying Proposition I.4.11. in [3], we get that every bounded $\mathscr{B}$-measurable function is contained in $\mathscr{H}$. The choice $f = I_B \in \mathscr{H}$, $B \in \mathscr{B}$ gives $P = Q$, which is the statement of Lemma 1. □

We use the technology of the proof of Proposition given by Theorem 1. First, we introduce some more definitions that are needed to understand this Theorem. Fix $x \in \mathbb{C}$, $x(0) = 0$ and denote

(7) $$\tau_c^x = \inf\{s \geq 0, |x(s)| \geq c\}$$

the first entry of the function $|x|$ into the set $\{-c, c\}$ and

(8) $$\alpha_c(x) = (x(t \wedge \tau_c^x), t \geq 0) \in \mathbb{C}$$

the function x stopped at time τ_c^x. Then the mapping $\alpha_c^0 : x \in \mathbb{C} \mapsto \alpha_c(x - x(0)) \in \mathbb{C}$ is Borel measurable. We recall that a real-valued continuous $\mathscr{F}_t$-adapted process Y is an $\mathscr{F}_t$-local martingale iff for all $c > 0$ $\alpha_c^0(Y)$ is an $\mathscr{F}_t$-martingale.

Theorem 1. *Let Y_n, Y be real-valued continuous processes such that $Y_n \to Y$ a.s. as $n \to \infty$ in $\mathbb{C}(\mathbb{R}^+)$. Then for $c > 0$ there are sequences $\delta_k \in (0, c)$ and $n_k \in \mathbb{N}$ such that*

(9) $$\alpha_{c-\delta_k}^0(Y_{n_k}) \to \alpha_c^0(Y), k \to \infty \textit{ a.s. in } \mathbb{C}(\mathbb{R}^+).$$

Theorem 2. *Let X, X_n be real-valued continuous processes such that $X_n \to X$ in distribution as $n \to \infty$ in $\mathbb{C}(\mathbb{R}^+)$. If $R : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+)$ is a continuous mapping such that for $n \in \mathbb{N}$ $R(X_n)$ is a local $\mathscr{F}_t^{X_n}$-martingale and $R(X)$ is an $\mathscr{F}_t^X$-adapted process, then $R(X)$ is also an $\mathscr{F}_t^X$-local martingale. The assertion will stay to be valid if we replace the filtrations $\mathscr{F}_t^X$ and $\mathscr{F}_t^{X_n}$ by $\mathscr{F}_t^{X,P}$ and $\mathscr{F}_t^{X_n,P}$, respectively.*

Theorem 3. *α_c^0 is a continuous mapping at Y a.s. whenever Y is an $\mathscr{F}_t^Y$-local martingale and $c > 0$.*

Remark 2. Theorem 2 with the choice $R(f) = f$ and Theorem 3 provide an equivalent condition to the a.s. convergence of local martingales as follows:
If Y_n, Y are $\mathscr{F}_t^{Y_n}$ and $\mathscr{F}_t^Y$ local martingales, respectively, then

$$Y_n \to Y \text{ a.s. iff } \alpha_c^0(Y_n) \to \alpha_c^0(Y) \text{ a.s. for } c > 0 \;\&\; Y_n(0) \to Y(0) \text{ a.s. as } n \to \infty.$$

One may ask if we could leave out the assumption, $R(X)$ is an $\mathscr{F}_t^X$-adapted process, in Theorem 2. The following example shows that it is impossible.

Counterexample. We will find $\mathbb{R}$-valued continuous processes such that $X_n \to X$ as $n \to \infty$ everywhere and $R : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+)$ a continuous mapping such that for $n \in \mathbb{N}$ $R(X_n)$ is an $\mathscr{F}_t^{X_n}$-local martingale but $R(X)$ is not even adapted to the completed canonical filtration of the process X.

Denote by W one-dimensional Brownian motion and for $n \in \mathbb{N}$ put

$$X_n(t) = W(t \wedge 1)/n + W[(t-1)^+ \wedge 1] \to X(t) = W[(t-1)^+ \wedge 1], n \to \infty$$
$$\text{and } R : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+),\ R(f)(t) = f(t+1) - f(1).$$

Now it is enough to check that $R(X_n) = W(. \wedge 1)$ is an $\mathscr{F}_t^{X_n} = \mathscr{F}_{t \wedge 1}^W$-local martingale but $R(X) = W(. \wedge 1)$ at time 1 is a nontrivial random variable and it cannot be measurable with respect to a trivial σ-algebra $\mathscr{F}_t^{X,P} = \mathscr{F}_{(t-1)^+ \wedge 1}^{W,P}$ at time $t = 1$.

2. Proofs

Proof of Theorem 1. Let Y_n, Y be real-valued continuous processes on probability space $(\Omega, \mathcal{F}, P)$ such that $Y_n \to Y$ a.s. in $\mathbb{C}(\mathbb{R}^+)$. Fix $c > 0$ and check by the definition of α_c^0 that we may assume that $Y_n(0) = 0$ and $Y(0) = 0$ on Ω and work with α_c instead of with α_c^0.

I. Fix $\varepsilon, \eta > 0, t \geq 0$. We will show that there is $\delta \in (0, c)$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

(10) $$P(\|\alpha_{c-\delta}(Y_n) - \alpha_c(Y)\|_t > \eta) < \varepsilon,$$

where $\|y\|_t = \sup\{|y(s)|, s \leq t\}$ is a pseudonorm on $\mathbb{C}(\mathbb{R}^+)$. The continuity of Y implies that $w_Y^t(1/k) \to 0$ in probability as $k \to \infty$, where

(11) $$w_f^t(\vartheta) = \sup\{|f(u) - f(s)|, |s - u| \leq \vartheta, s, u \leq t\}$$

is the modul of continuity of function f up to time t. Hence, we can find a measurable set $F_1 \in \mathcal{F}$ and $k \in \mathbb{N}$ such that for $l \geq k$

(12) $$w_Y^t(1/l) < \eta/2 \text{ on } F_1 \text{ and } P(F_1^c) < \varepsilon/3.$$

Considering a real variable

(13) $$Z_k = \min\{\|Y_u| - c|, u \leq \tau_c^Y \wedge t - 1/k\},$$

which possesses only positive values, and by the continuity of measure P, we get a measurable set $F_2 \in \mathcal{F}$ and $\delta > 0$ such that

(14) $$Z_k > 2\delta \text{ on } F_2 \text{ and } P(F_2^c) < \varepsilon/3.$$

Applying the definition of τ_c^Y and Z_k, we get from (14)

(15) $$\|Y\|_{\tau_c^Y \wedge t - 1/k} < c - 2\delta \text{ on } F_2.$$

The assumption $Y_n \to Y$ a.s. as $n \to \infty$ gives us a measurable set $F_3 \in \mathcal{F}$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

(16) $$\|Y_n - Y\|_t < \eta/2 \wedge \delta \text{ on } F_3 \text{ and } P(F_3^c) < \varepsilon/3.$$

Now apply the definition of α_c to compute

(17) $$\|\alpha_{c-\delta}(Y_n) - \alpha_c(Y)\|_t \leq \max_{s \leq t} |Y_n(s \wedge \tau_{c-\delta}^{Y_n}) - Y(s \wedge \tau_{c-\delta}^{Y_n})| + \max_{s \leq t} |Y(s \wedge \tau_{c-\delta}^{Y_n}) - Y(s \wedge \tau_c^Y)| \leq \|Y_n - Y\|_t + w_Y^t(|t \wedge \tau_c^Y - t \wedge \tau_{c-\delta}^{Y_n}|).$$

A combination of the above results yields that it suffices to prove

(18) $$|t \wedge \tau_c^Y - t \wedge \tau_{c-\delta}^{Y_n}| \leq 1/k \text{ on } F_1 \cap F_2 \cap F_3.$$

Fix $\omega \in F_1 \cap F_2 \cap F_3$. We will show that

(19) $$\tau_c^Y \wedge t \geq \tau_{c-\delta}^{Y_n} \wedge t \geq \tau_c^Y \wedge t - 1/k.$$

Apply relations (15), (16) to check that

$$(20)\qquad \|Y_n\|_{\tau_c^Y\wedge t-1/k} \leq \|Y\|_{\tau_c^Y\wedge t-1/k} + \|Y_n - Y\|_t < c - 2\delta + \delta = c - \delta,$$

which by the definition of $\tau_{c-\delta}^{Y_n}$ easily implies the second inequality in (19). To prove the first one in (19), we may assume that $\tau_c^Y < t$, which implies $\|Y\|_{\tau_c^Y\wedge t} = c$. Now use the definition of $\tau_{c-\delta}^{Y_n}$ to check that the following inequality, using (16) and $\|Y\|_{\tau_c^Y\wedge t} = c$, proves the first one in (19).

$$(21)\qquad \|Y_n\|_{\tau_c^Y\wedge t} \geq \|Y\|_{\tau_c^Y\wedge t} - \|Y_n - Y\|_t > c - \delta.$$

II. For $m \in \mathbb{N}$ put $\varepsilon, \eta = 1/m, t = m$ and use (10) to get $\delta(m) \in (0, c)$, $n(m) \in \mathbb{N}$ such that

$$(22)\qquad P(\|\alpha_{c-\delta(m)}(Y_{n(m)}) - \alpha_c(Y)\|_m > 1/m) < 1/m.$$

Now it remains to realize that $\alpha_{c-\delta(m)}(Y_{n(m)})$ converge to $\alpha_c(Y)$ in probability in $\mathbb{C}$ as $m \to \infty$ and select m_k such that

$$(23)\qquad \alpha_{c-\delta(m_k)}(Y_{n(m_k)}) \to \alpha_c(Y) \text{ a.s. in } \mathbb{C}(\mathbb{R}^+),$$

which is the statement of Theorem 1, since we assumed that Y_n, Y started at 0. □

Now we introduce lemma, which characterizes the martingale property of an adapted process with respect to the canonical filtration of another process in terms of their distribution.

Lemma 2. *Let Y be a real-valued continuous process adapted to the canonical filtration of a process X, which possesses values in some separable metric space T. Then Y is an $\mathscr{F}_t^X$-martingale iff for all $0 \leq s < t$, $H \in \otimes_{[0,s]} C_b(T)$*

$$(24)\qquad EH(X)\,Y_s = EH(X)\,Y_t.$$

Proof. If Y is an $\mathscr{F}_t^X$-martingale and $0 \leq s \leq t$, then

$$(25)\qquad EH(X)\,Y_t = EE^{\mathscr{F}_s^X} H(X)\,Y_t = EH(X)\,E^{\mathscr{F}_s^X} Y_t = EH(X)\,Y_s.$$

Conversely, we are to show that $\mathscr{F}_s^X$ is a subset of $\mathscr{M} = \{F \in \mathscr{F}_s^X, EI_F Y_s = EI_F Y_t\}$, which is easily seen to be a Dynkin system. Now it remains to find a system $\mathscr{L} \subseteq \mathscr{M}$ closed under intersections, which generates σ-algebra $\mathscr{F}_s^X$, and use Dynkin Lemma. Put

$$(26)\qquad \mathscr{L} = \{[X_{s_i} \in G_i, i \leq n], s_i \leq s, G_i \text{ open in } T, i \leq n \in \mathbb{N}\}$$

and check that it is closed under intersections and every $F \in \mathscr{L}$ is a pointwise bounded limit of some $H_n(X)$, where $H_n \in \otimes_{[0,s]} C_b(T)$. Now use Dominated Convergence Theorem and (24) to get that $\mathscr{L} \subseteq \mathscr{M}$. □

Remark 3. Apply Lemma 2 to see that the $\mathscr{F}_t^Y$ (local) martingale property of a real continuous process depends only on its distribution and note that Lemma 2 will stay to be valid if we replace the canonical filtration of the process X by the completed one $\mathscr{F}_t^{X,P}$.

Lemma 3. *Let* $Z_n = (Z_n^1, Z_n^2)$, $n \in \mathbb{N} \cup \{\infty\}$ *be random variables with values in* $T_1 \times T_2$ *for some Polish spaces* T_1 *and* T_2 *such that for every* $f \in C_b(T_1) \otimes C_b(T_2)$

$$Ef(Z_n) \to Ef(Z_\infty) \text{ as } n \to \infty. \tag{27}$$

Then $Z_n \to Z_\infty$ *in distribution as* $n \to \infty$ *in* $T_1 \times T_2$.

Proof. Use Prochorov Theorem to get $\{\mathscr{L}(Z_n^i), n \in \mathbb{N}\}$ is a tight set of Borel probability measures on T_i, $(i = 1, 2)$, which easily implies that $\{\mathscr{L}(Z_n), n \in \mathbb{N}\}$ is also a tight set of Borel probability measures on $T_1 \times T_2$. Now use Prochorov Theorem again to get that $\{\mathscr{L}(Z_n), n \in \mathbb{N}\}$ is a relatively weakly compact set. Note that the system $C_b(T_1) \otimes C_b(T_2)$ contains only continuous functions and determines measure on $T_1 \times T_2$ to get by (27) that $\mathscr{L}(Z_\infty)$ is the only possible weak limit of each subsequence of $(\mathscr{L}(Z_n), n \in \mathbb{N})$. A combination of the above results yields $\mathscr{L}(Z_n) \to \mathscr{L}(Z_\infty)$ as $n \to \infty$ weakly, which is the statement of Lemma 3. □

We will use Skorochod Theorem on the representation of the convergence in distribution, Theorem 1 and Lemma 2 and 3 to prove the Proposition. The proof of Theorem 2 will be omitted, since it is a consequence of the Proposition.

Skorochod Theorem. If ϑ_n, ϑ are $\mathbb{S}$-valued random variables, where $\mathbb{S}$ is a separable metric space, such that $\vartheta_n \to \vartheta$ in distribution, then there are η_n, η on some probability space $(\Omega, \mathscr{F}, P)$ such that $\mathscr{L}(\vartheta_n) = \mathscr{L}(\eta_n)$, $\mathscr{L}(\vartheta) = \mathscr{L}(\eta)$ and $\eta_n \to \eta$ as $n \to \infty$ on Ω (see thm. 6.7. page 70 in [1]).

Proof of Proposition. We will work with the canonical filtrations of the processes X, X_n. The proof for the completed ones would be similar.

1. case (a) Fix $s \le t$ and $H \in \otimes_{[0,s]} C_b(T)$ and use assumption (iii) and Lemma 3 to get

$$(H(X_n), Y_n) \to (H(X), Y) \text{ as } n \to \infty \text{ in distribution in } \mathbb{R} \times \mathbb{C}. \tag{28}$$

Now use Skorochod Theorem and the separability of $\mathbb{R} \times \mathbb{C}$ to check that we may assume[2] that $Z_n := (H(X_n), Y_n) \to Z := (H(X), Y)$ everywhere on some common probability space. Fix $c > 0$ and apply Theorem 1 to get $n_k \in \mathbb{N}$ and $\delta_k \in (0, c)$ such that

$$(H(X_{n_k}), \alpha^0_{c-\delta_k}(Y_{n_k})) \to (H(X), \alpha^0_c(Y)) \text{ a.s. as } k \to \infty. \tag{29}$$

Then for every $v \in \mathbb{R}$

$$EH(X_{n_k})\, \alpha^0_{c-\delta_k}(Y_{n_k})_v \to EH(X)\, \alpha^0_c(Y)_v \text{ as } k \to \infty. \tag{30}$$

[2] A suspicious reader can imagine that Z_n, Z are defined on some other probability space given by Skorochod Theorem and look at $H(X_n)$, $H(X)$ and Y_n, Y as projections of Z_n, Z up to the relation (33), which depends only on distribution of $(H(X), Y)$.

Now use assumption (ii) and the equivalent definition of local martingale, using α_c^0, to see that for $k \in \mathbb{N}$

$$\alpha^0_{c-\delta_k}(Y_{n_k}) \text{ is an } \mathscr{F}_t^{X_{n_k}}\text{-martingale.} \tag{31}$$

By Lemma 2 and the previous relation, for $k \in \mathbb{N}$

$$EH(X_{n_k})\,\alpha^0_{c-\delta_k}(Y_{n_k})_s = EH(X_{n_k})\,\alpha^0_{c-\delta_k}(Y_{n_k})_t. \tag{32}$$

A combination of (30) for $v = s, t$ and relation (32) gives

$$EH(X)\,\alpha^0_c(Y)_s = EH(X)\,\alpha^0_c(Y)_t, \tag{33}$$

which is by Lemma 2 the martingale property of $\alpha_c^0(Y)$ with respect to the canonical filtration of the process X, since $\mathscr{F}_u^{\alpha_c^0(Y)} \subseteq \mathscr{F}_u^Y \subseteq \mathscr{F}_u^X$ and $s \le t$, $H \in \otimes_{[0,s]} C_b(T)$ were arbitrary. Now it remains to apply the equivalent definition of local martingale and assumption (i) again to see that Y is an $\mathscr{F}_t^X$-local martingale.

2. case (b). Let $s < t$ and $G \in \otimes_{[0,s]}$, say

$$G(u) = g_1(u(s_1)) \dots g_k(u(s_k)),\ u \in T^{[0,\infty)} \text{ for some } g_i \in C_b(T),\ s_i \le s,\ i \le k. \tag{34}$$

Choose $s_i^m \in S$ such that $s_i^m \downarrow s_i$ and put $H_m(u) = g_1(u(s_1^m)) \dots g_k(u(s_k^m)) \in \otimes_{[0,s^m] \cap S} C_b(T)$, where $s^m := s_1^m \vee \dots \vee s_k^m$. Then for sufficiently large m ($s^m < t$) and by part 1. up to the relation (33) with $H := H_m$ and $s := s^m$ we get for $c > 0$

$$EH_m(X)\,\alpha^0_c(Y)_{s^m} = EH_m(X)\,\alpha^0_c(Y)_t. \tag{35}$$

Now it remains to use the right continuity of process X and Dominated Convergence Theorem to get $EG(X)\,\alpha_c^0(Y)_s = EG(X)\,\alpha_c^0(Y)_t$ and to repeat arguments of part 1. following after the relation (33). □

Remark 4. For $f \in \mathbb{C}_0 = \{g \in \mathbb{C}, g(0) = 0\}$ and $c > 0$ we may define $\tau^f_{c+} = \lim_{\varepsilon \to 0+} \tau^f_{c+\varepsilon} = \inf\{t \ge 0, |f(t)| > c\}$ the time of the first entry of the function $|f|$ into (c, ∞). Then τ_c^f is a lower semi-continuous and τ^f_{c+} is an upper semi-continuous function on $\mathbb{C}_0$.

Now we introduce lemma which characterizes points of discontinuity of mapping α_c^0 in terms of τ_c^f, τ_{c+}^f.

Lemma 4. *Let $c > 0$, f be a continuous real function on $\mathbb{R}^+$. Then α_c^0 is continuous at f iff $g = f - f(0)$ is constant on interval $[\tau_c^g, \tau_{c+}^g]$.*[3]

Proof. See for the definition of α_c^0 to check that it is enough to prove that for $f \in \mathbb{C}$ with $f(0) = 0$ holds

$$f \text{ is constant on } [\tau_c^f, \tau_{c+}^f] \text{ iff } \alpha_c(f_n) \to \alpha_c(f) \text{ in } \mathbb{C}, \tag{36}$$

[3] By $[a, b]$ we understand the interval $\{x \in \mathbb{R},\ a \le x \le b\}$ even if $b = +\infty$.

whenever, $f_n \to f$ in $\mathbb{C}$ such that $f_n(0) = 0$ for $n \in \mathbb{N}$. If the left-hand side of (36) fails, then it is enough to put $f_n = f \cdot (1 - 1/n)$ and check that the right-hand side fails, too.

Conversely, we use the following inequality for $t \geq 0$

$$\|\alpha_c(f_n) - \alpha_c(f)\|_t = \sup_{s \leq t} |\alpha_c(f_n)(s) - \alpha_c(f)(s)| \leq \tag{37}$$

$$\sup_{s \leq t} |f_n(s \wedge \tau_c^{f_n}) - f(s \wedge \tau_c^{f_n})| + \sup_{s \leq t} |f(s \wedge \tau_c^{f_n}) - f(s \wedge \tau_c^{f})| \leq \tag{38}$$

$$\|f_n - f\|_t + \sup \{|f(u) - f(v)|, u, v \text{ between } \tau_c^f \wedge t, \tau_{c+}^f \wedge t\} \vee 0\,. \tag{39}$$

If f is constant on $[\tau_c^f, \tau_{c+}^f]$ and $f_n \to f$ in $\mathbb{C}_0 = \{g \in \mathbb{C}, g(0) = 0\}$, then the lower and upper semi-continuity of τ_c and τ_{c+} provide

$$\tau_c^f \wedge t \leq \liminf_{n \to \infty} \tau_c^{f_n} \wedge t \leq \limsup_{n \to \infty} \tau_{c+}^{f_n} \wedge t \leq \tau_{c+}^f \wedge t\,. \tag{40}$$

It implies that the last term in (39) is asymptotically less or equal to zero since f is constant on $[\tau_c^f \wedge t, \tau_{c+}^f \wedge t]$. □

Now we introduce lemma which says something about stability of the property of being a point of continuity of α_c^0 under transformation.

Lemma 5. *Let $c > 0, f, h \in \mathbb{C}$ such that there exists a non-decreasing continuous real function a such that $a(0) = 0$ and $f(t) = f(0) + h(a(t))$ for $t \geq 0$. If h is a point of continuity of α_c^0, then so is f.*

Proof. See for the definition of α_c^0 to check that we may assume that $f(0) = 0$. Denote by $a(\infty) = \lim_{t \to \infty} a(t)$ and use definitions of τ_c^f, τ_{c+}^f to check[4] that for $d > 0$

$$\tau_d^h = a(\tau_d^f) \text{ which implies } \tau_{c+}^h = a(\tau_{c+}^f) \text{ as } d \to c^+ \tag{41}$$

since a is continuous. By Lemma 4, we get that h is constant on $[\tau_c^h, \tau_{c+}^h]$ and we are to show the same for f. Let $u \in [\tau_c^f, \tau_{c+}^f]$, then $f(u) = h(a(u)) = h(\tau_c^h)$ does not depend on u, since $a(u) \in [a(\tau_c^f), a(\tau_{c+}^f)] = [\tau_c^h, \tau_{c+}^h]$. □

Proof of Theorem 3. I. If $Y = W$ is a Brownian motion, then $\tau_c^W = \tau_{c+}^W$ a.s. which by Lemma 4 implies that α_c^0 is continuous at W almost surely.

II. If Y is an $\mathscr{F}_t^Y$-local martingale with $\langle Y \rangle(\infty) = \infty$ a.s., use DDS Theorem to get a Brownian motion W on Ω such that

$$Y(t) = Y(0) + W(\langle Y \rangle)(t),\ t \geq 0 \text{ a.s.}, \tag{42}$$

where $\langle Y \rangle_\omega$ is a non-decreasing continuous real function with $\langle Y \rangle(0) = 0$ a.s., which implies that α_c^0 is continuous at Y a.s. by Lemma 5 and part I.

[4] Show that $h(a(\tau_d^f)) \in \{-d, d\}$ if $a(\tau_d^f) < \infty$ and use the properties of a to verify that $h(u) \in (-d, d)$ if $a(0) \leq u < a(\tau_d^f)$.

III. If Y is an $\mathcal{F}_t^Y$-local martingale, see for Remark 3 to check that both, the assumption and the conclusion of Theorem 3, depend only on distribution of the process Y. It means that we may assume that there is a Brownian motion W independent of Y. Now use Lemma 4 or the definition of α_c^0 to check that we may assume that $Y(0) = 0$. For $n \in \mathbb{N}$ we define an $\mathcal{F}_t^{(Y,W)}$-local martingale

$$Y_n(t) := Y(t) + W[(t-n)^+] \text{ with } \langle Y_n \rangle (t) = (t-n)^+ \to \infty \text{ a.s. as } t \to \infty .$$

By Lemma 4, we are to show that Y is a.s. constant on $[\tau_c^Y, \tau_{c+}^Y]$. Since $Y_n = Y$ on $[0, n]$ for $n \in \mathbb{N}$, it is sufficient[5] to show that for $n \in \mathbb{N}$ Y_n is a.s. constant on $[\tau_c^{Y_n}, \tau_{c+}^{Y_n}]$. Now it remains to use Lemma 4 and part II of this proof for Y_n since for its quadratic variation holds $\langle Y_n \rangle (\infty) = \infty$ almost surely. □

3. References

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[4] ŠTĚPÁN J., ŠEVČÍK P., *Local Martingales Measures*, Acta Universitatis Carolinae, Mathematica et Physica, vol. 41 NO. 1, Praha, 2000.

[5] If $u \in [\tau_c^Y, \tau_{c+}^Y]$ such that $Y(u) \neq Y(\tau_c^Y)$, then for $n > u$ hold: $\tau_c^Y = \tau_c^{Y_n}$ and $Y_n(u) = Y(u) \neq Y(\tau_c^Y) = Y_n(\tau_c^{Y_n})$.