Quasigroups of Fractions

T. KEPKA

Praha

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Various imbeddings of cancellative groupoids into quasigroups are studied.
Studují se rozmanitá vnoření groupoidů s krácením do kvazigrup.

I. Closed subgroupoids

I.1 Translations

1.1 A groupoid is a non-empty set together with a binary operation which is usually denoted multiplicatively.

1.2 Let G be a groupoid. For each $a \in G$ we define the left translation L_a $(= \mathrm{L}_{a,G})$ and the right translation R_a $(= \mathrm{R}_{a,G})$ by $\mathrm{L}_a(x) = xa$ and $\mathrm{R}_a(x) = xa$ for every $x \in G$. Now, the left multiplication monoid $\mathrm{Mul}_\mathrm{l}(G)$ and the right multiplication monoid $\mathrm{Mul}_\mathrm{r}(G)$ are the submonoids of the transformation monoid of G generated by the sets $\{\mathrm{L}_a; a \in G\}$ and $\{\mathrm{R}_a; a \in G\}$, resp. The multiplication monoid is generated by $\{\mathrm{L}_a, \mathrm{R}_a; a \in G\}$.

1.3 Let H be a subgroupoid of a groupoid G. We denote by $\mathrm{Mul}_\mathrm{l}(G, H)$ the submonoid of $\mathrm{Mul}_\mathrm{l}(G)$ generated by all $\mathrm{L}_{a,G}$, $a \in H$. The monoids $\mathrm{Mul}_\mathrm{r}(G, H)$ and $\mathrm{Mul}(G, H)$ are defined similarly.

1.4 Let G be a groupoid. We say that G is left (right) cancellative if L_a (R_a) is injective for each $a \in G$ and we say that G is cancellative if G is both left and right cancellative. We say that G is left (right) divisible if L_a (R_a) is projective for each

Department of Algebra, Charles University, Sokolovská 83, 18 675 Prague, Czech Republic

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$a \in G$ and we say that G is divisible if G is both left and right divisible. Finally, G is said to be a (left, right) quasigroup if G is both (left, right) cancellative and (left, right) divisible.

1.5 Proposition. *Let G be a groupoid.*

(i) If G is left (right) cancellative, then the monoid $\mathrm{Mul}_l(G)$ *(*$\mathrm{Mul}_r(G)$*) is left cancellative.*

(ii) If G is cancellative, then the monoid $\mathrm{Mul}(G)$ *is left cancellative.*

1.6 Proposition. *Let G be a groupoid.*

(i) If G is left (right) divisible, then the monoid $\mathrm{Mul}_l(G)$ *(*$\mathrm{Mul}_r(G)$*) is right cancellative.*

(ii) If G is divisible, then the monoid $\mathrm{Mul}(G)$ *is right cancellative.*

1.7 If G is a (left, right) quasigroup, then every (left, right) translation is a permutation and we denote by ($\mathrm{Mult}_l(G)$, $\mathrm{Mult}_r(G)$) $\mathrm{Mult}(G)$ the permutation group generated by $(\{\mathrm{L}_a, a \in G\}, \{\mathrm{R}_a, a \in G\})$ $\{\mathrm{L}_a, \mathrm{R}_a, a \in G\}$.

1.8 Let r be a relation defined on a groupoid G. We say that r is left (right) stable if $(a, b) \in r$ implies $(xa, xb) \in r$ $((ax, bx) \in r)$ for every $x \in G$ and we say that r is stable is it is both left and right stable. We say that r is left (right) cancellative if $a, b, c \in G$ and $(ca, cb) \in r$ $((ac, bc) \in r)$ implies $(a, b) \in r$ and we say that r is cancellative if it is both left and right cancellative. Finally, we say that r is compatible if $(a, b) \in r$ and $(c, d) \in r$ implies $(ac, bd) \in r$. (Clearly, if r compatible and reflexive, then r is stable and if r is stable and transitive, then r is compatible.)

1.9 If S is a non-empty subset of a groupoid G, then $\langle S \rangle_G$ means the subgroupoid generated by S. (Clearly, $\mathrm{card}(\langle S \rangle) \leq \aleph_0$ for S finite and $\mathrm{card}(\langle S \rangle) = \mathrm{card}(S)$ for S infinite.)

1.10 For a groupoid G we put:

$\mathrm{A}_l(G) = \{a \in G; \mathrm{L}_a \text{ is injective}\}$,

$\mathrm{A}_r(G) = \{a \in G; \mathrm{R}_a \text{ is injective}\}$,

$\mathrm{A}(G) = \mathrm{A}_l(G) \cap \mathrm{A}_r(G)$;

$\mathrm{B}_l(G) = \{a \in G; \mathrm{L}_a \text{ is projective}\}$,

$\mathrm{B}_r(G) = \{a \in G; \mathrm{R}_a \text{ is projective}\}$,

$\mathrm{B}(G) = \mathrm{B}_l(G) \cap \mathrm{B}_r(G)$;

$\mathrm{C}_l(G) = \mathrm{A}_l(G) \cap \mathrm{B}_l(G)$, $\mathrm{C}_r(G) = \mathrm{A}_r(G) \cap \mathrm{B}_r(G)$, $\mathrm{C}(G) = \mathrm{C}_l(G) \cap \mathrm{C}_r(G)$.

1.11 Let G be a groupoid. We define two equivalences p_G and q_G on G by $(a, b) \in \mathrm{p}_G$ and $(c, d) \in \mathrm{q}_G$ if and only if $\mathrm{L}_a = \mathrm{L}_b$ and $\mathrm{R}_c = \mathrm{R}_d$.

1.12 Let G be a groupoid. We define three transformations σ_G, τ_G and ρ_G of G by $\sigma_G(x) = x^2$, $\tau_G(x) = x \cdot x^2$ and $\rho_G(x) = x^2 \cdot x$ for every $x \in G$.

1.13 Proposition. *Let G be a groupoid such that σ_G (τ_G, ρ_G) is an injective endomorphism of G. Then there exists a groupoid K with the following properties:*

(i) G is a subgroupoid of K and G, K are equationally equivalent (i.e., they satisfy the same groupoid equations).
(ii) σ_K (τ_K, ρ_K) is an automorphism of K.
(iii) Every finitely generated subgroupoid of K is contained in a subgroupoid isomorphic to G.
(iv) K is (left, right) cancellative if and only if G is so.
(v) K is (left, right) divisible if and only if G is so.

Proof. Clearly, there is a subgroupoid H such that G is a subgroupoid of H, $G \cong H$ and $\sigma_H(H) = G$. Now, putting $G_0 = G$, $G_1 = H$, etc., we get the chain $G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots$ and $K = \bigcup_{i=0}^{\infty} G_i$. ▲

1.14 Let G be a groupoid. We denote by c_G $(\mathrm{c}_{\mathrm{l},G}, \mathrm{c}_{\mathrm{r},G})$ the smallest (left, right) cancellative congruence of G. Further, if G is non-trivial, then we denote by c_G^* $(\mathrm{c}_{\mathrm{l},G}^*, \mathrm{c}_{\mathrm{r},G}^*)$, the intersection of all non-identical (left, right) cancellative congruences of G. Clearly $\mathrm{c}_G \subseteq \mathrm{c}_G^*$ $(\mathrm{c}_{\mathrm{l},G} \subseteq \mathrm{c}_{\mathrm{l},G}^*, \mathrm{c}_{\mathrm{r},G} \subseteq \mathrm{c}_{\mathrm{r},G}^*)$ and, if G is not (left, right) cancellative, then $\mathrm{c}_G = \mathrm{c}_G^* \neq \mathrm{id}_G$ $(\mathrm{c}_{\mathrm{l},G} = \mathrm{c}_{\mathrm{l},G}^* \neq \mathrm{id}_G, \mathrm{c}_{\mathrm{r},G} = \mathrm{c}_{\mathrm{r},G}^* \neq \mathrm{id}_G)$.
A non-trivial (left, right) cancellative groupoid G will be said subdirectly c-irreducible (cl-irreducible, cr-irreducible) if $\mathrm{c}_G^* \neq \mathrm{id}_G$ $(\mathrm{c}_{\mathrm{l},G}^* \neq \mathrm{id}_G, \mathrm{c}_{\mathrm{r},G}^* \neq \mathrm{id}_G)$.

1.15 Proposition. *Let G be a non-trivial (left, right) cancellative groupoid. Then G is a subdirect product of subdirectly c-irreducible (cl-irreducible, cr-irreducible) (left, right) cancellative groupoids.*

Proof. For every $x = (a, b) \in G^{(2)}$, $a \neq b$, let r_x denote a cancellative congruence of G that is maximal with respect to $x \notin r_x$. Then G can be imbedded onto the product $\prod G/r_x$, $x \in G^{(2)} \setminus \mathrm{id}_G$. ▲

I.2 Closed subsets

2.1 Let G be a groupoid. For any subset S of G and all $n \geq 1$ we put:

$$(\alpha_{0,G}(S) =)\, \alpha_0(S) = S;$$
$$(\alpha_{n,G}(S) =)\, \alpha_n(S) = S = \{x \in G;\ a_1(a_2(\ldots (a_n x))) \in S \text{ for some } a_1, \ldots, a_n \in S\};$$
$$(\alpha_G(S) =)\, \alpha(S) = \bigcup_{m=0}^{\infty} \alpha_m(S);$$
$$(\beta_{0,G}(S) =)\, \beta_0(S) = S;$$
$$(\beta_{n,G}(S) =)\, \beta_n(S) = S = \{x \in G;\ (((x a_n) \ldots)\, a_2)\, a_1 \in S \text{ for some } a_1, \ldots, a_n \in S\};$$
$$(\beta_G(S) =)\, \beta(S) = \bigcup_{m=0}^{\infty} \beta_m(S);$$

2.2 A subset S of a groupoid G is said to be (left, right) closed in G if $(\alpha_1(S) \subseteq S$, $\beta_1(S) \subseteq S)$ $\alpha_1(S) \cup \beta_1(S) \subseteq S$.

2.3 The intersection of any non-empty family of (left, right) closed subsets of a groupoid G is again (left, right) closed. Hence, given a subset S of G, we denote by $([S]_{l,G}, [S]_{r,G})$ $[S]_G$ the smallest (left, right) closed subset containing S. Clearly, $[S]_l \cup [S]_r \subseteq [S]$.

2.4 Remark. Let S be a subset of a groupoid G.

(i) Put $S_0 = S$ and $S_{i+1} = \alpha_1(S_i) \cup S_i$ for every $i \geq 0$. Then $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and $[S]_l = \bigcup_{i=0}^{\infty} S_i$.

(ii) Put $S_0 = S$ and $S_{i+1} = \alpha_1(S_i) \cup \beta_1(S_i)$ for every $i \geq 0$. Again $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and $[S] = \bigcup_{i=0}^{\infty} S_i$.

(iii) Finally, put $S_0 = S$ and $S_{i+1} = \alpha_1(S_i) \cup S_i$ for every even $i \geq 0$ and $S_{i+1} = \beta_1(S_i) \cup S_i$ for every odd $i \geq 1$. Then $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and $[S] = \bigcup_{i=1}^{\infty} S_i$.

2.5 Lemma. *Let H be a subgroupoid of a groupoid G and S a subset of G.*

(i) If S is (left, right) closed in G, then $S \cap H$ is (left, right) closed in H.

(ii) If $S \subseteq H$, S is (left, right) closed in H, and H is (left, right) closed in G, then S is (left, right) closed in G.

(iii) If $S \subseteq H$, S is (left, right) closed in G, then S is (left, right) closed in H.

(iv) If $S \subseteq H$, then $([S]_{l,H} \subseteq [S]_{l,G}, [S]_{r,H} \subseteq [S]_{r,G})$ $[S]_H \subseteq [S]_G$.

2.6 Lemma. *Let S be a subset of a groupoid G. Then:*

(i) $\alpha(S) \subseteq [S]_l$ and $\beta(S) \subseteq [S]_r$.

(ii) $\alpha(S) \cup \beta(S) \subseteq [S]$.

I.3 Closed subgroupoids

3.1 Let G be a groupoid. The intersection of any non-empty family of (left, right) closed subgroupoids of G is either empty or a (left, right) closed subgroupoid. Hence, given a non-empty subset S of G, $(\langle S\rangle_{lc,G}, \langle S\rangle_{rc,G})$ $\langle S\rangle_{c,G}$ will denote the smallest (left, right) closed subgroupoid containing S. Clearly, $([S]_l \subseteq \langle S\rangle_{lc}, [S]_r \subseteq \langle S\rangle_{rc})$ $[S] \subseteq \langle S\rangle_c$. Moreover, $\langle S\rangle_{lc} \cup \langle S\rangle_{rc} \subseteq \langle S\rangle_c$.

3.2 Remark. Let S be a non-empty subset of a groupoid G.

(i) Put $S_0 = S$, $S_{i+1} = \{xy; x, y \in S_i\}$ for every even $i \geq 0$ and $S_{i+1} = \alpha_1(S_i) \cup S_i$ for every odd $i \geq 1$. Then $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and $\langle S\rangle_{lc} = \bigcup_{i=1}^{\infty} S_i$.

(ii) Put $S_0 = S$, $S_{i+1} = \{xy; x, y \in S_i\}$ for every even $i \geq 0$ and $S_{i+1} = \alpha_1(S_i) \cup \beta_1(S_i) \cup S_i$ for every odd $i \geq 1$. Again $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and $\langle S\rangle_c = \bigcup_{i=1}^{\infty} S_i$.

(iii) Finally, put $S_0 = S$, $S_{3i+1} = \{xy; x, y \in S_{3i}\}$, $S_{3i+2} = \alpha_1(S_{3i+1}) \cup S_{3i+1}$ and $S_{3i+3} = \beta_1(S_{3i+2}) \cup S_{3i+2}$ for every $i \geq 0$. Then $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and $\langle S \rangle_{\mathrm{c}} = \bigcup_{i=1}^{\infty} S_i$.

3.3 A non-empty subset S of a groupoid G is said to be left (right) strongly dense in G if $G = \langle S \rangle_{\mathrm{lc}}$ $(G = \langle S \rangle_{\mathrm{rc}})$ (we will also say that S lc/rc-generates G, etc.) and S is said to be strongly dense in G if it is both left and right stronly dense in G.

3.4 A non-empty subset S of a groupoid G is said to be dense in G if $\langle S \rangle_{\mathrm{c}} = G$ (we will also say that S c-generates G, etc.).

3.5 Lemma. *Let S be a non-empty subset of a groupoid G. Then:*
(i) S is left (right) strongly dense in $\langle S \rangle_{\mathrm{lc}, G}$ $(\langle S \rangle_{\mathrm{rc}, G})$.
(ii) S is dense in $\langle S \rangle_{\mathrm{c}, G}$.

3.6 Lemma. *Let H be a (left, right) strongly dense subgroupoid of a groupoid G and S a non-empty subset of H such that S (left, right) strongly dense in H. Then S is (left, right) strongly dense in G.*

Proof. We have $H = \langle S \rangle_{\mathrm{lc}, H} \subseteq \langle S \rangle_{\mathrm{lc}, G} = K$. But K is left closed in G and consequently $K = G$ and S is left strongly dense in G. ▲

3.7 Lemma. *Let H be a dense subgroupoid of a groupoid G and S a non-empty subset of H such that S is dense in H. Then S is dense in G.*

Proof. Similar to that of 3.6. ▲

3.8 Lemma. *Let H be a subgroupoid of a groupoid G. Then:*
(i) $\alpha_G(H) = \{x \in G; f(x) \in H$ for some $f \in \mathrm{Mul}_{\mathrm{l}}(G, H)\}$.
(ii) $\beta_G(H) = \{x \in G; f(x) \in H$ for some $f \in \mathrm{Mul}_{\mathrm{r}}(G, H)\}$.
(iii) $H \subseteq \alpha_G(H) \subseteq [H]_{\mathrm{l}, G} \subseteq \langle H \rangle_{\mathrm{lc}, G}$.
(iv) $H \subseteq \beta_G(H) \subseteq [H]_{\mathrm{r}, G} \subseteq \langle H \rangle_{\mathrm{rc}, G}$.

3.9 Let H be a subgroupoid of a groupoid G. We put $(\gamma_G(H) =)\, \gamma(H) = \{x \in G;$ $f(x) \in H$ for some $f \in \mathrm{Mul}(G, H)\}$.

3.10 Lemma. *Let H be a subgroupoid of a groupoid G. Then $H \subseteq \gamma_G(H) \subseteq [H]_G$.*

3.11 Lemma. *Let H be a subgroupoid of a groupoid G. Then $\alpha_{n,G}(H) \subseteq \alpha^n_{1,G}(H)$ and $\beta_{n,G}(H) \subseteq \beta^n_{1,G}(H)$ for every $n \geq 0$.*

3.12 Lemma. *Let φ, ψ be homomorphism of a groupoid G into a (left, right) cancellative groupoid K. Then set $\{x \in G; \varphi(x) = \psi(x)\}$ is either empty or a (left, right) closed subgroupoid of G.*

3.13 Corollary. *Let H be a (left strongly, right strongly) dense subgroupoid of a groupoid G and let φ be a homomorphism of H into a (left, right) cancellative groupoid K. Then φ can be extended to at most one homomorphism of G into K.*

4.1 Let H be a subgroupoid of a groupoid G. Define the following eleven conditions for H in G:

(A1) There exists $n \geq 1$ such that for all $q \geq 1$, $m = qn$, $a_1, \dots, a_m \in H$, $x, y \in G$, $a_1(\dots(a_m x)) \in H$, there exist $k \geq 0$, $l \geq 0$, $b_1, \dots, b_k, c_1, \dots, c_l \in H$ with $b_1(\dots(b_k \cdot xy)) = (a_1(\dots(a_m x)))(c_1(\dots(c_l y)))$;

(A2) If $m \geq 0$, $a, a_1, \dots, a_m \in H$ and $x \in G$, then there exist $k \geq 0$, $l \geq 0$, $b_1, \dots, b_k$, $c_1, \dots, c_l \in H$ such that $b_1(\dots(b_k \cdot ax)) = (c_1(\dots(c_l a)))(a_1(\dots(a_m x)))$;

(A3) For every $f \in \mathrm{Mul}_l(G, H)$ there exist g, h, u, v in $\mathrm{Mul}_l(G, H)$ such that $g(xy) = f(x)\,h(y)$ and $u(ay) = v(a)\,f(y)$ for all $a \in H$, $x, y \in G$, $f(x) \in H$;

(A4) For every $f \in \mathrm{Mul}_l(G, H)$ there exist $g \in \mathrm{Mul}_l(G, H)$ such that $g(xy) = f(x)\,f(y)$ for all $x, y \in G$;

(A5) There exist a transformation t of H such that $t(a) \cdot xy = ax \cdot ay$ for all $a \in H$ and x, y $\in G$;

(A6) For every $f \in \mathrm{Mul}_l(G, H)$ there exist g, h, u, v in $\mathrm{Mul}_l(G, H)$ such that $g(xy) = f(x)\,h(x)$ and $u(wz) = v(w)\,f(z)$ for all $x, y, w, z \in G$ with $f(x) \in H$ and $f(z) \in H$;

(A7) For all $f, g \in \mathrm{Mul}_l(G, H)$ there exist h, u in $\mathrm{Mul}_l(G, H)$ such that $hf = ug$ (i.e., the monoid $\mathrm{Mul}_l(G, H)$ is right uniform);

(A8) For all $f \in \mathrm{Mul}_l(G, H)$ and $a \in H$ there exist $g, h \in \mathrm{Mul}_l(G, H)$ such that $g\mathrm{L}_{a,G} = hf$;

(A9) For every $f \in \mathrm{Mul}_l(G, H)$ there exist g, h, u, v in $\mathrm{Mul}_l(G, H)$ such that $g(xy) = f(x)\,h(y)$ and $u(wz) = v(w)\,f(z)$ for all $x, y, w, z \in G$ such that $f(x) \in H$ and either $w \in H$ or $f(z) \in H$;

(A10) If $x, y \in G$, $a \in H$ and $b, c \in \langle a \rangle$, then $bx \cdot cy = bc \cdot xy$;

(A11) For every $f \in \mathrm{Mul}_l(G, H)$ there exist g, h in $\mathrm{Mul}_l(G, H)$ such that $f(xa) = g(x)\,h(a)$ for all $x \in G$ and $a \in H$.

4.2 Proposition. *Let H be a subgroupoid of a groupoid G. Then the following implications take place:*

(1) (A3) ⇒ (A1), (A2), (A7) and (A8);

(2) (A4) ⇒ (A1), (A2), (A3), (A6), (A7), (A8) and (A9);

(3) (A5) ⇒ (A1), (A2), (A3), (A4), (A6), (A7), (A8) and (A9);

(4) (A8) ⇒ (A7);

(5) (A9) ⇒ (A1), (A2), (A3), (A6), (A7) and (A8);

(6) (A10) ⇒ (A1), (A2), (A3), (A4), (A5), (A6), (A7), (A8) and (A9).

Proof. First, we show that (A5) ⇒ (A4). For, let $f \in \mathrm{Mul}_l(G, H)$. If $f = \mathrm{id}_G$, then we can take $g = f$. If $f \neq \mathrm{id}_G$, then $f = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_n}$ for a some $n \geq 1$ and $a_1, \dots, a_n \in H$ and we put $g = \mathrm{L}_{t(a_1)} \dots \mathrm{L}_{t(a_n)}$.

Now, we are going to show that (A8) ⇒ (A7). We have $g = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_n}$ for suitable $n \geq 0$ and $a_1, \dots, a_n \in H$. If $n = 0$, then we put $h = \mathrm{id}_G$ and $u = f$. If

$n = 1$, then (A8) applies. If $n \geq 2$, then we proceed by induction on n. Put $g_1 = \mathrm{L}_{a_2} \dots \mathrm{L}_{a_n}$. There are h_1, u_1 in $\mathrm{Mul}_l(G, H)$ such that $h_1 f = u_1 g_1$. Further, by (A8), $v\mathrm{L}_{a_1} = tu_1$ for some $v, t \in \mathrm{Mul}_l(G, H)$. Thus $hf = th_1 f = tu_1 g_1 = v\mathrm{L}_{a_1} g_1 = vg$, where $h = tu_1 \in \mathrm{Mul}_l(G, H)$.

The remaining implications are easy. ▲

4.3 Proposition. *Let H be a subgroupoid of a groupoid G such that the conditions (A1) and (A2) are satisfied. Then:*

(i) $\langle H \rangle_{lc, G} = [H]_{l, G} = \alpha_G(H)$.

(ii) $x \in \langle H \rangle_{lc}$ *if and only if there exist* $n \geq 1$ *and* $a_1, \dots, a_n \in H$ *such that* $a_1(\dots(a_n x)) \in H$.

Proof. We have $H \subseteq \alpha(H) \subseteq [H]_l \subseteq \langle H \rangle_{lc}$ and it suffices to show that $\alpha(H)$ is a left closed subgroupoid of G.

First, we check that $\alpha(H)$ is a subgroupoid of G. For, let $x, y \in \alpha(H)$. Then $g(x) \in H$ for some $g \in \mathrm{Mul}_l(G, H)$ and we can assume without loss of generality that $g = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_m}$, where $m = qn$, $q \geq 1$ and $a_1, \dots, a_m \in H$ are by (A1). There exist $k \geq 0$, $l \geq 0$ and $b_1, \dots, b_k, c_1, \dots, c_l$ in H such that $b_1(\dots(b_k \cdot xy)) = (a_1(\dots(a_m x)))(c_1(\dots(c_l y)))$. Further, there is $f \in \mathrm{Mul}_l(G, H)$ with $f(y) \in H$. Now, define two sequences $f_1, f_2, \dots$ and $f_1', f_2', \dots$ of transformations from $\mathrm{Mul}_l(G, H)$ as follows: $f_1 = f$; if $i \geq 1$, then $f_{i+1}, f_i' \in \mathrm{Mul}_l(G, H)$ are such that $f_{i+1}(az) = f_i'(a) f_i(z)$, all $a \in H$, $z \in G$ (by (A2)). Then we have $f_{l+2}(b_1(\dots(b_k \cdot xy))) = f_{l+1}' g(x) \cdot f_{l+1}(c_1(\dots(c_l y)))$ and $f_{l+1}(c_1(\dots(c_l y))) = f_l'(c_1) \cdot f_l(c_2(\dots(c_l y))) = \dots = f_l'(c_1)(f_{l-1}'(c_2)(\dots(f_1'(c_l) f(c)))) \in H$. Thus $h(xy) \in H$, where $h = f_{l+2} \mathrm{L}_{b_1} \dots \mathrm{L}_{b_k} \in \mathrm{Mul}_l(G, H)$.

It reamins to show that $\alpha(H)$ is left closed in G. Let $x, y \in G$, $x, xy \in \alpha(H)$. Then, similarly as above (by (A1)), there are $f, h, v \in \mathrm{Mul}_l(G, H)$ such that $v(xy) = f(x) h(y)$ and $f(x) \in H$. Further, $g(xy) \in H$ for some $g = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_n}$, $n \geq 0$, $a_1, \dots, a_n \in H$. Now, $v_n'(a_1)(v_{n-1}'(a_2)(\dots(v_1'(a_n) \cdot f(x) h(y)))) = v_n'(a_1)(v_{n-1}'(a_2)(\dots(v_1'(a_n) \cdot v(xy)))) = v_n'(a_1)(v_{n-1}'(a_2)(\dots(v_2'(a_{n-1}) \cdot v_2(a_n \cdot xy)))) = \dots = v_n'(a_1) v_n(a_2(a_3(\dots(a_n \cdot xy)))) = v_{n+1} g(xy) \in H$ (the sequences $v_1 = v, v_2, \dots, v_1', v_2', \dots$ are defined similarly as above). Hence $p(y) \in H$, where $p = \mathrm{L}_{v_n'(a_1)} \dots \mathrm{L}_{v_n'(a_n)} \mathrm{L}_{f(x)} \in \mathrm{Mul}_l(G, H)$ and it follows that $y \in \alpha(H)$. ▲

4.4 Proposition. *Let H be a right closed subgroupoid of a groupoid G such that the conditions (A9) and (A11) are satisfied. Then $\alpha_G(H)$ is closed subgroupoid of G.*

Proof. By 4.2(5) and 4.3, $K = \alpha(H)$ is a left closed subgroupoid. It remains to show that K is right closed.

For, let $x \in G$, $a, b \in K$ and $xa = b$. There are $f, g, h \in \mathrm{Mul}_l(G, H)$ such that $f(a) \in H$ and $g(b) = h(x) f(a)$. Further, by (A11), there are $u, v, p \in \mathrm{Mul}_l(G, H)$ such that $ug(b) \in H$ and $ug(b) = u(h(x) f(a)) = vh(x) pf(a)$. From this, $vh(x) \in H$, since H is right closed, and hence $x \in K$. We have proved that K is right closed. ▲

4.5 In the sequel, the right hand forms (or the duals) of the conditions (A1), ..., (A11) will be denoted by (B1), ..., (B11).

4.6 Theorem. *Let H be a subgroupoid of a groupoid G.*

(i) If (A9) is satisfied for H in G, then $\alpha_G(H) = \langle H\rangle_{\mathrm{lc},G} = [H]_{\mathrm{l},G}$. If, moreover, (B9) and (B11) are satisfied for $\alpha_G(H)$ in G, then $\delta_G(H) \subseteq \beta_G(\alpha_G(H)) = \langle H\rangle_{\mathrm{c},G} = \langle\langle H\rangle_{\mathrm{c},G}\rangle_{\mathrm{lc},G} = [[H]_{\mathrm{r},G}]_{\mathrm{l},G} = [H]_G$.

(ii) If (B9) is satisfied for H in G, then $\beta_G(H) = \langle H\rangle_{\mathrm{rc},G} = [H]_{\mathrm{r},G}$. If, moreover, (A9) and (A11) are satisfied for $\beta_G(H)$ in G, then $\delta_G(H) \subseteq \alpha_G(\beta_G(H)) = \langle H\rangle_{\mathrm{c},G} = \langle\langle H\rangle_{\mathrm{c},G}\rangle_{\mathrm{lc},G} = [[H]_{\mathrm{r},G}]_{\mathrm{l},G} = [H]_G$.

Proof. (i) By 4.3, $K = \alpha_G(H) = \langle H\rangle_{\mathrm{lc},G} = [H]_{\mathrm{l},G}$ and, by the right hand form of 4.3, we have $L = \beta_G(K) = \langle K\rangle_{\mathrm{rc},G}$. Now, by the right hand form of 4.4, L is closed in G, and so $L = \langle H\rangle_{\mathrm{c},G}$; clearly $\beta_G(\alpha_G(H)) \subseteq [H]_G \subseteq \langle H\rangle_{\mathrm{c},G}$.

(ii) Dual to (i). ▲

4.7 Corollary. *Let H be a subgroupoid of a groupoid G suh that (A9) is satisfied for H and $\beta_G(H)$ in G, (B9) is satisfied for H and $\alpha_G(H)$ in G, (A11) for $\beta_G(H)$ and (B11) for $\alpha_G(H)$ in G. Then $\beta_G(\alpha_G(H)) = \alpha_G(\beta_G(H)) = \langle\langle H\rangle_{\mathrm{c},G}\rangle_{\mathrm{rc},G} = \langle H\rangle_{\mathrm{c},G} = [[H]_{\mathrm{l},G}]_{\mathrm{r},G} = [[H]_{\mathrm{r},G}]_{\mathrm{l},G} = [H]_G$.*

I.5 The conditions (C1), ..., (C7)

5.1 Let H be a subgroupoid of a groupoid G. Define the following seven conditions for H in G:

(C1) For every $f \in \mathrm{Mul}(G, H)$ there exist $g, h \in \mathrm{Mul}(G, H)$ such that $g(xy) = f(x)\,h(y)$ for all $x, y \in G$, $f(x) \in H$;

(C2) For every $f \in \mathrm{Mul}(G, H)$ there exist $g, h \in \mathrm{Mul}(G, H)$ such that $g(xy) = h(x)\,f(y)$ for all $x, y \in G$, $f(y) \in H$;

(C3) For every $f \in \mathrm{Mul}(G, H)$ there exist $g, h \in \mathrm{Mul}(G, H)$ such that $g(xa) = f(x)\,h(a)$ for all $x \in G$, $a \in H$;

(C4) For every $f \in \mathrm{Mul}(G, H)$ there exist $g, h \in \mathrm{Mul}(G, H)$ such that $g(ax) = h(a)\,f(x)$ for all $x \in G$, $a \in H$;

(C5) For every $f \in \mathrm{Mul}(G, H)$ there exist $g \in \mathrm{Mul}(G, H)$ such that $g(xy) = f(x)\,f(y)$ for all $x, y \in G$;

(C6) For all $f \in \mathrm{Mul}(G, H)$ and $a \in H$ there exist $g, h, u, v \in \mathrm{Mul}(G, H)$ such that $g\mathrm{L}_{a,G} = hf$ and $u\mathrm{R}_{a,G} = vf$;

(C7) For all $f, g \in \mathrm{Mul}(G, H)$ there exist $h, u \in \mathrm{Mul}(G, H)$ such that $hf = ug$ (i.e., the monoid $\mathrm{Mul}(G, H)$ is right uniform).

5.2 Proposition. *Let H be a subgroupoid of a groupoid G. Then the following implications take place:*

(1) (C3) and (C4) ⇒ (C6) and (C7);

(2) (C5) ⇒ (C1), (C2), (C3), (C4), (C6) and (C7);

(3) (C6) ⇒ (C7);
(4) (A5) and (B5) ⇒ (C5).

Proof. We will prove only that (A5) and (B5) imply (C5). First, there are transformations p and q of H such that $p(a) \cdot xy = ax \cdot ay$ and $xy \cdot q(a) = xa \cdot ya$ for all $a \in H$ and $x, y \in G$. Now, given $f \in \mathrm{Mul}(G, H)$, $f = \mathrm{T}_{1,a_1} \dots \mathrm{T}_{n,a_n}$, $n \geq 1$, $a_1, \dots, a_n \in H$, $\mathrm{T}_1, \dots, \mathrm{T}_n \in \{\mathrm{L}, \mathrm{R}\}$, we put $b_i = p(a_i)$ if $\mathrm{T}_i = \mathrm{L}$ and $b_i = q(a_i)$ if $\mathrm{T}_i = \mathrm{R}$. Then $q(xy) = f(x)\, f(y)$ for all $x, y \in G$, where $g = \mathrm{T}_{1,b_1} \dots \mathrm{T}_{n,b_n}$. ▲

5.3 Theorem. *Let H be a subgroupoid of a groupoid G such that the conditions (C1) (resp. (C2)) and (C3), (C4) are satisfied. Then $\gamma_G(H)$ is a left (resp. right) closed subgroupoid of G.*

Proof. First, we check that $K = \gamma_G(H)$ is a subgroupoid of G. For, let $x, y \in K$. There are g, u, v in $\mathrm{Mul}(G, H)$ such that $g(x) \in H$ and $u(xy) = g(x)\, v(y)$; we have $v = \mathrm{T}_{1,a_1} \dots \mathrm{T}_{n,a_n}$, where $n \geq 0$, $a_i \in H$ and $\mathrm{T}_i \in \{\mathrm{L}, \mathrm{R}\}$. Further, $f(y) \in H$ for some $f \in \mathrm{Mul}(G, H)$. Now, define sequences $f_1, f_2, \dots, f_{n+2}$ and $f_1', f_2', \dots, f_{n+1}'$ of transformations form $\mathrm{Mul}(G, H)$ as follows: $f_1 = f$; if $n \geq i \geq 1$ and $\mathrm{T}_j = \mathrm{L}$, $j = n + 1 - i$, then f_{i+1} and f_i' are such that $f_{i+1}(az) = f_i'(a)\, f_i(z)$ for all $a \in H$, $z \in G$; if $n \geq i \geq 1$ and $\mathrm{T}_j = \mathrm{R}$, $j = n + 1 - i$, then f_{i+1} and f_i' are such that $f_{i+1}(za) = f_i(z)\, f_i'(a)$ for all $a \in H$, $z \in G$; f_{n+2} and f_{n+1}' are such that $f_{n+2}(az) = f_{n+1}'(a)\, f_{n+1}(z)$ for all $a \in H$, $z \in G$. Then we have $f_{n+2}u(xy) = f_{n+2}(g(x)\, v(y)) = f_{n+1}'g(x) \cdot f_{n+1}v(y)$ and $f_{n+1}v(y) = f_{n+1}(\mathrm{T}_{1,a_1} \dots \mathrm{T}_{n,a_n}(y)) = \mathrm{T}_{1,b_1} f_n(\mathrm{T}_{2,a_2} \dots \mathrm{T}_{n,a_n}(y)) = \dots = \mathrm{T}_{1,b_1} \dots \mathrm{T}_{n,b_n} f_1(y) \in H$, where $b_i = f_{n+1-i}'(a_i) \in H$. Thus $f_{n+2}u(xy) \in H$ and $xy \in K$.

Now, we are going to show that K is left closed in G. Let $x, y \in G$, $x, xy \in K$. Then $h(xy) = f(x)\, u(y)$, $f(x) \in H$, $g(xy) \in H$, $g = \mathrm{T}_{1,a_1} \dots \mathrm{T}_{n,a_n}$, for some $f, g, h, u \in \mathrm{Mul}(G, H)$, $n \geq 0$, $a_i \in H$, $\mathrm{T}_i \in \{\mathrm{L}, \mathrm{R}\}$. Define the sequences $h_1, \dots, h_{n+1}$, $h_1', \dots, h_{n+1}'$ similarly as above and put $b_i = h_{n+1-i}'(a_i) \in H$. Then we have $\mathrm{T}_{1,b_1} \dots \mathrm{T}_{n,b_n}(f(x)\, u(y)) = \mathrm{T}_{1,b_1} \dots \mathrm{T}_{n,b_n}(h(xy)) = \mathrm{T}_{1,b_1} \dots \mathrm{T}_{n-1,b_{n-1}} h_2(\mathrm{T}_{n,a_n}(xy)) = \dots = h_{n+1}g(xy) \in H$ and consequently $p(y) \in H$, where $p = \mathrm{T}_{1,b_1} \dots \mathrm{T}_{n,b_n} \mathrm{L}_{f(x)} u \in \mathrm{Mul}(G, H)$. It follows that $y \in K$. ▲

5.4 Corollary. *Let H be a subgroupoid of a groupoid G such that (C1), (C2), (C3) and (C4) are satisfied. The $\gamma_G(H) = \langle H \rangle_{c,G} = [H]_G$ and, moreover, if the assumptions of 4.7 are satisfied, then also $\gamma_G(H) = \alpha_G(\beta_G(H)) = \beta_G(\alpha_G(H))$.*

I.6 Existensions of cancellative congruences

6.1 Proposition. *Let H be a subgroupoid of a groupoid G such that $K = \alpha_G(H)$ is also a subgroupoid.*

(i) If $n \geq 1$ and $x_1, \dots, x_n \in K$, then there exists $f \in \mathrm{Mul}_l(G, H)$ such that $f(x_1), \dots, f(x_n) \in H$.

(ii) If r is an equivalence on H, then r can be extended to at most one left cancellative left stable equivalence on K.

Proof. (i) Since $x_1 \in K$, $f_1(x_1) \in H$ for suitable, $f_1 \in \mathrm{Mul}(G, H)$. But K is a subgroupoid of G, hence $f_1(x_2) \in K$ and $f_2 f_1(x_2) \in H$ for some $f_2 \in \mathrm{Mul}_l(G, H)$; clearly $f_2 f_1 \in \mathrm{Mul}_l(G, H)$ as well. Now, there are $f_3, \ldots, f_n \in \mathrm{Mul}_l(G, H)$ such that $f_3 f_2 f_1(x_3), \ldots, f_n f_{n-1} \ldots f_1(x_n) \in H$ and it suffies to put $f = f_n \ldots f_1$.

(ii) Let s, t be left cancellative left stable equivalences on K such that $s \cap H^{(2)} = r = t \cap H^{(2)}$. If $(x, y) \in s$, then $f(x), f(y) \in H$ for some $f \in \mathrm{Mul}_l(G, H)$ (by (i)) and we have $(f(x), f(y)) \in s \cap H^{(2)} = r \subseteq t$. Since t is left cancellative, we have also $(x, y) \in t$. Thus $s \subseteq t$ and, quite similarly, $t \subseteq s$. ▲

6.2 Proposition. *Let H be a subgroupoid of a groupoid G such that $K = \alpha_G(H)$ is again a subgroupoid and let r be a left stable equivalence on H. Define a relation s on K by $(x, y) \in s$ if and only if $(f(x), f(y)) \in r$ for some $f \in \mathrm{Mul}_l(G, H)$. Then:*

(i) $r \subseteq s$ and s is reflexive and symmetric.
(ii) If (A7) is satisfied for H in G, then s is an equivalence.
(iii) If (A6) is satisfied, then s is left stable.
(iv) If (A6) is satisfied and if r is right stable, then s is right stable.
(v) If (A6), (A7) are satisfied, then s is left cancellative.
(vi) If (A6), (A7) are satisfied and r is cancellative, then s is cancellative.
(vii) If r is left cancellative, then s extends r.

Proof. (i) Since r is symmetric, s is symmetric, too. If $x \in K$, then $f(x) \in H$ for some $f \in \mathrm{Mul}_l(G, H)$, and hence $(x, x) \in s$. The inclusion $r \subseteq s$ is obvous.

(ii) Let $(x, y), (y, z) \in s$. We have $(f(x), f(y)) \in r$ and $(g(y), g(z)) \in r$ for suitable, $f, g \in \mathrm{Mul}_l(G, H)$. By (A7), there are h, u in $\mathrm{Mul}_l(G, H)$ with $hf = ug$. Since r is left stable on H, we have $(hf(x), hf(y)) \in r$ and $(ug(y), uz(g)) \in r$. Therefore $(gf(x), hf(z)) \in r$ and $(x, z) \in s$.

(iii) Let $x, y, z \in K, (x, y) \in s$. Again, there are f, g, h in $\mathrm{Mul}_l(G, H)$ such that $(f(x), f(y)) \in z$ and $g(zx) = h(z)\, f(x), g(zy) = h(z)\, f(y)$. Further, there are u, p, q in $\mathrm{Mul}_l(G, H)$ with $uh(z) \in H$ and $p(vw) = u(v)\, q(w)$ for all $v, w \in G$, $u(v) \in H$. Now, $pg(zx) = p(h(z)\, f(x)) = uh(z) \cdot qf(x), pg(zy) = uh(z) \cdot qf(y)$. Since r is left stable, we have $(pg(zx), pg(zy)) \in r$, and so $(zx, zy) \in s$.

(iv) We can proceed similarly as in (iii).

(v) Let $x, y, z \in K, (zx, zy) \in s$. There is $f \in \mathrm{Mul}_l(G, H)$ such that $(f(zx), f(zy)) \in r$. Further, $g(x), g(y) \in H$ for some $g \in \mathrm{Mul}_l(G, H)$ and there are $u, v \in \mathrm{Mul}_l(G, H)$ such that $u(zx) = v(z)\, g(x)$ and $u(zy) = v(z)\, g(y)$. Similarly, $pv(z) \in H, p \in \mathrm{Mul}_l(G, H)$, and there are $q, t \in \mathrm{Mul}_l(G, H)$ such that $qu(zx) = q(v(z)\, g(x)) = pv(z) \cdot tg(x)$ and $qu(zy) = pv(z) \cdot tg(y)$ (all this by (A6)). Now, by (A7), $hf = wqu$ for some $h, w \in \mathrm{Mul}_l(G, H)$. Of course, $(hf(zx), hf(zy)) \in r$, since r is left stable. But $hf(zx) = wqu(zx) = w(pv(z) \cdot tg(x))$ and $hf(zy) = w(pv(z) \cdot tg(y))$. We have proved that $(e(x), e(y)) \in r$ for $e = w\mathrm{L}_{pv(z)} tg \in \mathrm{Mul}_l(G, H)$ and it follows $(x, y) \in s$.

(vi) By (v), s is left cancellative. Now, let $x, y, z \in K$, $(xz, yz) \in s$. Proceeding similarly as in (v), we can show that there are w, p, t, v, g in $\mathrm{Mul}_l(G, H)$ such that

$g(x), g(y), pu(z) \in H$ and $(w(tg(x) \cdot pv(z)), w(tg(y) \cdot pv(z)) \in r$. Since r is left cancellative, we have $(tg(x) \cdot pv(z), tg(y) \cdot pv(z)) \in r$. But r is also right cancellative and so $(tg(x), tg(y)) \in r$. Thus $(x, y) \in s$.

(vii) Obvious. ▲

6.3 Theorem. *Let H be a left strongly dense subgroupoid of a groupoid G such that the condition (A9) is satisfied. Then:*

(i) $G = \alpha_G(H)$, *i.e.,;for every* $x \in G$ *there exist* $n \geq 0$ *and* $a_1, \dots, a_n \in H$ *such that* $a_1(a_2(\dots(a_n x))) \in H$.

(ii) Every left cancellative congruence r of H can be extended in a unique way to a left cancellative congruence s of G; s is cancellative if an only if r is so.

Proof. (i) By 4.2(5), the conditions (A3), (A6) and (A7) are satisfied and we can use 4.3.

(ii) Use 6.2 and 6.1(ii). ▲

6.4 Remark. Let G be a groupoid satisfying the following condition: If $f \in \mathrm{Mul}_l(G)$, then there exist $g, h, u, v \in \mathrm{Mul}_l(G)$ such that $g(xy) = f(x)\,h(y)$ and $u(xy) = v(x)\,f(y)$ for all $x, y \in G$.

(i) Define a relation t on G by $(x, y) \in t$ if and only if $f(x) = f(y)$ for some $f \in \mathrm{Mul}_l(G)$. From 6.2 (for $H = G$ and $r = \mathrm{id}_G$) it follows that t is a congruence of G and it is easy to see that t is just the smallest left cancellative congruence of G, i.e. $t = \mathrm{c}_{l,G}$ (see 1.14).

(ii) Let H be a left strongly dense subgroupoid of G such that (A9) is satisfied and let r denote the smallest left cancellative congruence of H (r is the intersection of all left cancellative congruences of H). By 6.3, r can uniquely be extended to a left cancellative congruence s of G. Then $t \subseteq s$ (see (i)), $r' = t \cap H^{(2)}$ is a left cancellative congruence of $H, r \subseteq r' \subseteq s \cap H^{(2)} = r$, $r = r'$ and $s = t$. That is, t extends r.

6.5 Proposition. *Let H be a subgroupoid of a groupoid G such that* $K = \gamma_G(H)$ *is again a subgroupoid and let r be a congruence of H. Define a relation s on K by* $(x, y) \in s$ *if and only if* $(f(x), f(y)) \in r$ *for some* $f \in \mathrm{Mul}_l(G, H)$. *Then:*

(i) $r \subseteq s$ *and s is reflexive and symmetric.*

(ii) If (C7) is satisfied for H in G, then s is an equivalence.

(iii) If (C1) and (C3) are satisfied, then s is a congruence of K.

(iv) If (C1), (C2) and (C7) are satisfied, then s is cancellative.

(v) If r is cancellative, then s extends r.

Proof. Similar to that of 6.2. ▲

6.6 Theorem. *Let H be a dense subgroupoid of a groupoid G such that the conditions (C1), (C2), (C3) and (C4) are satisfied. Then:*

(i) $G = \gamma_G(H)$, *i.e., for every* $x \in G$ *there exist* $n \geq 0$, $a_1, \dots, a_n \in H$ *and* $\mathrm{T}_1, \dots, \mathrm{T}_n \in \{\mathrm{L}, \mathrm{R}\}$ *such that* $\mathrm{T}_{1,a_1} \cdots \mathrm{T}_{n,a_n}(x) \in H$.

(ii) Every cancellative congruence of H can be extended in a unique way to a cancellative congruence of G.

Proof. Combine 5.4 and 6.5. ▲

6.7 Remark. Let G be a groupoid satisfying the following condition: If $f \in \mathrm{Mul}(G)$, then there exist $g, h, u, v \in \mathrm{Mul}(G)$ such that $g(xy) = f(x)\,h(y)$ and $u(xy) = v(x)\,f(y)$ for all $x, y \in G$.

(i) Define a relation t on G by $(x, y) \in s$ if and only if $f(x) = f(y)$ for some $f \in \mathrm{Mul}(G)$. Then t is the smallest cancellative congruence of G, i.e. $t = \mathrm{c}_G$ (see 1.14).

(ii) Let H be dense subgroupoid of G such that (C1), (C2), (C3) and (C4) are satisfied and let r be the smallest cancellative congruence of H. Then t extends r.

I.7 Comments

The chapter is of introductory character, a basic terminology is introduced and the results are adapted mainly from [21] and [24].

II. Closed subgroupoids of cancellative groupoids

II.1 Left closed subgroupoids of left cancellative groupoids

1.1 Lemma. *Let H be a subgroupoid of a left cancellative groupoid G.*

(i) If H is a left quasigroup, then H is left closed in G.

(ii) If H is finite, then H is left closed in G.

1.2 Lemma. *Let G be a left quasigroup. A groupoid H of G is left closed in G if and only if H is also a left quasigroup.*

1.3 Corollary. *A subgroupoid H of a left quasigroup G is left strongly dense in G if and only if $K = G$ whenever K is a left subquasigroup of G such that $H \subseteq K$.*

1.4 Lemma. *Let H be a left strongly dense subgroupoid of a left quasigroup G, let H' be a subgroupoid of a left divisible groupoid G' and let $\varphi : G' \to G$ be a homomorphism such that $\varphi(H') = H$. Then:*

(i) $\varphi(G') = G$ and $\varphi(K') = G$, where $K' = \langle H' \rangle_{\mathrm{lc}, G'}$.

(ii) If G' is a left quasigroup, $\varphi \restriction H'$ is injective and $K' = \alpha_{G'}(H')$ (e.g., if (A3) is true for H' in G' — see I.4.1, I.4.2, I.4.3), then $\varphi \restriction K'$ is an isomorphism of K' onto G.

Proof. (i) Put $K = \varphi(K')$. Since K' is left closed in G', K' is left divisible, and hence K is a left quasigroup, $H \subseteq K \subseteq G$. Now, $K = G$ by 1.3.

(ii) Put $\psi = \varphi \restriction K'$. By (i), $\psi(K') = G$. On the other hand, $\ker(\psi)$ and $\mathrm{id}_{K'}$ are left cancellative congruences of K' and they extend $\mathrm{id}_{H'}$. By I.6.1(ii), $\ker(\psi) = \mathrm{id}_{K'}$. ▲

1.5 Corollary. *Let H be a left strongly dense subgroupoid of a left quasigroup Q such that (A3) is satisfied and let H be a subgroupoid of a left quasigroup G. If $\varphi : Q \to G$ is a homomorphism such that $\varphi \restriction H = \mathrm{id}_H$, then φ is an isomorphism of Q onto $\langle H \rangle_{\mathrm{lc},G}$.*

1.6 Lemma. *Let S be a subset of a left cancellative groupoid G and $R = \alpha_{1,G}(S)$.*
(i) If S is finite, then $\mathrm{card}(R) \le (\mathrm{card}(S))^2$.
(ii) If S is infinite, then $\mathrm{card}(R) \le \mathrm{card}(S)$.

Proof. We have $R = \bigcup_{a \in S} S_a$, $S_a = \{x \in G; ax \in S\}, \mathrm{card}(S_a) \le \mathrm{card}(S)$. ▲

1.7 Lemma. *Let S be a non-empty subset of a left cancellative groupoid G.*
(i) If S is finite, then $\mathrm{card}([S]_{1,G}) \le \mathrm{card}(\langle S \rangle_{\mathrm{lc},G}) \le \aleph_0$.
(ii) If S is infinite, then $\mathrm{card}([S]_{1,G}) = \mathrm{card}(\langle S \rangle_{\mathrm{lc},G}) = \mathrm{card}(S)$.

Proof. Combine 1.6 and I.2.4(i), I.3.2(i). ▲

1.8 Proposition. *Let H be a strongly dense subgroupoid of a left cancellative groupoid G. Then* $\mathrm{card}(H) = \mathrm{card}(G)$.

Proof. If H is infinite, then the result follows from 1.7(ii). If H is finite, then H is a left quasigroup, H is left closed in G, and so $H = G$. ▲

1.9 Proposition. *Let H be a left strongly dense subgroupoid of a left cancellative groupoid G such that (A9) is satisfied. If H is right cancellative, then G is cancellative.*

Proof. The result follows easily from I.6.3(ii). ▲

1.10 Theorem. *Let H be a left strongly dense subgroupoid of a left cancellative groupoid G such that (A10) is satisfied. Then the groupoids G and H are equationally equivalent (i.e., they satisfy the same groupoid equations or, in other words, they generate the same groupoid variety).*

Proof. Of course, every equation which is true for G is true for H. Now, let W denote the absolutely free groupoid of groupoid words over an infinite countable set X and let $u, v \in W$ be such that $u \triangleq v$ holds in H. Let $\varphi : W \to G$ be a homomorphism. We have to show that $\varphi(u) = \varphi(v)$.

Let $\{x_1, \dots, x_n\}, n \ge 1$, be the set of variables from X that occur in uv. There is $f \in \mathrm{Mul}_1(G, H)$ such that $f\varphi(x_1), \dots, f\varphi(x_n)$ are all in H and we have $f = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_m}$, $m \ge 1, a_1, \dots, a_m \in H$. For $1 \le i \le m$, let φ_i denote the homomorphism of W into H such that $\varphi_i(x) = a_i$ for each $x \in X$. Further, let $\psi : W \to H$ be a homomorphism such that $\psi(x_j) = f\varphi(x_j)$, $1 \le j \le n$.

We prove by induction on length of t that if $t \in W$ and $\mathrm{var}(t) \subseteq \{x_1, \dots, x_n\}$, then $\psi(t) = \mathrm{L}_{\varphi_1(t)} \dots \mathrm{L}_{\varphi_m(t)}(\varphi(t))$. This is clear for $t = x_j$, $1 \le j \le n$. Now, let $t = pq$, $p, q \in W$. We have $\psi(t) = \psi(p)\,\psi(q) = \mathrm{L}_{\varphi_1(p)} \dots \mathrm{L}_{\varphi_m(p)}(\varphi(p)) \cdot \mathrm{L}_{\varphi_1(q)} \dots \mathrm{L}_{\varphi_m(q)}(\varphi(q)) =$

$L_{\varphi_1(p)\varphi_1(q)} \cdots L_{\varphi_m(p)\varphi_m(q)}(\varphi(p)\,\varphi(q)) = L_{\varphi_1(t)} \cdots L_{\varphi_m(t)}(\varphi(t))$ by (A10) (both $\varphi_i(p)$ and $\varphi_i(q)$ are in $\langle a_i \rangle_G$). Now, since $u \triangleq v$ is true in H, we have $\psi(u) = \psi(v)$, i.e., $L_{\varphi_1(u)} \cdots L_{\varphi_m(u)}(\varphi(u)) = L_{\varphi_1(v)} \cdots L_{\varphi_m(v)}(\varphi(v))$. Again, $\varphi_j(u) = \varphi_j(v)$ and G is left cancellative. Thus $\varphi(u) = \varphi(v)$. ▲

1.11 Remark. Let H be a left strongly dense subgroupoid of a left cancellative groupoid G and let r be a left cancellative congruence of G such that H/r is a left quasigroup (e.g., H/r finite). Then $H/r = G/r$, i.e., for every $x \in G$ there is $a \in H$ with $(a, x) \in r$. In particular, if H/r is finite, then G/r is so and consequently G/r is a left quasigroup.

1.12 Remark. Let H be a left strongly dense subgroupoid of a left cancellative groupoid G. Assume that id_H and $H \times H$ are the only left cancellative congruences of H. Now, if r is a left cancellative congruence of G, then either $r \restriction H = \mathrm{id}_H$ or $r \restriction H = H \times H$, and hence $r = G \times G$.

II.2 Closed subgroupoids of cancellative groupoids

2.1 Lemma. *Let H be a subgroupoid of a cancellative groupoid G.*
(i) If H is a quasigroup, then H is closed in G.
(ii) If H is finite, then H is closed in G.

2.2 Lemma. *Let G be a quasigroup. A subgroupoid H of G is closed in G if and only if H is also a quasigroup.*

2.3 Corollary. *A subgroupoid H of a quasigroup G is dense in G if and only if $K = G$ whenever K is a subquasigroup of G such that $H \subseteq K$.*

2.4 Lemma. *Let H be a dense subgroupoid of a quasigroup G, let H' be a subgroupoid of a divisible groupoid G' and let $\varphi : G' \to G$ be a homomorphism such that $\varphi(H') = H$. Then:*
(i) $\varphi(G') = G$ and $\varphi(K') = G$, where $K' = \langle H' \rangle_{c,G'}$.
(ii) If G' is as quasigroup, $\varphi \restriction H'$ is injective and $K' = \gamma_{G'}(H')$ (e.g., if (C1), (C2), (C3) and (C4) are satisfied for H' in G' — see I.5.1 and I.5.4), then $\varphi \restriction K'$ is an isomorphism of K' onto G.

Proof. (i) Put $K = \varphi(K')$. Since K' is closed in G', K' is divisible groupoid, and hence K is a quasigroup. By 2.3, $K = G$.

(ii) Put $\psi = \varphi \restriction K'$. By (i), $\psi(K') = G$. On the other hand, $\ker(\psi)$ and $\mathrm{id}_{K'}$ are cancellative congruences of K' and they extend $\mathrm{id}_{H'}$. By I.6.6(ii), $\ker(\psi) = \mathrm{id}_{K'}$. ▲

2.5 Corollary. *Let H be a dense subgroupoid of a quasigroup Q such that (C1), (C2), (C3) and (C4) are satisfied and let H be a subgroupoid of a quasigroup G. If $\varphi : Q \to G$ is a homomorphism such that $\varphi \restriction H = \mathrm{id}_H$, then φ is an isomorphism of Q onto $\langle H \rangle_{c,G}$.*

2.6 Lemma. *Let S be a non-empty subset of a cancellative groupoid G.*
(i) If S is finite, then $\operatorname{card}([S]_G) \le \operatorname{card}(\langle S\rangle_{c,G}) \le \aleph_0$.
(ii) If S is infinite, then $\operatorname{card}([S]_G) = \operatorname{card}(\langle S\rangle_{c,G}) = \operatorname{card}(S)$.

Proof. Use 1.6, its dual and I.2.4(iii), I.3.2(iii). ▲

2.7 Proposition. *Let H be a dense subgroupoid of a cancellative groupoid G. Then* $\operatorname{card}(H) = \operatorname{card}(G)$.

Proof. If H is infinite, then the result follows from 2.6. If H is finite, then H is a quasigroup, and so $H = G$. ▲

2.8 Theorem. *Let H be a dense subgroupoid of a cancellative groupoid G such that (A10), (B10), (C1), (C2), (C3) and (C4) are satisfied. Then the groupoids H and G are equationally equivalent.*

Proof. Using I.6.6, we can proceed similarly as in the proof of 1.10. ▲

2.9 Remark. Let H be a dense subgroupoid of a cancellative groupoid G and let r be a cancellative congruence of G such that H/r is a quasigroup (e.g., H/r finite). Then $H/r = G/r$, i.e., for every $x \in G$ there is $a \in H$ with $(a, x) \in r$. In particular, if H/r is finite, then G/r is so, and consequently G/r is a quasigroup.

2.10 Remark. Let H be a dense subgroupoid of a cancellative groupoid G. Assume that id_H and $H \times H$ are the only cancellative congruences of H. Now, if r is a cancellative congruence of G, then either $r \restriction H = \mathrm{id}_H$ or $r \restriction H = H \times H$, and hence $r = G \times G$.

II.3 Reflexions in left cancellative groupoids

3.1 Let $\mathscr{K}$ be an abstract class of groupoids (i.e., K is closed under isomorphic images) and let $\mathscr{L}$ be a non-empty abstract subclass of $\mathscr{K}$. If $G \in \mathscr{K}$, then a homomorphism $\varphi : G \to L$, $L \in \mathscr{L}$, is said to be a reflexion of G in $\mathscr{L}$ if for every homomorphism $\psi : G \to K$, $K \in \mathscr{L}$, there exists just one homomorphism $\zeta : L \to K$ such that $\zeta\varphi = \psi$.

(i) Let $\varphi : G \to L$ be a reflexion of G in $\mathscr{L}$, $G \in \mathscr{K}$, $L \in \mathscr{L}$. A homomorphism $\varphi' : G \to L'$ is a reflexion of G in $\mathscr{L}$ if and only if there exists an isomorphism $\lambda : L \to L'$ with $\varphi = \lambda\varphi$.

(ii) Let $\varphi : G \to L$ be a reflexion of G in $\mathscr{L}$, $G \in \mathscr{K}$, $L \in \mathscr{L}$. Then φ is injective if and only if there exists at least one injective homomorphism $\psi : G \to K \in \mathscr{L}$.

(iii) If $G \in \mathscr{L}$, then $\mathrm{id}_G : G \to G$ is a reflexion of G in $\mathscr{L}$.

3.2 Example. Let $\mathscr{K}$ be an abstract class lf groupoids and let $\mathscr{L}$ be an abstract subclass of $\mathscr{K}$ such that $\mathscr{L}$ is closed under subgroupoids and cartesian products and that trivial groupoids are in L. Then every groupoid from $\mathscr{K}$ has a (projective) reflexion in $\mathscr{L}$; for $G \in \mathscr{K}$, the natural projection $G \to G/r$, r being the smallest congruence with $G/r \in \mathscr{L}$, is a reflexion of G in $\mathscr{L}$.

3.3 Example. Let $\mathscr{K}$ be the class of cancellative semigroups and $\mathscr{L}$ the class of those semigroups from $\mathscr{K}$ which contain an idempotent element (notice that if e is idempotent in $S \in \mathscr{K}$, then e is a neutral element of S). If $S \in \mathscr{L}$, then $\mathrm{id}_S : S \to S$ is a reflexion of S in $\mathscr{L}$. If $S \in \mathscr{K}$ and $S \notin \mathscr{L}$, then S contains no idempotent and $\mathrm{id}_S : S \to T$ is a reflexion of S in $\mathscr{L}$, where $T = S \cup \{e\}, e \notin S$ and e is neutral in T. Thus every semigroup from $\mathscr{K}$ has an (injective) reflexion in $\mathscr{L}$. Notice that this reflexion is an epimorphism in $\mathscr{K}$ (viewed as a category) and that $\mathscr{L}$ is closed under cartesian products but not under subsemigroups.

3.4 Proposition. *Let $\mathscr{L}$ be an abstract class of left cancellative groupoids such that $\mathscr{L}$ is closed under cartesian products and left closed subgroupoids and that $\mathscr{L}$ contains trivial groupoids. Then every groupoid G possesses a reflexion $\varphi : G \to L$ in $\mathscr{L}$ such that $\varphi(G)$ is a left strongly dense subgroupoid of L. Moreover, if G is left cancellative, then φ is an epimorphism in the category of left cancellative groupoids.*

Proof. It follows from 1.8 that there is a non-empty family $\varphi_i : G \to A_i$, $i \in I$, of homomorphisms such that the following two conditions are satisfied:

(1) If $i \in I$, then $A_i \in \mathscr{L}$ and $\varphi_i(G)$ is left strongly dense in A_i;

(2) If $B \in \mathscr{L}$ and $\psi : G \to B$ is a homomorphism such that $\psi(G)$ is left strongly dense in B, then there are $j \in I$ and an isomorphism $\zeta : A_j \to B$ with $\psi = \zeta\varphi_j$.

Now, put $A = \prod_{i \in I} A_i$, $\varphi(x) = (\varphi_i(x)) \in A$ for each $x \in G$ and $L = \langle \varphi(G) \rangle_{\mathrm{lc},A}$. Then $\varphi(G)$ is left strongly dense in $L \in \mathscr{L}$ and φ can be viewed as a homomorphism of G into L. We are going to check that $\varphi : G \to L$ is a reflexion of G in $\mathscr{L}$.

Let $C \in \mathscr{L}$ and let $\psi : G \to C$ be a homomorphism. Then $B = \langle \psi(G) \rangle_{\mathrm{lc},G} \in \mathscr{L}$ and $\psi(G)$ is left strongly dense in B. By (2), $\psi = \zeta\varphi_j$ for some $j \in I$ and an isomorphism $\zeta : A_j \to B$. Now, $\zeta\mu\varphi = \zeta\varphi_j = \psi$, $\zeta\mu : L \to B$, where μ means the restriction of the j-th projection $A \to A_j$ to L. The unicity of $\zeta\mu$ and the rest are clear from I.3.13. ▲

3.5 Remark. Let $\mathscr{L}$ be an abstract class of left quasigroups closed under cartesian products and left subquasigroups and containing trivial groupoids. By 3.4, every groupoid G has a reflexion $\varphi : G \to L$ in $\mathscr{L}$ such that $K = L$ whenever $\varphi(G) \subseteq K \subseteq L$ and K is a left subquasigroup of L.

3.6 Remark. Let $\mathscr{L}$ be an abstract class of left cancellative groupoids as in 3.4 and let G be a left strongly dense subgroupoid of a groupoid $L \in \mathscr{L}$ such that the imbedding $G \to L$ is a reflexion of G in $\mathscr{L}$. Finally, let φ be an endomorphism of G. Then there exists an endomorphism ψ of L with $\psi \upharpoonright G = \varphi$, i.e., ψ extends φ. Of course, ψ is determined uniquely.

(i) If φ is projective, then $G \subseteq \psi(L) \subseteq L$. Now, if L is a left quasigroup, then $\psi(L)$ is so, hence $\psi(L)$ is left closed in L and consequently $\psi(L) = L$. Thus ψ projective.

(ii) If φ is injective and G satisfies (A3) in L, then ψ is injective (we have $L = \alpha_L(G)$ and the result follows from I.6.1(ii)).

(iii) If φ is an automorphism of G, G satisfies (A3) in L and L is a left quasigroup, then ψ is an automorphism of L.

3.7 Remark. Let $\mathscr{V}$ be a variety of groupoids, $\mathscr{K}$ the class of left cancellative groupoids from $\mathscr{V}$, $\mathscr{L}$ the class of left quasigroups from $\mathscr{V}$ and $\mathscr{K}_1$ the class of subgroupoids of left quasigroups from $\mathscr{L}$.

(i) $\mathscr{L} \subseteq \mathscr{K}_1 \subseteq \mathscr{K} \subseteq \mathscr{V}$ and all the four classes are abstract and closed under cartesian products. Moreover, $\mathscr{V}$, $\mathscr{K}$ and $\mathscr{K}_1$ are closed under subgroupoids and $\mathscr{L}$ under left closed subgroupoids.

(ii) By 3.4, every groupoid G has a reflexion $\varphi : G \to L$ in $\mathscr{L}$ such that $\varphi(G)$ is left strongly dense in L; clearly, φ is injective if and only if $G \in \mathscr{K}_1$.

(iii) Now, suppose that (A3) is satisfied everywhere in $\mathscr{V}$, i.e. whenever $G \in \mathscr{V}$ and H is a subgroupoid of G.

Let $Q \in \mathscr{L}$ and let G be a left strongly dense subgroupoid of Q. From the existence of a reflexion of G in $\mathscr{L}$ and from 1.5 it follows that the imbedding $G \to Q$ is a reflexion of G in $\mathscr{L}$; in particular, Q is determined uniquely up to G-isomorphism. Moreover, if $P \in \mathscr{L}$ and G is a subgroupoid of P, then $\langle G \rangle_{\mathrm{lc},P}$ is G-isomorphic to Q.

(iv) Suppose, finally, that (A9) is satisfied everywhere in $\mathscr{V}$. By I.6.3(ii), the class $\mathscr{K}_1$ is closed under left cancellative homomorphic images (in particular, $\mathscr{K}_1 = \mathscr{K}$, provided that $\mathscr{K}$-free groupoids belong to $\mathscr{K}_1$). Moreover, by 1.9, if $Q \in \mathscr{L}$ and G is a left strongly dense subgroupoid of Q (see (iii)), then Q is cancellative if and only if G is so.

II.4 Reflexions in cancellative groupoids

4.1 Proposition. *Let $\mathscr{L}$ be an abstract class of cancellative groupoids such that $\mathscr{L}$ is closed under cartesian products and closed subgroupoids and that $\mathscr{L}$ contains trivial groupoids. Then every groupoid G possesses a reflexion $\varphi : G \to L$ such that $\varphi(G)$ is a dense subgroupoid of L. Moreover, if G is cancellative, then φ is an epimorphism in the category of cancellative groupoids.*

Proof. Quite similar to that of 3.4 ▲

4.2 Remark. Let $\mathscr{L}$ be an abstract class of quasigroups closed under cartesian products and subquasigroups and containing trivial groupoids. By 4.1, every groupoid G has a reflexion $\varphi : G \to L$ in $\mathscr{L}$ such that $K = L$ whenever $\varphi(G) \subseteq K \subseteq L$ and K is a subquasigroup of L.

4.3 Remark. Let $\mathscr{L}$ be an abstract class of cancellative groupoids as in 4.1 and Let G be a sense subgroupoid of a groupoid $L \in \mathscr{L}$ such that the imbedding $G \to L$ is a reflexion of G in $\mathscr{L}$. Finally, let φ be an endomorphism of G. Then there exists an endomorphism ψ of L with $\psi \restriction G = \varphi$.

(i) If φ is projective, then $G \subseteq \psi(L) \subseteq L$. Now, if L is a quasigroup, then $\psi(L) = L$. Thus ψ is projective.

(ii) If φ is injective and G satisfies (C1), (C2), (C3), (C4) in L, then ψ is injective.

(iii) If φ is an automorphism of G, G satisfies (C1), (C2), (C3), (C4) in L and L is a quasigroup, then ψ is an automorphism of L.

4.4 Remark. Let $\mathscr{V}$ be a variety of groupoids, $\mathscr{K}$ the class of cancellative groupoids from $\mathscr{V}$, $\mathscr{G}$ the class of quasigroups from $\mathscr{V}$ and $\mathscr{K}_1$ the class of subgroupoids of quasigroups from $\mathscr{G}$.

(i) $\mathscr{G} \subseteq \mathscr{K}_1 \subseteq \mathscr{K} \subseteq \mathscr{V}$ and all the four classes are abstract and closed under cartesian products. Moreover, $\mathscr{V}$, $\mathscr{K}$ and $\mathscr{K}_1$ are closed under subgroupoids and $\mathscr{G}$ under closed subgroupoids.

(ii) By 4.1, every groupoid G has a reflexion $\varphi : G \to Q$ in $\mathscr{G}$ such that $\varphi(G)$ is dense in Q; φ is injective if and only if $G \in \mathscr{K}_1$.

(iii) Now, suppose that (C1), (C2), (C3) and (C4) are satisfied everywhere in $\mathscr{V}$.

Let $Q \in \mathscr{G}$ and let G be a dense subgroupoid of Q. From the existence of a reflexion of $G \in \mathscr{G}$ and from 2.5 it follows that the imbedding $G \to Q$ is a reflexion of $G \in \mathscr{G}$; in particular, Q is determined uniquely up to G-isomorphism. Moreover, if $P \in \mathscr{G}$ and G is a subgroupoid of P, then $\langle G \rangle_{c,P}$ and Q are G-isomorphic.

By I.6.6(ii), $\mathscr{K}_1$ is closed under cancellative homomorphic images. In particular, $K = \mathscr{K}_1$, provided that $\mathscr{K}$-free groupoids belong to $\mathscr{K}_1$.

II.5 Comments

Still being of introductory character, the results of this chapter are based on [7], [22], [23] and [24].

III. Imbeddings of (left/right) cancellative groupoids into (left/right) quasigroups

III.1 Imbeddings into (left/right) quasigroups in some classes

1.1 Consider the following three conditions defined for an abstract class $\mathscr{K}$ of groupoids:

(UCH) If G is a groupoid such that $G = \bigcup_{i \in I} G_i$, where I is a non-empty linearly ordered index set, $G_i \in \mathscr{K}$ are subgroupoids of G and $G_i \subseteq G_j$ for $i \leq j$, then $G \in \mathscr{K}$;

(LDE) If $G \in \mathscr{K}$ and $a, b \in G$, then there exist $K \in \mathscr{K}$, $u \in K$ and an injective homomorphism $\varphi : G \to K$ such that $\varphi(a)\, u = \varphi(b)$;

(RDE) If $G \in \mathscr{K}$ and $a, b \in G$, then there exist $K \in \mathscr{K}$, $v \in K$ and an injective homomorphism $\varphi : G \to K$ such that $v\varphi(a) = \varphi(b)$.

1.2 Proposition. *Let $\mathscr{K}$ be an abstract class of left cancellative groupoids satisfying the conditions (UCH) and (LDE). Then every groupoid from $\mathscr{K}$ can be imbedded into a left quasigroup from $\mathscr{K}$.*

Proof. Let $G \in \mathscr{K}$. Put $\mathrm{I} = G^{(2)}$ and suppose that I is well ordered. Further, define a chain G_i, $i \in \mathrm{I}$, of groupoids from $\mathscr{K}$ such that G_i is a subgroupoid of G_j for $i \leq j$. First, let $i_0 = (a_0, b_0)$ be the smallest element of I. We choose a groupoid $K = G_{i_0}$ in $\mathscr{K}$ that G is a subgroupoid of K and $a_0 u = b_0$ for some $u \in K$ (by (LDE)).

Now, let $i > i_0$, $i = (a, b)$, and let $H = \bigcup_{j<i} G_j$. Then H is a groupoid, all G_j are subgroupoids of $H \in \mathscr{K}$ (by (UCH)) for i limit and $H = G_{i-1} \in \mathscr{K}$ for i non-limit. Again, H is a subgroupoid of a groupoid $G_i \in \mathscr{K}$ such that $au = b$ for some $u \in G_i$.

Put $\varepsilon(G) = \bigcup_{i \in \mathrm{I}} G_i$. Then $\varepsilon(G) \in \mathscr{K}$, G is a subgroupoid of $\varepsilon(G)$ and, if $a, b \in G$, then $au = b$ for some $u \in \varepsilon(G)$. Repeating the operator ε, we come by a countable chain $G \subseteq \varepsilon(G) \subseteq \varepsilon^2(G) \subseteq \dots$ of groupoids and we put $L = \bigcup_{n=0}^{\infty} \varepsilon^n(G)$. Then $L \in \mathscr{K}$ and L is a left quasigroup. ▲

1.3 Remark. Let $\mathscr{K}$ be an abstract class of left cancellative groupoids closed under left closed subgroupoids and satisfying (UCH), (LDE). Using a slightly different method, we show the same as in 1.2.

Let $G \in \mathscr{K}$. If G is finite, then G is a left quasigroup itself, and hence assume that G is infinite and that G is a subset of a set S such that $\mathrm{card}(G) < \mathrm{card}(S)$. Denote by $\mathfrak{A}$ the set of ordered pairs $(R, *)$, where R is a subset of S with $G \subseteq R$ and $*$ is a binary operation defined on R such that $R(*) \in \mathscr{K}$ and G is a left strongly dense subgroupoid of $R(*)$. Then the set $\mathfrak{A}$ is non-empty and is ordered by $(R_1, *_1) \leq (R_2, *_2)$ if and only if $R_1(*_1)$ is a subgroupoid of $R_2(*_2)$.

Now, let $(R_i, *_i)$, $i \in \mathrm{I}$, be a chain of elements from $\mathfrak{A}$ and let $R(*) = \bigcup_{i \in \mathrm{I}} R_i(*_i)$. Then $R(*) \in \mathscr{K}$ and G is left strongly dense in $R(*)$, so that $R(*) \in \mathfrak{A}$ and $\mathfrak{A}$ contains maximal elements.

Let $(T, *)$ be maximal in $\mathfrak{A}$. We show that $T(*)$ is a left quasigroup. First, $\mathrm{card}(T) = \mathrm{card}(G)$ by 1.8. Now, take $a, b \in T$. By (LDE) there are a groupoid $K \in \mathscr{K}$, an injective homomorphism $\varphi : T(*) \to K$ and an element $u \in K$ such that $\varphi(a)\, u = \varphi(b)$. Put $L = \langle \varphi(T) \rangle_{\mathrm{lc}, K}$; then $L \in \mathscr{K}$, $u \in L$, $\varphi(T)$ is left strongly dense in L and $\mathrm{card}(L) < \mathrm{card}(S)$. Consequently, taking a copy of L in S, we get a groupoid $P(\circ) \in \mathscr{K}$ such that $T \subseteq P \subseteq S$, $T(*)$ is a left strongly dense subgroupoid of $P(\circ)$ and $a \circ v = b$ for some $v \in P$. Then $(P, \circ) \in \mathfrak{A}$, $(T, *) \leq (P, \circ)$, hence $(T, *) = (P, \circ)$. Thus we have proved that $T(*)$ is a left quasigroup.

1.4 Proposition. *Let $\mathscr{K}$ be an abstract class of cancellative groupoids satisfying the conditions (UCH), (LDE) and (RDE). Then every groupoid from $\mathscr{K}$ can be imbedded into a quasigroup from $\mathscr{K}$.*

Proof. By 1.2 and its dual, there are groupoids $\varrho(G)$ and $\tau(G)$ in $\mathscr{K}$ such that G is a subgroupoid of both $\varrho(G)$ and $\tau(G)$ and for all $a, b \in G$ there are $u \in \varrho(G)$ and $v \in \tau(G)$ with $au = b = va$. Repeating these operators, we get the following countable chain:

$$G \subseteq \varrho(G) \subseteq \tau(\varrho(G)) \subseteq \varrho(\tau(\varrho(G))) \subseteq \dots .$$

Let Q be the union of the chain. Then $Q \in \mathscr{K}$, G is a subgroupoid of Q and Q is a quasigroup. ▲

1.5 Remark. Let $\mathscr{K}$ be an abstract class of cancellative groupoids closed under closed subgroupoids and satisfying (UCH), (LDE) and (RDE). Using 2.7 and proceeding similarly as in 1.3 we may show the same as in 1.4.

1.6 Remark. Let $\mathscr{K}$ be an abstract class of (left, right) cancellative groupoids closed under subgroupoids and filtered products and such that every finitely generated groupoid from $\mathscr{K}$ can be imbedded into a (left, right) quasigroup from $\mathscr{K}$. We show that then every groupoid from $\mathscr{K}$ can be imbedded into a (left, right) quasigroup from $\mathscr{K}$.

Let $G \in \mathscr{K}$. Denote by $\mathscr{A}$ the set of non-empty finite subsets of G and for every $A \in G$ put $G_A = \langle A \rangle_G$ and choose $w_A \in A$. Now, there is a (left, right) quasigroup Q_A in $\mathscr{K}$ such that G_A is a subgroupoid of Q_A and we put $Q = \prod_{A \in \mathscr{A}} Q_A$. Define a relation r on Q by $((x_A), (y_A) \in r$ if and only if there is $A_1 \in \mathscr{A}$ such that $x_B = y_B$ whenever $B \in \mathscr{A}$ and $A_1 \subseteq B$. Then r is a congruence of Q, $P = Q/r$ is again a (left, right) quasigroup from $\mathscr{K}$ and we denote by $\pi : Q \to P$ the natural projection. Finally, define $\sigma : G \to Q$ by $\sigma(x) = (x_A) \in Q$ where $x_A = x$ if $x \in G_A$ and $x_A = w_A$ otherwise. Then $\varphi = \pi\sigma : G \to P$ is an injective homomorphism.

1.7 Remark. Let $\mathscr{V}$ be a variety of groupoids, $\mathscr{K}$ the class of left cancellative groupoids from $\mathscr{V}$ and $\mathscr{L}$ the class of left quasigroups from $\mathscr{V}$; we have $\mathscr{L} \subseteq \mathscr{K} \subseteq \mathscr{V}$.

(i) If $Q \in \mathscr{L}$, then we can define a binary operation, say $*$, on Q by $x(x * y) = y$ for all $x, y \in Q$. The algebra $\bar{Q} = Q(\cdot, *)$ with two binary operations satisfies the equations $x(x * y) \triangleq y \triangleq x * (xy)$ and the class $\bar{\mathscr{L}} = \{\bar{Q}; Q \in \mathscr{L}\}$ is a variety of algebras with two binary operations; the correspondence $Q \leftrightarrow \bar{Q}$ is an equivalence between $\mathscr{L}$ and $\bar{\mathscr{L}}$.

(ii) Let $F \in \mathscr{V}$ be a free groupoid over a (non-empty) set X of free generators. Suppose that F is left cancellative and that F is a subgroupoid of a left quasigroup $Q \in \mathscr{L}$. Let $\bar{E} \in \bar{\mathscr{L}}$ be a free algebra in $\bar{\mathscr{L}}$ over X. Denote by G the subgroupoid of $E \in \mathscr{L}$ generated by X; we have $G \in \mathscr{K}$. There exists a homomorphism $\varphi : E \to Q$

such that $\varphi \upharpoonright X = \mathrm{id}_X$. Then $\psi(G) = F$, $\psi = \varphi \upharpoonright G$, and, since F is free in $\mathscr{V}$, there is a homomorphism $\eta : F \to G$ with $\eta \upharpoonright X = \mathrm{id}_X$. Now, $\eta\psi = \mathrm{id}_G$, $\psi\eta = \mathrm{id}_F$ and we see that both ψ and η are isomorphisms. Thus G is free over X in $\mathscr{V}$.

1.8 Remark. Let $\mathscr{V}$ be a variety of groupoids, $\mathscr{K}$ the class of cancellative groupoids from $\mathscr{V}$ and $\mathscr{G}$ the class of quasigroups from $\mathscr{V}$; we have $\mathscr{G} \subseteq \mathscr{K} \subseteq \mathscr{V}$.

(i) If $Q \in \mathscr{K}$, then we can define binary operations $*$ and $\circ$ on Q by $x(x * y) = y = (y \circ x)x$ for all $x, y \in Q$. The algebra $\bar{Q} = Q(\cdot, *, \circ)$ with three binary operations satisfies the equations $x(x * y) \triangleq y \triangleq x * (xy)$ and $(y \circ x)x \triangleq y \triangleq (yx) \circ x$ and the class $\bar{\mathscr{G}} = \{\bar{Q};\, Q \in \mathscr{G}\}$ is a variety of algebras with three binary operations; the correspondence $Q \leftrightarrow \bar{Q}$ is an equivalence between $\mathscr{G}$ and $\bar{\mathscr{G}}$.

(ii) Let $F \in \mathscr{V}$ be a free groupoid over a (non-empty) set X of free generators. Suppose that F is a subgroupoid of a quasigroup $Q \in \mathscr{G}$. Let $\bar{E} \in \bar{\mathscr{G}}$ be a free algebra over X. Then the subgroupoid $G = \langle X \rangle_E \in \mathscr{K}$ is free over X in $\mathscr{V}$.

1.9 Lemma. *Let $\mathscr{K}$ be an abstract class of groupoids closed under cartesian products. Suppose that for all $a, b \in G$, $a \neq b$, there exist $K \in \mathscr{K}$ and a homomorphism $\varphi : G \to K$ such that $\varphi(a) \neq \varphi(b)$. Then G can be imbedded into a groupoid from $\mathscr{K}$.*

Proof. For each $i \in G^{(2)} \setminus \mathrm{id}_G$, $i = (a, b)$, choose $K_i \in \mathscr{K}$ and $\varphi_i : G \to K$, $\varphi_i(a) \neq \varphi_i(b)$. Now, put $K = \prod K_i$ and $\varphi(x) = (\varphi_i(x)) \in K$, $x \in G$. ▲

III.2 Minimal imbeddings of left cancellative groupoids into left quasigroups

2.1 Let G be a left cancellative groupoid. For $a \in G$, we put $(\mathrm{M}_a(G) =)\, \mathrm{M}_a = \{(a, b);\, b \in G \setminus aG\}$; clearly, $\mathrm{M}_a = \emptyset$ if and only if $a \in \mathrm{C}_1(G)$, i.e., the left translation L_a is bijective.

The sets M_a, $a \in G$, are pair-wise disjoint and we put $(\mathrm{M}(G) =)\, \mathrm{M} = \bigcup_{a \in G} \mathrm{M}_a$. Further, we put $(\mathrm{N}_a(G) =)\, \mathrm{N}_a = \mathrm{M} \setminus \mathrm{M}_a = \bigcup_{b \neq a} \mathrm{M}_b$ for every $a \in G$. (Clearly, $\mathrm{M} = \emptyset$ if and only if G is a left quasigroup.)

Finally, let $(\mathrm{P}(G) =)\, \mathrm{P}$ denote the set of $a \in G$ such that $\mathrm{card}(\mathrm{N}_a) < \mathrm{card}(\mathrm{M})$; clearly, $\mathrm{P} \cap \mathrm{C}_1(G) = \emptyset$.

2.1.1 Lemma. *If* M *is finite, then* $\mathrm{P} = G \setminus \mathrm{C}_1(G)$ *is finite.*

Proof. Obvious. ▲

2.1.2 Lemma. *If* M *is infinite, then either* $\mathrm{P} = \emptyset$ *or* $\mathrm{card}(\mathrm{P}) = 1$. *If the latter is true and* $\mathrm{P} = \{a\}$, *then* $\mathrm{card}(\mathrm{M}) = \mathrm{card}(\mathrm{M}_a)$ *and* $\mathrm{card}(\mathrm{C}_1(G)) = \mathrm{card}(G)$.

Proof. Assume that $a \in \mathrm{P}$. Then $\mathrm{card}(\mathrm{N}_a) < \mathrm{card}(\mathrm{M})$, M is the disjoint union of N_a and M_a and M is infinite; hence $\mathrm{card}(\mathrm{M}_a) = \mathrm{card}(\mathrm{M})$. Further, for $b \neq a$, we have $\mathrm{M}_b \subseteq \mathrm{N}_a$, $\mathrm{card}(\mathrm{M}_b) < \mathrm{card}(\mathrm{M})$, and so $\mathrm{card}(\mathrm{N}_b) = \mathrm{card}(\mathrm{M} \setminus \mathrm{M}_b) = \mathrm{card}(\mathrm{M})$. Consequently, $\mathrm{P} = \{a\}$.

Now, put $R = G\setminus C_l(G)$. If $b \in R$, $b \neq a$, then $\mathrm{card}(M_b) < \mathrm{card}(M)$. On the other hand, $\mathrm{card}(M_a) \leq \mathrm{card}(G)$, $\mathrm{card}(R) - 1 \leq \mathrm{card}(N_a)$ and $\mathrm{card}(R) < \mathrm{card}(M)$. It follows that $\mathrm{card}(C_l(G)) = \mathrm{card}(G)$. ▲

2.1.3 Lemma. P *is a finite set. Moreover, if either* $\mathrm{card}(C_l(G)) < \mathrm{card}(G)$ *or* $C_l(G)$ *is finite, then* $P = \emptyset$.

Proof. With respect to 2.1.2, we may assume that M is finite and G infinite (if G is finite, then G is a left quasigroup and $P = \emptyset$). By 2.1.1, $P = G\setminus C_l(G)$ is finite, and therefore $\mathrm{card}(C_l(G)) = \mathrm{card}(G)$, a contradiction. ▲

2.2 Example. (i) $P(\mathbb{N}(+)) = \emptyset = C_l(\mathbb{N}(+))$, where $\mathbb{N}(+)$ is the additive semigroup of positive integers.

(ii) Define an operation $*$ on $\mathbb{N}$ by $1 * j = 2j$, $2 * j = j + 1$ and $i * j = j$ for all $i, j \in \mathbb{N}$, $i \geq 3$. Then $\mathbb{N}(*)$ is a left cancellative groupoid, $M(\mathbb{N}(+)) = \{(1, 2k+1); k \geq 0\} \cup \{(2,1)\}$, $C_l(\mathbb{N}(*)) = \mathbb{N}\setminus\{1,2\}$ and $P(\mathbb{N}(*)) = \{1\}$.

(iii) Define an operation $\circ$ on $\mathbb{N}$ by $1 \circ j = j + 1$, $2 \circ j = j + 1$ and $i \circ j = j$ for all $i, j \in \mathbb{N}$, $i \geq 3$. Then $\mathbb{N}(\circ)$ is a left cancellative groupoid, $M(\mathbb{N}(\circ)) = \{(1,1), (2,1)\}$, $C_l(\mathbb{N}(\circ) = \mathbb{N}\setminus\{1,2\}$ and $P(\mathbb{N}(\circ)) = \{1,2\}$.

2.3 Theorem. *The following conditions are equivalent for a left cancellative groupoid G:*

(i) $P(G) = \emptyset$.

(ii) There exists a left quasigroup Q such that G is a subgroupoid of Q and $Q = \alpha_{1,Q}(G)$ *(i.e., for every* $x \in G$ *there exists* $a \in G$ *with* $ax \in G$*).*

If these conditions are satisfied, then Q may be choosen in such a way that every element from $Q\setminus G$ *is left neutral in Q and G is strongly dense in Q.*

Proof. (i) implies (ii). Assume that $G \cap M = \emptyset$ and put $Q = G \cup M$. Since $P = \emptyset$, for every $a \in G$ there is a bijective mapping $f_a : N_a \to M$. Now, define a binary operation $*$ on Q as follows:

(1) $x * y = xy$ for all $x, y \in G$;

(2) $u * b = v$ for all $u \in M$ and $v \in Q$;

(3) $a * (a, b) = b$ for all $a, b \in G$, $(a, b) \in M_a$;

(4) $a * u = f_a(u)$ for all $a \in G$, $u \in N_a$.

One sees easily that $Q(*)$ is a left quasigroup satisfying the conditions from (ii).

(ii) implies (i). Suppose, on the contrary, that $P(G) \neq \emptyset$ and put $R = Q\setminus G$. If $(a, b) \in M$ $(= M(G))$, then $ax = b$ for suitable $x \in R$, and hence we have a mapping $\sigma : M \to R$, $\sigma(a, b) = x$. With regard to (ii), $\sigma(M) = R$ and $\sigma \restriction M_a$ is injective, $a \in G$. In particular, $\mathrm{card}(M_a) \leq \mathrm{card}(R) \leq \mathrm{card}(M)$.

Now, assume that M is infinite. By 2.1.2, $P = \{a_0\}$; we have $\mathrm{card}(M_{a_0}) = \mathrm{card}(M)$, and therefore $\mathrm{card}(R) = \mathrm{card}(M)$. On the other hand, $M = M_{a_0} \cup N_{a_0}$, $\mathrm{card}(N_{a_0}) < \mathrm{card}(M)$ and $R = S \cup T$, where $S = \sigma(M_{a_0})$, $\mathrm{card}(S) = \mathrm{card}(R)$, and $T = \sigma(N_{a_0})$, $\mathrm{card}(T) < \mathrm{card}(R)$. But $Q = a_0Q = a_0G \cup a_0S \cup a_0T$, $a_0G \subseteq G$ and $a_0S \subseteq G$. Thus $R \subseteq a_0T$ and $\mathrm{card}(R) \leq \mathrm{card}(T)$, a contradiction.

Next, assume that M is finite. By 2.1.1, $P = G \setminus C_1(G)$ is finite. For $a_0 \in P$, put $S = \sigma(M_{a_0})$ and $T = R \setminus S$. Similarly as above, we have $Q = a_0Q = a_0G \cup a_0S \cup a_0T$, $a_0G \cup a_0S \subseteq G$ and $R \subseteq a_0T$, which yields $\mathrm{card}(R) = \mathrm{card}(T)$. But R is finite and $T \subseteq R$. Consequently, $T = R$, $S = \emptyset$ and $M_{a_0} = \emptyset$, a contradiction. ▲

2.4 Theorem. *Let G be a left cancellative groupoid. Then there exists a left quasigroup Q such that G is a strongly dense subgroupoid of Q, $Q = \alpha_{2,Q}(G)$ (i.e., for every $x \in Q$ there are $a, b \in G$ with $a(bx) \in G$) and every element from $Q \setminus G$ is left neutral in Q.*

Proof. First, assume that M is infinite. Let K be a set with $\mathrm{card}(K) = \mathrm{card}(M)$; we also assume that the sets G, M, K are pair-wise disjoint and we put $L = M \cup K$ and $Q = G \cup L$. Furher, there are bijective mappings $f_a : N_a \cup K \to L$, $a \in G$, such that $f_a(K) = N_a$ if $a \notin P$. Now, define a binary operation $*$ on Q as follows:

(1) $x * y = xy$ for all $x, y \in G$;

(2) $u * v = v$ for all $u \in L$ and $v \in Q$;

(3) $a * (a, b) = b$ for all $a, b \in G$, $(a, b) \in M_a$;

(4) $a * u = f_a(u)$ for all $a \in G$, $u \in N_a \cup K$.

Clearly, $Q(*)$ is a left quasigroup and G is a subgroupoid of $Q(*)$. Finally, by 2.1.2, $P \neq G$, and if $a \in G \setminus P$, then $a * K \subseteq N_a \subseteq M$. Thus $K \subseteq \alpha_{2,Q(*)}(G)$.

Now, assume that M is finite and non-empty. By 2.1.1, $P = G \setminus C_1(G)$ is finite and also non-empty. Put $Q = G \cup M \cup \mathbb{N}$ (we assume that G, M, $\mathbb{N}$ are pair-wise disjoint) and, for every $a \in G$, let $f_a : N_a \cup \mathbb{N} \to M \cup \mathbb{N}$ be a bijection.

Finally, let $a_0 \in P$. We may assume that for every $i \in \mathbb{N}$ there is $b \in C_1(G)$ with $f_b(i) \in M_{a_0}$. Now, define $*$ similarly as above. Then $Q(*)$ is a left quasigroup, G is a subgroupoid of $Q(*)$ and for every $i \in \mathbb{N}$ there is $b \in C_1(G)$ such that $a_0 * (b * i) \in G$. The rest is clear. ▲

III.3 Free imbeddings of (left/right) cancellative groupoids into (left/right) quasigroups

3.1 By a partial groupoid we mean a non-empty set together with a partial binary operation (possibly empty).

3.2 Let G be a partial groupoid. We put $N(G) = \{(a, b); a, b \in G, ab \text{ is defined}\}$. Now, G is said to be left (right) cancellative if $b = c$ whenever $(a, b), (a, c) \in N(G)$ $((b, a), (c, a) \in N(G))$ and $ab = ac$ $(ba = ca)$. G is said to be cancellative if it is both left and right cancellative.

3.3 A partial groupoid $H(*)$ is said to be a partial subgroupoid of a partial groupoid $G(\circ)$ if $H \subseteq G$, $N(H(*)) \subseteq N(G(\circ))$ and $a * b = a \circ b$ for each $(a, b) \in N(H(*))$.

3.4 Lemma. *Let G be a left cancellative partial groupoid. Then there exists a left cancellative partial groupoid K such that G is a partial subgroupoid of K and $G^{(2)} = N(K)$. Moreover, K is cancellative if and only if G is so.*

Proof. Put $K = G \cup A$, $A = G^{(2)} \setminus \mathrm{N}(G)$ (we assume $K \cap A = \emptyset$) and define a partial multiplication on K in such a way that G is a partial subgroupoid of K and $ab = (a, b) \in \mathrm{N}(G)$ for each $(a, b) \in \mathrm{N}(G)$. ▲

3.5 Lemma. *Let G be a left cancellative partial groupoid. Then there exists a left cancellative groupoid L such that G is a partial subgroupoid of L. Moreover, L is cancellative if and only if G is so.*

Proof. L is the union of the countable chain of partial groupoids constructed in 3.4. ▲

3.6 Lemma. *Let G be a left cancellative groupoid. Then there exists a left cancellative partial groupoid P such that G is a (partial) subgroupoid of P and $G = aP$ for every $a \in G$. Moreover, P is cancellative if and only if G is so.*

Proof. Put $P = G \cup G^{(2)}$ (again, assume $G \cap G^{(2)} = \emptyset$) and define $a(a, b) = b$ for all $a, b \in G$. ▲

3.7 Theorem. *Let G be a (left, right) cancellative groupoid. Then there exists a (left, right) quasigroup Q such that G is (left, strongly, right strongly) dense subgroupoid of Q.*

Proof. Just combine 3.5 and 3.6 (and the dual of 3.6) in the countable chain. ▲

3.8 Remark. Let G be a (left strongly, right strongly) dense subgroupoid of a (left, right) quasigroup Q. Clearly, $\mathrm{id}_Q \in \mathcal{N}$, where $\mathcal{N}$ is the set of (left, right) cancellative congruence r of Q such that $r \restriction G = \mathrm{id}_G$, and the set $\mathcal{N}$ is upwords inductive. Consequently, the set $\mathcal{M}$ of maximal elements of $\mathcal{N}$ is non-empty and, given $s \in \mathcal{M}$, G imbeds as a (left strongly, right strongly) dense subgroupoid into the (left, right) quasigroup Q/s. Moreover, if t is a (left, right) cancellative congruence of Q/s such that $t \restriction G = \mathrm{id}_G$, then $t = \mathrm{id}_Q$. In this respect, Q/s is a minimal (left, right) quasigroup envelope of G.

3.9 Remark. Let G be a (left strongly, right strongly) dense subgroupoid of a (left, right) quasigroup Q such that Q is a minimal (left, right) quasigroup envelope of G (see 3.8). That is, $r = \mathrm{id}_G$ whenever r is a (left, right) cancellative congruence of Q such that $r \restriction G = \mathrm{id}_G$. Now, assume that id_G and $G \times G$ are the only (left, right) cancellative congruences of G. If s is a (left, right) cancellative congruence of Q such that $s \neq \mathrm{id}_Q$, then $s \restriction G \neq \mathrm{id}_G$, $s \restriction G = G \times G$ and, finally, $s = Q \times Q$.

III.4 Comments and open problem

The first and the last sections of the chapter are folklore more or less and the second section is an improved version of [23, § 9]. The following problems remain open:

Characterize the cancellative groupoids G such that G is a subgroupoid of a quasigroup Q and the following condition (1) ((2), (3), (4), resp.) is satisfied:

(1) For every $x \in Q$ there is $a \in G$ with $ax \in G$.

(2) For every $x \in Q$ there are $a, b \in G$ with $a \cdot bx \in G$.

(3) For every $x \in Q$ there are $a, b \in G$ with $ax \cdot b \in G$.

(4) For every $x \in Q$ there is $a \in G$ such that at least one of the elements ax, xa is in G.

For which (left, right) cancellative groupoids G are the minimal (left, right) quasigroup envelopes (see 3.8) determined uniquely up to a G-isomorphism?

IV. Cancellative selfdistributive groupoids

IV.1 Dense subgroupoids of left distributive groupoids

1.1 A groupoid is said to be left (right) distributive if it satisfies the equation $x \cdot yz \triangleq xy \cdot xz$ $(zy \cdot x \triangleq zx \cdot yx)$. A groupoid is said to be (bi-) distributive if it is both left and right distributive.

Notice that, if G is (left, right) distributive, then $(\mathrm{Mul_l}(G) \subseteq \mathrm{End}(G), \mathrm{Mul_r}(G) \subseteq \mathrm{End}(G))$ $\mathrm{Mul}(G) \subseteq \mathrm{End}(G)$.

1.2 Lemma. *Let H be a subgroupoid of a left distributive groupoid G. Then:*

(i) For all $f, g \in \mathrm{Mul_l}(G, H)$ there exists $h \in \mathrm{Mul_l}(G, H)$ such that $fg = hf$.

(ii) $\mathrm{Mul}(G, H) = \mathrm{Mul_r}(G, H) \cdot \mathrm{Mul_l}(G, H)$.

(iii) If G is left divisible, then $\mathrm{Mul}(G, H) = \mathrm{Mul_l}(G, H) \cdot \mathrm{Mul_r}(G, H)$.

Proof. (i) We have $g = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_n}$, $n \geq 0$, $a_i \in H$ and since f is an endomorphism of G and $f(H) \subseteq H$, we can put $h = \mathrm{L}_{f(a_1)} \dots \mathrm{L}_{f(a_n)} \in \mathrm{Mul_l}(G, H)$.

(ii) and (iii). The asertions follow easily from the fact that $\mathrm{L}_a\mathrm{R}_b = \mathrm{R}_{ab}\mathrm{L}_a$ for all $a, b \in G$. ▲

1.3 Corollary. *Let G be a left distributive groupoid. Then:*

(i) For all $f, g \in \mathrm{Mul_l}(G)$ there exists $h \in \mathrm{Mul_l}(G)$ such that $fg = hf$.

(ii) $\mathrm{Mul}(G) = \mathrm{Mul_r}(G) \cdot \mathrm{Mul_l}(G)$.

(iii) If G is left divisible, then $\mathrm{Mul}(G) = \mathrm{Mul_l}(G) \cdot \mathrm{Mul_r}(g)$.

1.4 Proposition. *Let H be a subgroupoid of a left distributive groupoid. Then:*

(i) The condition (A1), ..., (A9) and (A11) (see I.4.1) are satisfied for H in G.

(ii) $\langle H \rangle_{\mathrm{lc}, G} = [H]_{\mathrm{l}, G} = \{x \in G;\ f(x) \in H$ for some $f \in \mathrm{Mul_l}(G, H)\} = \alpha_G(H)$.

(iii) $\gamma_G(H) \subseteq \beta_G(\alpha_G(H))$.

(iv) If G is left divisible, then $\gamma_G(H) \subseteq \alpha_G(\beta_G(H))$.

Proof. (i) Immediately clear from the fact that $\mathrm{Mul_l}(G, H) \subseteq \mathrm{End}(G)$.

(ii) If $f(x), g(y) \in H$, then $fg(y) \in H$ and $fg = hf$ for some $h \in \mathrm{Mul_l}(G, H)$. Now, $fg(xy) = fg(x)\,fg(y) = hf(x)\,fg(y) \in H$. Similarly, if $f(x), g(xy) \in H$, then

$hf(x)\,fg(y) = fg(x)\,fg(y) = fg(xy) \in H$; then $k(y) \in H$, where $k = L_{hf(x)}fg \in \mathrm{Mul_l}(G, H)$. We have shown that $\alpha_G(H)$ is a left closed subgroupoid of G and the rest is clear.

(iii) and (iv). Use 1.2(ii) and (iii). ▲

1.5 Proposition. *Let H be a subgroupoid of a left distributive groupoid G and $K = \langle H \rangle_{\mathrm{lc}, G}$. If r is a left cancellative congruence of H, then r can be extended in a unique way to a left cancellative congruence s of K; s is cancellative if and only if r is so.*

Proof. Define s by $(x, y) \in s$ if and only if $(f(x), f(y)) \in r$ for some $f \in \mathrm{Mul_l}(G, H)$. Then s is an equivalence extending r (use 1.2(i)) and if $(x, y) \in s$, $z \in K$, $(f(x), f(y)) \in r$, then $gf(x), gf(y), gf(z) \in H$ for some $g \in \mathrm{Mul_l}(G, H)$ and we have $gf(zx) = gf(z)\,gf(x), gf(zy) = gf(z)\,gf(y), (gf(zx), gf(zy)) \in r$ and $(zx, zy) \in s$. Quite similarly, $(xz, yz) \in s$.

Finally, let $x, y, z \in K$, $f \in \mathrm{Mul_l}(G, H)$ and $(f(zx), f(zy)) \in r$. Again, $gf(x)$, $gf(y)$, $gf(z) \in H$, $(gf(z)\,gf(x), gf(z)\,gf(y)) \in r$, $(gf(x), gf(y)) \in r$ and $(x, y) \in s$. The rest is similar. ▲

1.6 Proposition. *Let H be a subgroupoid of a left cancellative left distributive groupoid G and $K = \langle H \rangle_{\mathrm{lc}, G}$. Then the groupoids H and K are equationally equivalent.*

Proof. Let W denote an absolutely free groupoid over an infinite countable set X of variables and let $u, v \in W$ be such that $u \triangleq v$ holds in H. Now, let $\varphi : W \to K$ be a homomorphism. Then $f\varphi(x) \in H$ for each $x \in \mathrm{var}(uv)$ and some $f \in \mathrm{Mul_l}(G, H)$ and there is a homomorphism $\psi : W \to H$ such that $\psi(x) = f\varphi(x)$ for each $x \in \mathrm{var}(uv)$. Now, $f\varphi(u) = \psi(u) = \psi(v) = f\varphi(v)$ and, since f is injective, we have $\varphi(u) = \varphi(v)$. ▲

1.7 Theorem. *Let H be a left strongly dense subgroupoid of a left distributive groupoid G. Then*

(i) For every $x \in G$ there exists $f \in \mathrm{Mul_l}(G, H)$ such that $f(x) \in H$.

(ii) Every (left) cancellative congruence of H can uniquely be extended to a (left) cancellative congruence of G.

(iii) If H is cancellative and G is left cancellative, then G is cancellative.

(iv) If H is right divisible and G is a left quasigroup, then G is right divisible.

(v) If G is left cancellative, then the groupoids H and G are equationally equivalent.

Proof. (i), (ii) and (iii). See 1.4 and 1.5.

(iv) Let $x, y \in G$. Then $f(x), f(y) \in H$ for some $f \in \mathrm{Mul_l}(G, H)$ and, since H is right divisible, $af(x) = f(y)$ for some $a \in H$. Further, since G is left divisible f is a projective transformation of G and $a = f(z)$ for suitable $z \in G$. Now, $f(y) = af(x) = f(z)\,f(x) = f(zx)$ and $y = zx$, f being injective.

(v) See 1.6. ▲

IV.2 Imbeddings of left cancellative left distributive groupoids into left distributive left quasigroups

2.1 Proposition. *Let G be a left distributive groupoid. Put* $\mathrm{A_l}(G) = \{a \in G;$ L_a injective$\}$ *and* $\widetilde{\mathrm{A}}_\mathrm{l}(G) = \{a \in \mathrm{A_l}(G);\ aa = aa \cdot a\}$. *Then:*

(i) $\mathrm{A_l}(G)$ *is either empty or a left closed subgroupoid of G.*

(ii) $\widetilde{\mathrm{A}}_\mathrm{l}(G)$ *is either empty or a left closed subgroupoid of G.*

(iii) $\mathrm{A_l}(G)\setminus\widetilde{\mathrm{A}}_\mathrm{l}(G)$ *is either empty or a left ideal of* $\mathrm{A_l}(G)$.

(iv) $ax = aa \cdot x$ for all $a \in \widetilde{A}_l(G)$ *and* $x \in G$.

Proof. (i) It suffices to take into account that $\mathrm{L}_a\mathrm{L}_b = \mathrm{L}_{ab}\mathrm{L}_a = \mathrm{L}_{ab\cdot a}\mathrm{L}_{ab}$ for all $a, b \in G$.

(ii) and (iii). If $a, b \in \widetilde{\mathrm{A}}_\mathrm{l}(G)$, then $ab \cdot ab = a \cdot bb = a(bb \cdot b) = (ab \cdot ab)(ab)$, and so $ab \in \widetilde{\mathrm{A}}_\mathrm{l}(G)$ by (i). Similarly, if $a, b \in G$, $a \in \mathrm{A_l}(G)$ and $ab \in \widetilde{\mathrm{A}}_\mathrm{l}(G)$, then $a \cdot bb = (ab \cdot ab)(ab) = a(bb \cdot b)$ implies $b \in \widetilde{\mathrm{A}}_\mathrm{l}(G)$.

(iv) $aa \cdot ax = (aa \cdot a)(aa \cdot x) = (aa)(aa \cdot x)$, and so $ax = aa \cdot x$. ▲

2.2 Proposition. *Let G be a left distributive groupoid. Then:*

(i) If G is left cancellative and satisfies the equation $xx \triangleq xx \cdot x$, then the monoid $\mathrm{Mul_l}(G)$ *is cancellative.*

(ii) If $\mathrm{Mul_l}(G)$ *is right cancellative, then G satisfies $xx \triangleq xx \cdot x$, the equivalence p_G (see I.1.11) is a congruence of G and the factor G/p_G is idempotent.*

(iii) If $\mathrm{Mul_l}(G)$ *is cancellative, then the factor G/p_G is left cancellative.*

Proof. (i) Since G is left cancellative, the monoid $\mathrm{Mul_l}(G)$ is left cancellative as well. Let $f, g, h \in \mathrm{Mul_l}(G)$ be such that $fh = gh$. Proceeding by induction on $n \geq 0$, $h = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_n}$, we are going to show that $f = g$. The result is clear for $n = 0$, and hence let $n \geq 1$, $k = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_{n-1}}$ and $a = a_n$. We have $fk(ax) = gk(ax)$ for each $x \in G$. Consequently, $b = fk(aa) = gk(aa)$ and $bfk(x) = fk(aa \cdot x) = fk(ax) = gk(ax) = bgk(x)$. Thus $fk(x) = gk(x)$, $fk = gk$ and, finally, $f = g$ by induction.

(ii) We have $\mathrm{L}_a\mathrm{L}_a = \mathrm{L}_{aa}\mathrm{L}_a$ for every $a \in G$, and so $\mathrm{L}_a = \mathrm{L}_{aa}$. Further, if $(a, b) \in p_G$ and $c \in G$, then $\mathrm{L}_{ca}\mathrm{L}_c = \mathrm{L}_c\mathrm{L}_a = \mathrm{L}_c\mathrm{L}_b = \mathrm{L}_{cb}\mathrm{L}_c$, so that $\mathrm{L}_{ca} = \mathrm{L}_{cb}$ and $(ca, cb) \in p_G$. Since $(ac, bc) \in p_G$ by the definition of p_G, we see that p_G is a congruence of G.

(iii) If $(ca, cb) \in p_G$, then $\mathrm{L}_c\mathrm{L}_a = \mathrm{L}_{ca}\mathrm{L}_c = \mathrm{L}_c\mathrm{L}_b$, and therefore $\mathrm{L}_a = \mathrm{L}_b$. ▲

2.3 Theorem. *Let G be left cancellative left distributive groupoid. Then G can be imbedded into a left distributive left quasigroup if and only if G satisfies the equation $xx \triangleq xx \cdot x$.*

Proof. (i) Let Q be a left distributive left quasigroup. If $a, b \in Q$, then $b = ac$ for some $c \in Q$ and $ab = a \cdot ac = aa \cdot ac = aa \cdot b$. Thus Q (and so every subgroupoid of Q) satisfies $xy \triangleq xx \cdot y$.

(ii) Let K be the quasivariety of left cancellative left distributive groupoids satisfying $xx \triangleq xx \cdot y$, see 2.1 (iv)). By (i), every left distributive left quasigroup is in K. Clearly, K is an abstract class and K satisfies (UCH) (see III.1.1). We show that K satisfies also (LDE).

Let $G \in K$ and $a, b \in G$. Put $\varphi = \mathrm{L}_{a,G}$, so that φ is an injective endomorphism of G and $\varphi(a)\, b = aa \cdot b = ab = \varphi(b)$.

We have checked both (UCH) and (LDE) and the result now follows from III.1.2. ▲

2.4 Theorem. *Let G be a left cancellative left distributive groupoid satisfying $xx \triangleq xx \cdot x$. Then there exists a left distributive left quasigroup Q such that:*

(i) G is a left strongly dense subgroupoid of Q (i.e., $P = Q$ whenever P is a left subgroupoid of Q, $G \subseteq P$).

(ii) For every $x \in Q$ there are $n \geq 0$ and $a_1, \dots, a_n \in G$ such that $a_1(\dots(a_n x)) \in G$.

(iii) Every (left) cancellative congruence of G can uniquely be extended to a (left) cancellative congruence of Q.

(iv) If G is cancellative, then Q is cancellative.

(v) If G is right divisible, then Q is right divisible.

(vi) If G is a right quasigroup, then Q is a right quasigroup.

(vii) The groupouds G and Q are equationally equivalent.

(viii) The imbedding $G \hookrightarrow Q$ is a reflexion of G in the class of left distributive left quasigroups.

Proof. Combine 2.3, 1.7 and II.3.7(iii). ▲

2.5 Example. ([4]) Let $\mathscr{I}$ denote the set of injective transformations f of the set $\mathbb{N}$ of positive integers such that $f(\mathbb{N}) \neq \mathbb{N}$. Define an operation $*$ on $\mathscr{I}$ by $f * g = h$, where $h(i) = fgf^{-1}(i)$ for $i \in f(\mathbb{N})$ and $h(i) = i$ for $i \in \mathbb{N} \setminus f(\mathbb{N})$. Then $\mathscr{I}(*)$ is a left cancellative left distributive groupoid and $f * f \neq (f * f) * f$ for every $f \in \mathscr{I}$.

2.6 Remark. Let $\mathscr{V}$ denote the variety of left distributive groupoids, $\mathscr{L}$ the class of left cancellative left distributive groupoids, $\mathscr{G}$ the class of left distributive left quasigroup and $\mathscr{K}$ the class of subgroupoids of left quasigroups from $\mathscr{G}$. Then $\mathscr{G} \subseteq \mathscr{K} \subseteq \mathscr{L} \subseteq \mathscr{V}$, $\mathscr{K} = \{G \in \mathscr{L};\ xx \triangleq xx \cdot x \text{ is true in } G\}$ and $\mathscr{G} \neq \mathscr{K} \neq \mathscr{L} \neq \mathscr{V}$ (see 2.5).

2.7 Remark. Let G be a left distributive groupoid and $a \in \widetilde{\mathrm{A}}_1(G)$. According to the proof of 2.3, there exists a left distributive groupoid $\mu_a(G) = H$ and an element $b \in H$ such that G is a subgroupoid of H, $b \in \widetilde{\mathrm{A}}_1(H)$, $a = bb$ and $G = aH = bH$. The mapping $\mathrm{L}_{a,H} = \mathrm{L}_{b,H}$ is an isomorphism of H onto G, G is left strongly dense in H and $\mathrm{A}_1(G) \subseteq \mathrm{A}_1(H)$, $\widetilde{\mathrm{A}}_1(G) \subseteq \widetilde{\mathrm{A}}_1(H)$.

New, we get a chain $G \subseteq G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$, where $G_{i+1} = \mu_a(G_i)$, and there are elements $a_i \in G_i$ such that $a_0 = a$, $a_{i+1}^2 = a_i$ and $G_i = a_{i+1} G_{i+1} = a G_{i+1}$. Thus $\mathrm{L}^i_{a,G_i}(G_i) = G_0 = G$. All the groupoids G_i are isomorphic to G.

Put $K = \bigcup_{i=0}^{\infty} G_i$. Then G is left strongly dense subgroupoid of K, $K = aK$, $\mathrm{A}_l(G) \subseteq \mathrm{A}_l(K)$ and $\widetilde{\mathrm{A}}_l(G) \subseteq \widetilde{\mathrm{A}}_l(K)$.

Using the operator μ_a and standard transfinite construction, we imbed G as a left strongly dense subgroupoid into a left distributive groupoid P such that $\widetilde{\mathrm{A}}_l(G) \subseteq \mathrm{C}_l(P) = \widetilde{\mathrm{A}}_l(P)$.

2.8 Remark. Let G be a left cancellative left distributive groupoid satisfying $xx \triangleq xx \cdot x$ and suppose that G is countably lc-generated, i.e., there exists a subset S of G such that $G = \langle S \rangle_{\mathrm{lc},G}$ and $1 \leq \mathrm{card}(S) \leq \aleph_0$.

There is a bijective mapping $f : S \times \mathbb{N} \to \mathbb{N}$, $\mathbb{N}$ being the set of positive integers, and $f^{-1}(i) = (g(i), h(i))$, $g(i) \in S$, $h(i) \in \mathbb{N}$, $i \in \mathbb{N}$. Now, consider the chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots$, where $G_{i+1} = \mu_{g(i)}(G_i)$ for each $i \geq 0$ (see 2.7) and put $Q = \bigcup_{i=0}^{\infty} G_i$. Then each of the groupoids G_i is isomorphic to G, Q is a left cancellative left distributive groupoid and G is a left strongly dense subgroupoid of Q. We are going to show that Q is a left quasigroup (cf. 2.4).

Let $a \in S$ and $x \in Q$. Then $x \in G_i$ for some $i \geq 0$ and, of course, there is $j \geq 1$ sich that $f(a, j) = k \geq i$. We have $g(k) = a$, and so $aG_{k+1} = G_k$. But $x \in G_k$, $ay = x$ for some $y \in G_{k+1}$ and we have checked that $a \in \mathrm{C}_l(Q)$. Now, $S \subseteq \mathrm{C}_l(Q)$ and $G = \langle S \rangle_{\mathrm{lc},G} \subseteq \langle S \rangle_{\mathrm{lc},Q} \subseteq \mathrm{C}_l(Q)$, since $\mathrm{C}_l(Q)$ is a left closed subgroupoid of Q (see the proof of 2.1). On the other hand, G is left strongly dense in Q. Thus $Q = \mathrm{C}_l(Q)$ and it means that Q is a left quasigroup.

2.9 Remark. Let G be a left strongly dense subgroupoid of a left distributive left quasigroup Q and let $n \geq 1$ and $x_1, \ldots, x_n \in Q$. Then $f(x_1), \ldots, f(x_n) \in G$ for some $f \in \mathrm{Mul}_l(Q, G)$ and we put $K = f^{-1}(G) = \{x \in; f(x) \in G\}$. Then K is a subgroupoid of Q, $G \subseteq K$ and $x_1, \ldots, x_n \in K$; hence $\langle G, x_1, \ldots, x_n \rangle_Q \subseteq K$. We have $f(K) = G$ (since Q is a left quasigroup), and so $K \cong G$. In particular, we have shown that every finitely generated subgroupoid of Q is isomorphic to a subgroupoid of G. Consequently, by III.1.6, Q can be imbedded into a filtered product of (finitely generated) subgroupoids of G.

2.10 Remark. Let G be a left cancellative left distributive groupoid. Considering finitely generated subgroupoids of G and combining 2.8 and III.1.6 we get another proof of 2.3.

IV.3 Imbeddings of cancellative left distributive groupoids into divisible left distributive left quasigroups

3.1 Theorem. *Let G be a cancellative left distributive groupoid. Then G is a dense subgroupoid of a left distributive left quasigroup Q such that Q is a divisible groupoid.*

Proof. Since G is right cancellative, G is idempotent $(x \cdot xx = xx \cdot xx)$, $p_G = \mathrm{id}_G$ and, in view of 2.4, we can assume that G is already a left quasigroup. Now, denote by R the set of rational numbers of the form $n/2^m$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}$, $m \geq 0$, and define an operation $\circ$ on R by $x \circ y = (x + y)/2$. Obviously, $R(\circ)$ is a commutative idempotent distributive quasigroup, and so $P = G \times R(\circ)$ is an idempotent left distributive groupoid and a left quasigroup. Furthermore, $p_P = \mathrm{id}_P$ and $\mathrm{L}^k_{x,P} \neq \mathrm{id}_P$ for all $x \in P$ and $k \geq 1$.

Let $\mathscr{G}$ be the permutation group generated by all $\mathrm{L}_{x,P}$, $x \in P$, and let $\varphi(x) = \mathrm{L}_{x,P} \in \mathscr{G}$ for every $x \in P$. Define an operation $*$ on $\mathscr{G}$ by $f * g = fgf^{-1}$ for all $f, g \in \mathscr{G}$. Then $\mathscr{G}(*)$ is a left distributive left quasigroup and $\varphi : P \to \mathscr{G}(*)$ is an injective (groupoid) homomorphism. Now, $\mathscr{G}$ is a subgroup of a simple group $\mathscr{H}$ such that any two elements of the same order are conjugate in $\mathscr{H}$. Consequently, $\varphi(P) = \mathscr{K}(*)$ is a subgroupoid on $\mathscr{H}(*)$ and any two elements from $\mathscr{K}$ are conjugate in $\mathscr{H}$; in particular $\mathscr{K} \subseteq \mathscr{H}(*) * f$ for every $f \in \mathscr{K}$. Further, since $\mathscr{H}$ is simple, we have $p_{\mathscr{H}(*)} = \mathrm{id}_{\mathscr{H}}$.

We have proved that G is a subgroupoid of an idempotent left distributive left quasigroup H such that $p_H = \mathrm{id}_H$ and $G \subseteq Hx$ for every $x \in G$. Now, we put $G_0 = G$, $G_1 \subseteq H$ and $Q = \bigcup_{i=0}^{\infty} G_i$. ▲

IV.4 Dense subgroupoids of distributive groupoids

4.1 Let G be a distributive groupoid. Then $\mathrm{Mul}(G) \subseteq \mathrm{End}(G)$.

4.2 Lemma. *Let H be a subgroupoid of a distributive groupoid G. Then:*
(i) For all $f, g \in \mathrm{Mul}(G, H)$ there exists $h \in \mathrm{Mul}_l(G, H)$ such that $fg = hf$.
(ii) $\mathrm{Mul}(G, H) = \mathrm{Mul}_l(G, H) \cdot \mathrm{Mul}_r(G, H) = \mathrm{Mul}_r(G, H) \cdot \mathrm{Mul}(G, H)$.

Proof. (i) Use the fact that $\mathrm{Mul}(G) \subseteq \mathrm{End}(G)$.
(ii) Apply 1.2(ii) and its dual. ▲

4.3 Corollary. *Let G be a distributive groupoid. Then:*
(i) For all $f, g \in \mathrm{Mul}(G)$ there exists $h \in \mathrm{Mul}(G)$ such that $fg = hf$.
(ii) $\mathrm{Mul}(G) = \mathrm{Mul}_l(G) \cdot \mathrm{Mul}_r(G) = \mathrm{Mul}_r(G) \cdot \mathrm{Mul}_l(G)$.

4.4 Proposition. *Let H be a subgroupoid of a distributive groupoid G. Then:*
(i) The conditions (A1), ..., (A11), (B1), ..., (B11) and (C1), ..., (C7) are satisfied for H in G (see I.4.1, I.4.5 and I.5.1).
(ii) $\langle H \rangle_{c,G} = [H]_G = \{x \in G;\ f(x) \in H$ *for some* $f \in \mathrm{Mul}(G, H)\} = \gamma_G(H)$.
(iii) $\gamma_G(H) \subseteq \alpha_G(\beta_G(H)) \cap \beta_G(\alpha_G(H))$.

Proof. We can proceed similarly as in 1.4 (the conditions (A10) and (B10) follow from [11, Theorem IV.2.2]).

4.5 Theorem. *Let H be a dense subgroupoid of a distributive groupoid G. Then:*
(i) For every $x \in G$ there exists $f \in \mathrm{Mul}(G, H)$ such that $f(x) \in H$.

(ii) Every cancellative congruence of H can uniquely be extended to a cancellative congruence of G.

(iii) If G is cancellative, then the groupoids H and G are equationally equivalent.

Proof. Using 4.4, we can proceed similarly as in the proof of 1.7. ▲

IV.5 Imbeddings of cancellative distributive groupoids into distributive quasigroups

5.1 Proposition. *Let G be a distributive groupoid. Then:*

(i) If $A_l(G) \neq \emptyset (A_r(G) \neq \emptyset)$, *then* $A_l(G)\,(A_r(G))$ *is a left (right) closed sub-groupoid of G.*

(ii) If $A_l(G) \neq \emptyset \neq A_r(G)$ *(e.g.,* $A(G) \neq \emptyset$*), then* $A_l(G) = A_r(G) = A(G)$ *is a closed subgroupoid of G.*

(iii) If $C_l(G) \neq \emptyset\ (C_r(G) = \emptyset)$, *then* $C_l(G)\,(C_r(G))$ *is a left (rihgt closed sub-groupoid of G.*

(iv) If $C_l(G) \neq \emptyset \neq C_r(G)$, *then* $C_l(G) = C_r(G) = C(G)$ *is a closed subgroupoid of G.*

Proof. (i) We have $L_aL_b = L_{ab}L_a = L_{ab\cdot a}L_{ab}$, $L_aR_b = R_{ab}L_a = L_{a\cdot ab}R_{ab}$, $R_aR_b = R_{ba}R_a = R_{a\cdot ba}R_{ba}$, $R_aL_b = L_{ba}R_a = L_{ba\cdot a}L_{ba}$ for all $a, b \in G$. Now, it follows easily that $A_l(G)\ (A_r(G))$ is either empty or a left (right) closed subgroupoid of G.

(ii) Let $a \in A_l(G)$ and $b \in A_r(G)$. Since $L_aR_b = L_{a\cdot ab}R_{ab}$, R_{ab} is injective and $ab \in A_r(G)$. But $A_r(G)$ is right closed, and hence $a \in A_r(G)$. Thus $A_l(G) \subseteq A_r(G)$ and, quite similarly, $A_r(G) \subseteq A_l(G)$.

(iii) and (iv). We can proceed similarly as above. ▲

5.2 Proposition. *Let G be a cancellative distributive groupoid. Then G is idempotent and the moniod* $\mathrm{Mul}(G)$ *is cancellative.*

Proof. First, $a \cdot aa = aa \cdot aa$, and therefore $a = aa$ for each $a \in G$. Further, the transformations from $\mathrm{Mul}(G)$ are injective and it follows that $\mathrm{Mul}(G)$ is left cancellative.

Let $f, g, h \in \mathrm{Mul}(G)$, $fh = gh$. Then $h = T_{1,a_1} \cdot T_{n,a_n}$, $n \geq 0$, $a_1 \in G$, $T_i \in \{L, R\}$ and we proceed by induction on n. Nothing has to be proved for $n = 0$, and so let $n \geq 1$, $k = T_{1,a_1} \cdot T_{n-1,a_{n-1}}$, $a = a_n$; we assume $T_n = L$, the other case being similar. Now, $fk(ax) = gk(ax)$ for every $x \in G$, $fk(a) = gk(a)$, $fk = gk$ and $f = g$ by induction. ▲

5.3 Theorem. *Every cancellative distributive groupoid can be imbedded into a distributive quasigroup.*

Proof. Let $\mathscr{K}$ designate the class of cancellative distributive groupoids. Then $\mathscr{K}$ is an abstract class satisfying (UCH) (see III.1.1) and we show that $\mathscr{K}$ satisfies also (LDE) and (RDE).

Let $G \in \mathscr{K}$ and $a, b \in G$. Put $\varphi = \mathrm{L}_{a,G}$ and $\psi = \mathrm{R}_{a,G}$. Then φ, ψ are injective endomorphism of G and $\varphi(a)\, b = aa \cdot b = ab = \varphi(b)$, $b\psi(a) = b \cdot aa = ba = \psi(b)$.

Now, our results follow from III.1.4. ▲

5.4 Theorem. *Let G be a cancellative distributive groupoid. Then there exists a distributive quasigroup Q such that:*

(i) G is a dense subgroupoid of Q (i.e., $P = Q$ whenever P is a quasigroup of $Q, G \subseteq P$).

(ii) Every cancellative congruence of G can uniquely be extended to a cancellative congruence of Q.

(iii) The groupoids G and Q are equationally equivalent.

(iv) The imbedding $G \hookrightarrow Q$ is a reflexion of G in the class of distributive quasigroups.

Proof. Combine 5.3, 4.5 and II.4.4(iii). ▲

5.5 Remark. Let G be a distributive groupoid and $a \in \mathrm{A}(G)$. Then $aa \in \mathrm{A}(G)$ and $a \cdot aa = aa \cdot aa$ implies $a = aa$. The transformation $f = \mathrm{L}_{a,G}$ is an injective endomorphism of G and $f(a) = a$, $f(G) = a(G)$. Moreover, $G \cong f(G)$ and there exists a distributive groupoid $H = \mu_a(G)$ such that G is a subgroupoid of $H, G = aH$, $G \cong H$ and $L_{a,H}$ is an isomorphism of H onto G. Clearly, $\mathrm{A}(G) \subseteq \mathrm{A}(H)$.

Dually, there exists a distributive groupoid $K = \nu_a(G)$ such that G is a subgroupoid of K, $G = K_a$, $G \cong K, \mathrm{R}_{a,K}$ is an isomorphism of K onto G and $\mathrm{A}(G) \subseteq \mathrm{A}(K)$.

Now, we get the chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots$, where $G_{i+1} = \mu_a(G_1)$ for $i \geq 0$ even and $G_{i+1} = \nu_a(G_i)$ for $i \geq 1$ odd; $G_i \cong G$. Put $P = \bigcup_{i=0}^{\infty} G_i$. Then P is a distributive groupoid, G is a dense subgroupoid of P, $\mathrm{A}(G) \subseteq \mathrm{A}(P)$ and $a \in \mathrm{C}(P)$. Using the operators μ_a, ν_a and standard transfinite construction we can show that G is a dense subgroupoid of a distributive groupoid R such that $\mathrm{A}(G) \subseteq \mathrm{A}(R) = \mathrm{C}(R)$.

5.6 Remark. Let G be a countably c-generated distributive groupoid, i.e., $G = \langle S \rangle_{c,G}$ for a subset $S \subseteq G$, $1 \leq \mathrm{card}(S) \leq \aleph_0$.

There is a bijective mapping $f : S \times \mathbb{Z} \to \mathbb{N}$, $\mathbb{Z}$ being the set of integers and $\mathbb{N}$ that of positive integers. We have $f^{-1}(i) = (g(i), h(i))$, $g(i) \in S$, $h(i) \in \mathbb{Z}$ for every $i \in \mathbb{N}$. Now, consider the chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots$, where $G_{i+1} = \mu_{g(i)}(G_i)$ for $i \geq 0$ and $h(i) > 0$ and $G_{i+1} = \nu_{g(i)}(G_i)$ for $i \geq 0$ and $h(i) \leq 0$ (see 5.5); $G_i \triangleq G$. Put $Q = \bigcup_{i=0}^{\infty} G_i$. Then Q is a cancellative distributive groupoid and G is dense subgroupoid of H. Proceeding similarly as in 2.8, we can show, that $S \subseteq \mathrm{C}(Q)$. By 5.1, $\mathrm{C}(Q)$ is a closed subgroupoid of Q and $G = \langle S \rangle_{c,G} \subseteq \langle S \rangle_{c,H} \subseteq$

C(H). Since G is dense in Q, we have $Q = $ C(Q), i.e., Q is a quasigroup. As we have shown, Q is the union of countable chain of subgroupoids isomorphic to G.

5.7 Remark. Let G be a dense subgroupoid of a distributive quasigroup Q. If $x_1, \dots, x_n \in Q$, $n \geq 1$, then there exists a subgroupoid K of Q such that $\langle G, x_1, \dots, x_n \rangle_Q \subseteq K$ and $K \cong G$. Consequently, every finitely generated subgroupoid of Q is isomorphic to a subgroupoid of G and Q can be imbedded into a filtered product of (finitely) generated subgroupoid of G (see III.1.6).

5.8 Remark. Let G be a cancellative distributive groupoid. Considering finitely generated subgroupoids of G and combining 5.6 and III.1.6, we get another proof of 5.3.

5.9 Let G be a cancellative distributive groupoid. Define a relation μ_G on G by $(a, b) \in \mu_G$ if and only if $ab \cdot xy = ax \cdot by$ for all $x, y \in G$. By [11, § IV.3], μ_G is a cancellative congruence of G and, if $(a, b) \in \mu_G$, then the subgroupoid $\langle a, b, x, y \rangle_G$ is medial for all $x, y \in G$.

(i) Assume that G is a dense subgroupoid of a cancellative distributive groupoid H. Clearly, $\mu_H \restriction G \subseteq \mu_G$ and we are going to show that, in fact, $\mu_G = \mu_H \restriction G$. For, let $(a, b) \in \mu_G$ and $x, y \in H$. Then $f(x), f(y) \in G$ for some $f \in \mathrm{Mul}(G, H)$ and we have $f(a), f(b) \in \mu_G$. Thus the subgroupoid $\langle f(a), f(b), f(x), f(y) \rangle_G$ is medial and, in particular, $f(ab \cdot xy) = f(ax \cdot by)$.

(ii) Let φ be an endomorphism of H such that $\varphi(G) \subseteq G$ and $(a, \varphi(a)) \in \mu_G$ for each $a \in G$. Put $K = \{x \in H; (x, \varphi(x)) \in \mu_H\}$. Then $G \subseteq K$ and, since μ_H is a cancellative congruence of H, we see easily that K is a closed subgroupoid of H. Then $K = H$ and we have checked that $(x, \varphi(x)) \in \mu_H$ for each $x \in H$.

IV.6 Comments and open problems

The first proof of the basic Theorem 5.3 may be found in [1] (see also [28] and [11]). Theorem 2.3 was proved in [18] and Theorem 3.1 in [20]. The important Example 2.5 is taken from [4]. It remains an open question whether every cancellative left distributive groupoid can be imbedded into a left distributive quasigroup.

V. Cancellative semigroups (and also groupoids satisfying short linear equations)

V.1 Dense uniform subsemigroups

1.1 A semigroup S is said to be left (right) uniform if $Sa \cap Sb \neq \emptyset$ $(aS \cap bS \neq \emptyset)$ for all $a, b \in S$ and S is said to be (bi-)uniform if it is both left and right uniform.

1.2 Proposition. *Let S be a subsemigroup of a semigroup R. Then:*

(i) $\alpha_{1,R}(S) = \alpha_R(S)$ and $\beta_{1,R}(S) = \beta_R(S)$.

(ii) If S is left uniform, then $T = \alpha_R(S) = [S]_{1,R} = \langle S\rangle_{\mathrm{lc},R}$ and the semigroup T is again left uniform.

Proof. (i) The equalities follow immediately from the associative law.

(ii) Put $T = \alpha_R(S)$. If $x, y \in T$, $a, b, c, d \in S$, $ax = b$, $cy = d$, then $eb = fc$ for some $e, f \in S$ and we have $eaxy = eby = fcy = fd \in S$. Thus $xy \in T$ and T is a subsemigroup of R.

Now, let $u, v \in T$, $x \in R$, $ux = v, a, b, c, d \in S$ and $au = b$, $cv = d$. Again, $ea = fc$ for some $e, f \in S$ and we have $ebx = eaux = eav = fcv = fd \in S$ and $x \in T$. We have shown that T is left closed in R.

Finally, let $u, v \in T, a, b, c, d \in S$, $au = b$, $cv = d, e, f \in S$, $eb = fd$. Then $eau = eb = fd = fcv$, $Su \cap Sv \neq \emptyset$ and $Tu \cap Tv \neq \emptyset$. ▲

1.3 Proposition. *Let S be a subsemigroup of a right cancellative semigroup R. If S is left uniform, then $\alpha_R(S) = \langle S\rangle_{\mathrm{lc},R} = \langle S\rangle_{\mathrm{c},R}$* and *$\beta_R(S) \subseteq \gamma_R(S) = \alpha_R(S)$.*

Proof. With respect to 1.2, it is enough to show that $\alpha(S)$ is right closed in R. For, let $u, v \in \alpha(S)$, $x \in R$, $xu = v$. By the proof of 1.2, $au = bv$ for some $a, b \in S$ and we have $bxu = bv = au$ and $bx = a$. Thus $x \in \alpha(S)$. ▲

1.4 Corollary. *Let S be a subsemigroup of a cancellative semigroup R. If S is uniform, then $\alpha_{1,R}(S) = \beta_{1,R}(S) = \alpha_R(S) = \beta_R(S) = \gamma_R(S) = [S]_{1,R} = [S]_{\mathrm{r},R} = [S]_R = \langle S\rangle_{\mathrm{lc},R} = \langle S\rangle_{\mathrm{rc},R} = \langle S\rangle_{\mathrm{c},R} = T$ and the semigroup T is again uniform.*

1.5 Corollary. *Let S be a left strongly dense subsemigroup of a semigroup R. If S is left uniform, then R is left uniform and for every $x \in R$ there exists $a \in S$ such that $ax \in S$.*

1.6 Corollary. *Let S be a dense semigroup of a cancellative semigroup R. If S is uniform, then R is uniform and for every $x \in R$ there exist $a, b \in S$ such that $ax \in S$ and $xb \in S$.*

1.7 Proposition. *Let S be a subsemigroup of a group G such that S generates G as a group (i.e., $G = \langle S\rangle_{\mathrm{c},G}$). Then:*

(i) $S \subseteq \alpha_G(S) = \{a^{-1}b;\, a, b \in S\}$.

(ii) $S \subseteq \beta_G(S) = \{ab^{-1};\, a, b \in S\}$.

(iii) $G = \langle S\rangle_{\mathrm{lc},G} = \langle S\rangle_{\mathrm{rc},G} = \langle S\rangle_{\mathrm{c},G}$, *(i.e., S is strongly dense in G).*

1.8 Proposition. *Let S be a subsemigroup of a group G such that $\langle S\rangle_{\mathrm{c},G} = G$. The following conditions are equivalent:*

(i) $\alpha_G(S)\ (\beta_G(S))$ *is a subsemigroup of G.*

(ii) $\beta_G(S) \subseteq \alpha_G(S)\ (\alpha_G(S) \subseteq \beta_G(S))$.

(iii) S is left (right) uniform.

If these conditions are satisfied, then $G = \alpha_G(S)\ (G = \beta_G(S))$.

1.9 Example. ([3, § 1.10] and [29]) Let S be the set of ordered pairs (n, m) of non-negative integers together with a binary operation $(n, m) * (k, l) = (n + k, 2^n l + m)$. Then $S(*)$ is a cancellative semigroup possesing a neutral element and $S(*)$ is left uniform but not right uniform.

1.10 Example. Let F be a free semigroup of rank at least 2. Then F is cancellative and F can be imbedded into a group (in fact, a free group). On the other hand, F is neither left nor right uniform.

1.11 Example. Let F denote the free monoid of words over an eight-element alphabet $A = \{a, b, c, d, e, f, g, h\}$ and let S be the set of all the words $w \in F$ such that neither cd or cf or hf is a subword of w. Now, define a binary operation $*$ on S according to the following rules:

Let $u, v \in S$. If

(1) $uv \in S$, then $u * v = uv$;

(2) $u = u_1 c$ and $v = dv_1$, then $u * v = u_1 abv_1$;

(3) $u = u_1 c$ and $v = fv_1$, then $u * v = u_1 aev_1$;

(4) $u = u_1 h$ and $v = fv_1$, then $u * v = u_1 gev_1$.

It is easy to check that $S(*)$ is a cancellative monoid, $A \subseteq S$, $g * b \neq h * d$, $c * d = a * b$, $c * f = a * e$ and $h * f = g * e$. Consequently, if φ is a homomorphism of $S(*)$ into a group G, then the equalities $\varphi(c)\,\varphi(d) = \varphi(a)\,\varphi(b)$, $\varphi(c)\,\varphi(f) = \varphi(a)\,\varphi(e)$ and $\varphi(h)\,\varphi(f) = \varphi(g)\,\varphi(e)$ yield the equality $\varphi(g)\,\varphi(b) = \varphi(h)\,\varphi(d)$. In particular, $S(*)$ cannot be imbedded into a group.

V.2 The Ore's construction

2.1 Throughout this section, let S be a cancellative semigroup such that S is left uniform. We put $H = S \times S$ and $\varrho(a, b) = \{(u, v) \in H; ua = vb\}$ for all $(a, b) \in H$; since S is left uniform, $\varrho(a, b) \neq \emptyset$.

2.2 Lemma. *If $(a, b), (c, d) \in H$ are such that $\varrho(a, b) \cap \varrho(c, d) \neq \emptyset$, then $\varrho(a, b) = \varrho(c, d)$.*

Proof. There are $u, v \in S$ with $ua = vb$ and $uc = vd$. If $w, z \in S$, $wa = zb$, then $xu = yw$ for some $x, y \in S$ and we have $xvb = xua = ywa = yzb$, $xv = yz$, and $ywc = xuc = xvd = yzd$, $wc = zd$. Thus $\varrho(a, b) \subseteq \varrho(c, d)$ and, quite similarly, $\varrho(c, d) \subseteq \varrho(a, b)$. ▲

2.3 We define a relation $\sim$ on H by $(a, b) \sim (c, d)$ if and only if $\varrho(a, c) = \varrho(b, d)$ (or $\varrho(a, c) \cap \varrho(b, d) \neq \emptyset$, i.e., there exist $u, v \in S$ with $ua = vc$ and $ub = vd$).

2.4 Lemma. *The relation $\sim$ is an equivalence on H.*

Proof. (i) For each $x \in S$, $\varrho(x, x) = \{(y, y); y \in S\} = \mathrm{id}_S$. Thus, for all $a, b \in S$, $\varrho(a, a) = \varrho(b, b)$, which means $(a, b) \sim (a, b)$. It follows that $\sim$ is reflexive.

(ii) For all $a, b \in S$, $\varrho(b, a) = \{(v, u); (u, v) \in \varrho(a, b)\}$. Now, it is clear that $\sim$ is symmetric.

(iii) Let $(a, b) \sim (c, d) \sim (e, f)$ and let $ua = vc$, $ub = vd$, $rc = se$, $rd = sf$. Then $pv = qr$ for some p, q and we have $pua = pvc = qse$, $pub = pvd = qrd = qsf$. Thus $\varrho(a, e) \cap \varrho(b, f) \neq \emptyset$ and $(a, b) \sim (e, f)$. We have shown that $\sim$ is transitive. ▲

2.5 Lemma. *(i)* $(a, a) \sim (b, b)$ *for all* $a, b \in S$.
(ii) $(a, a^2) \sim (b, b^2)$ *for and only if* $a = b$.

2.6 Define a binary multioperation $*$ on H by $(a, b) * (c, d) = \varrho(b, c) \cdot (a, b) = \{(ua, vd); ub = vc\}$.

2.7 Lemma. *If* $(a, b), (c, d) \in H$, *then the set* $(a, b) * (c, d)$ *is contained in one of the blocks of* $\sim$.

Proof. Let $ub = vc$, $wb = zc$. If $ru = sw$, then $rvc = rub = swb = szc$ and $rv = sz$. Consequently, $(ua, vd) \sim (wa, zd)$. ▲

2.8 Lemma. *Let* $(a, b), (c, d), (e, f) \in H$. *Then the set* $(a, b) * ((c, d) * (e, f)) \cup ((a, b) * (c, d) * (e, f)$ *is contained one of the blocks of* $\sim$.

Proof. We have $(a, b) * (c, d) = \{(ua, vd); ub = vc\}$, $(c, d) * (e, f) = \{(xc, yf); xd = ye\}$, $(ua, vd) * (e, f) = \{(wua, zf); wvd = ze\}$, $(a, b) * (xc, yf) = \{(pa, qyf); pb = qxc\}$. Further, $rz = sqy$, $\alpha x = \beta wv$ and $\gamma r = \delta\beta$ for some $r, s, \alpha, \beta, \gamma, \delta \in S$. Now, $\alpha ye = \alpha xd = \beta wvd = \beta ze$, and so $\alpha y = \beta z$. Consequently, $\delta\alpha y = \delta\beta z = \gamma rz = \gamma sqy$ and $\delta\alpha = \gamma sq$. Finally, $\gamma spb = \gamma sqxc = \delta\alpha xc = \delta\beta wvc = \delta\beta wub = \gamma rwub$ and $sp = rwu$. Thus $\varrho(z, qy) \cap \varrho(wu, p) \neq \emptyset$. But then also $\varrho(zf, qyf) \cap \varrho(wua, pa) \neq \emptyset$ and it follows $(wua, zf) \sim (pa, qyf)$. ▲

2.9 Lemma. *Let* $(a, b), (c, d), (e, f) \in H$, $(a, b) \sim (c, d)$. *Then the set* $((a, b) * (e, f) \cup ((c, d) * (e, f))$ *is contained in one of the blocks of* $\sim$.

Proof. Let $au = vc$, $ub = vd$, $rb = se$ and $pd = qe$. We have to show that $\varrho(ra, pc) \cap \varrho(sf, qf) \neq \emptyset$. For this purpose, let $xs = yq$ and $wxr = zu$. Then $zvd = zub = wxrb = wxse = wyqe = wypd$, and so $zv = wyp$ and $wypc = zvc = zua = wxza$ and $ypc = xza$. ▲

2.10 Lemma. *Let* $(a, b), (c, d), (e, f) \in H, (a, b) \sim (c, d)$. *Then the set* $((e, f) * (a, b)) \cup ((e, f) * (c, d))$ *is contained in one of the blocks of* $\sim$.

Proof. Let $ua = vc$, $ub = vd$, $rf = sa$, $pf = qc$, $xr = yp$ and $wxs = zu$. Then $zvc = zua = wxsa = wxrf = wypf = wyqc$, $zv = wyq$, $wyqd = zvd = zub = wxsb$, $yqd = xsb$ and $(x, y) \in \varrho(re, pe) \cap \varrho(sb, qd)$. ▲

2.11 Lemma. *Let* $a, b, c \in S$. *Then* $((a, b) * (c, c)) \cup ((c, c) * (a, b))$ *is a subset of the block of* $\sim$ *that contains* (a, b).

Proof. First, let $ub = vc$ and $pua = qa$. Then $pu = q$, $pvc = pub = qb$ and $(ua, vc) \sim (ab)$; $(ua, vc) \in (a, b) * (c, c)$.

Now, let $rc = sa$ and $xsb = yb$. Then $xs = y$, $xrc = xsa = ya$ and $(rc, sb) \sim (a, b)$; $(rc, sb) \in (c, c) * (a, b)$. ▲

2.12 Lemma. *Let $a, b, c \in S$. Then $(a, b) * (b, a)$ is a subset of the block of $\sim$ that contains (c, c).*

2.13 Lemma. *Let $a, b \in S$. Then the set $((a, a^2) * (b, b^2))$ is a subset of the block of $\sim$ that contains $(ab, (ab)^2)$.*

Proof. Let $ua^2 = vb$, $rvb^2 = s(ab)^2$. Then $rua^2 = rvb = saba$, $rua = sab$ and $(r, s) \in \varrho(ua, ab) \cap \varrho(vb^2, (ab)^2)$. ▲

2.14. Lemma. *Let $a, b \in S$. Then the set $(a^2, a) * (b, b^2)$ is a subset of the block of $\sim$ that contains (a, b).*

Proof. Let $ua = vb$ and $ra = sua^2$. Then $r = sua$ and $rb = suab = svb^2$. Thus $(r, s) \in \varrho(a, ua^2) \cap \varrho(b, vb^2)$ and $(a, b) \sim (ua^2, vb^2)$. ▲

2.15 It follows from 2.9 and 2.10 that the equivalence $\sim$ is a congruence of the multigroupoid $H(*)$. Now, denote by G the corresponding factor-multigroupoid, the (multi)operation of G being denoted multiplicatively. Let $\pi : H(*) \to G$ be the natural projection. With respect to 2.7, the operation of G is single-valued, so that G becomes a groupoid; by 2.8, G is a semigroup and, by 2.5(i) and 2.11, G is a monoid. In fact, by 2.11, G is a group. Now, define a mapping $\varphi : S \to G$ by $\varphi(a) = \pi(a, a^2)$ for each $a \in S$. Then, by 2.5(ii) and 2.13, φ is an injective homomorphism. Moreover, by 2.12 and 2.14, $G = \alpha_{1,G}(\varphi(S))$.

Finally, let ψ be an endomorphism of S. Define a transformation ς of H by $\varsigma(a, b) = (\psi(a), \psi(b))$. It is easy to see that, for all $(a, b), (c, d) \in H$, the intersection $\varsigma((a, b) * (c, d)) \cap (a, b) * \varsigma(c, d))$ is non-empty and that $(a, b) \sim (c, d)$ implies $\varsigma(a, b) \sim \varsigma(c, d)$. Consequently, ς induces an endomorphism ξ of G. If $a \in S$, then $\xi\varphi(a) = \xi\pi(a, a^2) = \varphi(\psi(a), \psi(a)^2) = \varphi\psi(a)$, i.e., $\xi\varphi = \varphi\psi$.

2.16 Theorem. *The following conditions are equivalent for a semigroup S:*

(i) S is cancellative and (left, right) uniform.

(ii) There exists a group G such that S is a subsemigroup of G and $(G = \alpha_G(S)$, $G = \beta_G(S))$ $G = \alpha_G(S) = \beta_G(S)$.

Proof. Combine 1.8 and 2.15. ▲

V.3 Cancellative permutable groupoids

3.1 A groupoid G is said to be
- left permutable if it satisfies the equation $x \cdot yz \triangleq y \cdot xz$;
- right permutable if it satisfies the equation $zy \cdot x \triangleq zx \cdot y$;
- permutable (or bi-permutable) if it is both left and right permutable.

3.2 Let G be a left permutable groupoid. Then the monoid $\mathrm{Mul}_l(G)$ is commutative and, if G is left cancellative, then $\mathrm{Mul}_l(G)$ is cancellative.

3.3 Theorem. *Let G be a left cancellative left permutable groupoid. Then there exists a left permutable left quasigroup Q such that G is a subgroupoid of Q and for every $x \in Q$ there exists $f \in \mathrm{Mul}_l(Q, G)$ with $f(x) \in G$ (so that G is left strongly dense in Q).*

Proof. To avoid confusion, we denote the operation of the groupoid G by $*$. Now, let $S = \mathrm{Mul}_l(G(*))$. By 3.2, S is a cancellative commutative monoid and we denote by T the (abelian) group of fractions of S. Further, we can define a structure of a cancellative S-semimodule on G by setting $ax = a(x)$ for all $a \in S$ and $x \in G$ and we also put $\lambda(x) = \mathrm{L}_{x, G(*)} \in S$ for each $x \in G$. Then λ is a mapping of G into S and $x * y = \lambda(x)\, y$ for all $x, y \in G$.

Now, define a relation r on $S \times G$ by $((a, x), (b, y)) \in r$ if and only if $ay = bx$. Then r is on equivalence on $S \times G$ and we denote by Q the corresponding factor-set $(S \times G)/r$.

If $((a, x), (b, y)) \in r$ and $c, d \in S$, then $ay = bx$, $dacy = dbcx$ and $((da, cx), (db, cy)) \in r$.

If $c, d, e, f \in S$, $cd^{-1} = ef^{-1}$, $a \in S$ and $x \in G$, then $cf = de$, $daex = facx$ and $((da, cx), (fa, ex)) \in r$.

Using the above observations, we can define a T-scalar multiplication on Q by $cd^{-1}(a, x)/r = (da, cx)/r$ and so Q becomes a T-module. If $a, b \in S$ and $x \in G$, then $((a, ax), (b, bx)) \in r$ and we put $\varphi(x) = (a, axd)/r$. For $c \in S$, we have $\varphi(cx) = (acx)/r = c(a, ax)/r = c\varphi(x)$. Thus φ is an S-semimodule homomorphism of G into Q. If $x, y \in G$ are such that $\varphi(x) = \varphi(y)$, then $((a, ax), (a, ay)) \in r$, so that $a^2x = a^2y$ and $x = y$. This means that φ is injective and we will identify G and $\varphi(G)$.

Let $\varrho : Q \to T$ be a mapping such that $\varrho \restriction G = \lambda$. Define a binary operation $\circ$ on Q by $u \circ v = \varrho(u)\, v$ for all $u, v \in Q$. We immediately see that $G(*)$ is a subgroupoid of $Q(\circ)$ and we have $u \circ (v \circ w) = \varrho(u)\, \varrho(v)\, w = \varrho(v)\, \varrho(u)\, w = v \circ (u \circ w)$, so that $Q(\circ)$ is left permutable. Since T is a group, $Q(\circ)$ is a left quasigroup

If $u \in Q$, then $y = au \in G$ for some $a \in S$, $u = a^{-1}y$. On the other hand, $a = \lambda(x_1) \dots \lambda(x_n)$ for some $n \geq 0$ and $x_1, \dots, x_n \in G$. Consequently, $y = au = \lambda(x_1) \dots \lambda(x_n)\, u = x_1 \circ (x_2 \circ (\dots (x_n \circ u)))$. ▲

3.4 Example. Define a permutation p on $\mathbb{Z}(+)$ (the additive group of integers) by $p(n) = n + 1$ for $n \leq -2, p(-1) = 1$, $p(n) = n + 2$ for $n \geq 1$ odd and $p(n) = n$ for $n \geq 0$ even. Further, put $x * y = p(x) + y$ for all $x, y \in \mathbb{Z}$. Then $Q = \mathbb{Z}(*)$ is a left permutable left quasigroup and the set G of non-negative even numbers is a subgroupoid of Q; G is a left cancellative left permutable groupoid. Clearly, $\alpha_Q(G) = \alpha_{1,Q}(G)$ is just the set of even integers. We have $-2 \in \alpha_Q(G)$, $0 \in G$ and $-2 * 0 = -1 \notin \alpha_Q(G)$, and so $\alpha_Q(G)$ is not a subgroupoid of Q. On the other hand, $\alpha_Q(G) = \alpha_{1,Q}(G) \neq \alpha_{1,Q}^2(G) = [G]_{l,Q} = \langle G \rangle_{lc,Q} = Q$, the condition (A2) is satisfied and the condition (A1) not (see I.4.1). Notice also that $\mathrm{Mul}_l(Q) \cong \mathbb{Z}(+)$, $\mathrm{Mul}_l(Q, G) \cong G(+)$ and that both (A7) and (A8) arte satisfied.

3.5 Remark. Let $\mathcal{K}$ denote the class of left cancellative left permutable groupoids and $\mathcal{G}$ the class of left quasigroups from $\mathcal{K}$. By 3.4, there exists $Q \in \mathcal{G}$ and a left strongly dense subgroupoid G of Q such that $Q \neq \alpha_Q(G)$. By 3.3, G is a subgroupoid of $P \in \mathcal{G}$ such that $P = \alpha_P(G)$. Finally, let $G \hookrightarrow R$ be a reflexion of G in $\mathcal{G}$ (G is left strongly dense in R). Then there are projective homomorphism $\varphi : R \to Q$ and $\psi : R \to P$ extending id_G. Of course, $\varphi(\alpha_R)(G)) \subseteq \alpha_Q(G) \neq Q$, and therefore $\alpha_R(G) \neq R$. From this, it follows that the imbedding $G \hookrightarrow P$ is not a reflexion of G in $\mathcal{G}$. Finally, ψ is not an isomorphism and $\ker(\psi)$, id_R are two different left cancellative congruences of R, each of them extending id_G.

3.6 Remark. (i) Let $G(*)$ be a left permutable left quasigroup. The group S generated by L_x, $x \in G$, is abelian and G is an S-module. We have $x * y = \lambda(x)\, y$ for all $x, y \in G$, $\lambda(x) = \mathrm{L}_x \in S$.

(ii) Let S be an abelian group, G an S-module, $\lambda : G \to S$ a mapping and $x * y = \lambda(x)\, y$ for all $x, y \in G$. Then $G(*)$ is a left permutable left quasigroup.

(iii) Let $G(+)$ be an abelian group and t a transformation of G. Setting $xy = t(x) + y$ for all $x, y \in G$, we get a left permutable left quasigroup G; G is right cancellative (right divisible) if and only if t is injective (projective).

3.7 Remark. Let G be a left permutable left quasigroup, $a \in G$, $f = \mathrm{L}_a$ and $x * y = f^{-1}(xy)$ for all $x, y \in G$. Then $G(*)$ is a left permutable left quasigroup, a is a left neutral element of $G(*)$ and $xy = f(x * y) = x * f(y)$ for all $x, y \in G$.

3.8 Lemma. *Let G be a left permutable groupoid.*

(i) If $a, b \in G$, $ab = b$ and R_b is projective, then a is a left neutral element of G.

(ii) If G is right divisible (right quasigroup), then G is divisible (quasigroup) and G contains at least one left neutral element.

Proof. (i) We have $a \cdot xb = x \cdot ab = sb$.

(ii) We have $\mathrm{L}_x\mathrm{R}_y = \mathrm{R}_{xy}$ for all $x, y \in G$. ▲

3.9 Proposition. *The following conditions are equivalent for a groupoid G:*

(i) G is left permutable and there exist elements $a, b \in G$ such that $ab = b$ and both R_a and R_b are projective.

(ii) G is left permutable and there exists a left neutral element $e \in G$ such that R_e is projective.

(iii) There exist a commutative semigroup $G(+)$ with a neutral element 0 and a projective transformation f of G such that $f(0) = 0$ and $xy = f(x) + y$ for all $x, y \in G$.

Proof. (i) implies (ii). See 3.8(i).

(ii) implies (iii). There is a transformation g of G such that $g(e) = e$ and $g(x)\, e = x$ for every $c \in G$. Put $x + y = g(x)\, y$ for all $x, y \in G$ and $0 = e$. Then $x + 0 = g(x)\, e = x$, $0 + x = g(e)\, x = ex = x$ and 0 is a neutral element of

$G(+)$. Further, one may check easily that $G(+)$ is again left permutable. Consequently, $x + y = x + (y + 0) = y + (x + 0) = y + x$ and we see that $G(+)$ is a commutative semigroup. Moreover, $g(xe) \cdot ye = y(g(xe)\, e) = y \cdot xe = x \cdot ye$ for all $x, y \in G$. Thus, for $f = R_{1,G}$, we have $f(x) + y = g(xe)\, y = g(xe) \cdot g(y)\, e = x \cdot g(y)\, e = xy$ for all $x, y \in G$.

(iii) implies (i). Clearly, G is left permutable and it remains to put $a = 0 = b$. ▲

3.10 Proposition. *The following conditions are equivalent for a groupoid G:*

(i) G is a right divisible left permutable groupoid.

(ii) G is left permutable left quasigroup and G is divisible.

(iii) There exist an abelian $G(+)$ and a projective transformation f of G such that $f(0) = 0$ and $xy = f(x) + y$ for all $x, y \in G$.

Proof. (i) implies (ii). The condition 3.9(i) is satisfied, and hence, by 3.9(iii), there exist a commutative semigroup $G(+)$ with a neutral element 0 and a projective transformation f of G such $f(0) = 0$ and $xy = f(x) + y$. since G is right divisible, $G(+)$ is so, and therefore $G(+)$ is a group. ▲

3.11 Corollary. *Every right divisible left permutable groupoid is a left quasigroup.*

3.12 Proposition. *The following conditions are equivalent for a groupoid G:*

(i) G is a left permutable right quasigroup.

(ii) G is a left permutable quasigroup.

(iii) There exist an abelian group $G(+)$ and a permutation f of G such that $f(0) = 0$ and $xy = f(x) + y$ for all $x, y \in G$.

Proof. Use 3.10. ▲

3.13 Remark. (i) In [8], there is constructed a cancellative left permutable groupoid G such that G cannot be imbedded into a left permutable quasigroup. According to 3.12, G cannot be imbedded into a left permutable right quasigroup (with respect to 3.10 and [8, Lemma 5.4], G cannot be imbedded into a right divisible left permutable groupoid either).

(ii) Every free left permutable groupoid is cancellative and can be imbedded into a left permutable quasigroup (see [8]).

3.14 Remark. (i) Let G be a permutable groupoid. Then $xy \cdot uv = u(xy \cdot v) = u(xv \cdot y) = xv \cdot uy = (x \cdot uy)\, v = (u \cdot xy)\, v = uv \cdot xy$ for all $x, y, u, v \in G$. Consequently, if $G = GG$, then G is a commutative semigroup.

(ii) Let F be a free permutable groupoid. By [8], F is cancellative and not commutative. Consequently, F cannot be imbedded into a permutable groupoid G satisfying $G = GG$. In particular, F cannot be imbedded into a permutable (left, right) quasigroup.

V.4 Cancellative groupoids satisfying some short linear equations

4.1 Proposition. *(i) Every left cancellative groupoid satisfying* $x \cdot yz \triangleq x \cdot zy$ *is commutative.*

(ii) Every left (right) cancellative groupoid satisfying $x \cdot yz \triangleq zy \cdot x$ *is commutative.*

(iii) Every groupoid G *satisfying* $x \cdot yz \triangleq yz \cdot x$ *and* $G = GG$ *is commutative.*

(iv) Every left (right) cancellative groupoid satisfying $x \cdot yz \triangleq y \cdot zx$ *is commutative.*

(v) Every left (right) cancellative groupoid satisfying $x \cdot yz \triangleq zx \cdot y$ *is commutative.*

4.2 Remark. (i) Every free groupoid satisfying $x \cdot yz \triangleq x \cdot zy$ is right cancellative ([10, 3.3]).

(ii) Every free groupoid satisfying $x \cdot yz \triangleq yz \cdot x$ is cancellative and cannot be imbedded into a (left, right) quasigroup satisfying the same equation (4.1(iii) and [10, 7.1]).

4.3 Proposition. *Every left (right) cancellative groupoid satisfying* $x \cdot yz \triangleq yx \cdot z$ *is a semigroup.*

Proof. Let G be a groupoid satisfying $x \cdot yz \triangleq yx \cdot z$. Then, for all $x, y, z, u \in G$, we have $z(xy \cdot u) = z(y \cdot xu) = yz \cdot xu = (x \cdot yz)\, u = (yx \cdot z)\, u = z(yx \cdot u) = z(x \cdot yu) = xz \cdot yu = (y \cdot xz)\, u = (xy \cdot z)\, u$. In particular, $z(xy \cdot u) = z(x \cdot yu)$ and $(x \cdot yz)\, u = (xy \cdot z)\, u$. Now, if G is either left or right cancellative, then G is a semigroup.

4.4 Remark. Let G be a semigroup satisfying $xyz \triangleq yxz$. If G is right cancellative, then G is commutative (and hence it can be imbedded into abelian group). If G is left cancellative, then G/p_G is a cancellative commutative semigroup.

(i) Now, assume that G is a left quasigroup. Then G/p_G is an abelian group. Moreover, for every $x \in G$, there is a uniquely determined element $e(x) \in G$ such that $xe(e) = x$. Of course, $e(xy) = e(x)\, e(y)$ and $e(x)/p_G = 0_{G/p_G}$. Consequently, $(ze(x), z) \in p_G$ for every $z \in G$ and it follows that $ze(x)\, u = zu$ and $e(x)\, u = u$ for every $u \in G$, i.e., $e(x)$ is a left neutral element of G. That set $\mathrm{Id}(G)$ of idempotents is a subsemigroup of G, $\mathrm{Id}(G)$ is a semigroup of left units ($ef = f$ for all, $e, f \in \mathrm{Id}(G)$) and the mapping $x \to (x/p_G, e(x))$ is an isomorphism of G onto the product $G/p_G \times \mathrm{Id}(G)$ (if $x \in G$ and $e \in \mathrm{Id}(G)$, then $xe \cdot e = xe$ and $(x, xe) \in p_G$). Notice also that $e(x) = e(y)$ if and only if $x^2 = y^2$ (or $xy = yx$).

(ii) Assume that G is left cancellative and define relations r and s on G by $(x, y) \in r$ if and only if $xy = yx$ and $(x, y) \in s$ if and only if $x^2 = y^2$. Then both r and s are congruences of G and G/r is a semigroup of left units (we have $(x, xyx) \in r$ for all $x, y \in G$). Moreover, if $(x, y) \in r \cap p_G$, then $y^2 = x^2 = xx = yx$, and therefore $x = y$ (since G is left cancellative). Thus $r \cap p_G = \mathrm{id}_G$

and G can be imbedded into the product $G/p_G \times G/r$. On the other hand, G/p_G is a cancellative commutative semigroup, it can be imbedded into an abelian group and consequently G can be imbedded into a left quasigroup Q such that Q is associative and satisfies $xyz \triangleq yxz$ (see (i)).

4.5 Remark. (i) A groupoid satisfies $x \cdot yz \triangleq z \cdot xy$ if and only if it satisfies $x \cdot yz \triangleq y \cdot zx$ (see 4.1(iv)).

(ii) Every groupoid satisfying $x \cdot yz \triangleq z \cdot yx$ is medial (see VIII.2).

(iii) Groupoids satisfying $x \cdot yz \triangleq xz \cdot y$ are dual to those satisfying $x \cdot yz \triangleq yx \cdot z$ (see 4.3 and 4.4).

V.5 Comments

The idea of the well known Ore's construction goes back to [25] (but see also [5], [6] and [26]). Permutable groupoids are studied in [8] and groupoids satisfying short linear equations in [9] and [10].

VI. Cancellative semimedial groupoids

VI.1 Introduction

1.1 A groupoid G is said to be

– medial if it satisfies the equation $xy \cdot uv \triangleq xu \cdot yv$;

– left (right) semimedial if it satisfies the equations $xx \cdot yz \triangleq xy \cdot xz$ $(zy \cdot xx \triangleq zx \cdot yx)$;

– semimedial if it is both left and right semimedial;

– middle semimedial if it satisfies the equation $xy \cdot zx \triangleq xz \cdot yx$;

– strongly semimedial if it is both semimedial and middle semimedial.

1.2 Lemma. *Let G be a left semimedial groupoid and $a, b \in G$. Then:*

(1) $\mathrm{L}_{aa}\mathrm{L}_b = \mathrm{L}_{ab}\mathrm{L}_a$;

(2) $\mathrm{L}_{aa}\mathrm{R}_b = \mathrm{R}_{ab}\mathrm{L}_a$;

(3) $\mathrm{L}_{a^2 \cdot a^2}\mathrm{L}_{b^2} = \mathrm{L}_{ab \cdot a^2}\mathrm{L}_{ab}$.

Proof. $\mathrm{L}_{aa}\mathrm{L}_b(x) = aa \cdot bx = ab \cdot ax = \mathrm{L}_{ab}\mathrm{L}_a(x)$, $\mathrm{L}_{aa}\mathrm{R}_b(x) = aa \cdot xb = ax \cdot ab = \mathrm{R}_{ab}\mathrm{L}_a(x)$, $\mathrm{L}_{a^2 \cdot a^2}\mathrm{L}_{b^2}(x) = (a^2a^2)(b^2x) = (a^2b^2)(a^2x) = (ab)^2 \cdot a^2x = (ab \cdot a^2)(ab \cdot x) = \mathrm{L}_{ab \cdot a^2}\mathrm{L}_{ab}(x)$. ▲

1.3 Lemma. *Let G be a right semimedial groupoid and $a, b \in G$. Then:*

(1) $\mathrm{R}_{aa}\mathrm{R}_b = \mathrm{R}_{ba}\mathrm{R}_a$;

(2) $\mathrm{R}_{aa}\mathrm{L}_b = \mathrm{L}_{ba}\mathrm{R}_a$;

(3) $\mathrm{R}_{a^2 \cdot a^2}\mathrm{R}_{b^2} = \mathrm{R}_{a^2 \cdot ba}\mathrm{R}_{ba}$.

Proof. Dual to that of 1.2. ▲

1.4 Lemma. *Let G be a middle semimedial groupoid and $a, b \in G$. Then* $\mathrm{L}_{ab}\mathrm{R}_a = \mathrm{R}_{ab}\mathrm{L}_a$.

Proof. $\mathrm{L}_{ab}\mathrm{R}_a(x) = ab \cdot xa = ax \cdot ba = \mathrm{R}_{ba}\mathrm{L}_a(x)$. ▲

1.5 Lemma. *Let G be a semimedial groupoid and $a, b \in G$. Then:*
(1) $\mathrm{R}_{b^2 \cdot b^2}\mathrm{L}_{a^2} = \mathrm{R}_{ab \cdot b^2}\mathrm{L}_{ab}$;
(2) $\mathrm{L}_{b^2 \cdot b^2}\mathrm{R}_{a^2} = \mathrm{L}_{b^2 \cdot ba}\mathrm{R}_{ba}$.

Proof. $\mathrm{R}_{b^2 \cdot b^2}\mathrm{L}_{a^2}(x) = (a^2x)(b^2b^2) = (a^2b^2)(xb^2) = (ab)^2 \cdot xb^2 = (ab \cdot x)(ab \cdot b^2) = \mathrm{R}_{ab \cdot b^2}\mathrm{L}_{ax}(x)$ and $\mathrm{L}_{b^2 \cdot b^2}\mathrm{R}_{a^2}(x) = (b^2b^2)(xa^2) = (b^2x)(b^2a^2) = b^2x \cdot (ba)^2 = (b^2 \cdot ba)(x \cdot ba) = \mathrm{L}_{b^2 \cdot ba}\mathrm{R}_{ba}(x)$. ▲

1.6 Lemma. *Let H be a subgroupoid of a groupoid G. Then* $\mathrm{Mul}(G, H) = \mathrm{Mul}_\mathrm{l}(G, H) \cdot \mathrm{Mul}_\mathrm{r}(G, H)$ $(\mathrm{Mul}(G, H) = \mathrm{Mul}_\mathrm{r}(G, H) \cdot \mathrm{Mul}_\mathrm{l}(G, H))$ *in each of the following three cases:*
(1) G is left (right) semimedial and H is left (right) divisible;
(2) G is right (left) semimedial and $\sigma_H(H) = H$;
(3) G is middle semimedial and H is right (left) divisible.

Proof. The assertions follow easily from 1.2(2), 1.3(2) and 1.4, resp. ▲

1.7 Lemma. *Let H be a subgroupoid of a groupoid G. Then* $\mathrm{Mul}(G, H) = \mathrm{Mul}_\mathrm{l}(G, H) \cdot \mathrm{Mul}_\mathrm{r}(G, H) = \mathrm{Mul}_\mathrm{r}(G, H) \cdot \mathrm{Mul}_\mathrm{l}(G, H)$ *in each of the following cases:*
(1) G is semimedial and $\sigma_H(H) = H$;
(2) G is middle semimedial and H is divisible;
(3) G is left semimedial, H is left divisible and $\sigma_H(H) = H$;
(4) G is right semimedial, H is right divisible and $\sigma_H(H) = H$;
(5) G is both left and middle semimedial and H is left divisible;
(6) G is both right and middle semimedial and H is right divisible;
(7) G is both left and middle semimedial, H is right divisible and $\sigma_H(H) = H$;
(8) G is both right and middle semimedial, H is left divisible and $\sigma_H(H) = H$;

Proof. Use 1.7. ▲

1.8 Lemma. *Let G be a left semimedial groupoid such that* $\mathrm{Id}(G) = \{a \in G;\ aa = a\}$ *is non-empty. Then:*
(i) $\mathrm{Id}(G)$ *is a subgroupoid of G, $\mathrm{L}_{e,G}$ is an endomorphism of G and* $\mathrm{L}_{e,G}\mathrm{R}_{e,G} = \mathrm{R}_{e,G}\mathrm{L}_{e,G}$ *for every $e \in \mathrm{Id}(G)$.*
(ii) If G is (left) cancellative, then $\mathrm{Id}(G)$ is left closed in G.

Proof. (i) If $a, b \in \mathrm{Id}(G)$, then $ab \cdot ab = aa \cdot bb = ab$, $a \cdot xa = ax \cdot a$ and $a \cdot xy = ax \cdot ay$ for all $x, y \in G$.
(ii) If $a, b \in G$, $a, ab \in \mathrm{Id}(G)$, then we have $ab = ab \cdot ab = aa \cdot bb = a \cdot bb$ und $b = bb$. ▲

1.9 Remark. Let G be a left semimedial groupoid and let $e \in \mathrm{Id}(G)$ be such that $\mathrm{L}_{e,G}$ is injective. Proceeding similarly as in IV.2.7, we can show that there exists a left semimedial groupoid K such that G is a left strongly dense subgroupoid of K, $\mathrm{L}_{e,K}$ is an automorphism of K and K is the union of a countable chain of subgroupoids isomorphic to G. Moreover, if $\mathrm{R}_{e,G}$ is injective (resp. projective), then the same is true for $\mathrm{R}_{e,K}$.

1.10 Remark. Let G be a semimedial groupoid and let $e \in \mathrm{Id}(G)$ be such that $\mathrm{L}_{e,G}$ and $\mathrm{R}_{e,G}$ are injective. Proceeding similarly as in IV.5.5, we can show that there exists a semimedial groupoid K such that G is a dense subgroupoid of K, $\mathrm{L}_{e,K}$ and $\mathrm{R}_{e,K}$ are automorphisms of K and K is the union of a countable chain of subgroupoids isomorphic to G.

VI.2 Dense subgroupoids of left semimedial groupoids

2.1 Lemma. *Let G be a left semimedial groupoid and $f \in \mathrm{Mul}_l(G)$. Then there exists $g \in \mathrm{Mul}_l(G)$ such that $g(xy) = f(x)\,f(y)$ for all $x, y \mathrel{+} G$. Moreover:*
(i) If $G = GG$, then g is determined uniquely by f.
(ii) If H is a subgroupoid of G and $f \in \mathrm{Mul}_l(G, H)$, then $g \in \mathrm{Mul}_l(G, H)$.

Proof. There are $n \geq 0$ and $a_1, \ldots, a_n \in G$ such that $f = \mathrm{L}_{a_1} \ldots \mathrm{L}_{a_n}$ and we put $g = \mathrm{L}_{a_1^2} \ldots \mathrm{L}_{a_n^2}$. Then $g(xy) = a_1^2(\ldots(a_n^2 \cdot xy)) = a_1^2(\ldots(a_{n-1}^2(a_nx \cdot a_ny))) = \ldots = (a_1(\ldots(a_nx)))(a_1(\ldots(a_ny))) = f(x)\,f(y)$ for all $x, y \in G$. The rest is clear. ▲

2.2 Corollary. *Let H be a subgroupoid of a left semimedial groupoid G. Then all the conditions (A1), ..., (A9) (see I.4.1) are satisfied for H in G. In particular, the monoid $\mathrm{Mul}_l(G, H)$ is right uniform.*

2.3 Corollary. *Let G be a left semimedial groupoid. Then the monoid $\mathrm{Mul}_l(G)$ is right uniform.*

2.4 Remark. Let H be a subgroupoid of a left semimedial groupoid G. By 2.1, for each $f \in \mathrm{Mul}_l(G, H)$ there is a transformation $f' \in \mathrm{Mul}_l(G, H)$ such that $f'(xy) = f(x)\,f(y)$ for all $x, y \in G$; we will also use the notation $f^{(0)} = f$, $f^{(1)} = f', \ldots, f^{(i+1)} = (f^{(i)})'$.

Now, suppose that $G = GG$. Then the transformation f' is determined uniquely by f and we have $g'f'(xy) = g'(f(x)\,f(y)) = gf(x)\,gf(y)$ and $g'f' = (gf)'$. The mapping $f \to f'$ is an endomorphism of the monoid $\mathrm{Mul}_l(G, H)$; this endomorphism is projective, provided that $\sigma_H(H) = \{x^2; x \in H\} = H$.

2.5 Remark. Let H be a subgroupoid of a left semimedial groupoid G and let $f, g \in \mathrm{Mul}_l(G, H)$, $g = \mathrm{L}_{a_1} \ldots \mathrm{L}_{a_n}$, $n \geq 0$, $a_1, \ldots, a_n \in H$. Then $f^{(n)}g(x) = f^{(n)}(a_1(\ldots(a_nx))) = f^{(n-1)}(a_1) \cdot f^{(n-1)}(a_2(\ldots(a_nx))) = \ldots = f^{(n-1)}(a_1)(f^{n-2}(a_2)(\ldots(f^{(1)}(a_{n-1})(f^{(0)}(a_n)\,f(x))))) = hf(x)$, $h \in \mathrm{Mul}_l(G, H)$; we have $f^{(n)}g = hf$.

2.6 Proposition. *Let H be a subgroupoid of a left semimedial groupoid G and $K = \langle H \rangle_{lc,G}$. Then:*

(i) $K = [H]_{l,G} = \alpha_G(H)$ (i.e., $x \in K$ if and only if $a_1(\dots(a_n x)) \in H$ for some $n \geq 0$ and $a_1, \dots, a_n \in H$).

(ii) Every left cancellative congruence r of H can be extended in unique way to a left cancellative congruence s of K; s is cancellative if and only if r is so.

Proof. Combine 2.2, I.4.3(i) and I.6.2. ▲

2.7 Let W be an absolutely free groupoid over an infinite countable set X of variables. By induction, we define quasisymmetric terms from W: Each variable is a quasisymmetric term; if $u, v \in W$ are quasisymmetric terms of the same length, then uv is quasisymmetric.

A (groupoid) equation $u \triangleq v, u, v \in W$, will be caled quasisymmetric if both u and v are quasisymmetric of the same length.

2.8 Proposition. *Let H be a subgroupoid of a left cancellative left semimedial groupoid G. Then the groupoids H and $K = \langle H \rangle_{lc,G}$ satisfy the same quasisymmetric equations.*

Proof. Let $\varphi : W \to K$ be a homomorphism and let $u \triangleq v$ be quasisymmetric equation that is true in H. We have to show that $\varphi(u) = \varphi(v)$.

First, let $\{x_1, \dots, x_n\} = \mathrm{var}(uv)$. Then $f(x_1), \dots, f(x_n)$ are all in H for some $f \in \mathrm{Mul}_l(G, H)$ and we consider any homomorphism $\psi : W \to H$ such that $\psi(x_i) = f\varphi(x_i)$, $i = 1, \dots, n$. Now, by induction on $\mathrm{l}(w) = 2^m$, $m \geq 0$, we show that $\psi(w) = f^{(m)}\varphi(w)$, whenever $w \in W$ is quasisymmetric and $\mathrm{var}(w) \subseteq \{x_1, \dots, x_n\}$. This is clear for $m = 0$ and, if $m \geq 1$, then $w = pq$, where both p and q are quasisymmetric and of length 2^{m-1}. Then $\psi(w) = \psi(p)\,\psi(q) = f^{(m-1)}\varphi(p) \cdot f^{(m-1)}\varphi(q) = f^{(m)}(\varphi(p)\,\varphi(q)) = f^{(m)}\varphi(pq) = f^{(m)}\varphi(w)$.

Now, $f^{(k)}\varphi(u) = \psi(u) = \psi(v) = f^{(k)}\varphi(v)$, $\mathrm{l}(u) = 2^k = \mathrm{l}(v)$, and, since $f^{(k)}$ is an injective transformation, we have $\varphi(u) = \varphi(v)$. ▲

2.9 Theorem. *Let H be a left strongly dense subgroupoid of a left semimedial groupoid G. Then:*

(i) For every $x \in G$ there exist $n \geq 0$ and $a_1, \dots, a_n \in H$ such that $a_1(\dots(a_n)) \in H$.

(ii) Every left cancellative congruence r of H can uniquely be extended to a left cancellative congruence s of G; s is cancellative if and only if r is so.

(iii) If H is cancellative and G is left cancellative, then G is concellative.

(iv) If H is right divisible and G is a left quasigroup, then G is right divisible.

(v) If G is left cancellative, then the groupoids H and G satisfy the same quasisymmetric groupoid equations.

(vi) If G is left cancellative, then G is right semimedial (middle semimedial) if and only if H is so.

Proof. (i) and (ii). See 2.6.

(iii) Since G is left cancellative, id_G is a left cancellative congruence extending id_H. But id_H is cancellative, and therefore id_G is cancellative by (ii).

(iv) Let $x, y \in G$. Since G is left divisible, we have $y = yu$ for some $u \in G$. Further, there are $n \geq 0$ and $a_1, \ldots, a_n \in H$ such that $f(x), f(y), f(u) \in H$, $f = \mathrm{L}_{a_1} \ldots \mathrm{L}_{a_n}$. Now, $f'(y) = a_1^2(\ldots (a_n^2 y)) = f(y) f(u) \in H$ and, since H is right divisible, we have $af(x) = f'(y)$ for some $a \in H$. Further, we can also find $z \in G$ with $f(z) = a$ and thus we have $f'(y) = af(x) = f(z) f(x) = f'(zx)$. But the transformation f' is injective and consequently $y = zx$.

(v) See 2.8.

(vi) All the envolved equations are quasisymmetric. ▲

2.10 Lemma. *Let H be a right closed subgroupoid of a left semimedial groupoid G such that $K = KK$, where $K = \langle H \rangle_{\mathrm{lc}, G}$. Then $K = \langle H \rangle_{\mathrm{c}, G}$ (i.e., K is closed in G).*

Proof. It sufficies to show that K is right closed. For, let $x, y \in G$, $xy, y \in K$. We have $xy = uv$ for some $u, v \in K$ and there is $f \in \mathrm{Mul_l}(G, H)$ with $f(u), f(v), f(y) \in H$. Now, $f'(xy) = f'(uv) = f(u) f(v) \in H$, and hence also $f(x) f(y) = f'(xy) \in H$. But $f(y) \in H$ and H is right closed in G. Thus $f(x) \in H$ and $x \in K$. ▲

2.11 Lemma. *Let H be a subgroupoid of a left semimedial groupoid G such that $KK = K$, where $K = \alpha_G\, (\langle H \rangle_{\mathrm{lc}, G})$. Then $K = \langle H \rangle_{\mathrm{c}, G}$.*

Proof. The groupoid $\mathrm{L} = \langle H \rangle_{\mathrm{rc}, G}$ is right closed in G and the result follows from 2.10. ▲

2.12 Proposition. *Let G be a left semimedial groupoid such that for every $x \in G$ there exists $y \in G$ with $xy = x$. If H is a subgroupoid of G, then $\langle H \rangle_{\mathrm{c}, G} = \alpha_G\, (\langle H \rangle_{\mathrm{rc}, G})$.*

Proof. Let $x \in K = \alpha_G\, (\langle H \rangle_{\mathrm{rc}, G})$, $y = G$ and $xy = x$. Since K is left closed in G, we have $y \in K$ and it follows that $K = KK$. Now, we can use 2.11. ▲

2.13 Corollary. *Let G be a left semimedial groupoid such that for every $x \in G$ there exists $y \in G$ with $xy = x$. If H is a dense subgroupoid of G, then $G = \alpha_G\, (\langle H \rangle_{\mathrm{rc}, G})$.*

2.14 Lemma. *Let H be a left strongly dense subgroupoid of a left cancellative left semimedial groupoid G. Then:*

(i) If σ_H is injective, then σ_G is injective.

(ii) If G is a left quasigroup and σ_H is projective, then σ_G is projective.

Proof. (i) Let $x, y \in G$ be such that $xx = yy$. We have $f(x), f(y) \in H$ for some $f \in \mathrm{Mul_l}(G, H)$ and $f(x) f(x) = f'(xx) = f'(yy) = f(y) f(y)$, $f(x) = f(y)$. Since G is left cancellative, we have also $x = y$.

(ii) Let $x \in G$, $f \in \mathrm{Mul}_1(G, H)$, $f(x) \in H$. Then $f = g'$ for some $g \in \mathrm{Mul}_1(G, H)$, $f(x) = aa$, $a \in H$ and $a = g(y)$, $y \in G$. Now, $f(x) = g'(x) = aa = g(y)\,g(y) = g'(yy)$. But g' is injective, and so $x = yy$. ▲

VI.3 Dense subgroupoids of semimedial groupoids

3.1 Lemma. *Let G be a semimedial groupoid and $f \in \mathrm{Mul}(G)$. Then there exists $g \in \mathrm{Mul}(G)$ such that $g(xy) = f(x)\,f(y)$ for all $x, y \in G$. Moreover:*

(i) If $G = GG$, then g is determined uniquely by f.

(ii) If H is a subgroupoid of G and $f \in \mathrm{Mul}(G, H)$, then $g \in \mathrm{Mul}(G, H)$.

Proof. We have $f = \mathrm{T}_{1,a_1} \dots \mathrm{T}_{n,a_n}$, $n \geq 0$, $a_i \in H$, $\mathrm{T}_i \in \{\mathrm{L}, \mathrm{R}\}$ and we put $g = \mathrm{T}_{1,a_1^2} \dots \mathrm{T}_{n,a_n^2}$. ▲

3.2 Corollary. *Let H be a subgroupoid of a semimedial groupoid G. Then all the conditions (A1), ..., (A9), (B1), ..., B(9) and (C1), ..., (C7) are satisfied for H in G (see I.4.1, I.4.5 and I.5.1). In particular, the monoid $\mathrm{Mul}(G, H)$ is right uniform.*

3.3 Corollary. *Let G be a semimedial groupoid. Then the monoid $\mathrm{Mul}(G)$ is right uniform.*

3.4 Remark. Let H be a subgroupoid of a semimedial groupoid G. By 3.1, for each $f \in \mathrm{Mul}(G, H)$, there is a transformation $f' \in \mathrm{Mul}(G, H)$ such that $f'(xy) = f(x)\,f(y)$ for all $x, y \in G$; we will also use the notation $f^{(0)} = f$, $f^{(1)} = f'$, ..., $f^{(i+1)} = (f^{(i)})'$.

Now, suppose that $G = GG$. Then the transformation f' is determined uniquely by f and the mapping $f \to f'$ is an endomorphism of $\mathrm{Mul}(G, H)$; this endomorphism is projective, provided that $\sigma_H(H) = H$.

3.5 Remark. Let H be a subgroupoid of a semimedial groupoid G and let $f, g \in \mathrm{Mul}(G, H)$, $g = \mathrm{T}_{1,a_1} \dots \mathrm{T}_{n,a_n}$, $n \geq 0$, $a_i \in H$, $\mathrm{T}_i \in \{\mathrm{L}, \mathrm{R}\}$. Then $f^{(n)}g = hf$ for $h = \mathrm{T}_{1,b_1} \dots \mathrm{T}_{n,b_n} \in \mathrm{Mul}(G, H)$, $b_i = f^{(n-i)}(a_i)$.

3.6 Proposition. *Let H be a subgroupoid of semimedial groupoid G and $K = \langle H \rangle_{c,G}$. Then:*

(i) $K = [H]_G = \gamma_G(H)$.

(ii) Every cancellative congruence of H can be extended in unique way to a cancellative congruence of K.

(iii) If G is cancellative, then the groupoids H and K satisfy the same quasisymmetric groupoid equations.

Proof. (i) and (ii). Combine 3.2 and I.5.4, I.6.6(ii).

(iii) We can proceed similarly as in the proof of 2.8 (or, we can use (i), 2.8 and its dual). ▲

3.7 Theorem. *Let H be a dense subgroupoid of a semimedial groupoid G. Then:*

(i) For every $x \in G$, there exists $f \in \mathrm{Mul}(G, H)$ such that $f(x) \in H$.

(ii) Every cancellative congruence of H can uniquely be extended to a cancellative congruence of G.

(iii) If G is cancellative, then the groupoids H and G satisfy the same quasisymmetric groupoid equations.

Proof. The theorem follows immediately from 3.6. ▲

3.8 Remark. Let H be a dense subgroupoid of a semimedial groupoid G and let $x \in G$. By 3.7(i), $f(x) \in H$ for some $f \in \mathrm{Mul}(G, H)$. Now, assume that either H is left divisible or $\sigma_H(H) = H$. Then, by 1.6, we have $f = pq$ for some $p \in \mathrm{Mul}_l(G, H)$ and $q \in \mathrm{Mul}_r(G, H)$, and hence there are $a_1, \ldots, a_n \in H$ and $b_1, \ldots, b_m \in H$ such that $a_1(\ldots (a_n(((xb_1)\ldots) b_m))) \in H$.

3.9 Proposition. *Let H be a dense subgroupoid of a semimedial groupoid G such that at least one of the following conditions is satisfied:*

(1) H is divisible;

(2) $\sigma_H(H) = H$;

(3) For every $x \in G$ there exist $y, z \in G$ with $yx = x = xz$.

Then $G = \alpha_G(\beta_G(H)) = \beta_G(\alpha_G(H))$.

Proof. See 2.13, 3.8 and their duals. ▲

3.10 Remark. Let H be a subgroupoid of a semimedial groupoid G such that $G = \alpha_G(L)$, $L = \beta_G(H)$ (see 3.9). If $x \in G$, then there are $u_1, \ldots, u_n \in L$ such that $y = u_1(\ldots (u_n x)) \in L$. Futher, there is $f_1 \in \mathrm{Mul}_r(G, H)$ such that $f_1(u_n) \in H$ and we have $f_1^{(n)}(y) = f_1^{n-1}(u_1)(\ldots (f_1(u_n) f_1(x)))$, etc. Now, we find $f, g \in \mathrm{Mul}_r(G, H)$ and $a_1, \ldots, a_n \in H$ such that $g(y) = a_1(\ldots (a_n f(x)))$. Of course, $g(y) \in L$, and consequently there exist $b_1, \ldots, b_m \in H$ and $c_1, \ldots, c_k \in H$ such that $(((a_1(\ldots (a_n(((xb_1)\ldots) b_m)))) c_1)\ldots) c_k \in H$.

3.11 Lemma. *Let H be a dense subgroupoid of a cancellative semimedial groupoid G. Then:*

(i) If σ_H is injective, then σ_G is injective.

(ii) If G is a quasigroup and σ_H is projective, then σ_G is projective.

Proof. We can proceed similarly as in 2.14. ▲

VI.4 Cancellative semimedial groupoids without irreducible elements

4.1 Lemma. *Let G be a left semimedial groupoid.*

(i) If $\mathrm{Mul}_l(G)$ is right cancellative, then p_G is a congruence of G.

(ii) If $\mathrm{Mul}_l(G)$ is left cancellative, then p_G is left cancellative.

Proof. Both assertions are clear from the fact that $\mathrm{L}_{aa}\mathrm{L}_b = \mathrm{L}_{ab}\mathrm{L}_a$ for all $a, b \in G$. ▲

4.2 Proposition. *Let G be a left cancellative left semimedial groupoid such that $GG = G$. Then the monoid* $\mathrm{Mul}_l(G)$ *is cancellative.*

Proof. Since G is left cancellative, $\mathrm{Mul}_l(G)$ is also left cancellative. Now, let $f, g, h \in \mathrm{Mul}_l(G)$ be such that $fh = gh$. We have $h = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_n}$, $n \geq 0$, $a_i \in G$ and we proceed by induction on n to show $f = g$.

The result is clear for $n = 0$, and so let $n = 1$ and $a = a_1$. Then $f(ax) = g(ax)$ for each $x \in G$ and, in particular, $f(aa) = y(aa)$. Further, $f(aa)\,f(xy) = f'(aa \cdot xy) = f'(ax \cdot ay) = f(ax)\,f(ay) = g(ax)\,g(ay) = g'(ax \cdot ay) = g'(aa \cdot xy) = g(aa)\,g(xy)$. Since $f(aa) = g(aa)$ and G is left cancellative, we have $f(xy) = g(xy)$ for all $x, y \in G$. Since $G = GG$, we have also $f = g$.

Now, let $n \geq 2$ *and* $p = \mathrm{L}_{a_1} \dots \mathrm{L}_{a_{n-1}}$. Then $fp\mathrm{L}_{a_n} = fh = gh = gp\mathrm{L}_{a_n}$, $fp = gp$ by the preceding part of the proof and, finally, $f = g$ by induction. ▲

4.3 Proposition. *Let G be a left cancellative left semimedial groupoid such that $G = GG$. Then p_G is a left cancellative congruence of G.*

Proof. Combine 4.1 and 4.2. ▲

4.4 Remark. Let φ be an endomorphism of a group G. Define a binary operation $*$ on G by $x * y = \varphi(x)\,yx^{-1}$ for all $x, y \in G$. Then:

(i) $G(*)$ is a left semimedial left quasigroup and $\sigma_{G(*)} = \varphi$.

(ii) $(a, b) \in p_{G(*)}$ if and only if $a^{-1}b \in Z(G)$ (the centre) and $\varphi(a^{-1}b) = a^{-1}b$.

(iii) If $G(*)$ is right (or middle) semimedial, then $\varphi \restriction G' = \mathrm{id}_{G'}$ (G' denotes the commutator subgroup).

4.5 Proposition. *Let G be a left cancellative left semimedial groupoid such that $G = GG$ and $p_G = \mathrm{id}_G$. Then there exists a left semimedial left quasigroup Q with the following properties:*

(1) G is a left strongly dense subgroupoid of Q (for every $x \in Q$ there exist $n \geq 0$ and $a_1, \dots, a_n \in G$ such that $a_1(\dots(a_nx)) \in G$);

(2) Q is cancellative if and only if G is so;

(3) Q is right divisible, provided that G is right divisible;

(4) The groupoids G and Q satisfy the same quasisymmetric groupoid equations;

(5) Q is right semimedial (middle semimedial) if and only if G is so.

Proof. By 4.2, the monoid $S = \mathrm{Mul}_l(G)$ is cancellative and, by 2.3, S is right uniform. Now, due to V.2.15, there is a group R such that S is a subsemigroup of R and $R = \{f^{-1}g;\ f, g \in S\}$.

By 2.4, the mapping $\Phi : f \to f'$ is an endomorphism of S and, by V.2.15, Φ can be extended to an endomorphism Ψ of R. Now, define a binary operation $*$ on R by $x * y = \Psi(x)\,yx^{-1}$ for all $x, y \in R$. According to 4.4, $R(*)$ is a left semimedial left quasigroup.

For $a \in G$, put $\varphi(a) = \mathrm{L}_{a,G} \in S \subseteq R$. Then $\varphi(aa) = \mathrm{L}_{aa} = \Phi(\mathrm{L}_a) = \Psi(\mathrm{L}_a) = \Psi(\varphi(a))$ and $\Psi(\varphi(a)) \cdot \varphi(b) = \varphi(aa) \cdot \varphi(b) = \mathrm{L}_{aa}\mathrm{L}_b = \mathrm{L}_{ab}\mathrm{L}_a = \varphi(ab) \cdot \varphi(a)$ for all

$a, b \in G$. Consequently, $\varphi(ab) = \Psi(\varphi(a)) \cdot \varphi(b) \cdot \varphi(a)^{-1} = \varphi(a) * \varphi(b)$. Thus φ is a homomorphism of the groupoid G into the groupoid $R(*)$. Clearly, $\ker(\varphi) = p_G = \mathrm{id}_G$, and so φ is injective.

Finally, we identify G and $\varphi(G)$, we put $Q = \langle G \rangle_{\mathrm{lc},\, R(*)}$ and we take into account 2.9. ▲

4.6 Theorem. *Let G be a cancellative semimedial groupoid such that $G = GG$. Then there exists a semimedial quasigroup Q with the following properties:*

(1) G is a dense subgroupoid of Q and $Q = \alpha_Q(\beta_Q(G)) = \beta_Q(\alpha_Q(G))$ (cf. 3.10);

(2) The groupoids G and Q satisfy the same quasisymmetric groupoid equations.

Proof. By 4.5, G is a left strongly dense subgroupoid of a semimedial left quasigroup P such that $P = \alpha_P(G)$ and P is cancellative. By the right hand form of 4.5, P is a right strongly dense subgroupoid of a semimedial quasigroup $Q = \beta_Q(P)$. The rest is clear. ▲

4.7 Remark. Let G be a groupoid such that G is isomorphic to a subgroupoid of GG. Then there exists a groupoid H isomorphic to G such that G is a subgroupoid of H and $G \subseteq HH$. Now, it is clear that there exists a chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$ of groupoids such that $G_i \subseteq G_{i+1}G_{i+1}$ and $G_i \cong G$. Setting $K = \bigcup G_i$ we get a groupoid such that G is a subgroupoid of K and $K = KK$. Notice that the groupoids G and K are equationally equivalent and that K is (left, right) cancellative if and only if G is so.

4.8 Proposition. *Let G be a left cancellative left semimedial groupoid such that $p_G = \mathrm{id}_G$ and at least one of the following conditions is satisfied:*

(1) σ_G is injective (or projective);

(2) G contains at least one idempotent element.

Then G can be imbedded into a left semimedial left quasigroup.

Proof. If σ_G is injective, then σ_G is an injective endomorphism of G and we have $\sigma_G(G) \subseteq GG \subseteq G$ and $\sigma_G(G) \cong G$. Now, the result follows from 4.7 and 4.5. Similarly, if $ee = e$ for some $e \in G$, then $\varphi = \mathrm{L}_{e,G}$, is an injective endomorphism of G and again, $\varphi(G) \subseteq GG \subseteq G$, $\varphi(G) \cong G$. ▲

4.9 Theorem. *Let G be a cancellative semimedial groupoid such that at least one of the following conditions is satisfied:*

(1) σ_G is injective (or projective);

(2) G contains at least one idempotent element.

Then G can be imbedded into a semimedial quasigroup.

Proof. Similar to that of 4.8. ▲

4.10 Remark. Let G be a left semimedial groupoid containing a neutral element e.

(i) We have $xx \cdot y = xx \cdot ye = xy \cdot xe = xy \cdot x$, $xx \cdot y = xx \cdot ey = x \cdot xy$, and so $xy \cdot x = x \cdot xy$ for all $x, y = G$.

(ii) If G is left divisible, then G is commutative.

(iii) If $(a, b) \in p_G$, then $a = ae = be = e$, so that $p_G = \mathrm{id}_G$.

(iv) Assume that G is left cancellative. By 4.5, G is a left strongly dense subgroupoid of a left semimedial left quasigroup Q. If $x \in Q$, then $a_1(\dots(a_n x)) = b \in G$ for some $n \geq 0$ and $a_i \in G$ and we have $b = eb = e(a_1 \dots (a_n x)) = (ea_1)(e(a_2 \dots (a_n(x))) = a_1(e(a_2(\dots(a_n x)))) = \dots = a_1(\dots(a_n \cdot ex))$. Consequently $x = ex$ and we have shown that e is a left neutral element of Q.

Finally, if G is also right semimedial, then Q is so and e is a neutral element of Q $(a_1(\dots(a_n x)) = b = be = a_1(\dots(a_n \cdot xe)))$. By (ii), Q is commutative and consequently Q is a commutative Moufang loop and G is also commutative.

VI.5 Cancellative unipotent semimedial groupoids

5.1 A groupoid G is said to be unipotent if $aa = o = bb$ for all $a, b \in G$.

5.2 Lemma. *Let G be a unipotent left semimedial groupoid. Then:*

(i) $\mathrm{L}_o^2 = \mathrm{R}_o^2$ and $\mathrm{L}_o\mathrm{R}_o = \mathrm{R}_o\mathrm{L}_o$.

(ii) L_o *is injective (resp. projective) if and only if* R_o *is so.*

Proof. We have $o \cdot ox = (xx)(xx \cdot x) = (x \cdot xx)(xx) = xo \cdot o$ for every $x \in G$. ▲

5.3 Theorem. *The following conditions are equivalent for a groupoid G:*

(i) G is unipotent and left semimedial and at least one of the transformations $\mathrm{L}_{o,G}$, $\mathrm{R}_{o,G}$ is injective and at least one of them is projective.

(ii) There exist a group $G(\circ)$ and an automorphism φ of $G(\circ)$ such that $xy = \varphi(x^{-1} \circ y)$ for all $x, y \in G$.

Moreover, if these conditions are satisfied, then G is a quasigroup and G is right semimedial if and only if G is middle semimedial (equivalently, $G(\circ)$ is abelian).

Proof. First, we show that (i) implies (ii). By 5.2, both the translations L_o and R_o are bijections $\mathrm{L}_o^2 = \mathrm{R}_o^2$, $\mathrm{L}_o\mathrm{R}_o = \mathrm{R}_o\mathrm{L}_o$, $\psi = \mathrm{L}_o\mathrm{R}_o^{-1} = \mathrm{L}_o^{-1}\mathrm{R}_o = \mathrm{R}_o^{-1}\mathrm{L}_o = \mathrm{R}_o\mathrm{L}_o^{-1} = \psi^{-1}$. Moreover, L_o is an automorphism of the groupoid G. Now, define a binary operation $\circ$ on G by $x \circ y = \mathrm{R}_o^{-1}(x) \cdot \mathrm{L}_o^{-1}(y)$ for all $x, y \in G$. Clearly, o is a neutral element of the groupoid $G(\circ)$ and $xy = \mathrm{R}_o(x) \circ \mathrm{L}_o(y) = (xo) \circ (oy)$ for all $x, y \in G$. Since G is left semimedial, we have $\mathrm{R}_o(\mathrm{R}_o(x) \circ \mathrm{L}_o(x)) \circ \mathrm{L}_o(\mathrm{R}_o(y) \circ \mathrm{L}_o(z)) = xx \cdot yz = xy \cdot xz = \mathrm{R}_o(\mathrm{R}_o(x) \circ \mathrm{L}_o(y)) \circ \mathrm{L}_o(\mathrm{R}_o(x) \circ \mathrm{L}_o(z))$ for all $x, y, z \in G$ and consequently $\mathrm{R}_o(x \circ \psi(x)) \circ \mathrm{L}_o(y \circ z) = \mathrm{R}_o(x \circ \psi(y)) \circ \mathrm{L}_o(x \circ z)$ for all $x, y, z \in G$. Further, $\mathrm{L}_o(u \circ v) = \mathrm{L}_o(\mathrm{R}_o^{-1}(u) \cdot \mathrm{L}_o^{-1}(v)) = \mathrm{L}_o\mathrm{R}_o^{-1}(u) \cdot v = \psi(u)\, v = \mathrm{R}_o\psi(u) \circ \mathrm{L}_o(v) = \mathrm{L}_o(u) \circ \mathrm{L}_o(v)$ showing that L_o is an automorphism of $G(\circ)$. Now, $\psi(x \circ \psi(x)) \circ (y \circ z) = \mathrm{L}_o^{-1}(\mathrm{R}_o(x \circ \psi(x)) \circ \mathrm{L}_o(y \circ z)) = \mathrm{L}_o^{-1}(\mathrm{R}_o(x \circ \psi(y)) \circ \mathrm{L}_o(x \cdot z)) = \psi(x \circ \psi(y)) \circ (x \circ z)$ for all $x, y, z \in G$. Substituting $y = z = o$ into the latter equality, we get $\psi(x \circ \psi(x)) = \psi(x) \circ x$, and therefore $(\psi(x) \circ x) \circ (y \circ z) = \psi(x \circ \psi(y)) \circ (x \circ z)$. On the other hand, $\psi(x) \circ x = \mathrm{R}_o^{-1}(\mathrm{R}_o\mathrm{L}_o^{-1}(x)) \cdot \mathrm{L}_o^{-1} = \mathrm{L}_o^{-1}(x) \cdot \mathrm{L}_o^{-1}(x) = o$ and $x \circ \psi(x) = \mathrm{R}_o^{-1}(x) \cdot \mathrm{L}_o^{-1}(\mathrm{L}_o\mathrm{R}_o^{-1}(x)) = \mathrm{R}_o^{-1}(x) \cdot \mathrm{R}_o^{-1}(x) = o$.

Thus $y \circ z = o \circ (y \circ z) = (\psi(x) \circ x) \circ (y \circ z) = \psi(x \circ \psi(y)) \circ (x \circ z)$. We have proved that

$(*)$ $y \circ z = \psi(x \circ \psi(y)) \circ (x \circ z)$

for all $x, y, z \in G$.

Setting $y = o$ in $(*)$, we get the equality $\psi(x) \circ (x \circ z) = z$ for all $x, z \in G$. Then also $x \circ (\psi(x) \circ z) = z$ and we see that $G(\circ)$ is a left quasigroup. However, then G is a left quasigroup as well.

Setting $z = o$ in $(*)$, we get the equality $y = \psi(x \circ \psi(y)) \circ x$. Then $y = \psi(x \circ \psi(y)) \circ x = \psi(\psi(x) \circ \psi(y)) \circ \psi(x) = \psi(x) \circ (x \circ y) = x \circ (\psi(x) \circ y)$ for all $x, y \in G$. Moreover, $\psi(y) = \psi(x \circ y) \circ x = \psi(\psi(x) \circ y) \circ \psi(x) = \psi(x) \circ (x \circ \psi(y)) = x \circ (\psi(x) \circ \psi(y))$.

Now, $(x \circ y) \circ \psi(y) = \psi^2(x \circ y) \circ (\psi(x \circ y) \circ x) = x$ for all $x, y \in G$. Consequently, $(x \circ \psi(y)) \circ y = x$ and we see that both $G(\circ)$ and G are right quasigroups. Thus $G(\circ)$ is a loop and G is a quasigroup. Since $\psi(x) \circ x = o = x \circ \psi(x)$, we have $\psi(x) = x^{-1}$ (the inverse of x in $G(\circ)$) and we have already proved that $x^{-1} \circ (x \circ y) = x \circ (x^{-1} \circ y) = (y \circ x) \circ x^{-1} = (y \circ x^{-1}) \circ x = y$, $(x \circ y^{-1})^{-1} \circ x = y$ and $y \circ z = (x \circ y^{-1})^{-1} \circ (x \circ z)$. From this, $y \circ x^{-1} = ((x \circ y^{-1})^{-1} \circ x) \circ x^{-1} = (x \circ y^{-1})^{-1}$ for all $x, y \in G$ and this equality can also be written as $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$. That is, the permutation $x \to x^{-1}$ is an antiautomorphism of the loop $G(\circ)$. Now, $(*)$ can be rewritten as

$(**)$ $y \circ z = (y \circ x^{-1}) \circ (x \circ z)$

for all $x, y, z \in G$.

For $y = u \circ x$, we get $(u \circ x) \circ z = y \circ z = (y \circ x^{-1}) \circ (x \circ z) = ((u \circ x) \circ x^{-1}) \circ (x \circ z) = u \circ (x \circ z)$ for all $x, z, u \in G$. This means that $G(\circ)$ is a group. The permutation $\varphi = \mathrm{L}_{o,G}$ is an automorphism of $G(\circ)$ and $\mathrm{R}_{o,G}(x) = \varphi\psi(x) = \varphi(x^{-1})$. Thus $xy = \varphi(x^{-1} \circ y)$ for all $x, y \in G$.

If G is right semimedial, then $\mathrm{R}_{o,G}$ and ψ are automorphisms of both G and $G(\circ)$, and therefore $G(\circ)$ is an abelian group and G is medial. Similarly, if G is middle semimedial, then $G(\circ)$ is abelian.

Finally, if (ii) is true, then $xx = o$, o being the unit element of $G(\circ)$, $\mathrm{L}_{o,G} = \varphi$ and $xx \cdot yz = \varphi(\varphi(x^{-1} \circ x)^{-1} \circ \varphi(y^{-1} \circ z)) = \varphi^2(y^{-1} \circ z) = \varphi^2(y^{-1} \circ x \circ x^{-1} \circ z) = \varphi(\varphi(x^{-1} \circ y)^{-1} \circ \varphi(x^{-1} \circ z)) = xy \cdot xz$ for all $x, y, z \in G$. ▲

5.4 Theorem. *Let G be a unipotent left semimedial groupoid such that at least one of the translations $\mathrm{L}_{o,G}$ and $\mathrm{R}_{o,G}$ is injective. Then G is cancellative and, if G is right (middle) semimedial, then G is medial.*

Proof. By 5.2(ii), both $\mathrm{L}_{o,G}$ and $\mathrm{R}_{o,G}$ are injective. Now, by 1.9, there exists a unipotent left semimedial groupoid K such that G is a (strongly dense) subgroupoid of K and $\mathrm{L}_{o,K}$ is an automorphism of K. Now, the result follows from 5.2. ▲

5.5 Corollary. *Let G be a unipotent left semimedial groupoid such that G is either left or right cancellative. Then G is cancellative and can be imbedded into a (unipotent) left semimedial quasigroup.*

6.1 Proposition. *Let φ be an endomorphism of a groupoid G. Define a new binary operation $*$ on G by $x * y = \varphi(xy)$ for all $x, y \in G$. Then:*

(i) $(x * y) * (u * v) = \varphi^2(xy \cdot uv)$ *for all* $x, y, u, v \in G$.

(ii) $G(*)$ is (left, right, middle) semimedial, provided that $G(*)$ is so.

(iii) $G(*)$ is medial, provided that G is so.

(iv) $G(*)$ is (left, right) cancellative, provided that G is so and φ is injective.

(v) $G(*)$ is (left, right) divisible, provided that G is so and φ is projective.

(vi) φ is endomorphism of $G(*)$.

6.2 Proposition. *Let G be a left semímedial groupoid such that $\varphi = \sigma_G$ is a bijection (then φ is an automorphism of G). Define a new binary operation $*$ on G by $x * y = \varphi^{-1}(xy)$ for all $x, y \in G$. Then:*

(i) $G(*)$ is an idempotent left distributive groupoid.

(ii) φ is an automorphism of $G(*)$ and $xy = \varphi(x * y)$.

(iii) $G(*)$ is right distributive (resp. middle semimedial, medial) if and only if G is right semimedial (middle semimedial, medial).

(iv) $G(*)$ is (left, right) cancellative if and only if G is so.

(v) $G(*)$ is (left, right) divisible if and only if G is so.

6.3 Corollary. *The following conditions are equivalent for a groupoid G:*

(i) G is left semimedial and σ_G is bijective.

(ii) There exist an idempotent left distributive groupoid $G(*)$ and an automorphism φ of $G(*)$ such that $xy = \varphi(x * y)$ for all $x, y \in G$.

If these conditions are satisfied, then $\varphi = \sigma_G$, G is right semimedial (middle semimedial, medial) if and only if $G(*)$ is right distributive (middle semimedial, medial) and G is (left, right) cancellative (or divisible) if and only if $G(*)$ is so.

6.4 Theorem. *Let G be a left cancellative left semimedial groupoid such that σ_G is injective. Then these exists a left semimedial left quasigroup Q with the following properties:*

(i) G is a left strongly dense subgroupoid of Q.

(ii) σ_Q is injective.

(iii) σ_Q is projective, provided that σ_G is so.

(iv) Q is right (middle) semimedial if and only if G is so.

(v) Q is cancellative if and only if G is so.

(vi) Q is right divisible, provided that G is so.

Proof. By I.1.13, G can be imbedded into a groupoid K such that K is left cancellative left semimedial, $\varphi = \sigma_K$ is an automorphism of K, etc. Put $x * y = \varphi^{-1}(xy)$ for all $x, y \in K$ (see 6.2); then $K(*)$ is a left cancellative idempotent left distributive groupoid and φ is an automorphism of $K(*)$. By IV.2.4, there exists an idempotent left distributive left quasigroup $P(*)$ such that $K(*)$ is a left strongly dense subgroupoid of $P(*)$ and the imbedding $K(*) \hookrightarrow P(*)$ is a reflexion of $K(*)$ in the class of left distributive left quasigroups. Now, by II.3.6, φ can be extended to an automorphism ψ of $K(*)$. Again, put $xy = \psi(x * y)$ for all $x, y \in K$. Then K, together with this multiplication, becomes a left semimedial left quasigroup. Of course, G and K are subgroupoids of P, and hence G is left strongly dense subgroupoid of $Q = \langle G \rangle_{\mathrm{lc}, P}$; Q is a left semimedial left quasigroup. For $x \in P$, $\sigma_P(x) = xx = \psi(x * x) = \psi(x)$, and therefore σ_P is bijective. Consequently, σ_Q is injective and we have proved (i) and (ii). The asertions (iv), (v) and (vi) are proved in 2.9.

Finally, assume that σ_G is projective (i.e., bijective). Then $G = K$ and it is enough to show that $Q = P$. If $x \in P$, then $a_1 * (a_2 * (\dots (a_n * x))) \in G$, $a_i \in G$, and it follows that $\psi^{-1}(a_1)(\psi^{-1}(a_2)(\dots(\psi^{-n}(a_n)\psi^{-m}(x)))) \in G$. But $\psi = \sigma_P$ is bijective and $\psi \restriction G = \varphi$ is also bijective. Consequently, $\psi^{-1}(a_i) \in G$, $\psi^{-i}(a_i) \in G$, $\psi^{-n}(x) \in Q$ and $x = \psi^n(\psi^{-n}(x)) \in Q$. ▲

6.5 Theorem. *Let G be a cancellative semimedial groupoid such that σ_G is injective. Then there exists a semimedial quasigroup Q with the following properties:*

(i) G is a dense subgroupoid of Q.

(ii) σ_Q is injective.

(iii) σ_Q is projective, provided that σ_G is so.

(iv) Q is medial if and only if Q is so.

Proof. We can proceed similarly as in the proof of 6.4 (using I.1.13, 6.2, IV.5.4 and II.4.3) or we can simply use 4.9 and 3.11. ▲

VI.7 Cancellative semimedial groupoids with projective σ_G

7.1 Proposition. *Let G be a left semimedial groupoid such that σ_G is projective. Then there exists a left semimedial groupoid K with the following properties:*

(i) G is a homomorphic image of K.

(ii) σ_K is a bijection (and hence an automorphism of K).

(iii) The groupoids G and K are equationally equivalent.

(iv) K is (left, right) cancellative, provided that G is so.

Proof. K will be the set of sequences $(x_0, x_1, x_2, \dots) \in G^{\aleph_0}$ such that $x_{i+1}^2 = x_i$ for every $i \geq 0$. Then K is a subgroupoid of the power $G^{\aleph_0}$ and G is a homomorphic image of K ($(x_i) \to x_0$). Also the remaining assertions are clear. ▲

7.2 Proposition. *Let G be a left cancellative left semimedial groupoid such that σ_G is projective. Then G can be imbedded into a left semimedial left quasigroup.*

Proof. By 7.1, there exists a projective homomorphism $\varphi : K \to G$, where K is a left cancellative left semimedial groupoid and σ_K is an automorphism of K. Further, by 6.4, K is a left strongly dense subgroupoid of a left semimedial left quasigroup P (such that σ_P is an automorphism of P). Now, the congruence $r = \ker(\varphi)$ of K is left cancellative and it can be extended to a left cancellative congruence s of P (see 2.9). Then the factor $Q = P/s$ is again a left quasigroup and G can be imbedded into Q. ▲

7.3 Theorem. *Let G be a cancellative semimedial groupoid such that σ_G is projective. Then G can be imbedded into a semimedial quasigroup.*

Proof. We can proceed similarly as in the proof of 7.2 (using 7.1, 6.5 and 3.7) or we can just use 4.9. ▲

VI.8 Cancellative semimedial groupoids where τ_G (ϱ_G) is an endomorphism

8.1 Proposition. *Let G be a left cancellative groupoid such that the transformations σ_G and ϱ_G are endomorphisms of G. Then there exist groupoids H and K with the following properties:*

(i) Both H and K are left cancellative, σ_H is an automorphism of H and ϱ_K is an automorphism of K (see I.1.12).

(ii) There exists an injective homomorphism $G \to H \times K$.

(iii) The groupoids G and $H \times K$ are equationally equivalent.

Proof. With respect I.1.15, we can assume that G is non-trivial and subdirectly lc-irreducible. If $(a, b) \in \ker(\sigma_G) \cap \ker(\varrho_G)$, then $aa = bb$ and $aa \cdot a = bb \cdot b$. Since G is left cancellative, we have $a = b$, and so $\ker(\sigma_G) \cap \ker(\varrho_G) = \mathrm{id}_G$. Since σ_G and ϱ_G are endomorphism of G, both $\ker(\sigma_G)$ and $\ker(\varrho_G)$ are left cancellative congruences of G. But G is subdirectly lc-irreducible. Thus either $\ker(\sigma_G) = \mathrm{id}_G$ and σ_G is injective or $\ker(\varrho_G) = \mathrm{id}_G$ and σ_G is injective. The rest is clear from I.1.13. ▲

8.2 Proposition. *Let G be a cancellative groupoid such that both σ_G and ϱ_G are endomorphisms of G. Then there exist groupoids H and K with the following properties:*

(i) Both H and K are cancellative, σ_H is an automorphism of H and ϱ_K of K.

(ii) There exists an injective homomorphism $G \to H \times K$.

(iii) The groupoids G and $H \times K$ are equationally equivalent.

Proof. Similar to that of 8.1. ▲

8.3 Theorem. *Let G be a cancellative semimedial groupoid such that either ϱ_G or τ_G is an endomorphism of G (see I.1.12).* Then *G can be imbedded into a semimedial quasigroup.*

Proof. Combine 8.2 and 4.6. ▲

VI.9 Imbeddings of cancellative semimedial groupoids into semimedial quasigroups

9.1 Theorem. *Let G be a left cancellative left semimedial groupoid such that at least one of the following conditions is satisfied:*

(1) $G = GG$;
(2) σ_G *is injective;*
(3) σ_G *is projective;*
(4) σ_G *is constant (i.e., G is unipotent);*
(5) G *contains at least one idempotent element.*
Then G can be imbedded into a left semimedial left quasigroup.

Proof. (i) With respect to I.1.15, we can assume that G is subdirectly cl-irreducible. By 4.3, p_G is a left cancellative congruence of G. Now, if $(a, b) \in p_G \cap \ker(\sigma_G)$, then $aa = bb$ and $ax = bx$ for every $x \in G$. Hence $ab = bb = aa$ and $a = b$, since G is left cancellative. Thus $p_G \cap \ker(\sigma_G) = \mathrm{id}_G$ and either $p_G = \mathrm{id}_G$ or σ_G is injective. If $p_G = \mathrm{id}_G$, then we can use 4.5. If σ_G is injective, then 6.4 shows the result.
(ii) See 6.4.
(iii) See 7.2.
(iv) See 5.4.
(v) Combine (i) and 4.7. ▲

9.2 Theorem. *Let G be a left cancellative left semimedial groupoid satisfying at least one of the conditions (1), ..., (5) from 9.1. Then there exists a left semimedial left quasigroup Q with the following properties:*
(i) G is a strongly dense subgroupoid of Q and for every $x \in Q$ *there exist* $n \geq 0$ *and* $b, a_1, \dots, a_n \in G$ *such that* $a_1(\dots(a_n x)) = b \in G$.
(ii) Q is right cancellative if and only if G is so.
(iii) Q is right divisible, provided that G is so.
(iv) The groupoids G and Q satisfy the same quasisymmetric groupoid equations.

Proof. Combine 9.1 and 2.9. ▲

9.3 Theorem. *Let G be a cancellative semimedial groupoid such that at least one of the following conditions is satisfied:*
(1) $G = GG$;
(2) σ_G *is injective;*
(3) σ_G *is projective;*
(4) σ_G *is constant;*
(5) τ_G *is an endomorphism of G;*
(6) ϱ_G *is an endomorphism of G;*
(7) G *contains at least one idempotent element.*
Then G can be imbedded into a semimedial quasigroup.

Proof. Use 4.6, 4.9, 6.5, 7.3 and 8.3. ▲

9.4 Theorem. *Let G be a cancellative semimedial groupoid satisfying at least one of the conditions (1), ..., (7) from 9.3. Then there exists a semimedial quasigroup Q with the following properties:*

(i) G is a dense subgroupoid of Q and for every $x \in Q$ there exist $n \geq 0$, b, $a_1, \ldots, a_n \in G$ and $\mathrm{T}_1, \ldots, \mathrm{T}_n \in \{\mathrm{L}, \mathrm{R}\}$ such that $\mathrm{T}_{1,a_1} \ldots \mathrm{T}_{n,a_n}(x) = b \in G$.

(ii) The groupoids G and Q satisfy the same quasisymmetric groupoid equations.

Proof. Combine 9.3 and 3.7. ▲

9.5 Remark. Let G be a (left) cancellative (left) semimedial groupoid that can be imbedded into a (left) semimedial quasigroup (see 9.1, 9.2, 9.4 and 9.4). Then G is a (left strongly) dense subgroupoid of a (left) semimedial (left) quasigroup Q, Q is determined uniquely up to G-isomorphism, G and Q satisfy the same quasisymmetric groupoid equations (Q is right cancellative or divisible, provided that G is so) and the imbedding $G \hookrightarrow Q$ is a reflexion of G in the class of (left) semimedial (left) quasigroups (cf. II.3.5, II.3.6 and II.4.4).

VI.10 When semimedial groupoids are strongly semimedial

10.1 Theorem. *A semimedial groupoid G is strongly semimedial, provided that at least one of the following conditions is satisfied:*

(1) G is cancellative;

(2) σ_G is injective;

(3) σ_G is projective;

(4) G is commutative.

Proof. (i) If $a, b, c \in G$, then we have $((cc \cdot ca)(ba \cdot ba))((ac \cdot ba)(aa \cdot ba)) = ((cc \cdot ca)(ba \cdot ba))((ac \cdot aa)(ba \cdot ba)) = ((cc \cdot ca)(ba \cdot ba))((aa \cdot ca)(ba \cdot ba)) = ((cc \cdot ca)(aa \cdot ca))((ba \cdot ba)(ba \cdot ba)) = ((cc \cdot aa)(ca \cdot ca))((ba \cdot ba)(ba \cdot ba)) = ((ca \cdot ca)(ca \cdot ca))((ba \cdot ba)(ba \cdot ba)) = ((ca \cdot ca)(ca \cdot ca))((bb \cdot aa)(ba \cdot ba)) = ((ca \cdot ca)(ca \cdot ca))((bb \cdot aa)(bb \cdot aa)) = ((ca \cdot ca)(ca \cdot ca))((bb \cdot bb)(aa \cdot aa)) = ((ca \cdot ca)(bb \cdot bb))((ca \cdot ca)(aa \cdot aa)) = ((ca \cdot bb)(ca \cdot bb))((ca \cdot ca)(aa \cdot aa)) = ((ca \cdot bb)(ca \cdot bb))((ca \cdot aa)(ca \cdot aa)) = ((cb \cdot ab)(ca \cdot bb))((ca \cdot aa)(ca \cdot aa)) = ((cb \cdot ab)(cb \cdot ab))((ca \cdot aa)(ca \cdot aa)) = ((cb \cdot cb)(ab \cdot ab))((ca \cdot aa)(ca \cdot aa)) = ((cb \cdot cb)(ca \cdot aa))((ab \cdot ab)(ca \cdot aa)) = ((cb \cdot ca)(cb \cdot aa))((ab \cdot ab)(ca \cdot aa)) = ((cc \cdot ba)(cb \cdot aa))((ab \cdot ab)(ca \cdot aa)) = ((cc \cdot ba)(ca \cdot ba))((ab \cdot ab)(ca \cdot aa)) = ((cc \cdot ca)(ba \cdot ba))((ab \cdot ab)(ca \cdot aa)) = ((cc \cdot ca)(ba \cdot ba))((ab \cdot ca)(ab \cdot aa)) = ((cc \cdot ca)(ba \cdot ba))((ab \cdot ca)(aa \cdot ba))$.

(ii) With respect to I.1.13, we can assume that $\varphi = \sigma_G$ is an automorphism of G. Now, consider the idempotent distributive groupoid $G(*)$ constructed in 6.2. By [11, IV.1.4], $G(*)$ is middle semimedial, and therefore G is middle semimedial by 6.2(iii).

(iii) Combine (ii) and 7.1.

(iv) Obvious. ▲

10.2 Remark. Let F be a free semimedial groupoid. Then F is neither left nor right cancelative. In fact, it is easy to see that F is not middle semimedial (e.g., $(x \cdot xx)((x \cdot xx)x) \neq (x(x \cdot xx))(xx \cdot x)$ for every free generator $x \in F$) and consequently F is not cancellative (by 10.1). However, F is isomorphic to its opposite groupoid and it follows that F is neither left nor right cancellative.

VI.11 Cancellable (injective) elements of semimedial groupoids

11.1 Let G be a groupoid. An element $a \in G$ is said to be left (resp. right) injective (or cancellable) if the left (resp. right) translation $\mathrm{L}_{a,G}$ (resp. $\mathrm{R}_{a,G}$) is injective.

We denote by $\mathrm{A}_l(G)$ (resp. $\mathrm{A}_r(G)$) the set of left (resp. right) injective elements and we put $\mathrm{A}(G) = \mathrm{A}_l(G) \cap \mathrm{A}_r(G)$.

11.2 Lemma. *Let G be a left semimedial groupoid and $a, b \in G$.*

(i) If $a, ab \in \mathrm{A}_l(G)$, then $b \in \mathrm{A}_l(G)$.

(ii) If $aa, b \in \mathrm{A}_l(G)$, then $a \in \mathrm{A}_l(G)$.

(iii) If $a \in \mathrm{A}_l(G)$ and $ab \in \mathrm{A}_r(G)$, then $b \in \mathrm{A}_r(G)$.

(iv) If $aa \in \mathrm{A}_l(G)$ and $b \in \mathrm{A}_r(G)$, then $a \in \mathrm{A}_l(G)$.

Proof. By 1.2, $\mathrm{L}_{aa}\mathrm{L}_b = \mathrm{L}_{ab}\mathrm{L}_a$ and $\mathrm{L}_{aa}\mathrm{R}_b = \mathrm{R}_{ab}\mathrm{L}_a$. Now, it sufficies to use these equalities and the following well known observation: Let f, g be transformations of G. If f, g are injective, then fg is so. If fg is injective, then g is so. ▲

11.3 Proposition. *Let G be a left semimedial groupoid. Then:*

(i) $\mathrm{A}_l(G)$ is a left closed subset of G; if $a \in G$ and $\sigma_G(a) = aa \in \mathrm{A}_l(G)$, then $a \in \mathrm{A}_l(G)$.

(ii) $\mathrm{A}(G)$ is a left closed subset of G; if $a \in G$ and $\sigma_G(a) \in \mathrm{A}(G)$, then $a \in \mathrm{A}(G)$.

Proof. (i) The assertion follows immediately from 11.2(i), (ii).

(ii) Let, $a, b \in G$, $a, ab \in \mathrm{A}(G)$. Then $a, ab \in \mathrm{A}_l(G)$ and $b \in \mathrm{A}_l(G)$ by (i). Further, $a \in \mathrm{A}_l(G)$, $ab \in \mathrm{A}_r(G)$ and $b \in \mathrm{A}_r(G)$ by 11.2(iii). Thus $b \in \mathrm{A}(G)$ and we have proved that $\mathrm{A}(G)$ is left closed. Finally, let $a \in G$ and $aa \in \mathrm{A}(G)$. Then $aa = \mathrm{A}_l(G)$, and hence $a \in \mathrm{A}_l(G)$ by (i). Consequently, $a \in \mathrm{A}_l(G)$, $aa \in \mathrm{A}_r(G)$, and so $a \in \mathrm{A}_r(G)$ by 11.2(iii). Thus $a \in \mathrm{A}(G)$. ▲

11.4 Proposition. *Let G be a left cancellative left semimedial groupoid.*

(i) $\mathrm{A}_r(G) = \mathrm{A}(G)$ and if $a, b \in G$ and $ab \in \mathrm{A}(G)$, then $b \in \mathrm{A}(G)$.

(ii) If $\mathrm{A}(G) \neq G$, then $G \setminus \mathrm{A}(G)$ is a left ideal of G.

Proof. We have $G = \mathrm{A}_l(G)$ and it is enough to use 11.2(iii). ▲

11.5 Proposition. *Let G be a semimedial groupoid.*

(i) $\mathrm{A}(G)$ is a closed subset of G.

(ii) If G is left (resp. right) cancellative and if $a, b \in G$ are such that $ab \in \mathrm{A}(G)$, then $a, b \in \mathrm{A}(G)$. In other words, $G \setminus \mathrm{A}(G)$ is either empty or an ideal of G.

Proof. (i) This follows from 11.3(ii) and its right hand form.

(ii) Let $ab \in \mathrm{A}(G)$. By 11.4 (i), $b \in \mathrm{A}(G)$. However, by the dual of 11.3(ii), $\mathrm{A}(G)$ is right closed. Hence $a \in \mathrm{A}(G)$ as well. ▲

11.6 Proposition. *Let G be a left (right) quasigroup such that $\mathrm{A_r}(G) \neq \emptyset$ $(\mathrm{A_l}(G) \neq \emptyset)$.*

(i) If G is left semimedial, then $\mathrm{A}(G)$ is a left ideal of G. If, moreover, $\mathrm{A}(G) \neq G$, then $G \setminus \mathrm{A}(G)$ is a left ideal.

(ii) If G is semimedial, then G is cancellative.

Proof. (i) We have $\mathrm{A}(G) = \mathrm{A_r}(G) \neq \emptyset$ and, by 11.4, $G \setminus \mathrm{A}(G)$ is either empty or a left ideal of G. Now, let $a \in G$ and $b \in \mathrm{A}(G)$. We have $\mathrm{L}_{aa}\mathrm{R}_b = \mathrm{R}_{ab}\mathrm{L}_a$, and so $\mathrm{R}_{ab} = \mathrm{L}_{aa}\mathrm{R}_b\mathrm{L}_a^{-1}$ is injective. Thus $ab \in \mathrm{A_r}(G) = \mathrm{A}(G)$.

(ii) Let $a \in G$, $b \in \mathrm{A}(G)$. By (i), $ab \in \mathrm{A}(G)$. However, then $a \in \mathrm{A}(G)$ by 11.5(ii). Therefore $G = \mathrm{A}(G)$. ▲

11.7 Remark. Let G be an ideal-simple left (right) cancellative semimedial groupoid. Put $I = G \setminus \mathrm{A}(G)$. If $I = \emptyset$, then G is cancellative. If $I = G$, then $\mathrm{A}(G) = \emptyset$ and hence $\mathrm{A_r}(G) = \emptyset$ $(\mathrm{A_l}(G) = \emptyset)$. The last possibility is $I \neq \emptyset, G$. Since G is ideal-simple, $I = \{e\}$ is then a one-element set. Then, of course, e is an absorbing element of G and, since G is left (right) cancellative, $G = \{e\}$ and $I = G$, a contradiction.

11.8 Let G be a groupoid. Denote by $\mathrm{A_l^*}(G)$ the set of $a \in G$ such that $\sigma_G^n(a) \in \mathrm{A_l}(G)$ for each $n \geq 0$. The set $\mathrm{A_r^*}(G)$ is defined dually and $\mathrm{A^*}(G) = A_l^*(G) \cap A_r^*(G) = \{a \in G; \sigma_G^n(a) \in \mathrm{A}(G) \text{ for each } n \geq 0\}$. Clearly, $\mathrm{A_l^*}(G) \subseteq \mathrm{A_l}(G)$, $A_r^*(G) \subseteq \mathrm{A_r}(G)$, $\mathrm{A^*}(G) \subseteq \mathrm{A}(G)$ and the sets $\mathrm{A_l^*}(G)$, $\mathrm{A_r^*}(G)$, $\mathrm{A^*}(G)$ are closed under σ_G.

11.9 Proposition. *Let G be a left semimedial groupoid. Then $\mathrm{A_l^*}(G)$ is either empty or a left closed subgroupoid of G (in that case, $\mathrm{A_l^*}(G)$ is a left cancellative groupoid). Moreover, if $a \in G$ and $aa \in \mathrm{A_l^*}(G)$ then $a \in \mathrm{A_l^*}(G)$.*

Proof. First, let $a, b \in \mathrm{A_l^*}(G)$. By 1.2, $\mathrm{L}_{a^2 \cdot a^2}\mathrm{L}_{b^2} = \mathrm{L}_{ab \cdot a^2}\mathrm{L}_{ab}$. We have $a^2 \cdot a^2$, $b^2 \in \mathrm{A_l}(G)$, and so L_{ab} is injective and $ab \in \mathrm{A_l}(G)$. Now, for every $n \geq 0$, $\sigma_G^n(a), \sigma_G^n(b) \in \mathrm{A_l^*}(G)$, and so $\sigma_G^n(ab) = \sigma_G^n(a)\,\sigma_G^n(b) \in \mathrm{A_l}(G)$. Consequently, $ab \in A_l^*(G)$ and we have proved that $\mathrm{A_l^*}(G)$ is a subgroupoid (if non-empty).

Next, let $a, b \in G$ and $a, ab \in \mathrm{A_l^*}(G)$. Then, for each $n \geq 0$, $\sigma_G^n(ab) = \sigma_G^n(a)\,\sigma_G^n(b) \in \mathrm{A_l}(G)$ and $\sigma_G^n(a) \in \mathrm{A_l}(G)$. But $\mathrm{A_l}(G)$ is left closed, and hence $\sigma_G^n(b) \in \mathrm{A_l}(G)$ and $b \in \mathrm{A_l^*}(G)$.

Finally, let $aa \in \mathrm{A_l^*}(G)$. Then $\sigma_G^n(aa) = \sigma_G^n(a)\,\sigma_G^n(a) \in \mathrm{A_l}(G)$ for each $n \geq 0$. By 11.3(i), $\sigma_G^n(a) \in \mathrm{A_l}(G)$, i.e. $a \in \mathrm{A_l^*}(G)$. ▲

11.10 Lemma. *Let G be a left semimedial groupoid. Then $\mathrm{A^*}(G)$ is a left closed subset of G.*

Proof. Let $a, ab \in \mathrm{A^*}(G)$. Then $\sigma_G^n(a), \sigma_G^n(a)\,\sigma_G^n(b) \in \mathrm{A}(G)$ for each $n \geq 0$. By 11.3(ii), $\sigma_G^n(b) \in \mathrm{A}(G)$ and we see that $b \in \mathrm{A^*}(G)$. ▲

11.11 Lemma. *Let G* be a semimedial groupoid. *If* $aa \in \mathrm{A_l}(G)$ and $bb \cdot bb \in \mathrm{A_r}(G)$, then $ab \in \mathrm{A_l}(G)$ and $b \in \mathrm{A}(G)$.

Proof. By 11.3(ii) and its dual, $a \in A_l(G)$ and $b \in A_r(G)$. Further, by 1.3, $R_{b^2 \cdot b^2} L_{a^2} = R_{ab \cdot b^2} L_{ab}$, and hence L_{ab} is injective and $ab \in A_l(G)$. Since $A_l(G)$ is left closed, we have also $b \in A_l(G)$. Then, of course, $b \in A(G)$. ▲

11.12 Theorem. *Let G be a semimedial groupoid. Then:*

(i) $A(G)$ *is a closed subset of G.*

(ii) $A^*(G)$ *is either empty or a closed subgroupoid of G (in that case,* $A^*(G)$ *is a cancellative groupoid).*

(iii) If $A_l^*(G) \neq \emptyset \neq A_r^*(G)$ *(e.g. if* $A^*(G) \neq \emptyset$*), then* $A_l^*(G) = A_r^*(G) = A^*(G)$.

(iv) If $a \in G$ *and* $aa \in A(G)$ *(resp.* $aa \in A^*(G)$*), then* $a \in A(G)$ *(resp.* $a \in A^*(G)$*).*

Proof. (i) See 11.5(i).

(ii) By 11.10 and its dual, $A^*(G)$ is a closed subset of G. Let $A^*(G) \neq \emptyset$; we have $A^*(G) = A_l^*(G) \cap A_r^*(G)$ and $A^*(G)$ is a subgroupoid of G by 12.9 and its right hand form.

(iii) This assertion follows easily from 12.11 and its dual.

(iv) See 11.3(ii) and 11.9 and its right hand form. ▲

VI.12 Divisible (projective) elements of semimedial groupoids

12.1 Let G be a groupoid. An element $a \in G$ is said to be left (resp. right) projective (or divisible) if the left (resp. right) translation $L_{a,G}$ (resp. $R_{a,G}$) is projective.

We denote by $B_l(G)$ (resp. $B_r(G)$) the set of left (resp. right) projective elements and we put $B(G) = B_l(G) \cap B_r(G)$.

12.2 Lemma. *Let G be a left semimedial groupoid and* $a, b \in G$.

(i) If $a, ab \in B_l(G)$, *then* $aa \in B_l(G)$.

(ii) If $aa, b \in B_l(G)$, *then* $ab \in B_l(G)$.

(iii) If $a \in B_l(G)$ *and* $ab \in B_r(G)$, *then* $aa \in B_l(G)$.

(iv) If $aa \in B_l(G)$ *and* $b \in B_r(G)$, *then* $ab \in B_r(G)$.

Proof. Use 1.2 and the following simple fact: Let f, g be transformation of G. If f, g are projective, then fg is so. If fg is projective, then f is so. ▲

12.3 Proposition. *Let G be a left semimedial groupoid.*

(i) $B_l(G)$ *is either empty or a subgroupoid of G.*

(ii) If G is left divisible, then $B(G)$ *is either empty or a left ideal of G.*

(iii) If G is a left quasigroup, then $G \setminus B(G)$ *is either empty or a left ideal of G.*

Proof. (i) Let $a \in B_l(G)$ and $x \in G$. There are $y, z, v \in G$ such that $ay = a$, $az = v$ and $av = x$. Now, $aa \cdot yz = ay \cdot az = av = x$ and we have proved that $aa \in B_l(G)$. The result now follows from 12.2(ii).

(ii) We have $B(G) = B_r(G)$ and 12.2(iv) applies.

(iii) Let $a, b \in G$ and $ab \in B(G)$. We have $R_b = L_{aa}^{-1} R_{ab} L_a$, and hence R_b is projective and $b \in B(G)$. ▲

12.4 Corollary. *Let G be a semimedial groupoid. Then* $\mathrm{B}(G)$ *is either empty or a subgroupoid of G.*

VI.13 Bijective elements of semimedial groupoids

13.1 Let G be a groupoid. An element $a \in G$ is said to be left (resp. right) bijective if the left (resp. right) translation $\mathrm{L}_{a,G}$ (resp. $\mathrm{R}_{a,G}$) is bijective. We denote by $\mathrm{C}_\mathrm{l}(G)$ (resp. $\mathrm{C}_\mathrm{r}(G)$) the set of left (resp. right) bijective elements and we put $\mathrm{C}(G) = \mathrm{C}_\mathrm{l}(G) \cap \mathrm{C}_\mathrm{r}(G)$.

13.2 Lemma. *Let G be a left semimedial groupoid and* $a, b \in G$.

(i) *If* $a, aa, ab \in \mathrm{C}_\mathrm{l}(G)$, *then* $b \in \mathrm{C}_\mathrm{l}(G)$.
(ii) *If* $aa, b, ab \in \mathrm{C}_\mathrm{l}(G)$, *then* $a \in \mathrm{C}_\mathrm{l}(G)$.
(iii) *If* $a, aa, b \in \mathrm{C}_\mathrm{l}(G)$, *then* $ab \in \mathrm{C}_\mathrm{l}(G)$.
(iv) *If* $a, b, ab \in \mathrm{C}_\mathrm{l}(G)$, *then* $aa \in \mathrm{C}_\mathrm{l}(G)$.
(v) *If* $a, aa \in \mathrm{C}_\mathrm{l}(G)$ *and* $ab \in \mathrm{C}_\mathrm{r}(G)$, *then* $b \in \mathrm{C}_\mathrm{r}(G)$.
(vi) *If* $a, aa \in \mathrm{C}_\mathrm{l}(G)$ *and* $b \in \mathrm{C}_\mathrm{r}(G)$, *then* $ab \in \mathrm{C}_\mathrm{r}(G)$.
(vii) *If* $aa \in \mathrm{C}_\mathrm{l}(G)$ *and* $b, ab \in \mathrm{C}_\mathrm{r}(G)$, *then* $a \in \mathrm{C}_\mathrm{l}(G)$.
(viii) *If* $a \in \mathrm{C}_\mathrm{l}(G)$ *and* $b, ab \in \mathrm{C}_\mathrm{r}(G)$, *then* $aa \in \mathrm{C}_\mathrm{l}(G)$.

Proof. We have $\mathrm{L}_{aa}\mathrm{L}_b = \mathrm{L}_{ab}\mathrm{L}_a$ and $\mathrm{L}_{aa}\mathrm{R}_b = \mathrm{R}_{ab}\mathrm{L}_a$.

13.3 Let G be a groupoid. Put $\mathrm{C}_\mathrm{l}^*(G) = \{a \in G;\ \sigma_G^n(a) \in \mathrm{C}_\mathrm{l}(G) \text{ for each } n \geq 0\}$, $\mathrm{C}_\mathrm{r}^*(G) = \{a \in G;\ \sigma_G^n(a) \in \mathrm{C}_\mathrm{r}(G) \text{ for each } n \geq 0\}$ and $\mathrm{C}^*(G) = \mathrm{C}_\mathrm{l}^*(G) \cap \mathrm{C}_\mathrm{r}^*(G) = \{a \in G;\ \sigma_G^n(a) \in \mathrm{C}(G) \text{ for each } n \geq 0\}$.

13.4 Proposition. *Let G be a left semimedial groupoid. Then:*

(i) $\mathrm{C}_\mathrm{l}^*(G)$ *is either empty or a left closed subgroupoid of G (in that case,* $\mathrm{C}_\mathrm{l}^*(G)$ *is a left quasigroup).*

(ii) $\mathrm{C}^*(G)$ *is either empty or a left closed subgroupoid of G (in that case,* $\mathrm{C}^*(G)$ *is a quasigroup).*

Proof. (i) Use 13.2(i), (iii).
(ii) Use (i) and 13.2(v), (vi). ▲

13.5 Corollary. *Let G be a semimedial groupoid. Then* $\mathrm{C}^*(G)$ *is either empty or a closed subgroupoid of G (in that case,* $\mathrm{C}^*(G)$ *is a quasigroup).*

VI.14 Comments and open problems

Many results of this chapter are new and some other results on semimedial groupoids and quasigroups can be found in [7], [14] and [16]. The main open problem is whether every cancellative semimedial groupoid is a subgroupoid of a semimedial quasigroup.

VII. Cancellative trimedial groupoids

VII.1 Introduction

1.1 A groupoid G is said to be
- monomedial if every one-generated subgroupoid of G is medial;
- dimedial if every subgroupoid generated by at most two elements is medial;
- trimedial if every subgroupoid generated by at most three elements is medial;
- left near-trimedial if $ab \cdot xy = ax \cdot by$ for all $a, b, x, y \in G$ such that $a, b \in \langle c \rangle_G$ for some $c \in G$;
- right near-trimedial if $xy \cdot ab = xa \cdot yb$ for $a, b, x, y \in G$ such that $a, b \in \langle c \rangle_G$ for some $c \in G$;
- near-trimedial if G is both left and right near-trimedial.

1.2 In the sequel, we will consider the following groupoid equations:

(GE1) $(x \cdot xx)(yz) \triangleq (xy)(xx \cdot z)$;
(GE2) $(zy)(xx \cdot x) \triangleq (z \cdot xx)(yx)$;
(GE3) $(xx \cdot x)(yz) \triangleq (xx \cdot y)(xz)$;
(GE4) $(zy)(x \cdot xx) \triangleq (zx)(y \cdot xx)$;
(GE5) $(yx)(xx \cdot z) \triangleq (y \cdot xx)(xz)$;
(GE6) $(xy)(xy \cdot z) \triangleq (x \cdot xy)(yz)$;
(GE7) $(z \cdot xy)(xy) \triangleq (zx)(xy \cdot y)$;
(GE8) $(xy)(z \cdot xy) \triangleq (xz)(y \cdot xy)$;
(GE9) $(xy \cdot z)(xy) \triangleq (xy \cdot x)(zy)$;
(GE10) $((xy \cdot xy)(x \cdot xy))((xy \cdot y)(xy \cdot z)) \triangleq ((xy \cdot xy)(xy \cdot y))((x \cdot xy)(xy \cdot z))$;
(GE11) $((z \cdot xy)(x \cdot xy))((xy \cdot y)(xy \cdot xy)) \triangleq ((z \cdot xy)(xy \cdot y))((x \cdot xy)(xy \cdot xy))$;

1.3 Proposition. *Let G be a left (right) near-trimedial groupoid. Then:*

(i) G is left (right) semimedial.

(ii) Both τ_G and ϱ_G are endomorphisms of G.

(iii) G satisfies (GE1) and (GE3) ((GE2) and (GE4)).

(iv) G is monomedial.

(v) The condition (A10) ((B10)) is satisfied for any subgroupoid H of G (see I.4.1 and I.4.5).

1.4 Proposition. *Let G be a trimedial groupoid. Then:*

(i) G is near-trimedial and strongly semimedial.

(ii) G satisfies all the equations (GE1), ..., (GE11).

1.5 Lemma. *Let G be a groupoid and $a, b \in G$.*

(i) If G satisfies (GE6), then $L^2_{ab} = L_{a \cdot ab}L_b$.

(ii) If G satisfies (GE7), then $R^2_{ab} = R_{b \cdot ab}R_a$.

(iii) *If G satisfies (GE8), then* $\mathrm{L}_{ab}\mathrm{R}_{ab} = \mathrm{R}_{b\cdot ab}\mathrm{L}_a$.

(iv) *If G satisfies (GE9), then* $\mathrm{R}_{ab}\mathrm{L}_{ab} = \mathrm{L}_{ab\cdot a}\mathrm{R}_b$.

(v) *If G satisfies (GE6), (GE10) and G is left semimedial, then* $\mathrm{L}_c\mathrm{L}_{ab}^2 = \mathrm{L}_d\mathrm{L}_{a\cdot ab}\mathrm{L}_{ab}$, *where* $c = (ab)^2 \cdot (ab)^2$ *and* $d = (ab)^2 \cdot (ab \cdot b)$.

(vi) *If G satisfies (GE7), (GE11) and G is right semimedial, then* $\mathrm{R}_c\mathrm{R}_{ab}^2 = \mathrm{R}_d\mathrm{R}_{ab\cdot b}\mathrm{R}_{ab}$, *where* $c = (ab)^2 \cdot (ab)^2$ and $d = (a \cdot ab) \cdot (ab)^2$.

Proof. (v) Put $e = ab$. Then $(e^2e^2)(e \cdot ex) = (ee \cdot ee)((ab)(ab \cdot x)) = (ee \cdot ee)((a \cdot ab)(bx)) = (ee \cdot ee)(ae \cdot bx) = (ee \cdot ae)(ee \cdot bx) = (ee \cdot ae)(eb \cdot ex) = ((ab \cdot ab)(a \cdot ab))((ab \cdot b)(ab \cdot x)) = ((ab \cdot ab)(ab \cdot b))((a \cdot ab)(ab \cdot x)) = (ee \cdot eb)(ae \cdot ex)$. ▲

1.6 We will say that a groupoid G is

- a left (right) F-groupoid if $a \cdot bc = ab \cdot ec$ $(cb \cdot a = ce \cdot ba)$ whenever $a, b, c, e \in G$ and $ae = a$ $(ea = a)$;
- an F-groupoid if it is both a left and right F-groupoid;
- a left (right) E-groupoid if $a \cdot bc = eb \cdot ac$ $(cb \cdot a = ca \cdot be)$ whenever $a, b, c, e \in G$ and $ea = a$ $(ae = a)$;
- an E-groupoid if it is both a left and right E-groupoid.

1.7 Proposition. *(i) Every left cancellative left semimedial groupoid satisfying (GE1) is a left F-groupoid.*

(ii) Every left cancellative semimedial groupoid satisfying (GE3) and (GE5) is a right E-groupoid.

Proof. If $a, b, c, e \in G$, $a = ea$, then $(a \cdot aa)(a \cdot bc) = (aa)(aa \cdot bc) = (aa)(ab \cdot ac) = (a \cdot ab)(a \cdot ac) = (a \cdot ab)(ae \cdot ac) = (a \cdot ab)(aa \cdot ec) = (a \cdot aa)(ab \cdot ec)$, and hence $a \cdot bc = ab \cdot ec$.

(ii) If $a, b, c, e \in G$, $a = ae$, then $(aa \cdot a)(cb \cdot a) = (aa \cdot cb)(aa) = (ac \cdot ab)(aa) = (ac \cdot a)(ab \cdot a) = (ac \cdot a)(ab = ae) = (ac \cdot a)(aa \cdot be)$ and $(aa \cdot a)(ca \cdot be) = (aa \cdot ca)(a \cdot be) = (ac \cdot aa)(a \cdot be) = (ac \cdot a)(aa \cdot be)$. Consequently, $(aa \cdot a)(cb \cdot a) = (aa \cdot a)(ca \cdot be)$ and $cb \cdot a = ca \cdot be$. ▲

1.8 Corollary. *Let G be a cancellative semimedial groupoid.*

(i) If G satisfies (GE1) and (GE2), then G is an F-groupoid.

(ii) If G satisfies (GE3), (GE4) and (GE5), then G is an E-groupoid.

1.9 Remark. Let F be a free monomedial (resp., dimedial, trimedial, medial) groupoid. Then F is neither left nor right cancellative. Indeed, if $x \in F$ is a free generator and $y = xx$, then $((x \cdot xy)(yx))((x \cdot yx)(xy \cdot x)) = ((x \cdot xy)(yx))((x \cdot xy)(yx \cdot x) = ((x \cdot xy)(x \cdot xy))((yx)(yx \cdot x)) = ((xx)(xy \cdot xy))((yx)(yx \cdot x)) = ((xx)(xx \cdot yy))((yx)(yx \cdot x)) = (y(y \cdot yy))((xx \cdot x)(yx \cdot x) = (y(y \cdot yy))((xx \cdot yx)(xx)) = (y(y \cdot yy))((xy \cdot xx)(xx)) = (y(xy \cdot xx))((y \cdot yy)(xx)) = ((xx)(xy \cdot xx))((y \cdot yy)(xx)) = ((x \cdot xy)(x \cdot xx))((yx)(yy \cdot x)) = ((x \cdot xy)(yx))((x \cdot xx)(yy \cdot x)) = ((x \cdot xy)(yx))((xy)(yy \cdot x))$. On the other hand, it is easy to see that $(x \cdot yx)(xy \cdot x) \neq (xy)(yy \cdot x)$.

2.1 Let H be a subgroupoid of a left near-trimedial groupoid G and let $a \in H$ and $a = \langle a \rangle_H$. Put $\delta_a(H) = \delta_{a,G}(H) = \{x \in G;\ bx \in H$ for some $b \in A\}$.

(i) If $b, c \in A$ and $bx, cy \in H$, then $bc \cdot xy = bx \cdot cy \in H$, which shows that $\delta_a(H)$ is a subgroupoid of G; clearly, $H \subseteq \delta_a(H)$.

(ii) Assume that for all $b \in A$ and $u \in H$, there exists a uniquely determined $x \in G$ with $bx = u$ (e.g., if G is a left quasigroup). Then the mapping $\varphi : (b, u) \to x$ is a projective homomorphism of $A \times H$ onto $\delta_a(H)$. If $((b, u), (c, v)) \in \ker(\varphi)$, then $bx = u$, $cx = v$ for some $x \in \delta_a(H)$, and therefore $db \cdot v = db \cdot cx = dc \cdot bx = dc \cdot u$ for every $d \in A$. Conversely, if $b, c \in A$, $u, v \in H$, $eb \cdot v = ec \cdot u$, $e \in A$ and if $bx = u$, $cy = v, x, y \in \delta_a(H)$, then $ec \cdot bx = ec \cdot u = eb \cdot v = eb \cdot cy = ec \cdot by$. But $ec \cdot bx = ec \cdot u \in H$ and $ec \in A$. Thus $bx = u = by$ and $x = y$. We have proved that $((b, u), (c, v)) \in \ker(\varphi)$ if and only if $db \cdot v = dc \cdot u$ for some (and then for each) $d \in A$.

(iii) Now, assume that $\mathrm{L}_{b,G}$ is injective for every $b \in A$ and put $K = \{(b,u) \in A \times H;\ bx = u$ for some $x \in H\}$. Then K is a subgroupoid of $A \times H$ and $\varphi : K \to \delta_a(H)$, $\varphi(b, u) = x$, is a projective homomorphism of K onto $\delta_a(H)$. Again, $((b, u), (c, v) \in \ker(\varphi)$ if and only if $db \cdot v = dc \cdot u$ for some (and then for each) $d \in A$.

2.2 Let H be a subgroupoid of a left near-trimedial groupoid G, let $a \in H$ and $A = \langle a \rangle_H$. Put $\varepsilon_a(H) = \varepsilon_{a,H}(H) = \{x \in G;\ f(x) \in H$ for some $f \in \mathrm{Mul}_l(G, A)\}$.

(i) First, we check that $\varepsilon_a(H)$ is a subgroupoid of G. If $n, m \geq 0$, $a_i \in A$, $b_j \in A$, $x, y \in \varepsilon(H)$, $a_1(\dots(a_n x)) \in H$ and $b_1(\dots(b_m y)) \in H$, then we may assume $n = m$ and we get $(a_1 b_1)(\dots(a_n b_n \cdot xy)) = (a_1(\dots(a_n x)))(b_1(\dots(b_n y))) \in H$. Thus $xy \in \varepsilon_a(H)$ and we have proved that $\varepsilon_a(H)$ is a subgroupoid of G; clearly, $H \subseteq \varepsilon_{a,G}(H) \subseteq \alpha_G(H)$.

(ii) Put $H_0 = H$ and $H_{i+1} = \delta_{a,G}(H_i)$ for every $i \geq 0$ (see 2.1). Then $H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$ is a countable chain of subgroupoids of $\varepsilon_{a,G}(H)$ and it is easy to see that $\varepsilon_{a,G}(H) = \bigcup_{i=0}^{\infty} H_i$.

(iii) Now, assume that the left translation $\mathrm{L}_{b,G}$ is injective for every $b \in A$. According to 2.1(iii), there exist subgroupoids K_i of $A^{(i)} \times H$ (here, $A^i = A \times A \times \dots \times A$ i-times) and projective homomorphisms $\varphi : K_i \to H_i$. In particular, the groupoids H and $\varepsilon_{a,G}(H)$ are equationally equivalent.

(iv) We have $L = \langle H \rangle_{\mathrm{lc},G} = [H]_{\mathrm{l},G} = \alpha_G(H)$ (see I.4.3). If $x \in L$, then $a_1(\dots(a_n x)) \in H$ for some $n \geq 0$ and $a_1, \dots, a_n \in H$ and we have also $a_2(\dots(a_n x)) \in \delta_{a_1}(H)$, $a_3(\dots(a_n x)) \in \delta_{a_2}(\delta_{a_1}(H)), \dots, x \in \delta_{a_n}(\dots(\delta_{a_1}(H)))$. Now, it is clear that L is just the union of subgroupoids of the form $\delta_{b_1,G}(\delta_{b_2,G}(\dots(\delta_{b_m,G}(H))))$, $m \geq 0$, $b_1, \dots, b_m \in H$. Moreover, if all the left translations $L_{b,G}$, $b \in H$, are injective, than the subgroupoid $\delta_{b_1}(\delta_{b_2}(\dots(\delta_{b_m}(H))))$ is a homomorphic image of a subgroupoid of the $(m+1)$-th power $H^{(m+1)}$. In particular, the groupoids H and L are equationally equivalent.

2.3 Lemma. *Let H be a left strongly dense subgroupoid of a left cancellative left near-trimedial groupoid G and let $n \geq 1$ and $x_1, \dots, x_n \in G$. Then there is $k \geq 1$ such that the subgroupoid $\langle H, x_1, \dots, x_n \rangle_G$ is a homomorphism image of a subgroupoid of the k-th power $H^{(k)}$.*

Proof. Use 2.2(iv). ▲

2.4 Proposition. *Let H be a left strongly dense subgroupoid of a left cancellative left near-trimedial groupoid G. Then the groupoids H and G are equationally equivalent.*

Proof. See 2.2(iv) (or combine 1.3(v) and II.1.10). ▲

2.5 Proposition. *Let G be a left cancellative trimedial groupoid. Then:*

(i) The subgroupoid $\langle a, b, c \rangle_{\mathrm{lc}, G}$ is medial for all $a, b, c \in G$.

(ii) If $a, b \in G$ and $a = ab$, then the subgroupouid $\langle a, b, x, y \rangle_G$ is medial for all $x, y \in G$.

(iii) G is a left F-groupoid and a right E-groupoid.

Proof. (i) $H = \langle a, b, c \rangle_G$ is medial, and hence $K = \langle H \rangle_{\mathrm{lc}, G} = \langle a, b, c \rangle_{\mathrm{lc}, G}$ is medial by 2.4.

(ii) $\langle a, b, x, y \rangle_G \subseteq \langle a, x, y \rangle_{\mathrm{lc}, G}$ and (i) applies.

(iii) This follows immediately from (ii) (or see also 1.7). ▲

2.6 Let H be a subgroupoid of a near-trimedial groupoid G. Then $\langle H \rangle_{\mathrm{c}, G}$ is just the union of subgroupoids of the form $K = \kappa_{1, a_1, G}(\kappa_{2, a_2, G}(\dots (\kappa_{n, a_n, G}(H))))$, where $n \geq 0$, $a_1, \dots, a_n \in H$ and $\kappa_1, \dots, \kappa_n \in \{\delta, \bar{\delta}\}$ ($\bar{\delta}_a$ is the operator dual to δ — see 2.1 and 2.2).

If all the translations $\mathrm{L}_{a, G}$ and $\mathrm{R}_{a, G}$, $a \in H$, are injective, then each of the above subgroupoids K is a homomorphic image of a finite cartesian power of H.

2.7 Proposition. *Let H be a dense subgroupoid of a cancellative near-trimedial groupoid G. Then the groupoids H and G are equationally equivalent.*

Proof. See 2.6 (or II.2.8). ▲

2.8 Proposition. *Let G be a cancellative trimedial groupoid. Then:*

(i) The subgroupoid $\langle a, b, c \rangle_{\mathrm{c}, G}$ is medial for all $a, b, c \in G$.

(ii) G is both an F-groupoid and an E-groupoid.

Proof. Similar to that of 2.5. ▲

VII.3 When cancellative semimedial groupoids are trimedial

3.1 Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) Q is semimedial and satisfies (GE1).

(ii) Q is semimedial and satisfies (GE2).

(iii) Q satisfies $((xx \cdot yz)(vw \cdot uu))((p \cdot pp)(st)) \triangleq ((xy \cdot xz)(vu \cdot wu))((ps)(pp \cdot t))$.

Proof. See [17, Theorem 6].

3.2 Proposition. *Let G be a cancellative semimedial groupoid satisfying (GE1) (or (GE2)) and such that σ_G is injective. Then G is trimedial.*

Proof. With respect to I.1.13, we can assume that $\varphi = \sigma_G$ is an automorphism of G. Now, put $x * y = \varphi^{-1}(xy)$ for all $x, y \in G$. Then $G(*)$ becomes a cancellative idempotent distributive groupoid, φ is an automorphism of $G(*)$ becomes a cancellative idempotent distributive groupoid, φ is an automorphism of $G(*)$ and $xy = \varphi(x * y)$ for all $x, y \in G$. Further, $\varphi^2((x * \varphi(x)) * (y * z) = (x \cdot xx)(yz) = (xy)(xx \cdot z) = \varphi^2((x * y) * (\varphi(x) * z))$, and hence $(x * \varphi(x)) * (y * z) = (x * y) * (\varphi(x) * z)$ for all $x, y, z \in G$; equivalently, $(x, \varphi(x)) \in \mu_{G(*)}$ (see IV.5.9).

By IV.5.4, $G(*)$ is a dense subgroupoid of a distributive quasigroup $Q(*)$ and φ extends to an automorphism ψ of $Q(*)$. By IV.5.9, $(v, \psi(v)) \in \mu_{Q(*)}$ for every $v \in Q$. Now, define a multiplication on Q by $uv = \psi(u * v)$. Then Q is a semimedial quasigroup and G is a subgroupoid of Q. For $u, v, w \in Q$, we have $(u \cdot uu)(vw) = \psi^2((u * \psi(u)) * (v * w)) = \psi^2((u * v) * (\psi(u) * v)) = (uv)(uu \cdot w)$. We have shown that Q satisfies (GE1). It remains to apply 3.1. ▲

3.3 Proposition. *Let G be a cancellative semimedial groupoid satisfying (GE1) (or (GE2)) and containing at least one idempotent element. Then G is trimedial.*

Proof. Since G satisfies (GE1), the transformation τ_G is an endomorphism of G and then, by VI.8.3, G is a subgroupoid of a semimedial quasigroup Q. Now, let $e \in \mathrm{Id}(G)$. Then $\varphi = \mathrm{R}_{e,Q}$ and $\psi = \mathrm{R}_{e,Q}$ are automorphism of Q, $\varphi(e) = e = \psi(e)$ and $\varphi\psi = \psi\varphi$. Put $x + y = \varphi^{-1}(x)\,\psi^{-1}(y)$ for all $x, y \in Q$. Then $Q(+)$ is a loop ($e = 0$ is the neutral element), φ, ψ are automorphisms of $Q(+)$ and $xy = \varphi(x) + \psi(y)$ for all $x, y \in Q$.

By VI.10.1, Q is strongly semimedial. Consequently, $\psi(x)\,\varphi(y) = ex \cdot ye = ey \cdot xe = \psi(y)\,\varphi(x)$ for all $x, y \in Q$ and it follows that $x + y = \varphi^{-1}(x)\,\psi^{-1}(y) = \varphi^{-1}\psi^{-1}(\psi(x)\,\varphi(y)) = \varphi^{-1}\psi^{-1}(\psi(y)\,\varphi(x)) = \varphi^{-1}(y)\,\psi^{-1}(x) = y + x$. We have proved that $Q(+)$ is a commutative loop. Now, $Q(+)$ is a commutative Moufang loop and $\varphi(x) - \psi(x) \in Z(Q(+))$ (the centre) for every $x \in Q$.

If H is a subgroupoid of Q such that $0 \in H$, then $\psi(H) = 0 \cdot H \subseteq H$ and $\varphi(H) = H \cdot 0 \subseteq H$. Thus $H \subseteq \psi^{-1}(H) \cap \varphi^{-1}(H)$. But, φ, ψ are automorphism of Q, and so both $\psi^{-1}(H)$ and $\varphi^{-1}(H)$ are subgroupoids of Q and, of course, $\psi^{-1}(H) \cong H \cong \varphi^{-1}(H)$.

We have $0 \in G$ and $G \subseteq \varphi^{-1}(G) \subseteq \psi^{-1}\varphi^{-1}(G) \subseteq \psi^{-1}\varphi^{-2}(G) \subseteq \psi^{-2}\varphi^{-2}(G) \subseteq \ldots$ is a countable chain of subgroupoids isomorphic to H; denote by K the union of this chain. Then K is a subgroupoid of Q, K satisfies (GE1) and $\psi(K) = K = \varphi(K)$.

Now, put $L = \{a + u;\ a \in K,\ u \in Z(Q(+))\}$. If $a + u \in L$ and $b + v \in L$, then $(a + u)(b + v) = (\varphi(a) + \varphi(u)) + (\psi(b) + \psi(v)) = (\varphi(a) + \psi(b)) + (\varphi(u) + \psi(v)) = ab + w$, $w = \varphi(u) + \psi(v) \in Z(Q(+))$ and $ab \in K$. We see that L is a subgroupoid of Q; clearly $K \subseteq L$. Further, $\varphi(L) = \varphi(K) + \varphi(Z(Q(+))) = K + Z(Q(+)) = L$

and, similarly $\psi(L) = L$. Consequently, if $x, y \in L$, then $\varphi^{-1}(x)$, $\psi^{-1}(y) \in L$ and $x + y = \varphi^{-1}(x)\,\psi^{-1}(y) \in L$, and so $L(+)$ is a subgroupoid of $Q(+)$. On the other hand, $-3x \in Z(Q(+)) \subseteq L$, $-2x = x + (-3x) \in L$ and $-x = x + (-2x) \in L$ for every $x \in L$. This means that $L(+)$ is a subloop of $Q(+)$ and L a subquasigroup of Q.

Finally, let $a, b, c \in K$ and $u, v, w \in Z(Q(+))$, $x = a + u$, $y = b + v$, $z = c + w$. Then $(x \cdot xx)(yz) = (\varphi^2(x) + (\psi\varphi^2(x) + \psi^2\varphi(x))) + (\psi\varphi(y) + \psi^2(z)) = ((\varphi^2(a) + (\psi\varphi^2(a) + \psi^2\varphi(a))) + (\psi\varphi(b) + \psi^2(c)) + t = ((a \cdot aa)(bc) + t$, where $t = \varphi^2(u) + \psi\varphi^2(u) + \psi^2\varphi(u) + \psi\varphi(v) + \psi^2(w) \in Z(Q(+))$. Quite similarly, $(xy)(xx \cdot z) = ((ab)(aa \cdot c)) + s$, where $s = \varphi^2(u) + \psi\varphi(v) + \psi\varphi^2(u) + \psi^2\varphi(u) + \psi^2(w) \in Z(Q(+))$. However, $(a \cdot aa)(bc) = (ab)(aa \cdot c)$, and therefore $t = s$. Thus $(x \cdot xx)(yz) = (xy)(xx \cdot z)$ and the quasigroup L satisfies (GE1). By 3.1, L (and hence G) is trimedial. ▲

3.4 Proposition. *Let G be a cancellative semimedial groupoid satisfying (GE1) (or (GE2)) and containing at least one element e such that both* $\mathrm{L}_{e,Q}$ *and* $\mathrm{R}_{e,Q}$ *are bijections. Then G is trimedial.*

Proof. By IV.4.6, G is a subgroupoid of a semimedial quasigroup Q. Now, by [17, Theorem 3], there exist a commutative Moufang loop $Q(+)$, automorphisms γ, ψ of $Q(+)$ and an element $q \in Q$ such that $\varphi\psi = \psi\varphi$, $\varphi(x) - \psi(x) \in Z(Q(+))$ and $xy = (\varphi(x) + \varphi(y)) + q$ for all $x, y \in Q$. Without loss of generality, we may assume that $e = 0$ is the neutral element of $Q(+)$.

Since $\mathrm{L}_{0,G}$ and $\mathrm{R}_{0,G}$ are bijections of G, we have $\varphi^{-1}(x - q)$, $\psi^{-1}(x - q) \in G$ for every $x \in G$ (use the fact that $0 \cdot \psi^{-1}(x - q) = x = \varphi^{-1}(x - q) \cdot 0$). Now, if $x, y \in G$, then $(x + y) - q = ((x - q) + (y - q)) + q = \varphi^{-1}(x - q) \cdot \psi^{-1}(y - q) \in G$.

Put $L = \{a + u;\, a \in G, u \in Z(Q(+))\}$. Proceding similarly as in the proof of 3.3, we can show that L is a subgroupoid of Q and that L satisfies (GE1). If $a, b \in G$, then $(a + b) - q \in G$. In particular, since $0 \in G$, we have $a - q \in G$ and $b - q \in G$. Thus $(a + b) - 3q = ((a - q) + (b - q)) - q \in G$. But $3q \in Z(Q(+))$, so that $a + b \in L$. Now, it is easy to see that $L(+)$ is a subgroupoid of $Q(+)$ and, since $Z(Q(+)) \subseteq L$, in fact a subloop of $Q(+)$ (see the proof of 3.3).

If $x \in L$, then $\varphi(x) + q = x \cdot 0 \in L$, and so $\varphi(x) \in L$ $(q = 0 \cdot 0 \in G \subseteq L)$ and $\varphi(L) \subseteq L$. Similarly, $\psi(L)$ and, further, $\varphi^{-1}(a - q) \in G$ for each $a \in G$. Thus $\varphi^{-1}(-q) \in G$, $\varphi^{-1}(a) = \varphi^{-1}(a - q) - \varphi^{-1}(-q) \in L$ and $\varphi^{-1}(G) \subseteq L$. From this, $\varphi^{-1}(L) \subseteq L$ and $\varphi(L) = L$; similarly, $\xi\psi(L) = L$. It follows that L is a subquasigroup of Q and L is trimedial by 3.1. ▲

3.5 Theorem. *Let G be a cancellative semimedial groupoid satisfying (GE1) (or (GE2)). Then G is trimedial (and can be imbedded into a trimedial quasigroup) in each of the following cases:*

(i) The transformation σ_G *is injective.*

(ii) The transformation σ_G *is projective.*

(iii) *G contains at least one idempotent element.*

(iv) *G contains at least one element e such that both* $\mathrm{L}_{e,G}$ *and* $\mathrm{R}_{e,G}$ *are bijections of G.*

Proof. See 3.2, 3.3, 3.4 and VI.7.1. ▲

3.6 Remark. Let G be a cancellative semimedial groupoid satisfying (GE1). By VI.8.2 (in fact, the dual form of VI.8.2), there exist cancellative semimedial groupoids H and K satisfying (GE1) such that σ_H is an automorphism of H, τ_K of K and G can be imbedded into the product $H \times K$. Now, by 3.5, H is trimedial. In particular, G is trimedial if and only if K is so.

3.7 Remark. Let G be a cancellative semimedial groupoid satisfying (GE1) and such that $\varphi = \tau_G$ is an automorphism of G (cf. 3.6). Put $x * y = \varphi^{-1}(xy)$ for all $x, x \in G$. Then $G(*)$ is a cancellative semimedial groupoid, $x * (xx) = x$ for every $x \in G$ and $G(*)$ is a left F-groupoid (see 1.6).

3.8 Remark. Let G be a groupoid such that $\varphi = \tau_G$ and $\psi = \varrho_G$ are automorphisms of G (see I.1.12, I.1.13). Then $\varphi\psi = \psi\varphi$ (we have $\varphi\psi = \varphi(xx \cdot x) = \varphi(x)\,\varphi(x) \cdot \varphi(x) = \psi\varphi(x)$) and we will define a binary operation $*$ on G by $x * y = \varphi^{-1}(x)\,\psi^{-1}(y)$ for all $x, y \in G$. Then $G(*)$ is a groupoid, φ, ψ are automorphism of $G(*)$ and $xy = \varphi(x) * \psi(y)$ for all $x, y \in G$.

(i) For every $x \in G$, we have $x * \psi(\varphi^{-1}(x)\,\varphi^{-1}(x)) = \varphi^{-1}(x) \cdot \varphi^{-1}(x)\,\varphi^{-1}(x) = \varphi\varphi^{-1}(x) = x$ and $\varphi(\psi^{-1}(x)\,\psi^{-1}(x)) * x = \psi^{-1}(x)\,\psi^{-1}(x) \cdot \psi^{-1}(x) = \psi\psi^{-1}(x) = x$.

(ii) $G(*)$ is left (right) semimedial if and only if G satisfies the equation $(xx \cdot x)(x \cdot xx))(xy) \doteq ((xx \cdot x)\,y)((x \cdot xx)\,z)$ $((zy)((xx \cdot x)(x \cdot xx)) \doteq (z(xx \cdot x))(y(x \cdot xx)))$.

(iii) If G satisfies (GE1) ((GE2)), then $G(*)$ is a left (right) F-groupoid. Indeed, $x * (y * z) = \varphi^{-1}(x) \cdot (\psi^{-1}\varphi^{-1}(y)\,\varphi^{-2}(z)) = \varphi^{-2}((x \cdot xx)\,(\psi^{-1}\varphi(y)\,\psi^{-2}\varphi^{2}(z)) = \varphi^{-2}(x\psi^{-1}\varphi(y))\,(xx \cdot \psi^{-2}\varphi^{-2}(z))) = (x * y) * (\psi(\varphi^{-1}(x)\,\varphi^{-1}(x)) * z)$ (see (i)). The other case is similar.

(iv) $G(*)$ is left (right) near-trimedial, provided that G is so.

(v) $G(*)$ is medial if and only if G is so.

(vi) Let H be a subgroupoid of G. Then $\varphi(H) \subseteq H$ and $H \subseteq K = \varphi^{-1}(H)$. Clearly, K is a subgroupoid of G and $\varphi \restriction K$ is an isomorphism of K onto H. Similarly, $H \subseteq \mathrm{L} = \psi^{-1}(H)$ and $\psi \restriction \mathrm{L}$ is an isomorphism of L onto H. Now, we get a countable chain of subgroupoids $H = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$, where $H_i = H_i = \varphi^{-(i+1)/2} \cdot \psi^{-(i-1)/2}(H)$ for $i \geq 1$ odd and $H_i = \varphi^{-i/2}\psi^{-i/2}(H)$ for $i \geq 0$ even. If $P = \bigcup_{i=0}^{\infty} H_i$, then P is a subgroupoid of G, P and G are equaationally equivalent $\varphi(P) = P$, $\psi(P) = P$ and $P(*)$ is a subgroupoid of $G(*)$.

(vii) Suppose that G is trimedial, let $x, y, z \in G$ and $H = \langle x, y, z \rangle_G$. Now, consider the subgroupoid P constructed in (vi). Then P is medial, and hence $P(*)$ is also medial. But $\langle x, y, z \rangle_{G(*)} \subseteq P$. We have shown that the groupoid $G(*)$ is trimedial.

(viii) $G(*)$ is (left, right) cancellative if and only if G is so.
(ix) $G(*)$ is (left, right) divisible if and only if G is so.

VII.4 Imbeddings of cancellative trimedial groupoids into trimedial quasigroups (the Sholander's construction)

4.1 Let G be a groupoid and let A be a subgroupoid of G such that $L_{a,G}$ is injective for each $a \in A$ and that $ab \cdot xy = ax \cdot by$ for all $a, b \in A$ and $x, y \in G$ (it follows that A is a left cancellative medial groupoid). Now, put $H = A \times G$ and define a relation r on H by $((a, u), (b, v)) \in r$ if and only if $ca \cdot v = cb \cdot u$ for every $c \in A$.

4.1.1 Lemma. *Let $a, b \in A$ and $u, v \in G$. The following conditions are equivalent:*
(i) $((a, u), (b, v)) \in r$.
(ii) $aa \cdot v = ab \cdot u$.
(iii) $ba \cdot v = bb \cdot u$.
(iv) $da \cdot v = db \cdot u$ *for some* $d \in A$.

Proof. It is enough to show that (iv) implies (i). If $c \in A$, then $(dd \cdot d)(ca \cdot v) = (dd \cdot ca)(dv) = (dc \cdot da)(dv) = (dc \cdot d)(da \cdot v) = (dc \cdot d)(db \cdot u) = (dc \cdot db)(du) = (dd \cdot cb)(du) = (dd \cdot d)(cb \cdot u)$, and hence $ca \cdot v = cb \cdot u$. ▲

4.1.2 Lemma. *r is an equivalence.*

Proof. Clearly, r is reflexive and symmetric. Now, let $((a, u), (b, v)) \in r$ and $((b, v), (c, w)) \in r$. Then $ba \cdot v = bb \cdot u$, $bc \cdot v = bb \cdot w$ and $(bb \cdot bb)(aa \cdot w) = (bb \cdot aa)(bb \cdot w) = (bb \cdot aa)(bc \cdot v) = (ba \cdot ba)(bc \cdot v) = (ba \cdot bc)(ba \cdot v) = (ba \cdot bc)(bb \cdot u) = (bb \cdot ac)(bb \cdot u) = (bb \cdot bb)(ac \cdot u)$, and so $aa \cdot w = ac \cdot u$. By 3.1.1, $((a, u), (c, w)) \in r$. ▲

4.1.3 Lemma. *r is a congruence of the groupoid H.*

Proof. Let $((a, u), (b, v)) \in r$ and $((c, w), (d, z)) \in r$. Then $(ac \cdot ac)(vz) = (aa \cdot cc)(vz) = (aa \cdot v)(cc \cdot z) = (aa \cdot v)(cd \cdot w) = (ab \cdot u)(cd \cdot w) = (ab \cdot cd)(uw) = (ac \cdot bd)(uw)$ and it follows that $((ac, uw), (bd, vz)) \in r$. ▲

4.1.4 Lemma. *Let $a, b \in A$ and $u, v \in G$. Then $((a, au), (b, bv)) \in r$ if and only if $u = v$.*

Proof. If $((a, au), (b, bv)) \in r$, then $aa \cdot bv = ab \cdot au = aa \cdot bu$, $bv = bu$ and $u = v$. Conversely, if $u = v$, then $aa \cdot bu = ab \cdot au$. ▲

Now, denote by K the factor-groupoid H/r and define a mapping $\varphi : G \to K$ by $\varphi(u) = (a, au)/r \in K$, $a \in A$. According to 3.1.4, φ is an injective mapping.

4.1.5 Lemma. *φ is an injective groupoid homomorphism.*

Proof. For $u, v \in G$, $\varphi(u)\varphi(v) = (a, au)(a, av)/r = (aa, aa \cdot uv)/r$, and $\varphi(uv)(a, a \cdot uv)/r = (aa, aa \cdot uv)/r$, $a \in A$. ▲

4.1.6 Lemma. *For all $a \in A$ and $u \in G$, $\varphi(a) \cdot (a, u)/r = \varphi(u)$.*

Proof. $\varphi(a) \cdot (a, u)/r = (a, aa)(a, u)/r = (aa, aa \cdot u)/r = \varphi(u)$. ▲

4.1.7 Lemma. *The translation $\mathrm{L}_{\varphi(a), K}$ is injective for every $a \in A$.*

Proof. Let $\varphi(a) \cdot (b, u)/r = \varphi(a) \cdot (c, v)/r$. Then $(ab, aa \cdot u)/r = (ac, aa \cdot v)/r$ and this means that $(ab \cdot ab)(aa \cdot v) = (ab \cdot ac)(aa \cdot u)$, $(ab \cdot aa)(ab \cdot v) = (ab \cdot aa)(ac \cdot u)$, $ab \cdot v = ac \cdot u$ and $(b, u)/r = (c, v)/r$. ▲

4.1.8 Lemma. *If G is left cancellative, then K is left cancellative.*

Proof. Let $(a, u)(b, v)/r = (a, u)(c, z)/r$. Then, for $d \in A$, $(du)(ab \cdot z) = (d \cdot ab)(uz) = (d \cdot ac)(uv) = (du)(ac \cdot v)$ (since $((ab, uv), (ac, uz)) \in r$), and hence $ab \cdot z = ac \cdot v$ and $(b, v)/r = (c, z)/r$. ▲

4.2 Proposition. *Let G be a left cancellative left near-trimedial groupoid, let $a \in G$ and $A = \langle a \rangle_G$. Then there exists a groupoid K with the following properties:*

(i) G is a left strongly dense subgroupoid of K and $K = \alpha_{1,K}(G)$ (in fact, for every $x \in K$ there exists $b \in A$ with $bx \in G$).

(ii) $G \subseteq bK$ for every $b \in A$.

(iii) K is left cancellative.

(iv) K is right cancellative if and only if G is so.

(v) K is a homomorphic image of the product $A \times G$.

(vi) K is left near-trimedial.

Proof. See 4.1 and VI.2.9. ▲

4.3 Proposition. *Let G be a left cancellative left near-trimedial groupoid, $a \in G$ and $A = \langle a \rangle_G$. Then there exists a groupoid P with the following properties:*

(i) G is a left strongly dense subgroupoid of G.

(ii) For every $b \in A$, the left translation $\mathrm{L}_{a,P}$ is a bijection of P.

(iii) P is a left cancellative left near-trimedial groupoid.

(iv) P is right cancellative if and only if G is so.

Proof. The result follows from 3.2; we get P as the union of countable chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$ of subgroupoids constructed by means of 4.2. Notice that G_i is a homomorphic image of $A^{(i)} \times G$. ▲

4.4 Theorem. *Let G be a left cancellative left near-trimedial groupoid. Then there exists a left near-trimedial left quasigroup Q with the following properties:*

(i) G is a left strongly dense subgroupoid of Q (cf. 2.1, 2.2 and 2.3) and Q is determined uniquely up to G-isomorphism.

(ii) Q is right cancellative if and only if G is so.

(iii) Q is right divisible, provided that G is so.

(iv) The groupoids G and Q are equationally equivalent.

(v) Q is trimedial if and only if G is so.

(vi) The imbedding $G \hookrightarrow Q$ is a reflexion of G in the class of left near-trimedial left quasigroups.

Proof. Let $\mathscr{K}$ denote the class of left cancellative left near-trimedial groupoids. Then $\mathscr{K}$ is an abstract class and $\mathscr{K}$ satisfies the condition (UCH) from III.1.1. Moreover, by 4.4, $\mathscr{K}$ satisfies (LDE) as well and the result follows from III.1.2 (see also III.1.3), VI.2.9 and 2.4. ▲

4.5 Theorem. *Let G be a cancellative near-trimedial groupoid. Then G is trimedial and G is a dense subgroupoid of a trimedial quasigroup Q (cf. 2.6); the groupoids G and Q are equationally equivalent. Moreover Q is determined uniquely up to G-isomorphism and the imbedding $G \hookrightarrow Q$ is a reflexion of G in the class of trimedial quasigroup.*

Proof. Using 4.4 and its dual (see also III.1.4), we can imbed G into a near-trimedial quasigroup Q. Now, by 3.1, Q is trimedial. ▲

4.6 Remark. Let G be a trimedial groupoid such that G is a dense subgroupoid of a cancellative semimedial groupoid H with $H = HH$. By VI.9.3, H is a dense subgroupoid of a semimedial quasigroup Q, and hence G is a dense subgroupoid of Q and the imbedding $G \hookrightarrow Q$ is a reflexion of G in the class of semimedial quasigroups (II.4.4(iii)). On the other hand, by 4.5, G is a dense subgroupoid of a trimedial quasigroup P and consequently, the quasigroups Q and P are G-isomorphic (II.4.4). In particular, Q and H are trimedial.

VII.5 Cancellable (injective) elements of trimedial groupoids

5.1 Lemma. *Let G be groupoid.*

(i) If G satisfies (GE6), then $G \setminus \mathrm{A}_l(G)$ is either empty or a left ideal of G (or, equivalently, if $a, b \in G$ and $ab \in \mathrm{A}_l(G)$, then $b \in \mathrm{A}_l(G)$).

(ii) If G satisfies (GE8) and if $a, b \in G$, $ab \in \mathrm{A}(G)$, then $a \in \mathrm{A}_l(G)$.

Proof. (i) Let $ab \in \mathrm{A}_l(G)$. By 1.5(i), $\mathrm{L}_{ab}^2 = \mathrm{L}_{a \cdot ab}\mathrm{L}_b$, and hence L_b is injective and $b \in \mathrm{A}_l(G)$.

(ii) By 1.5(iv), $\mathrm{L}_{ab}\mathrm{R}_{ab} = \mathrm{R}_{b \cdot ab}\mathrm{L}_a$, and hence L_a is injective and $a \in \mathrm{A}_l(G)$. ▲

5.2 Proposition. *Let G be a semimedial groupoid satisfying the equations (GE6) and (GE9) (resp. (GE7) and (GE8)). Then:*

(i) $G \setminus \mathrm{A}(G)$ is either empty or an ideal of G (or, equivalently, if $a, b \in G$ and $ab \in \mathrm{A}(G)$, then $a, b \in \mathrm{A}(G)$).

(ii) $G \setminus \mathrm{A}^*(G)$ is either empty or an ideal of G (or, equivalently, if $a, b \in G$ and $ab \in \mathrm{A}^*(G)$, then $a, b \in \mathrm{A}^*(G)$).

(iii) $G \setminus \mathrm{A}(G) \subseteq G \setminus \mathrm{A}^*(G)$.

Proof. (i) Let $a, b \in G$ and $ab \in \mathrm{A}(G)$. By VI.11.13(i), $b \in \mathrm{A}_l(G)$ and, by the dual of VI.11.14(ii), $b \in \mathrm{A}_r(G)$, so that $b \in \mathrm{A}(G)$. However, $\mathrm{A}(G)$ is closed by VI.11.12(i), and therefore $a \in \mathrm{A}(G)$ as well.

(ii) Using the endomorphism σ_G, we can proceed similarly as in (i).

(iii) This is obvious. ▲

5.3 Proposition. *Let G be a semimedial groupoid satisfying the equations (GE6) and (GE7) (resp. (GE8) and (GE9)). Then $G \setminus \mathrm{A}^*(G)$ is either empty or an ideal of G.*

Proof. Let $a, b \in G$ and $ab \in \mathrm{A}^*(G)$. Then, for each $n \geq 0$, $\sigma_G^n(a)\,\sigma_G^n(b) \in \mathrm{A}_\mathrm{l}(G)$, and hence $\sigma_G^n(b) \in \mathrm{A}_\mathrm{l}(G)$ by VI.11.13(i) and $\sigma_G^n(a) \in \mathrm{A}_\mathrm{r}(G)$ by the dual of VI.11.13(i). From this, $b \in \mathrm{A}_\mathrm{l}^*(G)$ and $a \in \mathrm{A}_\mathrm{r}^*(G)$. However, $\mathrm{A}_\mathrm{l}^*(G) = \mathrm{A}_\mathrm{r}^*(G) = \mathrm{A}^*(G)$ by VI.11.12(iii).

Similarly the other case (use VI.11.13(ii) and its dual).

VII.6 Divisible (projective) elements of trimedial groupoids

6.1 Lemma. *Let G a groupoid.*

(i) If G satisfies (GE6) and if $\mathrm{A}_\mathrm{l}(G) \neq \emptyset$, then $\mathrm{B}_\mathrm{r}(G) \subseteq \mathrm{A}_\mathrm{l}(G)$.

(ii) If G satisfies (GE7) and if $\mathrm{A}_\mathrm{r}(G) \neq \emptyset$, then $\mathrm{B}_\mathrm{l}(G) \subseteq \mathrm{A}_\mathrm{r}(G)$.

Proof. (i) Let $b \in \mathrm{B}_\mathrm{r}(G)$. Then $ab \in \mathrm{A}_\mathrm{l}(G)$ for some $a \in G$. By 1.5(i), $\mathrm{L}_{ab}^2 = \mathrm{L}_{a \cdot b}\mathrm{L}_b$, and hence L_b is injective and $b \in \mathrm{A}_\mathrm{l}(G)$.

(ii) Dual to (i). ▲

6.2 Lemma. *Let G be a groupoid.*

(i) If G satisfies (GE8) and if $\mathrm{A}(G) \neq \emptyset$, then $\mathrm{B}_\mathrm{l}(G) \subseteq \mathrm{A}_\mathrm{l}(G)$.

(ii) If G satisfies (GE9) and if $\mathrm{A}(G) \neq \emptyset$, then $\mathrm{B}_\mathrm{r}(G) \subseteq \mathrm{A}_\mathrm{r}(G)$.

Proof. (i) Let $a \in \mathrm{B}_\mathrm{l}(G)$. Then $ab \in \mathrm{A}(G)$ for some $b \in G$. By 1.5(iii), $\mathrm{L}_{ab}\mathrm{R}_{ab} = \mathrm{R}_{b \cdot ab}\mathrm{L}_a$, L_a is injective and $a \in \mathrm{A}_\mathrm{l}(G)$.

(ii) This is dual to (i). ▲

6.3 Corollary. *Let G be a groupoid satisfying (GE6), (GE7), (GE8) and (GE9) and such that $\mathrm{A}(G) \neq \emptyset$. Then $\mathrm{B}_\mathrm{l}(G) \cup \mathrm{B}_\mathrm{r}(G) \subseteq \mathrm{A}(G)$.*

6.4 Proposition. *Let G be a semimedial groupoid.*

(i) If G satisfies (GE6) and $\mathrm{A}_\mathrm{l}(G) \neq \emptyset$, then $\mathrm{B}_\mathrm{r}(G) \subseteq \mathrm{A}_\mathrm{l}^*(G)$.

(ii) If G satisfies (GE8) and $\mathrm{A}(G) \neq \emptyset$, then $\mathrm{B}_\mathrm{l}(G) \subseteq \mathrm{A}_\mathrm{l}^*(G)$.

Proof. Combine VI.12.3(i), its dual and 6.1, 6.2. ▲

6.5 Proposition. *Let G be a semimedial groupoid satisfying (GE6). If $\mathrm{A}_\mathrm{l}(G) \neq \emptyset \neq \mathrm{A}_\mathrm{r}^*(G)$, then $\mathrm{B}_\mathrm{r}(G) \subseteq \mathrm{A}^*(G)$.*

Proof. By 6.4(i), $\mathrm{B}_\mathrm{r}(G) \subseteq \mathrm{A}_\mathrm{l}^*(G)$. However, $\mathrm{A}_\mathrm{l}^*(G) = \mathrm{A}^*(G)$ by VI.11.12(iii). ▲

6.6 Proposition. *Let G be a semimedial groupoid satisfying (GE6) and (GE9) (resp. (GE7) and (GE8)). If $\mathrm{A}(G) \neq \emptyset$, then $\mathrm{B}_\mathrm{l}(G) \cup \mathrm{B}_\mathrm{r}(G) \subseteq \mathrm{A}^*(G)$.*

Proof. By 5.2(i), $B_l(G) \cup B_r(G) \subseteq A(G)$. However, $B_l(G)$ (resp. $B_r(G)$) is either empty or a subgroupoid. ▲

6.7 Proposition. *Let G be a semimedial groupoid satisfying (GE6) and (GE7) and such that* $A_l(G) \neq \emptyset \neq A_r(G)$ *and* $B_l(G) \neq \emptyset \neq B_r(G)$. *Then* $B_l(G) \cup B_r(G) \subseteq A^*(G) \neq \emptyset$.

Proof. By 6.1(i), (ii) and VI.12.3 (and its dual), $B_r(G) \cup B_l(G) \subseteq A_l^*(G)$. Now, the result follows from VI.11.12. ▲

VII.7 Bijective elements of trimedial groupoids

7.1 Proposition. *Let G be a left semimedial groupoid satisfying (GE6) and (GE10). Then* $G \setminus C_l^*(G)$ *is either empty or a left ideal of G (equivalently, if* $a, b \in G$ *and* $ab \in C_l^*(G)$, *then* $b \in C_l^*(G)$*).*

Proof. We have $\sigma_G^n(ab) = \sigma_G^{(n)}(a)\,\sigma_G^n(b) \in C_l^*(G)$ for each $n \geq 0$. Further, by (GE6), $L_{ab}^2 = L_{a\cdot ab}L_b$, and hence $b \in A_l(G)$ and $a \cdot ab \in B_l(G)$. Put $c = ab$. By 1.5(v), $L_{c^2\cdot c^2}L_c = L_{c^2\cdot cb}L_{ac}$, and so $ac \in A_l(G)$. But $ac \in B_l(G)$, and therefore $ac = a \cdot ab \in C_l(G)$. Since $L_c^2 = L_{ac}L_b$, L_b is bijective and $b \in C_l(G)$. Quite similarly, $\sigma_G^n(G) \in C_l(G)$ for each $n \geq 1$ and we have proved that $b \in C_l^*(G)$. ▲

7.2 Proposition. *Let G be a left semimedial groupoid satisfying (GE6) and (GE10). If* $C_l^*(G) \neq \emptyset$, *then* $B_r(G) \subseteq C_l^*(G)$.

Proof. If $b \in B_r(G)$, then $ab \in C_l^*(G)$ for some $a \in G$. By 7.1, $b \in C_l^*(G)$. ▲

7.3 Theorem. *Let G be a semimedial groupoid satisfying the equations (GE6), (GE7) and such that each of the sets* $A_l(G)$, $A_r(G)$, $B_l(G)$ *and* $B_r(G)$ *is non-empty. Then:*

(i) $B_l(G) = C_l(G) = C_l^*(G) \subseteq A^*(G)$, $C_l(G)$ *is a left closed subgroupoid of G and it is a cancellative left quasigroup.*

(ii) $B_r(G) = C_r(G) = C_r^*(G) \subseteq A^*(G)$, $C_r(G)$ *is a right closed subgroupoid of G and it is a cancellative right quasigroup.*

(iii) $B(G) = C(G) = C^*(G) \subseteq A^*(G)$ *and* $C(G)$ *is either empty or a closed subgroupoid of G (in that case,* $C(G)$ *is a quasigroup).*

(vi) If G satisfies the equation (GE10), then $C(G) = C_r(G) \subseteq C_l(G)$ *and* $G \setminus C_l(G)$ *is either empty or a left ideal.*

(v) If G satisfies the equation (GE11), then $C(G) = C_l(G) \subseteq C_r(G)$ *and* $G \setminus C_r(G)$ *is either empty or a right ideal.*

(vi) If G satisfies the equations (GE10) and (GE11), then $C_l(G) = C_r(G) = C(G) \neq \emptyset$ *is a closed subgroupoid of G and* $G \setminus C(G)$ *is either empty or an ideal.*

(vii) $A_l^*(G) = A_r^*(G) = A^*(G) \neq \emptyset$, $A^*(G)$ *is a closed subgroupoid of G and* $G \setminus A^*(G)$ *is either empty or an ideal of G.*

Proof. Clearly, $C_l^*(G) \subseteq C_l(G) \subseteq B_l(G)$. By 6.7 $B_l(G) \subseteq A^*(G)$. Consequently, $B_l(G) = C_l(G)$. However, by VI. 12.3(i), $B_l(G)$ is a subgroupoid of G. Thus $B_l(G) = C_l^*(G)$. Finally, $C_l(G)$ is left closed by VI.13.4(i).

(ii) This is dual to (i).

(iii) $B(G) = B_l(G) \cap B_r(G) = C_l(G) \cap C_r(G) = C(G) = C_l^*(G) \cap C_r^*(G) = C^*(G) \subseteq A^*(G)$ by (i) and (ii). By VI.13.5, $C(G)$ is either empty or a closed subgroupoid.

(vi) Use (ii) and 7.2.

(v) This is dual to (iv).

(vi) Use (iii), (iv) and (v).

(vii) See VI.11.12 and 5.3. ▲

7.4 Corollary. *Let G be a trimedial groupoid such that each of the sets $A_l(G)$, $A_r(G)$, $B_l(G)$ and $B_r(G)$ is non-empty. Then $\emptyset \neq B_l(G) = B_r(G) = B(G) = C_l(G) = C_r(G) = C(G) = C_l^*(G) = C_r^*(G) = C^*(G) \subseteq A^*(G) = A_l^*(G) = A_r^*(G)$, both $C(G)$ and $A^*(G)$ are closed subgroupoids of G, $G \setminus C(G)$ is either empty or an ideal and $G \setminus A^*(G)$ is either empty or an ideal of G.*

VII.8 Comments

The main portion of the results in this chapter is new and further results on trimedial quasigroups may be seen in [15]. In VII.4, a generalization of the Sholander's construction (see [27]) is used to show that every cancellative trimedial groupoid imbeds into a trimedial quasigroup.

VIII. Cancellative medial and paramedial groupoids

VIII.1 Dense subgroupoids of medial groupoids

1.1 (See VI.2 and VII.2.) Let H be a subgroupoid of a medial groupoid G.

(i) $\alpha_{1,G}(H) = \{x \in G; Hx \cap H \neq \emptyset\}$ is a subgroupoid of G and $H \subseteq \alpha_{1,G}(H)$. Moreover, if all the left translations $L_{a,G}$, $a \in H$, are injective (e.g., G left cancellative), then $\alpha_{1,G}(H)$ is a homomorphic image of a subgroupoid of the product $H \times H$.

(ii) $H = \alpha_{1,G}^0(H) \subseteq \alpha_{1,G}^1(H) \subseteq \alpha_{1,G}^2(H) \subseteq \dots$ is a countable chain of subgroupoids of G and the union is just the closure $\langle H \rangle_{lc,G}$.

(iii) $H \subseteq \alpha_{1,G}(H) \subseteq \beta_{1,G}\alpha_{1,G}(H) \subseteq \alpha_{1,G}\beta_{1,G}\alpha_{1,G}(H) \subseteq \dots$, is a countable chain of subgroupoids of G and the union is just the closure $\langle H \rangle_{c,G}$.

(iv) For every $n \geq 0$, $\alpha_{n,G}(H)$ (see I.2.1) is a subgroupoid of G and $\alpha_{1,G}(\alpha_{n,G}(H)) \subseteq \alpha_{2n+1,G}(H)$. Consequently, $\alpha_{1,G}^m(H) \subseteq \alpha_{2^m-1,G}(H)$ for every $m \geq 0$. Now, it is clear that $H = \alpha_{0,G}(H) \subseteq \alpha_{1,G}(H) \subseteq \alpha_{2,G}(H) \subseteq \dots$ and the union of this chain is just $\langle H \rangle_{lc,G}$.

1.2 Lemma. *Let H be a subgroupoid of a medial groupoid G such that $H = HH$. Define a relation r on G by $(x, y) \in r$ if and only if $xH = yH$. Then:*

(i) r is a congruence of G.

(ii) If H is right closed in G and H is a left divisible groupoid, then H is a block of r.

(iii) If G is cancellative and $aH = H = Hb$ for some $a, b \in H$, then r is cancellative.

Proof. (i) Obviously, r is an equivalence. Now, let $xH = yH$ and let $z \in G$. If $a \in H$, then $a = bc$ for some $b, c \in H$ and $xc = yd$ for some $d \in H$. Consequently, $zx \cdot a = zx \cdot bc = zb \cdot xc = zb \cdot yd = zy \cdot bd$ and $zxH \subseteq zyH$. Similarly, the converse inclusion, and so $zxH = zyH$, i.e., $(zx, zy) \in r$. Similarly, $(xz, yz) \in r$.

(ii) Since H is left divisible, $aH = H$ for each $a \in H$, and therefore H is contained in a block of r. If $xH = H$ for some $x \in G$, then $x \in H$, since H is right closed in G.

(iii) Let $zxH = zyH$. Then, for every $u \in H$ there is $v \in H$ with $za \cdot xu = zx \cdot au = zy \cdot av = za \cdot yv$, i.e., $xu = yv$ and $xH \subseteq yH$. Similarly, $yH \subseteq xH$ and $(x, y) \in r$. We have proved that r is left cancellative. Similarly, r is right cancellative. ▲

1.3 Corollary. *Let H be a subquasigroup of a cancellative medial groupoid G. Then H is a block of a cancellative congruence of G.*

1.4 Proposition. *Let H be a subquasigroup of a cancellative medial groupoid G. Then every cancellative congruence of H can be extended to a cancellative congruence of G.*

Proof. G is a subgroupoid of a medial quasigroup P (see VII.4.5). Now, let r be a cancellative congruence of H and $Q = \langle H \rangle_{c,P}$. By I.6.6(ii), r can be extended to a cancellative congruence s of Q and, by [7, 5.5.2], s can be extended to a cancellative congruence t of P. Then $t \restriction G$ extends r. ▲

VIII.2 Imbeddings of cancellative medial groupoids into medial quasigroups

2.1 Let G be a left cancellative medial groupoid. Put $H = G^{(2)}$ and define a relation r on H by $((x, y), (u, v)) \in r$ if and only $wx \cdot v = wu \cdot y$ for every $w \in G$. Then (see VII.4.1) r is a left cancellative congruence of the groupoid H and the factor-groupoid $K = H/r$ is a left cancellative medial groupoid. The maping $\varphi : x \to (y, yx)/r \in K$, $y \in G$, is an injective (groupoid) homomorphism of G into K and $\varphi(G)$ is a left strongly dense subgroupoid of K. Moreover, $\varphi(u) \cdot (u, v)/r = \varphi(v)$ for all $u, v \in G$. Thus $\varphi(G) \subseteq \varphi(u) K$.

Now, repeating this process, we imbed G as a left strongly dense subgroupoid into a medial left quasigroup P such that P is the union of a countable chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$ of subgroupoids where each G_i is a homomorphic image of $G^{(2^i)}$. The groupoids G and P are equationally equivalent, P is right cancellative if and only if G is so and P is right divisible, provided that G is so.

2.2 Every cancellative medial groupoid can be imbedded into a medial quasigroup. To show this, we can use the following methods:

(1) According to 2.1, we imbed G as a left strongly dense subgroupoid into a medial left quasigroup P which is also right cancellative. Now, applying the right hand form of 2.1, we find a medial right quasigroup such that P is a right strongly dense subgroupoid of Q. Then, however, G is a dense subgroupoid of Q and Q is a quasigroup.

(2) Using VII.4.1 and its right hand form, we construct a chain $G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$ of groupoids such that G_i is a left (for $i \geq 0$ even) or right (for $i \geq 1$ odd) strongly dense subgroupoid of G_{i+1} and $G_i \subseteq aG_{i+1}$ or $G_i \subseteq G_{i+1}a$ for each $a \in G$. Each of the groupoids G_i is cancellative and a homomorphic image of $G^{(2^i)}$. Now, $Q = \bigcup G_i$ is a medial quasigroup and G is a dense subgroupoid of Q.

(3) By VI.9.4, G can be imbedded into a semimedial quasigroup Q such that G and Q satisfy the same quasi-symmetric groupoid equations. Consequently, Q is medial.

(4) There exist a free cancellative medial groupoid F and a projective homomorphism $\varphi : F \to G$. Using the equational theory of cancellative medial groupoids, one can show that F imbeds into a medial quasigroup P (see [7, Theorem 5.3.1]). Further, the cancellative congruence $\ker(\varphi)$ can be extended to a cancellative congruence r of P, and so $Q = P/r$ is a medial quasigroup and G can be imbedded into it.

VIII.3 Bijective elements of medial groupoids

3.1 Lemma. *Let G be a medial groupoid such that $\mathrm{C}_\mathrm{l}^*(G) \neq \emptyset \neq \mathrm{A}_\mathrm{r}(G)$. Then $\mathrm{A}_\mathrm{l}(G) \subseteq \mathrm{A}_\mathrm{r}(G)$.*

Proof. Let $a \in \mathrm{C}_\mathrm{l}^*(G)$, $b \in \mathrm{A}_\mathrm{l}(G)$ and $c \in \mathrm{A}_\mathrm{r}(G)$. First, $\mathrm{L}_{aa}\mathrm{L}_b\mathrm{L}_a^{-1} = \mathrm{L}_{ab}$ and $aa \in \mathrm{C}_\mathrm{l}^*(G)$ implies $bc \in \mathrm{A}_\mathrm{r}(G)$. But $\mathrm{A}_\mathrm{r}(G)$ is right closed by the dual of VI.1.13(i). Therefore $b \in \mathrm{A}_\mathrm{r}(G)$. ▲

3.2 Lemma. *Let G be a medial groupoid such that $\mathrm{C}_\mathrm{l}^*(G) \neq \emptyset$. Then $\mathrm{A}_\mathrm{l}(G) \cdot \mathrm{A}_\mathrm{r}(G) \subseteq \mathrm{A}_\mathrm{r}(G)$.*

Proof. Let $a \in \mathrm{C}_\mathrm{l}^*(G)$, $b \in \mathrm{A}_\mathrm{l}(G)$ and $c \in \mathrm{A}_\mathrm{r}(G)$. Further, let $x, y \in G$, $x = au$, $y = av$. If $ab \cdot x = ab \cdot y$, then $aa \cdot bu = aa \cdot bv$, and hence $u = v$ and $x = y$. We have proved that $ab \in \mathrm{A}_\mathrm{l}(G)$. Similarly, if $x \cdot bc = y \cdot bc$, then $ab \cdot uc = au \cdot bc = av \cdot bc = ab \cdot vc$, $uc = vc$, $u = v$ and $x = y$. We have proved that $bc \in \mathrm{A}_\mathrm{r}(G)$. ▲

3.3 Theorem. *Let G be a medial groupoid such that each of the sets $\mathrm{A}_\mathrm{l}(G)$, $\mathrm{A}_\mathrm{r}(G)$, $\mathrm{B}_\mathrm{l}(G)$ and $\mathrm{B}_\mathrm{r}(G)$ is non-empty. Then $\mathrm{A}_\mathrm{l}(G) = \mathrm{A}_\mathrm{r}(G) = \mathrm{A}(G) = \mathrm{A}^*(G) = \mathrm{B}_\mathrm{l}(G) = \mathrm{B}_\mathrm{r}(G) = \mathrm{C}(G) = \mathrm{C}^*(G)$.*

Proof. By VII.7.4, $\mathrm{C}^*(G) \neq \emptyset$. Now, by 3.1 and its dual, $\mathrm{A}_l(G) = \mathrm{A}_r(G) = \mathrm{A}(G)$. Further, by 3.2, $\mathrm{A}(G)$ is a subgroupoid. Hence $\mathrm{A}(G) = \mathrm{A}^*(G)$. ▲

3.4 Remark. The foregoing result can also be proved by means of a semigroup representation:

Let G be a medial groupoid with non-empty $\mathrm{C}(G)$. By [7, Proposition 3.1.3], there exist a commutative semigroup $G(+)$ with a neutral element 0, commuting automorphisms φ, ψ of $G(+)$ and invertible element $e \in G(+)$ such that $xy = \varphi(x) + \psi(y) + e$ for all $x, y \in G$. Now, it follows immediately that $\mathrm{B}_l(G) = \mathrm{B}_r(G) = \mathrm{B}(G(+))$, $\mathrm{A}_l(G) = \mathrm{A}_r(G) = \mathrm{A}(G(+))$ and $\mathrm{C}_l(G) = \mathrm{C}_r(G) = \mathrm{C}(G(+))$. But $\mathrm{B}(G(+)) = \mathrm{C}(G(+))$ is the subgroup of invertible elements and $\mathrm{A}(G(+))$ is a subsemigroup of $G(+)$. Moreover, $e \in \mathrm{A}(G(+))$ and $\mathrm{A}(G(+))$ is closed under φ and ψ. Consequently, $\mathrm{A}(G)$ is a subgroupoid of G.

3.5 Example. Let $G(+)$ be the quasicyclic Prüfer 2-group. Put $xy = 2x - y$ for all $x, y \in G$. Then G becomes a divisible idempotent medial groupoid which is a left quasigroup and we have $\mathrm{A}_l(G) = \mathrm{A}_l^*(G) = \mathrm{B}_l(G) = \mathrm{C}_l(G) = \mathrm{C}_l^*(G) = G = \mathrm{B}_r(G) = \mathrm{B}(G)$, $\mathrm{A}_r(G) = \mathrm{C}_r(G) = \emptyset$.

3.6 Example. Let G be the free medial groupoid with one free generator x. Put $y = xx$. It is easy to see that both x and y belong to $\mathrm{A}(G)$. However, by VII.1.9, $(x \cdot xy)(yx) \notin \mathrm{A}_l(G)$, $\mathrm{A}^*(G) \neq \mathrm{A}(G)$. Similarly, $(xy)(yx \cdot x) \notin \mathrm{A}_r(G)$, $x \notin \mathrm{A}_r^*(G)$ and $\mathrm{A}_r^*(G) \neq \mathrm{A}_r(G)$.

VIII.4 Paramedial groupoids

4.1 A groupoid G is said to be paramedial if it satisfies the equation $xu \cdot vy \triangleq yu \cdot vx$.

4.2 Lemma. *Let G be a paramedial groupoid, $n \geq 1$ and $x, y, a_1, \dots, a_n, b_1, \dots, b_n \in G$. Then:*

(i) $(a_1(\dots(a_nx)))(b_1(\dots(b_ny))) = ((((yx)(b_na_n))(a_{n-1}b_{n-1}))\dots)(b_1a_1)$ *and* $(((yb_n)\dots)b_1)(((xa_n)\dots)a_1) = (a_1b_1)(\dots((b_{n-1}a_{n-1})((a_nb_n)(xy))))$ *for n odd.*

(ii) $(a_1(\dots(a_nx)))(b_1(\dots(b_ny))) = ((((xy)(a_nb_n))(b_{n-1}a_{n-1}))\dots)(b_1a_1)$ *and* $(((xa_n)\dots)a_1)(((yb_n)\dots)b_1) = (a_1b_1)(\dots((a_{n-1}b_{n-1})((b_na_n)(yx))))$ *for n even.*

Proof. Easy. ▲

4.3 Lemma. *Let H be a subgroupoid of a paramedial groupoid G. Put $A_n = \alpha_{n,G}(H)$ and $B_n = \beta_{n,G}(H)$ for every $n \geq 0$. Then:*

(i) $A_nA_n \subseteq B_n$ *and* $B_nB_n \subseteq A_n$.

(ii) $A_1 \subseteq A_2 \cap B_2$ *and* $B_1 \subseteq A_2 \cap B_2$.

(iii) $A_1B_1 \cup B_1A_1 \subseteq A_2 \cap B_2$.

(iv) $(A_1 \cup B_1)(A_1 \cup B_1) \subseteq A_2 \cup B_2$.

(v) $\alpha_{1,G}(A_1) \subseteq A_3$ *and* $\beta_{1,G}(B_1) \subseteq B_3$.

Proof. (i) Use 4.2(i), (ii).

(ii) Clearly, $A_1 \subseteq A_2$ and, if $ax = b \in H$, $a \in H$, $x \in G$, then $xa \cdot aa = aa \cdot ax = aa \cdot b \in H$. Thus $x \in B_2$.

(iii) By (i) and (ii), $A_1B_1 \subseteq A_2A_2 \subseteq B_2$. Similarly the other inclusions.

(iv) Combine (i), (ii) and (iii).

(v) Let $ux = v$, $u, v \in A_1$, $x \in G$. Then $au = c$, $bv = d$, $a, b, c, d \in H$, and we have $(fc)(b \cdot ex) = (ex \cdot c)(bf) = (ex \cdot au)(bf) = (ux \cdot ae)(bf) = (v \cdot ae)(bf) = (f \cdot ae)(bv) = (f \cdot ae)\, d \in H$ for all $e, f \in H$. Thus $x \in A_3$. ▲

4.4 Theorem. *([13]) Every cancellative paramedial groupoid can be imbedded into a paramedial quasigroup.*

VIII.5 Comments and open problems

The fundamental construction of the medial quasigroup of fractions for a cancellative medial groupoid (2.1, 2.2) was discovered by M. Sholander in [27].

The existence of the paramedial quasigroup of fractions for a cancellative paramedial groupoid is proved in [13] (see also [2] and [12]). To find a direct construction of this quasigroup (mimicking the Sholander's approach) is the main task to be done.

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