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Josef Štěpán; P. Ševčík

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Convex Analysis for Sets of Local Martingales Measures

J. ŠTĚPÁN and P. ŠEVČÍK

Praha*

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Convexity, measure convexity and the integral representation property for the sets of solutions of continuous local martingale problems are investigated in the framework introduced in [11]. The research is focused on the local martingale problems which solutions μ are uniquely determined by a boundary condition in the form $\mathscr{L}(H \mid \mu) = \nu$ where H is a random variable on $\mathbb{C}(\mathbb{R}^+)$ and ν a probability distribution on its target space. The results provide both a refinement and a generalization of Stroock-Varadhan Theorem (see 18.10. in [4]) on the weak uniqueness in the theory of the stochastic differential equations. They are presented in Section 3 as Corollaries to a more general theory on measure convex sets of probability distributions with a Polish domain that is developed in Section 2.

1 Introduction and notation

We continue the research on geometry and topology of sets of solutions to local martingale problems that was started in [11], the inspiration being delivered by M. Yor (1978, 1979) and by D. W. Stroock, S. R. S. Varadhan (1969). The mathematical tools, coming from Choquet theory for convex sets in measure spaces, are in a general form developed in Section 2. Supported by results of G. Winkler (1980), G. Winkler (1978), H. v. Weizsäcker, G. Winkler (1979), H. v. Weizsäcker, G. Winkler (1980) and J. Štěpán (1984) we study measure convex sets $\mathfrak{M}$ of Borel probability measures on a Polish space, especially the sets $\mathfrak{M}$, called here Choquet sets, that are generated as the measure convex hull of their respective

* Faculty of Mathematics and Physics, Charles University in Prague, Department of Probability and Statistics, 18600 Praha 8 - Karlín, Sokolovská 83, Czech Republic

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extremal boundaries. Theorems 2.3, 2.5 and Lemma 2.7 offer sufficient conditions for a set $\mathfrak{M}$ to be a Choquet set or even a Choquet simplex that are applied in Section 3 to the sets $\mathscr{W}_{\mathscr{G},B}$ of $(\mathscr{G}, B)$-local martingale problems introduced in [11]. Recall briefly the concepts and notations treated there:

Let $\mathbb{C} = \mathbb{C}(\mathbb{R}^+)$ be the set of all continuous functions defined on $\mathbb{R}^+ = [0, \infty)$, $\mathscr{B}(\mathbb{C}) = \mathscr{B}(\mathbb{C}(\mathbb{R}^+))$ the Borel σ-algebra of the space endowed with the Polish topology of uniform convergence on compact intervals and finally $\mathscr{P}(\mathbb{C}) = \mathscr{P}(\mathbb{C}(\mathbb{R}^+))$ the set of all Borel probability measures on $\mathbb{C}$. We write

$$\mathbf{x}(t, x) := x(t),\ x \in \mathbb{C},\ t \geq 0 \quad \text{and} \quad \mathscr{F}_t^{\mathbf{x}} := \sigma(\mathbf{x}(s), s \leq t),\ t \geq 0$$

to define the canonical stochastic process $\mathbf{x} = (\mathbf{x}(t), t \geq 0)$ and the canonical filtration of σ-algebras $(\mathscr{F}_t^{\mathbf{x}})$ on the measurable space $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$. As in [11] we denote by $\mathscr{C}$ the set of all continuous processes G with $G(0) = 0$ defined on $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$, i.e.

$$\mathscr{C} := \{G : \mathbb{C} \to \mathbb{C} \text{ a Borel map},\ G(0) \equiv 0\}.$$

Important subsets of $\mathscr{C}$ are

$$\mathscr{C}_a := \{G \in \mathscr{C} : G \text{ is } \mathscr{F}_t^{\mathbf{x}}\text{-adapted process}\},\ \mathscr{C}_c := \{G \in \mathscr{C} : G \text{ is a continuous map } \mathbb{C} \to \mathbb{C}\},$$
$$\mathscr{C}_{c,a} := \mathscr{C}_c \cap \mathscr{C}_a.$$

For a $\mu \in \mathscr{P}(\mathbb{C})$ we denote by

$$\mathscr{F}_t^{\mathbf{x},\mu} := \sigma(\mathscr{F}_t^{\mathbf{x}} \cup N_\mu),\ t \geq 0,\ N_\mu = \{B \in \mathscr{B}(\mathbb{C}), \mu(B) = 0\}$$

the μ-completion of the canonical filtration $(\mathscr{F}_t^{\mathbf{x}})$.

Given a set $\mathscr{G} \subset \mathscr{C}$ and a Borel set $B \subset \mathbb{C}(\mathbb{R}^+)$ we write

$$\mathscr{W}_{\mathscr{G},B} := \{\mu \in \mathscr{P}(\mathbb{C}) : G \text{ is an } \mathscr{F}_t^{\mathbf{x},\mu}\text{-local martingale on } (\mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu), G \in \mathscr{G}, \mu(B) = 1\}$$

where $\mathscr{B}(\mathbb{C})^\mu$ denotes the μ-completion of $\mathscr{B}(\mathbb{C})$. We say that B and $\mathscr{G}$ are **compatible** if all $G \in \mathscr{G}$ are $\mathscr{F}_t^{\mathbf{x},\mu}$-adapted processes for any $\mu \in \mathscr{P}(\mathbb{C})$ with $\mu(B) = 1$. We abbreviate $\mathscr{W}_{\mathscr{G}} := \mathscr{W}_{\mathscr{G},\mathbb{C}}$ and $\mathscr{W}_G := \mathscr{W}_{\mathscr{G}}$ if $\mathscr{G} = \{G\}$ is a singleton. A probability distribution $\mu \in \mathscr{W}_{\mathscr{G},B}$ is called a **solution to the $(\mathscr{G}, B)$-local martingale problem** while a continuous stochastic process $X = (X(t), t \geq 0)$ defined on a complete probability space $(\Omega, \mathscr{F}, P)$ such that $G(X)$ is an $\mathscr{F}_t^{X,P}$-local martingale[1] is referred to as a **strong solution to the $(\mathscr{G}, B)$-local martingale problem.** Lemma 2.3.(c) in [11] says that X is a **strong solution** if and only if $\mathscr{L}(X) \in \mathscr{W}_{\mathscr{G},B}$.

Theorem 3.1 in Section 3 states that $\mathscr{W}_{\mathscr{G},B}$ is a Borel[2] **Choquet set** with a Borel extremal boundary ex $\mathscr{W}_{\mathscr{G},B}$, i.e.

[1] $\mathscr{F}_t^{X,P} := \sigma(\sigma(X(s), s \leq t) \cup \{N \in \mathscr{F}, P(N) = 0\})$ denotes the P-completion of the canonical filtration $(\mathscr{F}_t^X)$ of a process X.

[2] Borel σ-algebra in $\mathscr{P}(\mathbb{C})$ is defined as $\sigma(F : F \subset \mathscr{P}(\mathscr{S})$ a weakly closed set).

$$(1) \qquad \mathscr{W}_{\mathscr{G},B} = \left\{ \int \mu \, p(\mathrm{d}\mu), \, p \text{ a Borel probability on ex } \mathscr{W}_{\mathscr{G},B} \right\},$$

provided that $B \in \mathscr{B}(\mathbb{C})$ and $\mathscr{G} \subset \mathscr{C}$ is at most countable set such that $\mathscr{G}$ and B are compatible. Especially, $\mathscr{W}_{\mathscr{G},B}$ is a **measure convex set**, i.e. $\int \mu \, p(\mathrm{d}\mu) \in \mathscr{W}_{\mathscr{G},B}$ for any Borel probability p supported by $\mathscr{W}_{\mathscr{G},B}$. Moreover, it is proved that $\mathscr{W}_{\mathscr{G},B}$ is a **simplex** (any $\mu_0 \in \mathscr{W}_{\mathscr{G},B}$ is represented by $\int_{\mathrm{ex}\, \mathscr{W}_{\mathscr{G},B}} \mu \, p(\mathrm{d}\mu)$ uniquely) if and only if it is a **Choquet simplex** ($\mathbb{R}^+ \cdot \mathscr{W}_{\mathscr{G},B}$ is lattice cone in its own order). The above information is already available for any $\mathscr{W}_{\mathscr{G}}$ with an arbitrary $\mathscr{G} \subset \mathscr{C}_{c,a}$ via Corollary 3.5.(b) in [11] because any convex weakly closed set in $\mathscr{P}(\mathbb{C})$ is a Choquet set and it is a simplex if and only if it is a Choquet simplex by our present Remark 2.4.

The results in [11] center around the sets

$$\mathscr{L}(\mathbb{A}^0) := \{\mathscr{L}(X, Y \mid P), \quad (X, Y, \Omega, \mathscr{F}, P) \in \mathbb{A}^0\},$$
$$\mathscr{L}(\mathbb{L}^0) := \{\mathscr{L}(X, Y \mid P), \quad (X, Y, \Omega, \mathscr{F}, P) \in \mathbb{L}^0\},$$

where $\mathbb{A}^0(\mathbb{L}^0)$ denote the set of $(X, Y, \Omega, \mathscr{F}, P)$ such that $(\Omega, \mathscr{F}, P)$ goes through all complete probability spaces and then (X, Y) through all pairs of continuous processes defined on $(\Omega, \mathscr{F})$ such that $Y(0) \overset{a.s.}{=} 0 \; [P]$ and Y is an $\mathscr{F}_t^{X,P}$-adapted process ($\mathscr{F}_t^{X,P}$-local martingale) on $(\Omega, \mathscr{F}, P)$. We denote by $\mathscr{L}(X, Y \mid P)$ the probability distribution of (X, Y) in $\mathbb{C}(\mathbb{R}^+) \times \mathbb{C}(\mathbb{R}^+)$ w.r.t. the measure P, especially $\mathscr{L}(X, Y \mid P)$ is a Borel probability measure on $\mathbb{C}(\mathbb{R}^+) \times \mathbb{C}(\mathbb{R}^+)$. Proposition in Section 3 of [11] says that $\mathscr{L}(\mathbb{L}^0)$ is a weakly closed set in $\mathscr{L}(\mathbb{A}^0)$ and it follows by 2.3 Lemma (a), (b) in [11] that if $\mathscr{L}(X, Y)$ is in $\mathscr{L}(\mathbb{A}^0)$ or in $\mathscr{L}(\mathbb{L}^0)$ for a pair of continuous processes defined on a complete probability space $(\Omega, \mathscr{F}, P)$, then $(X, Y, \Omega, \mathscr{F}, P)$ is in $\mathbb{A}^0$ and $\mathbb{L}^0$ respectively. Hence for $G \in \mathscr{C}$

$$(2) \quad \mathscr{W}_G = \{\mu \in \mathscr{P}(\mathbb{C}) : (\mathbf{x}, G, \mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu) \in \mathbb{L}^0\} = \{\mu \in \mathscr{P}(\mathbb{C}) : \mathscr{L}(\mathbf{x}, G \mid \mu) \in \mathscr{L}(\mathbb{L}^0)\},$$

holds. Thus, $\mathscr{W}_{\mathbf{x}-\mathbf{x}(0)}$ stays in our notation for the set of all $\mathscr{L}(M \mid (\Omega, \mathscr{F}, P)) \in \mathscr{P}(\mathbb{C})$ where M goes through all continuous $\mathscr{F}_t^M$-local martingales on $(\Omega, \mathscr{F}, P)$ without having fixed the probability space.

Examples 1.2 and 1.3 in [11] propose another important instances of local martingale problems: For $G, v \in \mathscr{C}$ such that v is a process of finite variation

$$(3) \qquad \mathscr{W}_{G,v} := \{\mu \in \mathscr{W}_G : \langle G \rangle = v \text{ a.s. } [\mu]\} = \mathscr{W}_{\{G, G^2 - v\}}$$

holds by the uniqueness part of Doob-Mayer theorem. If b and σ are $\mathscr{F}_t^{\mathbf{x}}$-progressive processes on $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$ we define

$$(4) \qquad B_{b,\sigma} = \left\{ x \in \mathbb{C} : \int_0^t |b| + \sigma^2 \, \mathrm{d}s < \infty \; \forall t \geq 0 \right\}$$

and

$$G_b := \left[\mathbf{x} - \mathbf{x}(0) - \int_0^{\cdot} b(\mathbf{x}) \, \mathrm{d}s \right] I_{B_{b,\sigma}}, \quad G_{b,\sigma} := \left[\mathbf{x}^2 - \mathbf{x}^2(0) - 2 \int_0^{\cdot} \mathbf{x} \, b(\mathbf{x}) + \sigma^2(\mathbf{x}) \, \mathrm{d}s \right] I_{B_{b,\sigma}}.$$

The set $B_{b,\sigma}$ is easily seen to be Borel and compatible with $\{G_b, G_{b,\sigma}, v_\sigma\} \subset \mathscr{C}$ where $v_\sigma := [\int_0^{\cdot} \sigma^2 \, ds] \, I_{B_{b,\sigma}}$. The elements of stochastic analysis, see Example 1.2 in [11], show that

(5) $\mathscr{W}_{b,\sigma} = \mathscr{W}_{\{G_b, G_{b,\sigma}\}, B_{b,\sigma}} = \mathscr{W}_{G_b, v_\sigma} \cap \{\mu \in \mathscr{P}(\mathbb{C}) : \mu(B_{b,\sigma}) = 1\} = \mathscr{W}_{\{G_b, G_b^2 - v_\sigma\}, B_{b,\sigma}}$,

where

(6)

$\mathscr{W}_{b,\sigma} := \{\mathscr{L}(X) \in \mathscr{P}(\mathbb{C}) : X$ **is a weak solution of** $dX(t) = b(X)\,dt + \sigma(X)\,dW(t)\}$.

Recall that

$$dX(t) = b(X)\,dt + \sigma(X)\,dW(t) \tag{7}$$

is the differential form for the (b, σ)-stochastic differential equation, shortly (b, σ)**-SDE**, and that $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W, X) =: X$ is **a weak solution of** (b, σ)**-SDE** if $(\Omega, \mathscr{F}, P)$ is a complete probability space, $(\mathscr{F}_t)$-a complete filtration, W an $\mathscr{F}_t$-Wiener process, X a continuous $\mathscr{F}_t$-semimartingale with $P[X \in B_{b,\sigma}] = 1$ and with the stochastic differential given by (7).

In fact Example 1.2. in [11] we referred to provides a more complex information than (5) and (6): A continuous process X is a strong solution to $(\mathscr{G}, B) := (G_b, G_{b,\sigma}, B_{b,\sigma})$-problem if and only if it is a strong solution to $(\{G_b, G_b^2 - v_\sigma\}, B_{b,\sigma})$-problem, any weak solution of (7) is a strong solution of the above local martingale problem and finally for any strong solution X of $(\{G_b, G_{b,\sigma}\}, B_{b,\sigma})$-problem there exists a weak solution X^e of (7) such that $\mathscr{L}(X^e) = \mathscr{L}(X)$.

Thus, Theorem 3.1 applies to prove that (5) and (6) define Borel Choquet (measure convex) sets with Borel extremal boundaries and that $\mathscr{W}_{G,v}$ in (3) posseses the properties also if $G, v \in \mathscr{C}_a$. Unfortunately, for a $G \in \mathscr{C}$ generally, $\mathscr{W}_G$ need not be even a convex set as it is shown by Example 3.2.

Recall that an SDE (7) is called to be **well-posed** for an initial probability distribution ν on $\mathbb{R}$ it there exists exactly one probability measure $\mu_\nu \in \mathscr{W}_{\mathscr{G},B}$ such that $\mathscr{L}(\mathbf{x}(0) \mid \mu_\nu) = \nu$. Stroock-Varadhan Theorem, see 18.10 in [4] says that (7) is a well-posed equation for any ν if and only if it is well posed for any deterministic initial condition $y \in \mathbb{R}$. In Corollary 3.9 we add that if it is so, then $\mathscr{W}_{\mathscr{G},B}$ is a simplex with a Borel extremal boundary $\{\mu_{\varepsilon_y}, y \in \mathbb{R}\}$, while Corollary 3.8 extends Stroock-Varadhan Theorem to the sets of solutions of $(\mathscr{G}, B)$-martingale problems generally if only $\mathscr{G} \subset \mathscr{C}$ is a countable set compatible with a $B \in \mathscr{B}(\mathbb{C})$. Theorems 3.6 and 3.7 present similar results for local martingale problems $\mathscr{W}_{\mathscr{G},B}$ constrained by asking for solutions $\mu \in \mathscr{W}_{\mathscr{G},B}$ such that $\mathscr{L}(H \mid \mu) \in \mathscr{D}$ holds where H is a given random variable and $\mathscr{D}$ is a given set of probability distributions. Theorems 3.6 and 3.7 scrutinize in detail relations between the sets $\mathscr{W}_{\mathscr{G},B}$ and $\{\mathscr{L}(H \mid \mu), \mu \in \mathscr{W}_{\mathscr{G},B}\}$ and their extremal boundaries. Their proofs depend heavily on a general theory developed by Theorems 2.8, 2.9 and by Corollaries 2.11, 2.13 for equations

$$\mathscr{L}(H \mid \mu) = \nu, \quad \mu \in \mathfrak{M}, \quad \nu \in \mathscr{D},$$

where $H : S \to T$ is a given Borel map, $\mathscr{D}$ a given set of probability distributions on T and μ a solution of the equation in a Choquet set $\mathfrak{M}$ of probability distributions on S, S, T being Polish spaces. Remark finally that the origins of the SDE-theory observed from the view of Choquet abstract convex analysis comes perhaps to M. Ersov ([2] and [3]).

2 Measure convex sets

We recall first the elements of convex analysis in measure spaces. We shall fix a space S with a metric topology s and denote by $\mathscr{B}(S)$, $\mathscr{U}(S)$ and $\mathscr{P}(S)$ its Borel σ-algebra, the σ-algebra of universally measurable sets[3] and the space of all Borel probability measures on S, respectively. If not stated otherwise the measure space $\mathscr{P}(S)$ is endowed by the standard weak topology $w = w(s)$ that is metric and inherits both the separability and completeness from the S. Especially, S is a Polish space if and only if $\mathscr{P}(S)$ posseses the property. Note that for a $B \subset S$ the spaces $\mathscr{P}(B)$ and $\{\mu \in \mathscr{P}(S) : \mu^*(B) = 1\}$ are homeomorphic, hence we may and shall identity them. Also note that

(8) $$B \in \mathscr{B}(S) \Rightarrow \mathscr{P}(B) \in \mathscr{B}(\mathscr{P}(S)), \qquad B \in \mathscr{U}(S) \Rightarrow \mathscr{P}(B) \in \mathscr{U}(\mathscr{P}(S))$$

because $\mu \to \mu(B)$ is obviously a Borel function on $\mathscr{P}(S)$ provided $B \in \mathscr{B}(S)$ and universally measurable function on $\mathscr{P}(S)$ provided $B \in \mathscr{U}(S)$ by Lemma 2.6. Hence, denoting by $\Sigma(\mathfrak{M})$ the minimal σ-algebra in $\mathfrak{M} \subset \mathscr{P}(S)$ that makes all the maps $\{\mu \to \mu(B), B \in \mathscr{B}(S)\}$ measurable, then

(9) $$\Sigma(\mathfrak{M}) \subset \mathscr{B}(\mathfrak{M}) \quad \text{and} \quad \Sigma(\mathfrak{M}) = \mathscr{B}(\mathfrak{M}) \ \text{if} \ \mathfrak{M} \subset \mathscr{P}(S) \ \text{is separable}$$

because in the latter case $\mathscr{B}(\mathfrak{M})$ is generated by a topological base of open sets that belong to $\Sigma(\mathfrak{M})$.

Handling some other metric topology either on S or on $\mathscr{P}(S)$, say t, we may be more specific in our notation writing $\mathscr{B}(S, t)$, $\mathscr{U}(S, t)$, $\mathscr{P}(S, t)$ and speaking about t-Borel sets, t-universally measurable sets and t-Borel probabilities on S. For example, $\mathscr{B}(\mathscr{P}(S, s), t)$ stays for the t-Borel σ-algebra of subsets of the space of s-Borel probabilities on S.

As usual for a $p \in \mathscr{P}(\mathscr{P}(S))$ we denote by $r(p)$ or by $\int \mu \, p(\mathrm{d}\mu)$ the measure in $\mathscr{P}(S)$ defined by

(10) $$r(p)(B) = \int_{\mathscr{P}(S)} \mu(B) \, p(\mathrm{d}\mu), \quad B \in \mathscr{B}(S)$$

and call it the **barycenter of** p, the legitimity of the definition being ensured by the first statement in (9). Further denote $\mathrm{co}_M(\mathfrak{M}) = r(\mathscr{P}(\mathfrak{M}))$, call it **the measure convex hull** of $\mathfrak{M}$ and say that $\mathfrak{M}$ is a **measure convex set** if $\mathrm{co}_M(\mathfrak{M}) \subset \mathfrak{M}$ ($\Rightarrow \mathrm{co}_M(\mathfrak{M}) = \mathfrak{M}$).

[3] $\mathscr{U}(S) := \bigcap\{\mathscr{B}(S)^\mu, \mu \in \mathscr{P}(S)\}$, where $\mathscr{B}(S)^\mu$ is the μ-completion of $\mathscr{B}(S)$.

We refer to [13] for a general theory of measure convex (hence convex) sets in locally convex spaces, namely to 12.5 Proposition that characterize measure convex spaces, namely to 12.5 Proposition that characterize measure convex sets $\mathfrak{M} \subset \mathscr{P}(S)$ as those for which

$$K \subset \mathfrak{M} \text{ weakly compact set } \Rightarrow \overline{\mathrm{co}}(K) \text{ a weakly compact subset of } \mathfrak{M},$$

where $\overline{\mathrm{co}}(K)$ denotes the weakly closed convex hull of K. In what follows we summarize and complement the properties of the measure convexity relevant to the sets of solutions of local martingale problems.

Note that for any $\mathfrak{M} \subset \mathscr{P}(S)$, $\Sigma(\mathfrak{M}) = \sigma(\mu \to \mu(f), f : S \to \mathbb{R}$ Borel bounded), putting $\mu(f) = \int f \, \mathrm{d}\mu$ and (10) can be therefore extended to

$$r(p)(f) = \int_{\mathscr{P}(S)} \mu(f) \, p(\mathrm{d}\mu), \quad f : S \to \mathbb{R} \text{ Borel bounded.} \tag{11}$$

Recall that $(p_x, x \in X)$ is a **stochastic kernel** from a measurable space (X, χ) to S if $(p_x) \subset \mathscr{P}(S)$ and $x \to p_x(B)$ is an χ-measurable function for any $B \in \mathscr{B}(S)$. Any probability measure α defined on χ will be referred here as a mixing measure of the kernel.

2.1. Lemma. *If $\mathfrak{M} \subset \mathscr{P}(S)$ is a separable set then*

$$\mathrm{co}_M(\mathfrak{M}) = \left\{ \int p_x \, \alpha(\mathrm{d}x), \quad (p_x) \subset \mathfrak{M} \textit{ a stochastic kernel, } \alpha \textit{ a mixing measure} \right\} \tag{12}$$

and $\mathfrak{M}$ is a measure convex set if and only if $\int p_x \, \alpha(\mathrm{d}x) \in \mathfrak{M}$ for any stochastic kernel $(p_x) \subset \mathfrak{M}$ and any mixing measure α.

Proof. The "$\subset$" part of (12) follows by (9) even for non separable sets: If $(p_x) \subset \mathfrak{M}$ is a stochastic kernel and an α its mixing measure then $\int p_x \, \alpha(\mathrm{d}x) = r(p)$ where p is the image of α under the map $x \to p_x$ that is $(X, \chi) \to (\mathfrak{M}, \mathscr{B}(\mathfrak{M}))$ measurable by (8) due to the separability of $\mathfrak{M}$. Hence, $p \in \mathscr{P}(\mathfrak{M})$ and $\int p_x \, \alpha(\mathrm{d}x) \in \mathrm{co}_M(\mathfrak{M})$. □

Recall that a subset $\mathfrak{M}$ of a Polish space is said to be **analytic** if it is a continuous image of some other Polish space. We refer to [1] for the details we may need later on, reminding only that Borel sets are analytic which sets are further universally measurable.

2.2. Lemma. *Assume that S is a Polish space and $\mathfrak{M}$ an analytic subset of $\mathscr{P}(S)$. Then there exists a stochastic kernel $(p_\mu, \mu \in \mathrm{co}_M(\mathfrak{M})) \subset \mathscr{P}(\mathfrak{M})$ from $(\mathrm{co}_M(\mathfrak{M}), \mathscr{U}(\mathrm{co}_M(\mathfrak{M})))$ to $\mathscr{P}(S)$ such that $r(p_\mu) = \mu$ holds for all $\mu \in \mathrm{co}_M(\mathfrak{M})$. Moreover, $\mathrm{co}_M(\mathfrak{M})$ is a measure convex analytic subset of $\mathscr{P}(S)$ and therefore the minimal measure convex set containing $\mathfrak{M}$.*

Proof. Observe first that assuming $\mathfrak{M}$ to be analytic we get $\mathscr{P}(\mathfrak{M}) \subset \mathscr{P}(\mathscr{P}(S))$ also as an analytic set and that

$$r : \mathscr{P}(\mathscr{P}(S)) \to \mathscr{P}(S) \text{ is a weakly continuous map.} \tag{13}$$

Thus, $\mathrm{co}_M(\mathfrak{M}) = r(\mathscr{P}(\mathfrak{M}))$ is an analytic set and we may apply 8.5.3 in [1] to prove the existence of a map $\pi : \mathrm{co}_M(\mathfrak{M}) \to \mathscr{P}(\mathfrak{M})$ that is universally measurable and such that $r \circ \pi$ is the identity on $\mathrm{co}_M(\mathfrak{M})$. Denoting $p_\mu := \pi(\mu)$ we construct a stochastic kernel with the desired properties. It remains to prove that $\mathrm{co}_M(\mathfrak{M})$ is a measure convex set: Choose $\alpha \in \mathscr{P}(\mathrm{co}_M(\mathfrak{M}))$ and check the validity of the formula

$$r(\alpha) = \int_{\mathrm{co}_M(\mathfrak{M})} (r \circ \pi)(\mu)\, \alpha(\mathrm{d}\mu) = r\left(\int_{\mathrm{co}_M(\mathfrak{M})} p_\mu\, \alpha(\mathrm{d}\mu)\right).$$

It follows that $r(\alpha) = r(p)$ for a $p \in \mathscr{P}(\mathfrak{M})$, hence $r(\alpha) \in \mathrm{co}_M(\mathfrak{M})$. □

Lemma 2.2 and the results presented in [11], namely Theorem 3.7, suggest an interest in **convex sets** $\mathfrak{M} \subset \mathscr{P}(S)$ generated by $\mathfrak{M} = \mathrm{co}_M(\mathrm{ex}\,\mathfrak{M})$. We agree to call such an $\mathfrak{M}$ a **Choquet set.** Note that a Choquet set $\mathfrak{M}$ with an analytic extremal boundary $\mathrm{ex}\,\mathfrak{M}$ is forced by Lemma 2.2 to be measure convex and analytic provided that S is a Polish space. Recall that a Choquet set $\mathfrak{M}$ for which $r : \mathscr{P}(\mathrm{ex}\,\mathfrak{M}) \to \mathfrak{M}$ is a bijective map, is called a **simplex** and it is called a **Choquet simplex** if and only if the convex cone $\mathbb{R}^+ \cdot \mathfrak{M}$ is a lattice with respect to its natural order given by $\mu \leq \nu \Leftrightarrow \nu - \mu \in \mathbb{R}^+ \cdot \mathfrak{M}$. Also note that any measure convex Choquet set $\mathfrak{M}$ is the minimal measure convex set among those containing its extremal boundary $\mathrm{ex}\,\mathfrak{M}$.

Weizsäcker and Winkler (1979), (1980) identified a property of a convex set $\mathfrak{M} \subset \mathscr{P}(S)$ which is easily recognized as a property of the sets of solutions of local martingale problems that forces $\mathfrak{M}$ to be Choquet set and to be a simplex if and only if it is a Choquet simplex provided that S is a Polish space.

Consider a set $\mathscr{F}$ of bounded universally measurable functions $f : S \to \mathbb{R}$ and agree to call $\mathfrak{M} \subset \mathscr{P}(S)$ and **$\mathscr{F}$-closed set** if for any net $(\mu_\alpha) \subset \mathfrak{M}$ and any measure $\mu \in \mathscr{P}(S)$

$$\int f\, \mathrm{d}\mu_\alpha \to \int f\, \mathrm{d}\mu, \quad f \in \mathscr{F} \;\Rightarrow\; \mu \in \mathfrak{M} \tag{14}$$

holds. The topology of $\mathscr{P}(S)$ defined by the family of all $\mathscr{F}$-closed sets or equivalently by the convergence (14) will be referred here as the **$\mathscr{F}$-topology of** $\mathscr{P}(S)$. Denote by $\mathbb{F} = \mathbb{F}(S)$ the class of all $\mathscr{F} \subset \mathbb{R}^S$ such that all f in $\mathscr{F}$ are bounded and universally measurable, such that all bounded continuous functions are in $\mathscr{F}$ and such that there are at most countable many $f \in \mathscr{F}$ that are not continuous. Denote also by $\mathbb{F}_b = \mathbb{F}_b(S)$ the class of all $\mathscr{F} \in \mathbb{F}(S)$ the elements of which are all Borel measurable. Further agree to say that a set $\mathfrak{M} \subset \mathscr{P}(S)$ is **$\mathbb{F}(S)$-closed** and **$\mathbb{F}_b(S)$-closed** if there exists $\mathscr{F} \in \mathbb{F}(S)$ and $\mathscr{F} \in \mathbb{F}_b(S)$ such that $\mathfrak{M}$ is an $\mathscr{F}$-closed set, respectively.

Recall that a metric space S is called a Radon space if all measures in $\mathscr{P}(S)$ are tight, i.e. if

$$\mu(B) = \sup\{\mu(K), K \subset B \text{ compact}\}, \quad B \in \mathscr{B}(S),\ \mu \in \mathscr{P}(S)$$

holds. Remark that both Polish and analytic S are Radon and there is a separable metric space S that is not a Radon space.

2.3. Theorem (Weizsäcker, Winkler). *Let S be a separable Radon metric space and $\mathfrak{M} \subset \mathscr{P}(S)$ a convex $\mathbb{F}_b(S)$-closed set, say $\mathscr{F}$-closed for an $\mathscr{F} \in \mathbb{F}_b(S)$. Then $\mathfrak{M}$ is a Borel measure convex Choquet set with a Borel extremal boundary* ex $\mathfrak{M}$. *It is a simplex if and only if it is a Choquet simplex.*

Proof. The assertion is basically a very special case to the results presented in [17]: $\mathfrak{M}$ and ex $(\mathfrak{M})$ are Borel sets in $\mathscr{P}(S)$ by Theorem 3a and by Theorem 3b, respectively because of the separability of S. The inclusion $\mathfrak{M} \subset \mathrm{co}_M(\mathrm{ex}\,\mathfrak{M})$ is exactly the statement of Theorem 1. To prove the measure convexity of $\mathfrak{M}$ consider $\mu = r(p)$ for a $p \in \mathscr{P}(\mathfrak{M})$. It follows by Theorem on the separation of convex sets in locally convex spaces that there is a net $(p_\alpha) \subset \mathscr{P}(\mathfrak{M})$ such that

$$p_\alpha = \sum_{i=1}^{k_\alpha} \lambda_i^\alpha \varepsilon_{\mu_i^\alpha}, \quad \mu_i^\alpha \in \mathfrak{M}, \quad p(g) = \lim_\alpha p_\alpha(g), \quad g : \mathfrak{M} \to \mathbb{R} \text{ Borel bounded.}$$

Especially, putting $\mu_\alpha = r(p_\alpha) = \sum_{i=1}^{k_\alpha} \lambda_i^\alpha \mu_i^\alpha$, it follows by (8) and (11) that $\mu(f) = \lim_\alpha \mu_\alpha(f)$ for all $f \in \mathscr{F}$ and therefore $\mu \in \mathfrak{M}$ because $\mathfrak{M}$ is a convex $\mathscr{F}$-closed set. Hence, $\mathfrak{M}$ is a measure convex set and $\mathfrak{M} \subset \mathrm{co}_M(\mathrm{ex}\,\mathfrak{M}) \subset \mathfrak{M}$ proves the first assertion. Finally, the simplex part of Theorem is implied by Theorem 2 in [17] because S is metric Radon space. □

2.4. Remark. *Choosing in Theorem 2.3 $\mathscr{F} = \mathbb{C}_b(S)$ the space of bounded continuous functions defined on S we prove that for a* **separable Radon metric space** *S any convex weakly closed set $\mathfrak{M}$ is measure convex Choquet set, hence the minimal measure convex set among those containing its extremal boundary* ex $(\mathfrak{M})$. *It follows that $\mathfrak{M} = \mathrm{co}_M(\mathrm{ex}\,\mathfrak{M}) = \overline{\mathrm{co}}(\mathrm{ex}\,\mathfrak{M})$ holds for all convex weakly closed sets $\mathfrak{M} \subset \mathscr{P}(S)$.*

By the method similar to that of Lemma in [17] we may extend Theorem 2.3 to $\mathbb{F}(S)$-closed convex sets.

2.5. Theorem. *Let S be a separable Radon space and $\mathfrak{M} \subset \mathscr{P}(S)$ a convex $\mathbb{F}(S)$-closed, say $\mathscr{F}$-closed, set. Then $\mathfrak{M}$ is a universally measurable measure convex Choquet set with a universally measurable extremal boundary* ex $\mathfrak{M}$. *It is a simplex if and only if it is a Choquet simplex.*

Proof. Let $\{f_1, f_2, \dots\}$ be the set of all $f \in \mathscr{F}$ that are not s-continuous where s is the topology of S. Consider the map $F : S \to S \times \mathbb{R}^{\mathbb{N}}$ defined by

$$F(k) = (k, f_1(k), f_2(k), \dots), \qquad k \in S$$

and the initial topology t of F in S. Obviously, $t = \{F^{-1}(U), U \subset S \times \mathbb{R}^{\mathbb{N}} \text{ open}\}$ is the minimal topology on S finer than s for which all $f \in \mathscr{F}$ are continuous functions. Because $S \times \mathbb{R}^{\mathbb{N}}$ is a separable metric space it follows that there is countable base for the topology t, hence (S, t) is a **separable metric space** by Urysohn metrization theorem (Theorem 17(a) in [5]). Since F is obviously an s-universally measurable map $S \to S \times \mathbb{R}^{\mathbb{N}}$ it also follows that

$$(15) \quad \mathscr{B}(S, s) \subset \mathscr{B}(S, t) = \sigma(t) \subset \mathscr{U}(S, s) \quad \text{and} \quad \mathscr{P}(S, t) := \{\mu^e, \mu \in \mathscr{P}(S, s)\},$$

where μ^e denotes the unique extension of a $\mu \in \mathscr{P}(S, s)$ from the σ-algebra $\mathscr{B}(S, s)$ to the σ-algebra $\mathscr{B}(S, t)$. Again, because $S \times \mathbb{R}^{\mathbb{N}}$ is a separable metric space, it follows by Theorem 5, p. 26 in [12] that F is Lusin μ-measurable map for any $\mu \in \mathscr{P}(S, s)$. It is exactly as to say

$$\mu(B) = \sup \{\mu(K) : K \subset B, K\, s\text{-compact}, F : K \to S \times \mathbb{R}^{\mathbb{N}} \text{ continuous}\},$$
$$B \in \mathscr{U}(S, s), \mu \in \mathscr{P}(S, s).$$

Since any s-compact set K for which $F \mid K$ is a continuous map is easily seen to be a t-compact set, it follows that any $\mu^e \in \mathscr{P}(S, t)$ is a t-tight measure. Hence, (S, t) is proved to be a separable Radon space and $\mathfrak{M}^e := \{\mu^e, \mu \in \mathfrak{M}\} \subset \mathscr{P}(S, t)$ becomes to be a convex and t-weakly closed set because the set $\mathscr{F}$ is contained in the space of all bounded t-continuous functions defined on S.

Denote by $e : \mathscr{P}(S, s) \to \mathscr{P}(S, t)$ the bijection defined by $e(\mu) = \mu^e$ and let $i := e^{-1}$. It follows that e **is a universally measurable** and e^{-1} **Borel measurable** map with respect to the corresponding weak topologies. Indeed, it follows by (8), by the separability of (S, s) that $i : \mathscr{P}(S, t) \to \mathscr{P}(S, s)$ is a Borel map if and only if $\{\nu \in \mathscr{P}(S, t) : \nu(B) \leq c\}$ is a Borel set in $\mathscr{P}(S, t)$ for any $B \in \mathscr{B}(S, s)$ and $c \geq 0$. It is of course an immediate consequence of (8) because $\mathscr{B}(S, s) \subset \mathscr{B}(S, t)$ by (15). Similarly, it follows by (8), by the separability of (S, t) that $e : \mathscr{P}(S, s) \to \mathscr{P}(S, t)$ is a universally measurable map if and only if $\{\mu \in \mathscr{P}(S, s), \mu(B) \leq c\}$ is a universally measurable set in $\mathscr{P}(S, s)$ for any $B \in \mathscr{B}(S, t)$ and $c \geq 0$. But it follows directly by Lemma 2.6 bellows because $\mathscr{B}(S, t) \subset \mathscr{U}(S, s)$ by (15).

Thus, $e : \mathscr{P}(S, s) \to \mathscr{P}(S, t)$ is a universally measurable affine bijection such that $i = e^{-1}$ is a Borel map. One can easily check that

$$\mathfrak{M} = e^{-1}\mathfrak{M}^e, \quad \operatorname{ex} \mathfrak{M} = e^{-1} \operatorname{ex} \mathfrak{M}^e, \quad \mu_1 \leq \mu_2 \text{ in } \mathbb{R}^+ \cdot \mathfrak{M} \Leftrightarrow \mu_1^e \leq \mu_2^e \text{ in } \mathbb{R}^+ \cdot \mathfrak{M}^e$$

and that

$$\mu = r(p) \Leftrightarrow \mu^e = r(e \circ p), \quad \mu \in \mathfrak{M}, \ p \in \mathscr{P}(\mathfrak{M})$$
$$\nu = r(q) \Leftrightarrow i(\nu) = r(i \circ q), \quad \nu \in \mathfrak{M}^e, \ q \in \mathscr{P}(\mathfrak{M}^e)$$

and finally that the map $\mathscr{P}(\operatorname{ex} \mathfrak{M}) \to \mathscr{P}(\operatorname{ex} \mathfrak{M}^e)$ defined by $p \to e \circ p$ is a bijection. Observing all the above properties of the extension map e we prove all the statements of Theorem by an application of corresponding statements of Theorem 2.3 to the convex weakly closed set $\mathfrak{M}^e \subset \mathscr{P}(S, t)$. □

2.6. Lemma. *Let S be a metric space and $U \subset S$ a universally measurable set. Then $\mu \to \mu(U)$ is a universally measurable map $\mathscr{P}(S) \to \mathbb{R}$.*

Proof. Let $p_0 \in \mathscr{P}(\mathscr{P}(S))$ be an arbitrary probability measure, put $\mu_0 = r(p_0)$ and let $\mu_0(U) = \inf\{\mu_0(G), G \in \mathscr{G}\} = \sup\{\mu_0(F), F \in \mathscr{F}\}$ where $\mathscr{F}$ and $\mathscr{G}$ are countable sets of closed sets $F \subset U$ and open sets $G \supset U$. Obviously, $\mu \to m(\mu) := \inf\{\mu(G), G \in \mathscr{G}\}$ and $\mu \to M(\mu) := \sup\{\mu(F), F \in \mathscr{F}\}$ are Borel functions $\mathscr{P}(S) \to \mathbb{R}$ such that $M(\mu) \leq \mu(U) \leq m(\mu)$ holds for any $\mu \in \mathscr{P}(S)$ and such that

$$\int m(\mu)\, p_0(\mathrm{d}\mu) \leq \inf_{G \in \mathscr{G}} \int \mu(G)\, p_0(\mathrm{d}\mu) = \mu_0(U) = \sup_{F \in \mathscr{F}} \int \mu(F)\, p_0(\mathrm{d}\mu) \leq \int M(\mu)\, p_0(\mathrm{d}\mu).$$

Hence, $M = m$ almost surely $[p_0]$ and $\mu \to \mu(U)$ is a p_0-measurable function. □

Observe that $\mathbb{F}$-closed sets are stable w.r.t. finite unions, countable intersections and Borel inverse images operations:

2.7. Lemma. *Let S be a metric space and $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ $\mathbb{F}(S)$-closed ($\mathbb{F}_b(S)$-closed) sets in $\mathscr{P}(S)$. Then the sets $\mathfrak{M}_1 \cup \mathfrak{M}_2$ and $\bigcap_{n=1}^{\infty} \mathfrak{M}_n$ are $\mathbb{F}(S)$ closed ($\mathbb{F}_b(S)$-closed) also. Moreover, if T is a separable metric space, $\mathfrak{M} \subset \mathscr{P}(T)$ an $\mathbb{F}(T)$-closed (an $\mathbb{F}_b(T)$-closed) set and $H : S \to T$ a universally measurable (Borel measurable) map then $H^{-1} \circ \mathfrak{M} := \{\mu \in \mathscr{P}(S) : H \circ \mu \in \mathfrak{M}\}$ is an $\mathbb{F}(S)$-closed ($\mathbb{F}_b(S)$-closed) set.*

Proof. The first part of Lemma is obvious, note only that $\mathbb{F}(S)$-closed ($\mathbb{F}_b(S)$-closed) sets are stable w.r.t. countable unions and intersections, that the convergence (14) with an $\mathscr{F} \in \mathbb{F}(S)$ ($\mathscr{F} \in \mathbb{F}_b(S)$) promotes Hausdorff, in fact metric, topology in $\mathscr{P}(S)$.

Because T is a separable space there is a countable set of bounded continuous functions defined on T is a separable space there is a countable set of bounded continuous functions defined on T that determines the weak convergence in $\mathscr{P}(T)$. Hence, there is a countable set $\mathscr{G}$ of bounded universally (Borel)-measurable functions $T \to \mathbb{R}$ such that the set $\mathfrak{M}$ is $\mathscr{G}$-closed. Denote by $\mathscr{F}$ the set of functions $S \to \mathbb{R}$ that consists of all bounded continuous functions and of all $g(H)$ where g is running through $\mathscr{G}$. Obviously $\mathscr{F} \in \mathbb{F}(S)$ ($\mathscr{F} \in \mathbb{F}_b(S)$) and $H^{-1} \circ \mathfrak{M}$ is an $\mathscr{F}$-closed set in $\mathscr{P}(S)$ because for any net $(\mu_\alpha) \subset H^{-1} \circ \mathfrak{M}$ and any $\mu \in \mathscr{P}(S)$

$$\int_S f\, \mathrm{d}\mu_\alpha \to \int_S f\, \mathrm{d}\mu \ \forall f \in \mathscr{F} \ \Rightarrow \ \int_T g\, \mathrm{d}(H \circ \mu_\alpha) \to \int_T g\, \mathrm{d}(H \circ \mu)\ \forall g \in \mathscr{G}.$$

□

We shall close the present section by listing some properties of the direct image $H \circ \mathfrak{M}$ of a measure convex (measure convex Choquet) set $\mathfrak{M} \subset \mathscr{P}(S)$. We shall fix a pair of **Polish spaces** S, T and $H : S \to T$ **a Borel map**.

Obviously, only the separability of T yields via (8) that

$$\bar{H} : \mathscr{P}(S) \to \mathscr{P}(T), \quad \bar{H}(\mu) := H \circ \mu, \quad \mu \in \mathscr{P}(S),$$

is a Borel map having denoted by $H \circ \mu$ the image of μ under H. A straighforward computation verifies the barycentrical formula

$$r(\bar{H} \circ p) = H \circ r(p), \quad p \in \mathscr{P}(\mathscr{P}(S)). \tag{16}$$

Because $\bar{H} \circ \mathscr{P}(\mathfrak{N}) \subset \mathscr{P}(H \circ \mathfrak{N})$ for any $\mathfrak{N} \subset \mathscr{P}(S)$ it follows by (16) that

$$(17) \qquad \mathfrak{M} \subset \mathrm{co}_M(\mathfrak{N}) \Rightarrow H \circ \mathfrak{M} \subset \mathrm{co}_M(H \circ \mathfrak{N}), \quad \mathfrak{M}, \mathfrak{N} \subset \mathscr{P}(S),$$

where $H \circ \mathfrak{M} := \{H \circ \mu, \mu \in \mathfrak{M}\}$. We may get (17) with the reversed inclusions only under an additional assumptions on the set $\mathfrak{N}$ in the form

$$(18) \quad \mathrm{co}_M \mathfrak{N} \subset \mathfrak{M} \Rightarrow \mathrm{co}_M(H \circ \mathfrak{N}) \subset H \circ \mathfrak{M}, \quad \mathfrak{M}, \mathfrak{N} \subset \mathscr{P}(S), \quad \mathfrak{N} \text{ analytic.}$$

To verify (18) observe first that

$$(19) \qquad \mathfrak{N} \subset \mathscr{P}(S) \text{ analytic} \Rightarrow \bar{H} \circ \mathscr{P}(\mathfrak{N}) = \mathscr{P}(H \circ \mathfrak{N}).$$

Indeed, because $\mathscr{P}(S)$ and $\mathscr{P}(T)$ are Polish spaces and $\mathfrak{N} \subset \mathscr{P}(S)$ **is an analytic set**, it follows by 8.5.3 in [1] again that **there exists**

$$(20) \qquad \begin{array}{l} \text{a universally measurable surjection } R : \bar{H}(\mathfrak{N}) \to \mathfrak{N} \text{ such that} \\ (\bar{H} \circ R)(\nu) = \nu \text{ holds for all } \nu \in \bar{H}(\mathfrak{N}). \end{array}$$

Now (19) follows easily because if $q \in \mathscr{P}(H \circ \mathfrak{N})$ then $p := R \circ q \in \mathscr{P}(\mathfrak{N})$ is a measure for which $\bar{H} \circ p = q$ holds by (20). Applying (16) and (19) we prove (18).

Finally, (17) and (18) may be combined to

$$(21) \qquad H \circ \mathrm{co}_M(\mathfrak{N}) = \mathrm{co}_M(H \circ \mathfrak{N}), \quad \mathfrak{N} \subset \mathscr{P}(S) \text{ analytic}$$

while (18) by itself proves

$$(22)$$
$$\mathfrak{M} \subset \mathscr{P}(S) \text{ measure convex analytic} \Rightarrow H \circ \mathfrak{M} \subset \mathscr{P}(T) \text{ measure convex analytic}.$$

We suspect that the direct image of a Choquet set need not be a Choquet set, hence we shall find useful the following simple device how to recognize the property.

2.8. Theorem. *Assume that S, T are Polish spaces, $H : S \to T$ a Borel map and $\mathfrak{M} \subset \mathscr{P}(S)$ is a Choquet set with an analytic extremal boundary* $\mathrm{ex}\, \mathfrak{M}$. *Then* $H \circ \mathfrak{M} = \mathrm{co}_M(H \circ \mathrm{ex}\, \mathfrak{M})$ *is an analytic measure convex set such that* $\mathrm{ex}\,(H \circ \mathfrak{M}) \subset H \circ \mathrm{ex}\, \mathfrak{M}$ *holds. If there is a Choquet set $\mathscr{D} \subset \mathscr{P}(T)$ such that* $\mathrm{ex}\, \mathscr{D} \subset H \circ \mathfrak{M}$ *then $\mathscr{D} \subset H \circ \mathfrak{M}$.*

Remark that the requirements on $\mathfrak{M}$ are satisfied when $\mathfrak{M}$ is an $\mathbb{F}_b(S)$-closed convex set by Theorem 2.3.

Proof. Because $\mathfrak{M}$ is a Choquet set with an analytical boundary $\mathrm{ex}\, \mathfrak{M}$ it follows by Lemma 2.2 that $\mathfrak{M}$ is measure convex and analytic. Thus, $H \circ \mathfrak{M}$ is measure convex and analytic set by (22) and the equality $H \circ \mathfrak{M} = \mathrm{co}_M(H \circ \mathrm{ex}\, \mathfrak{M})$ follows by (21).

Further, if $\nu \in \mathrm{ex}\,(H \circ \mathfrak{M})$ then $\nu \in \mathrm{co}_M(H \circ \mathrm{ex}\, \mathfrak{M})$ and therefore $\nu = r(q)$ for a $q \in \mathscr{P}(H \circ \mathrm{ex}\, \mathfrak{M})$. Because $H \circ \mathfrak{M}$ is already proved to be a measure convex set it follows by 15.5 Corollary in [13] that q equals to the point measure ε_ν and therefore $\nu \in H \circ \mathrm{ex}\,(\mathfrak{M})$. Finally, to verify the last statement check that the measure convexity of $H \circ \mathfrak{M}$ implies that

$$\mathscr{D} = \mathrm{co}_M(\mathrm{ex}\,\mathscr{D}) \subset \mathrm{co}_M(H \circ \mathfrak{M}) \subset H \circ \mathfrak{M}.$$

□

Even more can be stated assuming that $H \circ \mathfrak{M}$ a simplex.

2.9. Theorem. *Let S, T and H are as in Theorem 2.8. Assume that $\mathfrak{M} \subset \mathscr{P}(S)$ and $\mathscr{D} := H \circ \mathfrak{M} \subset \mathscr{P}(T)$ are Choquet sets with Borel extremal boundaries* ex $\mathfrak{M}$ *and* ex $\mathscr{D}$, *respectively. Moreover, assume that for any $\nu \in \mathrm{ex}\,\mathscr{D}$ there is a* **unique** *$\mu_\nu \in \mathrm{ex}\,\mathfrak{M}$ such that $H \circ \mu_\nu = \nu$ holds. Then*

(23) *$\nu \to \mu_\nu$ defines a stochastic kernel from $(\mathrm{ex}\,\mathscr{D}, \mathscr{B}(\mathrm{ex}\,\mathscr{D}))$ to S, hence a Borel injection $\mathrm{ex}\,\mathscr{D} \to \mathrm{ex}\,\mathfrak{M}$*

such that

$$\mathfrak{M}(\mathscr{D}) := \left\{\int \mu_\nu\, q(\mathrm{d}\nu),\ q \in \mathscr{P}(\mathrm{ex}\,\mathscr{D})\right\} = \mathrm{co}_M\{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\} \tag{24}$$

is a Choquet set with Borel extremal boundary $\mathrm{ex}\,\mathfrak{M}(\mathscr{D}) = \{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\}$.

Moreover,

(25) *$\mathscr{D}$ is a simplex if and only if $\mathfrak{M}(\mathscr{D})$ is a simplex and $\mu \to H \circ \mu$ is a Borel bijection of $\mathfrak{M}(\mathscr{D})$ onto $\mathscr{D}$.*

Remark that we get $\mathfrak{M}$ and $H \circ \mathfrak{M}$ as Choquet sets with Borel extremal boundaries assuming that $\mathfrak{M}$ and $H \circ \mathfrak{M}$ are convex $\mathbb{F}_b(S)$ and $\mathbb{F}_b(T)$-closed, respectively by Theorem 2.3

Remark also, that the uniqueness requirement asked by the Theorem is implied by

(26) for any $\nu \in \mathrm{ex}\,\mathscr{D}$ there is a unique $\mu_\nu \in \mathfrak{M}$ such that $H \circ \mu_\nu = \nu$ holds,

because $\mathrm{ex}\,\mathscr{D} \subset H \circ \mathrm{ex}\,\mathfrak{M}$ by Theorem 2.8.

Proof. It follows by the assumptions of Theorem that

$$\{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\} = (\bar{H}^{-1}\,\mathrm{ex}\,\mathscr{D}) \cap \mathrm{ex}\,\mathfrak{M} \quad \text{is a Borel set in } \mathscr{P}(S)$$

and $\bar{H} : \{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\} \to \mathrm{ex}\,\mathscr{D}$ a Borel bijection with the inverse given by $\nu \to \mu_\nu$. Hence, (23) is a true statement. It also follows immediately that $p \to \bar{H} \circ p$ defines a Borel bijection between $\mathscr{P}(\{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\})$ and $\mathscr{P}(\mathrm{ex}\,\mathscr{D})$, hence the equality (24) holds, because $r(p) = \int \mu_\nu(\bar{H} \circ p)(\mathrm{d}\nu)$ holds for any $p \in \mathscr{P}(\{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\})$. Further, because $\{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\}$ was already proved to be a Borel set, it follows by (24) and Lemma 2.2 that $\mathfrak{M}(\mathscr{D})$ is a measure convex (also analytic) set and will become a Choquet set with Borel extremal boundary if we shall be able to verify that $\mathrm{ex}\,\mathfrak{M}(\mathscr{D}) = \{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\}$ is true. Because

$$\{\mu_\nu, \nu \in \mathrm{ex}\,\mathscr{D}\} \subset \mathrm{ex}\,\mathfrak{M} \cap \mathfrak{M}(\mathscr{D}) \subset \mathrm{ex}\,\mathfrak{M}(\mathscr{D})$$

holds trivially we need only to choose a $\mu_0 \in \mathrm{ex}\, \mathfrak{M}(\mathscr{D})$ and apply the already proved measure convexity of $\mathfrak{M}(\mathscr{D})$ to conclude that $\mu_0 = \mu_{\nu_0}$ for a $\nu_0 \in \mathrm{ex}\,(\mathscr{D})$: Indeed, we have $\mu_0 = \int \mu_\nu\, q(\mathrm{d}\nu)$ for a $q \in \mathscr{P}(\mathrm{ex}\, \mathscr{D})$, hence $\mu_0 = r(p)$ where $p \in \mathscr{P}(\{\mu_\nu, \nu \in \mathrm{ex}\, \mathscr{D}\})$ is the image of q under the map $\nu \to \mu_\nu$. It follows by Corollary 15.5. in [13] that $p = \varepsilon_{\mu_0}$, hence $q = \varepsilon_{\nu_0}$ for a $\nu_0 \in \mathrm{ex}\, \mathscr{D}$ because $\nu \to \mu_\nu$ is a Borel bijection of ex $\mathscr{D}$ onto $\{\mu_\nu, \nu \in \mathrm{ex}\, \mathscr{D}\}$.

To prove (25) assume first that $\mathscr{D}$ **is a simplex** and consider $\mu = r(p_1) = r(p_2)$ for a pair $p_1, p_2 \in \mathscr{P}(\mathrm{ex}\, \mathfrak{M}(\mathscr{D})) = \mathscr{P}(\{\mu_\nu, \nu \in \mathrm{ex}\, \mathscr{D}\})$ and a $\mu \in \mathfrak{M}(\mathscr{D})$. Then it follows by (16) that $r(\bar{H} \circ p_1) = r(\bar{H} \circ p_2)$ where $\bar{H} \circ p_1, \bar{H} \circ p_2 \in \mathscr{P}\mathrm{ex}\, \mathscr{D})$ and therefore $\bar{H} \circ p_1 = \bar{H} \circ p_2$ because $\mathscr{D}$ is a simplex. Hence $p_1 = p_2$ because $p \to \bar{H} \circ p$ is a bijection between $\mathscr{P}(\mathrm{ex}\, \mathscr{D})$ and $\mathscr{P}(\mathrm{ex}\, \mathfrak{M}(\mathscr{D}))$. It follows that $\mathfrak{M}(\mathscr{D})$ **is a simplex.** Further, if $\mu_1 \neq \mu_2$ are measures in $\mathfrak{M}(\mathscr{D})$ then $\mu_1 = r(p_1)$ and $\mu_2 = r(p_2)$ where $p_1 \neq p_2$ are in $\mathscr{P}(\mathrm{ex}\, \mathfrak{M}(\mathscr{D}))$ as $\mathfrak{M}(\mathscr{D})$ was already proved to be a Choquet set. Hence, $H \circ \mu_1 = r(\bar{H} \circ p_1) \neq r(\bar{H} \circ p_2) = H \circ \mu_2$ by (16) because $p \to \bar{H} \circ p$ is a bijection between $\mathscr{P}(\mathrm{ex}\, \mathscr{D})$ and $\mathscr{P}(\mathrm{ex}\, \mathfrak{M}(\mathscr{D}))$ and because $\mathscr{D}$ is a simplex. It follows that $\mu \to H \circ \mu$ **defines a bijection** $\mathfrak{M}(\mathscr{D}) \to \mathscr{D}$ that is a Borel map.

Finally, assume that $\mathfrak{M}(\mathscr{D})$ **is a simplex** and that $\mu \to H \circ \mu$ **defines a bijection** of $\mathfrak{M}(\mathscr{D})$ onto $\mathscr{D}$. Let $\nu = r(q_1) = r(q_2)$ for a pair of measures $q_1, q_2 \in \mathscr{P}(\mathrm{ex}\, \mathscr{D})$. Denote by p_1 and p_2 the images of q_1 and q_2, respectively under the map $\bar{H}^{-1} : \mathrm{ex}\, \mathscr{D} \to \mathrm{ex}\, \mathfrak{M}(\mathscr{D})$ and observe that $H \circ r(p_1) = H \circ r(p_2) = \nu$. It follows that $r(p_1) = r(p_2)$ because $\bar{H} : \mathfrak{M}(\mathscr{D}) \to \mathscr{D}$ is assumed to be a bijection and then $p_1 = p_2$ because $\mathfrak{M}(\mathscr{D})$ is assumed to be a simplex. Hence, $q_1 = \bar{H} \circ p_1 = \bar{H} \circ p_2 = q_2$ and $\mathscr{D}$ **is proved to be a simplex**. □

2.10. Remark. *Theorems 2.8 and 2.9 may in general provide handy tools when trying to establish the existence (the unique existence) of a solution $\mu \in \mathfrak{M} \subset \mathscr{P}(S)$ of a stochastic equation $H \circ \mu = \nu$ where $H : S \to T$ and $\nu \in \mathscr{P}(T)$ are given.*

Theorem 2.8 simply say, that if $\mathfrak{M} \subset \mathscr{P}(S)$ and $\mathscr{D} \subset \mathscr{P}(T)$ are Choquet sets such that the equation has a solution $\mu_\nu \in \mathfrak{M}$ for an arbitrary right-hand coefficient ν in the extremal boundary ex $\mathscr{D}$ *then the equation has a solution $\mu_\nu \in \mathfrak{M}$ for any $\nu \in \mathscr{D}$, i.e. $\mathscr{D} \subset H \circ \mathfrak{M}$. A more complex Theorem 2.9 further informs that if $\mathscr{D} = H \circ \mathfrak{M}$ is a simplex such that for any $\nu \in ex\, \mathscr{D}$ there is a unique solution $\mu_\nu \in \mathfrak{M}$ (it is sufficient to ask $\mu_\nu \in \mathrm{ex}\, \mathfrak{M}$) then there exists a unique solution $\mu_\nu \in co_M\{\mu_\nu, \nu \in \mathrm{ex}\, \mathscr{D}\}$ for any $\nu \in \mathscr{D}$.*

We may be more specific about the measurability of a map $\nu \to \mu_\nu$ that selects a solution $\mu_\nu \in \mathfrak{M}$ of the equation with a right-hand measure coefficient $\nu \in H \circ \mathfrak{M}$.

2.11. Corollary. *Assume that S, T and H are as in Theorem 2.8. If $\mathfrak{M} \subset \mathscr{P}(S)$ and $\mathscr{D} := H \circ \mathfrak{M}$ are Choquet sets with Borel extremal boundaries then there exists a stochastic kernel $\{p_\nu, \nu \in \mathscr{D}\} \subset \mathscr{P}(\mathrm{ex}\, \mathfrak{M})$ from $(\mathscr{D}, \mathscr{U}(\mathscr{D}))$ to $\mathscr{P}(S)$ such that*

$$H \circ r(p_\nu) = \nu \text{ holds for all } \nu \in \mathscr{D}. \tag{27}$$

Assume further that $\mathfrak{M}$ and $\mathscr{D}$ are as in Theorem 2.9 such that $\mathfrak{M} = \mathfrak{M}(\mathscr{D})$ holds. Then $\mu \to H \circ \mu$ is a Borel bijection of $\mathfrak{M}$ onto $\mathscr{D}$. Denoting by $\nu \to \mu_\nu$ its inverse then $\{\mu_\nu, \nu \in \mathscr{D}\}$ is a stochastic kernel from $(\mathscr{D}, \mathscr{B}(\mathscr{D}))$ to S such that

$$(28) \qquad \nu_0 = \int_{\mathscr{D}} \nu \, dq \;\Rightarrow\; \mu_{\nu_0} = \int_{\mathscr{D}} \mu_\nu \, dq, \quad q \in \mathscr{P}(\mathscr{D})$$

holds. Especially, $\nu \to \mu_\nu$ is an affine Borel map. Moreover, (27) defines a stochastic kernel $\{p_\nu, \nu \in \mathscr{D}\} \subset \mathscr{P}(\text{ex}\,\mathfrak{M})$ uniquely and $\mu_\nu = r(p_\nu)$ holds for all $\nu \in \mathscr{D}$.

Proof. Because $\text{ex}\,\mathscr{D} \subset H \circ \text{ex}\,\mathfrak{M}$ by Theorem 2.8 it follows by (20) that there exists a universally measurable map $R : \text{ex}\,\mathscr{D} \to \text{ex}\,\mathfrak{M}$ such that $(\bar{H} \circ R)(\nu) = \nu$ holds for any $\nu \in \text{ex}\,\mathscr{D}$. Further, it follows by Lemma 2.2 that there is a stochastic kernel $\{q_\nu, \nu \in \mathscr{D}\} \subset \mathscr{P}(\text{ex}\,\mathscr{D})$ from $(\mathscr{D}, \mathscr{U}(\mathscr{D}))$ to $\mathscr{P}(S)$ such that $r(q_\nu) = \nu$ for all $\nu \in \mathscr{D}$. Denoting $p_\nu = R \circ q_\nu$, we obviously define a kernel for which (27) and $\{p_\nu, \nu \in \mathscr{D}\} \subset \mathscr{P}(\text{ex}\,\mathfrak{M})$ hold.

As for the latter part of Corollary 2.11 it follows by Remark 2.10 that $\mu \to H \circ \mu$ is a Borel bijection of $\mathfrak{M}$ onto $\mathscr{D}$ and that both $\mathfrak{M}$ and $\mathscr{D}$ are simplices with Borel extremal boundaries. It follows by (8) and Theorem 8.3.7 in [1] that $\mathfrak{M} = r(\mathscr{P}(\text{ex}\,\mathfrak{M}))$ and $\mathscr{D} = r(\mathscr{P}(\text{ex}\,\mathscr{D}))$ are Borel sets. Hence, the inverse map to $\bar{H} : \mathfrak{M} \to \mathscr{D}$, denoted by $\nu \to \mu_\nu$, is also Borel measurable, obviously such that (28) holds. It follows that $\mu_\nu = r(p_\nu)$ holds for all $\nu \in \mathscr{D}$ and for each stochastic kernel $\{p_\nu, \nu \in \mathscr{D}\} \subset \mathscr{P}(\text{ex}\,\mathfrak{M})$ that satisfies (27). Such a stochastic kernel is uniquely determined by (27) because $\mathfrak{M}$ is a simplex. □

Assumptions of Theorem 2.8 and 2.9 are fulfilled very easily in some cases. If S, T and H are as above, consider a $\mu \in \mathscr{P}(S)$ with $\nu := H \circ \mu$ and recall that a stochastic kernel $\{\mu_y^H\}, y \in T$ from $(T, \mathscr{B}(T))$ to S is called a **regular conditional distribution** (RCD) of μ given H if

$$\int_B \mu_y^H(A)\, \nu(dy) = \mu(A \cap H^{-1}B), \qquad B \in \mathscr{B}(T), \quad A \in \mathscr{B}(S).$$

Recall also that regular conditional distributions always exists in our setting and $\mu_y^H = \mu'_y$ almost surely $[\nu]$ for each pair $\{\mu_y^H, y \in T\}, \{\mu'_y, y \in T\}$ of RCD's of μ given H. Only elementary properties of the conditional expectations yields

$$(29) \qquad H \circ \mu_y^H = \varepsilon_y \;\; a.s.[\nu] \quad \text{where } \varepsilon_y(B) = I_B(y),\ y \in T,\ B \in \mathscr{B}(T).$$

Having also a set $\mathfrak{M} \subset \mathscr{P}(S)$ we shall say that a **measure μ is H-decomposable in $\mathfrak{M}$** if $\mu_y^H \in \mathfrak{M}$ almost surely $[\nu = H \circ \mu]$. If $\mathfrak{M}$ is a measure convex set and μ a measure H-decomposable in $\mathfrak{M}$ then μ necessarily belongs to $\mathfrak{M}$ by Lemma 2.1 because $\mu = \int \mu_y^H \, \nu(dy)$ holds.

2.12. Corollary. *Let S, T, H and $\mathfrak{M}$ satisfy the requirements of Theorem 2.8 and let $\mu \in \mathfrak{M}$ with $\nu = H \circ \mu$ be a measure H-decomposable in $\mathfrak{M}$. Then there* *is a* $B \in \mathscr{B}(T)$ *such that* $\nu(B) = 1$ *and* $H \circ \mathfrak{M} \supset \mathscr{P}(B)$ *holds. Especially,* $\nu' \in H \circ \mathfrak{M}$ *for any measure* $\nu' \in \mathscr{P}(T)$ *that is absolutely continuous w.r.t.* ν.

Proof. The set $\mathfrak{M} \subset \mathscr{P}(S)$ is analytic by Lemma 2.2, hence $\{y \in T : \mu_y^H \in \mathfrak{M}\} := B_0$ is a universally measurable set in T. It follows that there is a Borel set $B_1 \subset B_0$ with $\nu(B_1) = 1$. It follows by (29) that there is a Borel set $B \subset B_1$ with $\nu(B) = 1$ such that $\{\varepsilon_y, y \in B\} \subset H \circ \mathfrak{M}$ and we may apply Theorem 2.8 with $\mathscr{D} := \mathscr{P}(B)$ observing that $\mathscr{D}$ is a Choquet set with $\operatorname{ex} \mathscr{D} = \{\varepsilon_y, y \in B\}$. □

We suspect that a measure $\mu \in \mathscr{P}(S)$ is H-decomposable in an $\mathbb{F}(S)$-closed convex set $\mathfrak{M} \subset \mathscr{P}(S)$ if and only if $\mu(\cdot \mid H^{-1}B) \in \mathfrak{M}$ for any $B \in \mathscr{B}(T)$ with $\mu(H^{-1}B) > 0$.

2.13. Corollary. *Let* S, T, H *are as in Theorem 2.8 and* $\mathfrak{M} \subset \mathscr{P}(S)$ *a Choquet set with a Borel extremal boundary. Then*

$$(30) \quad H \circ \mathfrak{M} \supset \{\varepsilon_y, y \in T\} \Rightarrow H \circ \operatorname{ex} \mathfrak{M} \subset \{\varepsilon_y, y \in T\}, \quad H \circ \mathfrak{M} = \mathscr{P}(T).$$

If moreover any $\mu \in \mathfrak{M}$ *is* H*-decomposable in* $\mathfrak{M}$ *and*

$$(31) \qquad \textit{for any } y \in T \textit{ there is a unique } \mu_y \in \mathfrak{M} \textit{ with } H \circ \mu_y = \varepsilon_y$$

then $\{\mu_y, y \in T\}$ *is a stochastic kernel from* $(T, \mathscr{B}(T))$ *to* S *and*

$$(32) \quad \textit{for any } \nu \in \mathscr{P}(T) \textit{ there is a unique } \mu_\nu = \int_T \mu_y \, \nu(\mathrm{d}y) \in \mathfrak{M} \textit{ with } H \circ \mu_\nu = \nu .$$

Finally, $\mathfrak{M}$ *is a simplex with* $\operatorname{ex} \mathfrak{M} = (\mu_y, y \in T\}$.

Proof. (30) is proved by Theorem 2.8 with $\mathscr{D} = \mathscr{P}(T)$. Hence, (31) implies that $H \circ \mathfrak{M} = \mathscr{P}(T)$ is a simplex with $\operatorname{ex} \mathscr{P}(T) = \{\varepsilon_y, y \in T\}$ and we may apply Theorem 2.9 with $\mathscr{D} := \mathscr{P}(T)$. In this case we get $\mathfrak{M}(\mathscr{D}) = \{\int \mu_y \, \nu(\mathrm{d}y), \nu \in \mathscr{P}(T)\}$ because T and $\{\varepsilon_y, y \in T\}$ are homeomorphic each to other. Further, it follows by (29) and (31) that

$$\mu_y^H = \mu_y \text{ almost surely } [\nu], \quad \nu := H \circ \mu, \ \mu \in \mathfrak{M}$$

because each $\mu \in \mathfrak{M}$ is assumed to be H-decomposable in $\mathfrak{M}$. Hence, $\mathfrak{M}(\mathscr{D}) = \mathfrak{M}$ and (32) follows by (25) in Theorem 2.9 because $H \circ \int \mu_y \, \nu(\mathrm{d}y) = \nu$ holds for all $\nu \in \mathscr{P}(T)$. □

A way how to construct a Choquet set (a Choquet simplex) is suggested by

2.14. Example: Assume that S is an Abelian additive topological group such that the topology of S is Polish. Let $T \subset S$ be a closed subgroup and μ a measure in $\mathscr{P}(S)$. Then

$$\mathfrak{M}(\mu, T) := \{\mu * \alpha, \alpha \in \mathscr{P}(T)\} \subset \mathscr{P}(S)$$

is a weakly closed convex (hence Choquet) set by Theorem 2.3, with a weakly closed boundary $\operatorname{ex} \mathfrak{M}(\mu, T) = \{\mu_y = \mu * \varepsilon_y, y \in T\}$. We have denoted by $\mu * \alpha$ the convolution of μ and α given by $(\mu * \alpha)(B) = \int \mu(B - y) \, \alpha(\mathrm{d}y)$.

If, moreover, $\{0\} \subset T$ is the unique compact subgroup of T then $\mathfrak{M}(\mu, T)$ is a simplex if and only if equation $\mu * \alpha = \nu$ has at most one solution $\alpha \in \mathscr{P}(T)$ for any $\nu \in \mathscr{P}(S)$.

Indeed, if $\mu_n := \mu * \alpha_n$ is a sequence in $\mathfrak{M}(\mu, T)$ weakly convergent to a $\mu_0 \in \mathscr{P}(S)$ it follows by III.2.1. in [6] that the sequence (α_n) has a weak limit point $\alpha_0 \in \mathscr{P}(T)$. Hence, $\mu_0 = \mu * \alpha_0$ and $\mathfrak{M}(\mu, T)$ is a weakly closed set in $\mathscr{P}(S)$. A similar argument shows that $\{\mu_y, y \in T\}$ is a closed set also. It follows by 3.3 Theorem in [8] that $\mu_y \in \mathrm{ex}\, \mathfrak{M}(\mu, S)$ for any $y \in S$ and therefore $\{\mu_y, y \in T\} \subset \mathrm{ex}\, \mathfrak{M}(\mu, T)$. Let $\nu = \mu * \alpha \in \mathrm{ex}\, \mathfrak{M}(\mu, T)$ and denote by $p \in \mathscr{P}(\mathfrak{M}(\mu, T))$ the image of α under the continuous map $y \to \mu_y$. Then $\nu = r(p)$ and therefore p is a point measure on $\mathfrak{M}(\mu, S)$ again by Corollary 1.5.5. in [13] because $\mathfrak{M}(\mu, S)$ is a measure convex set. Hence, $\nu = \mu_y$ for any $y \in T$.

Assuming that there are no non-trivial compact subgroups in T it follows by 2.1. Lemma in [8] that $y \to \mu_y$ define a homeomorphism between T and $\{\mu_y, y \in T\} = \mathrm{ex}\, \mathfrak{M}(\mu, T)$. Hence, $\mathfrak{M}(\mu, T)$ is a simplex if and only if each equation $\nu = \mu * \alpha$, $\nu \in \mathscr{P}(S)$ has at most one solution $\alpha \in \mathscr{P}(T)$.

If T is a nontrivial compact subgroup of S and μ Haar probability measure on T then $\mathfrak{M}(\mu, T) = \{\mu\}$ and therefore it is a simplex. On the other hand $\mu = \mu * \alpha$ for any $\alpha \in \mathscr{P}(T)$ which shows that the above equivalence does not hold generally.

We close the present section with a simple result that extends Theorem 3.7 (c) in [11] (see also 3.2. in [15]) and provides a good reason to introduce and study the concept of a Choquet set. Denote by $\mathbb{B}(S)$ the set of all Borel bounded functions $g : S \to \mathbb{R}$.

2.15. Theorem. *Let $\mathfrak{M} \subset \mathscr{P}(S)$ be a Choquet set S, being an arbitrary metric space. Then*

$$\sup \{F(\mu), \mu \in \mathfrak{M}\} = \sup \{F(\mu), \mu \in \mathrm{ex}\, \mathfrak{M}\} =: s$$

for any convex map $F : \mathscr{P}(S) \to \bar{\mathbb{R}}$ that is lower bounded and lower semicontinuous in $\mathbb{B}(S)$-topology of $\mathscr{P}(S)$.

Note that $F : \mathscr{P}(\mathbb{R}) \to \mathbb{R}$ given by $F(\mu) := \mu(f)$ where $f : \mathbb{R} \to \mathbb{R}$ is a lower bounded Borel function defines an affine map that is lower bounded and lower semicontinuous in $\mathbb{B}(\mathbb{R})$-topology of $\mathscr{P}(\mathbb{R})$ (see Example 3.8 in [11]).

Proof. If $\nu \in \mathfrak{M}$ then $\nu = r(p)$ for a $p \in \mathscr{P}(\mathrm{ex}\, \mathfrak{M})$ and therefore

$$\nu(g) \leq \sup \{\mu(g), \mu \in \mathrm{ex}\, \mathfrak{M}\}, \qquad g \in \mathbb{B}(S)$$

by (11). It follows by Theorem on separation of convex sets in locally convex spaces (as in [9]) that there is a net (μ_α) of measures in the convex hull of $\mathrm{ex}\, \mathfrak{M}$ such that $\mu_\alpha \to \nu$ in $\mathbb{B}(S)$-topology. It follows by the convexity and lower semicontinuity of F that $F(\nu) \leq \liminf_\alpha F(\mu_\alpha) \leq s$. □

3 Convex analysis of local martingale problems

3.1. Theorem. *Let $\mathscr{G} \subset \mathscr{C}$ be countable set and $B \subset \mathbb{C}(\mathbb{R}^+)$ a Borel set such that B and $\mathscr{G}$ are compatible. Then $\mathscr{W}_{\mathscr{G},B}$ is an $\mathbb{F}_b(\mathbb{C})$-closed convex set, hence a measure convex Choquet set with a Borel extremal boundary* ex $\mathscr{W}_{\mathscr{G},B}$. *It is a simplex if and only if it is a Choquet simplex. Especially, $\mathscr{W}_{\mathscr{G},B}$ posseses all the above properties for any countable set $\mathscr{G} \subset \mathscr{C}_a$ and any $B \in \mathscr{B}(\mathbb{C})$.*

Proof. The set $\mathscr{W}_{\mathscr{G},B}$ is convex by Corollary 3.6 (a) in [11] and any $\mathscr{G} \subset \mathscr{C}_a$ is compatible with any $B \in \mathscr{B}(\mathbb{C})$. Hence, all statements will follow by Theorem 2.3 after having proved that $\mathscr{W}_{\mathscr{G},B}$ is an $\mathbb{F}_b(\mathbb{C})$-closed set: Fix first a $G \in \mathscr{G}$ and define a Borel map $H : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $H(x) := (x, G(x))$ and observe that it follows by (2) that $\mathscr{W}_G = \{\mu \in \mathscr{P}(\mathbb{C}) : H \circ \mu \in \mathscr{L}(\mathbb{L}^0)\}$. Thus, it follows by Proposition in [11] that

$$\mathscr{W}_G \cap \mathscr{P}(B) = H^{-1}\mathscr{L}(\mathbb{L}^0) \cap \mathscr{P}(B) = H^{-1}\overline{\mathscr{L}(\mathbb{L}^0)} \cap \mathscr{P}(B),$$

where $\overline{\mathscr{L}(\mathbb{L}^0)}$ denotes the weak closure of $\mathscr{L}(\mathbb{L}^0)$ in $\mathscr{P}(\mathbb{C}(\mathbb{R}^+) \times \mathbb{C}(\mathbb{R}^+)$, because $\mu \in \mathscr{P}(B)$ implies that G is an $\mathscr{F}_t^{\mathbf{x},\mu}$-adapted process. Hence, $H^{-1}\overline{\mathscr{L}(\mathbb{L}^0)}$ is an $\mathbb{F}_b(\mathbb{C})$-closed set in $\mathscr{P}(\mathbb{C})$ by 2.7 Lemma. Because $\mathscr{P}(B) = \{\mu \in \mathscr{P}(\mathbb{C}(\mathbb{R}^+)) : \mu(B) = 1\}$ is easily seen to be $\mathscr{F}$-closed choosing $\mathscr{F} := \mathbb{C}_b(\mathbb{C}) \cup \{I_B\} \in \mathbb{F}_b(\mathbb{C})$, it follows again by 2.7 Lemma that $\mathscr{W}_{\mathscr{G},B} = \bigcap_{\mathscr{G}}(\mathscr{W}_{\mathscr{G}} \cap \mathscr{P}(B))$ is an $\mathbb{F}_b(\mathbb{C})$-closed set as a countable intersection of $\mathbb{F}_b(\mathbb{C})$-closed sets. $\square$

It is important to stress that the **$\mathscr{W}_{\mathscr{G}}$-sets need not be convex generally** without assuming $\mathscr{G} \subset \mathscr{C}_a$, i.e. not assuming that all G's in $\mathscr{G}$ are $\mathscr{F}_t^{\mathbf{x}}$-adapted processes.

3.2. Example: Let $\mu_1 = \mathscr{L}(W)$ and $\mu_2 = \mathscr{L}(W^1)$ where W is a Wiener process and $W^1(t) = W(t \wedge 1)$ for $t \geq 0$. Then μ_1 and μ_2 are in $\mathscr{W}_{\mathbf{x}-\mathbf{x}(0)}$ and obviously singular such that $\mu_1(F) \equiv 1$ and $\mu_2(F) \equiv 1$ where F is the set of all $\mathbf{x} \in \mathbb{C}(\mathbb{R}^+)$ that are constant on $[1, \infty)$. It follows that $F \in \mathscr{F}_0^{\mathbf{x},\mu_1} \cap \mathscr{F}_0^{\mathbf{x},\mu_2}$ and therefore $G := (\mathbf{x} - \mathbf{x}(0))\, I_F \in \mathscr{C}$ is both an $(\mu_1, \mathscr{F}_t^{\mathbf{x},\mu_1})$ and $(\mu_2, \mathscr{F}_t^{\mathbf{x},\mu_2})$-local martingale on $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$. In other words, $\mu_1, \mu_2 \in \mathscr{W}_G$. Put $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ and check that $\mu \notin \mathscr{W}_{\mathscr{G}}$ or equivalently that G is not a $(\mu, \mathscr{F}_t^{\mathbf{x},\mu})$ local martingale. Indeed, **assuming the contrary** then G is an $\mathscr{F}_t^{\mathbf{x},\mu}$-measurable r.v. for any $t \geq 0$. It follows that $F \in \mathscr{F}_t^{\mathbf{x},\mu}$ for any $t > 0$ because for such a t, $\mu(\mathbf{x}(t) = 0) = 0$ implies $I_F = \frac{G(t)}{\mathbf{x}(t)}$ almost surely w.r.t. μ. Hence, assuming $\mu \in \mathscr{W}_{\mathscr{G}}$, the property F would be predictable up to any fixed time $0 < t < 1$ which intuitively produces a contradiction. A formal reasoning may be as follows: W and ξ are **independent processes** on a complete probability space $(\Omega, \mathscr{F}, P)$ such that $\mathscr{L}(W) = \mu_1$ and ξ is a right continuous 0, 1 valued non-decreasing process with $\xi_0 = 0$ and with a unique possible discontinuity at $t = 1$ such that $P[\xi_1 = 1] = \frac{1}{2} = P[\xi_1 = 0]$. Putting $X(t) = W(t \wedge [1 + \xi_t(t - 1)^+])$ for $t \geq 0$, we define a continuous stochastic process on $(\Omega, \mathscr{F}, P)$ such that $X = W$ if and only if $\xi_1 = 1$ and $X = W^1$ if and only if $\xi_1 = 0$ where $W^1(t) := W(t \wedge 1)$

as before. Obviously, it follows that $\mathscr{L}(X \mid P) = \mu$. Finally, fix a $t \in (0, 1)$ and assume that $F \in \mathscr{F}_t^{\mathbf{x},\mu}$. It follows by Lemma 2.2 in [11] that $I_F = g_t(\mathbf{x}(s), s \leq t)$ a.s. $[\mu]$ where $g_t : \mathbb{C}[0, t] \to \mathbb{R}$ is a Borel measurable map. Hence,

$$I_{[\xi_1 = 0]} = I_{[X \in F]} = g_t(X(s), s \leq t) = g_t(W(s), s \leq t) \text{ a.s. } [P]$$

and therefore $P[\xi_1 = 0]$ is either 0 or 1 because ξ and W are independent processes, hence a contradiction.

We insist that the only trouble maker when seeking for the convexity of a $\mathscr{W}_G$ with a $G \in \mathscr{C}$ is the requirement on the adaptive measurability of G. Indeed, a measure $\mu \in \mathscr{P}(\mathbb{C})$ is in $\mathscr{W}_G$ if and only if G is an $\mathscr{F}_t^{\mathbf{x},\mu}$-adapted process and G is an $\mathscr{F}_t^{\mathbf{x}}$-premartingale on $(\mathbb{C}, \mathscr{B}(\mathbb{C}), \mu)$, the latter property being defined by

$$\mathbb{E}_\mu^{\mathscr{F}_s^{\mathbf{x}}} G(t \wedge \tau^c) = \mathbb{E}_\mu^{\mathscr{F}_s^{\mathbf{x}}} G(s \wedge \tau^c) \text{ a.s. } [\mu], \quad s \leq t, \quad c \in \mathbb{N},$$

where τ^c denotes the first entry of $|\mathbf{x}|$ to $[c, +\infty]$. Because, for a fixed $G \in \mathscr{C}$, the set $\{\mu \in \mathscr{P}(\mathbb{C}) : G \text{ is an } \mathscr{F}_t^{\mathbf{x}}\text{-premartingale on } (\mathbb{C}, \mathscr{B}(\mathbb{C}), \mu)\}$ is easily seen to be a convex set, $\mathscr{W}_G$ may fail to be convex only because of the lack of convexity of the set $\{\mu \in \mathscr{P}(\mathbb{C}) : G \text{ is an } \mathscr{F}_t^{\mathbf{x},\mu}\text{-adapted process}\}$.

Remark that we could avoid the above "convexity catastrophe" considering a subset $\overline{\mathscr{W}_G} \subset \mathscr{W}_{\mathscr{G}}$ defined by

$$\overline{\mathscr{W}_G} := \{\mu \in \mathscr{P}(\mathbb{C}) : G \text{ is an } \overline{\mathscr{F}_t^{\mathbf{x},\mu}}\text{-local martingale on } (\mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)\},$$

where $\overline{\mathscr{F}_t^{\mathbf{x},\mu}}$ is the standard completion of $\mathscr{F}_t^{\mathbf{x}}$ with respect to the restriction of μ from $\mathscr{B}(\mathbb{C})$ to $\mathscr{F}_t^{\mathbf{x}}$. The set $\overline{\mathscr{W}_G}$ is easily seen to be convex but unfortunately only of a limited merit in stochastic analysis.

Examples 1.1, 1.2 and 1.3 in [11] propose positive applications of Theorem 3.1:

3.3. Corollary. *Consider $G, v \in \mathscr{C}_a$ such that v is a process of finite variation on $\mathbb{R}^+$ and $\mathscr{F}_t^{\mathbf{x}}$-progressive processes b and σ defined on $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$. Then the sets $\mathscr{W}_{G,v}$ and $\mathscr{W}_{b,\sigma}$ defined by (3) and (6), respectively are convex $\mathbb{F}_b(\mathbb{C})$-closed sets, i.e. Borel measure convex Choquet sets with Borel extremal boundaries. Either set is a simplex if and only if it is a Choquet simplex.*

The assertion follows by Theorem 3.1 applying (3) and (5), respectively.

Choquet sets and even Choquet simplices are of a frequent occurrence in stochastic analysis, indeed:

3.4. Example: Choosing $S := \mathbb{C}(\mathbb{R}^+)$ and $T := \{x \in \mathbb{C}(\mathbb{R}^+), x(t) = x(0), t \geq 0\}$ in Example 2.14 we prove that for any $\mu \in \mathscr{W}_{\mathbf{x}-\mathbf{x}(0)}$, denoting by $\mathbb{L}_\mu^0$ the set of all continuous local martingales M with $\mathscr{L}(M) = \mu$,

$$\mathscr{W}_\mu := \{\mathscr{L}(\xi + M), \xi \text{ and } M \text{ independent}, \xi \text{ an } \mathbb{R}\text{-valued r.v.}, M \in \mathbb{L}_\mu^0\} \subset \mathscr{W}_{\mathbf{x}-\mathbf{x}(0)}$$

is a weakly closed simplex with a weakly closed extremal boundary

$$\operatorname{ex} \mathscr{W}_\mu = \{\mathscr{L}(x + M), \quad x \in \mathbb{R}, \quad M \in \mathbb{L}_\mu^0\}.$$

It is enough to note that $\mathscr{W}_\mu = \mathfrak{M}(\mu, T)$ (see 2.14) and that $T \subset S$ is a closed subgroup, the only compact subgroup of which is $\{0\}$. Also, any equation $v = \mu * \alpha$, $v \in \mathscr{P}(\mathbb{C})$ has at most one solution $\alpha \in \mathscr{P}(T)$ because $v = \mu * \alpha$ implies $\mathscr{L}(\mathbf{x}(0) \mid v) = \alpha$.

Denoting by w the probability distribution of the standard Wiener process W, by $v(t) := t$ its quadratic variation and by $s(x) := \operatorname{sign}(x)$ for $x \in \mathbb{R}$ we get that $\mathscr{W}_w$ equals to the set of probability distributions of all continuous local martingales with the quadratic variation v and also to the set of probability distributions of all weak solutions of the $(0, s)$-SDE. In the notation we introduced in (3) and (6) the statement reads as

$$\mathscr{W}_w = \mathscr{W}_{\mathbf{x}-\mathbf{x}(0), v} = \mathscr{W}_{0,s}.$$

Indeed, the first equality follows by Levy's characterization theorem while the second one follows by (5) because $B_{0,s} = \mathbb{C}(\mathbb{R}^+)$ and $\int_0^t s^2(\mathbf{x})\,\mathrm{d}s = v(t)$. We have proved that the $(0, \operatorname{sign})$-SDE $\mathrm{d}X(t) = \operatorname{sign}(X(t))\,\mathrm{d}W(t)$ has a weak solution X for any given initial distribution $\mathscr{L}(X(0)) = \alpha \in \mathscr{P}(\mathbb{R})$ which probability distribution is uniquely determined by $\mathscr{L}(\bar{\alpha} * w)$, where $\bar{\alpha} \in \mathscr{P}(T)$ is the image of α under the natural homomorphismus of $\mathbb{R}$ onto T.

3.5. Example: Denoting

$$\tau_B(x) := \inf\{t \geq 0 : (t, x(t)) \in B\}, \quad x \in \mathbb{C}(\mathbb{R}^+)$$

the first debut of $x \in \mathbb{C}(\mathbb{R}^+)$ in a Borel set $B \subset \mathbb{R}^+ \times \mathbb{R}$ we note that $\tau_B : \mathbb{C}(\mathbb{R}^+) \to \mathbb{R}^+ \cup \{+\infty)$ is a universally measurable map because for any $T \geq 0$ the set $[\tau_B \leq T]$ is the projection of

$$\{(t, x) \in \mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+): \quad (t, x(t)) \in B\} \cap [0, T] \times \mathbb{C}(\mathbb{R}^+) \in \mathscr{B}(\mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+))$$

to $\mathbb{C}(\mathbb{R}^+)$. Hence, $H_B(x) := x(\tau_B(x))\, I_{[\tau_B < \infty]}(x)$ for $x \in \mathbb{C}(\mathbb{R}^+)$ defines also a universally measurable map $H_B : \mathbb{C}(\mathbb{R}^+) \to \mathbb{R}$. Considering a pair $B_2 \subset B_1$ of Borel sets in $\mathbb{R}^+ \times \mathbb{R}$ we may be interested in a "transport of a particle" from the boundary of B_1 to the boundary of B_2 by means of a continuous local martingale represented by its probability distribution $\mu \in \mathscr{W}_{\mathbf{x}-\mathbf{x}(0)}$ possibly such that $\mu[\tau_{B_1} < \tau_{B_2} < \infty] = 1$. A natural criterium for such a transport is given by a property $\mathscr{D} \subset \mathscr{P}(\mathbb{R}^{+2} \times \mathbb{R}^2)$ of the probability distribution $\mathscr{L}(H \mid \mu)$, where $H := (\tau_{B_1}, \tau_{B_2}, H_{B_1}, H_{B_2}) : \mathbb{C}(\mathbb{R}^+) \to \mathbb{R}^{+2} \times \mathbb{R}^2$ defines a map that is universally measurable. Hence, we are interested in a local martingale problem that is subjected to further boundary constraints given by

$$\begin{aligned}\mathfrak{M}_{B_1, B_2, H, \mathscr{D}} &:= \mathscr{W}_{\mathbf{x}-\mathbf{x}(0)} \cap \{\mu \in \mathscr{P}(\mathbb{C}) : \mu[\tau_{B_1} < \tau_{B_2} < \infty] = 1\} \cap \{\mu \in \mathscr{P}(\mathbb{C}) : \mathscr{L}(H \mid \mu) \in \mathscr{D}\} \\ &= \mathscr{W}_{\mathbf{x}-\mathbf{x}(0), [\tau_{B_1} < \tau_{B_2} < \infty]} \cup H^{-1} \circ \mathscr{D}.\end{aligned}$$

As suggested by Theorem 2.15 it is always profitable to deal with $\mathfrak{M} := \mathfrak{M}_{B_1, B_2, H, \mathscr{D}}$ that is a Choquet set whenever trying to perform an optimization on the set. **Assuming** that $\mathscr{D} \subset \mathscr{P}(\mathbb{R}^{+2} \times \mathbb{R}^2)$ **is convex** $\mathbb{F}(\mathbb{R}^{+2} \times \mathbb{R}^2)$**-closed set** we get the $\mathfrak{M}$ as a convex $\mathbb{F}(\mathbb{C})$-closed, hence a Choquet set by Theorem 2.5 with all "good" properties listed by the Theorem. Indeed, $\mathfrak{M}$ is an $\mathbb{F}(\mathbb{C})$-closed set by Lemma 2.7 as $\mathscr{W}_{\mathbf{x}-\mathbf{x}(0)}$ is an $\mathbb{F}(\mathbb{C})$-closed set by 3.1, $[\mu : \mu[\tau_{B_1} < \tau_{B_2} < \infty] = 1]$ is an $\mathbb{F}(\mathbb{C})$-closed set because τ_{B_1}, τ_{B_2} are universally measurable and finally it follows by 2.7 again that $H^{-1} \circ \mathscr{D}$ is an $\mathbb{F}(\mathbb{C})$-closed set due to the universal measurability of H.

For example, if we are required to organize a local martingale transport between the corresponding boundaries of B_1 and B_2 with the minimal mean velocity $\bar{v} = 1$ we should choose

$$\mathscr{D} = \left\{ v \in \mathscr{P}(\mathbb{R}^{+2} \times \mathbb{R}^2) : v\left[(t_1, t_2, h_1, h_2) : \frac{h_2 - h_1}{t_2 - t_1} \geq 1 \right] = 1 \right\}$$

that is obviously an $\mathbb{F}_b(\mathbb{R}^{+2} \times \mathbb{R}^2)$-closed convex set. This choice of $\mathscr{D}$ specifies $\mathfrak{M}_{B_1,B_2,H,\mathscr{D}}$ as the set of all probability distribution $\mathscr{L}(X) \in \mathscr{P}(\mathbb{C})$ where X goes through all continuous local martingales such that $0 \leq \tau_{B_1}(X) < \tau_{B_2}(X) < \infty$ and $\bar{v}(X) := \frac{X(\tau_{B_2}) - X(\tau_{B_1})}{\tau_{B_2} - \tau_{B_1}} \geq 1$ holds almost surely.

The above Example suggests to treat a local martingale problem $\mathscr{W}_{\mathscr{G},B}$ with a boundary condition given by an equation

$$\mathscr{L}(H \mid \mu) = v, \quad \mu \in \mathscr{W}_{\mathscr{G},B} \tag{33}$$

where the right-hand probability distribution v belongs to a fixed set of boundary conditions $\mathscr{D} \subset \mathscr{P}(T)$, T being a Polish set and $H : \mathbb{C} \to T$ a Borel map. Obviously there is a pair of important sets to be studied in connection with the equation (33). Namely,

$$H \circ \mathscr{W}_{\mathscr{G},B} := \{\mathscr{L}(H \mid \mu) : \mu \in \mathscr{W}_{\mathscr{G},B}\} \subset \mathscr{P}(T),$$

the set of all boundary conditions $v \in \mathscr{P}(T)$ for which (33) has a solution $\mu \in \mathscr{W}_{\mathscr{G},B}$ and for a $\mathscr{D} \subset \mathscr{P}(T)$

$$H^{-1}\mathscr{D} \cap \mathscr{W}_{\mathscr{G},B} = \{\mu \in \mathscr{W}_{\mathscr{G},B} : \mathscr{L}(H \mid \mu) \in \mathscr{D}\},$$

the set of all solutions μ of $(\mathscr{G}, B)$-local martingale problem that satisfy (33) with a boundary contidion $v \in \mathscr{D}$.

General results of Section 2 read in our context as follows.

3.6. Theorem. *If $\mathscr{G} \subset \mathscr{C}$ and $B \in \mathscr{B}(\mathbb{C})$ are compatible sets such that $\mathscr{G}$ is a countable set, if T is a Polish space, $H : \mathbb{C} \to T$ a Borel map and if $\mathscr{D} \subset \mathscr{P}(T)$ a Choquet set then $H \circ \mathscr{W}_{\mathscr{G},B} = \mathrm{co}_M(H \circ \mathrm{ex}\, \mathfrak{M})$ is an analytic measure convex set in $\mathscr{P}(T)$. Moreover, if (33) has a solution for any $v \in \mathrm{ex}\, \mathscr{D}$ then it has a solution for any $v \in \mathscr{D}$. The equation (33) with $v \in \mathrm{ex}\, \mathscr{D}$ has a solution $\mu \in \mathrm{ex}\, \mathscr{W}_{\mathscr{G},B}$. Finally if $H \circ \mathscr{W}_{\mathscr{G},B} =: \mathscr{D}$ is a Choquet set with a Borel extremal boundary then there is a universally measurable map $v \to \mu_v$ that maps $\mathscr{D}$ into $\mathscr{W}_{\mathscr{G},B}$ such that μ_v is a solution of (33) for any $v \in \mathrm{ex}\, \mathscr{D}$.*

The above assertions follow directly by 3.1, 2.8 and 2.11.

Remark that even though we suspect that $H \circ \mathscr{W}_{\mathscr{G},B}$ need not be a Choquet set generally, it is generated as the measure convex hull of the direct image $H \circ \operatorname{ex} \mathscr{W}_{\mathscr{G},B}$.

3.7. Theorem. *Assume $\mathscr{G}, B, T$ and H as in Theorem 3.6 such that $\mathscr{D} := H \circ \mathscr{W}_{\mathscr{G},B}$ is a simplex with a Borel extremal boundary* ex $\mathscr{D}$. *Denote*

$$\mathscr{W}_{\mathscr{G},B}(\mathscr{D}) := \operatorname{co}_M\{\mu \in \mathscr{W}_{\mathscr{G},B} : \mathscr{L}(H \mid \mu) \in \operatorname{ex} \mathscr{D}\}.$$

Then

(i) For any $\nu \in \operatorname{ex} \mathscr{D}$ there exists a unique solution μ_ν of (33) in $\mathscr{W}_{\mathscr{G},B}$.

(ii) For any $\nu \in \mathscr{D}$ there exists a unique solution μ_ν of (33) in $\mathscr{W}_{\mathscr{G},B}(\mathscr{D})$

are equivalent statements and each of them implies that $\mathscr{W}_{\mathscr{G},B}(\mathscr{D})$ is a simplex with a Borel extremal boundary, namely ex $\mathscr{W}_{\mathscr{G},B}(\mathscr{D}) = \{\mu_\nu, \nu \in \operatorname{ex} \mathscr{D}\}$. *Finally, if $\mathscr{W}_{\mathscr{G},B} = \mathscr{W}_{\mathscr{G},B}(\mathscr{D})$ then the map $\nu \to \mu_\nu$ defined by (ii) is a Borel affine bijection $\mathscr{D} \to \mathscr{W}_{\mathscr{G},B}$.*

All statements follows by 2.9 and 2.11 because $\mu_\nu \in \mathscr{W}_{\mathscr{G},B}$ defined by (i) for a $\nu \in \operatorname{ex} \mathscr{D}$ belongs to ex $\mathscr{W}_{\mathscr{G},B}$ according to (26).

Having rewritten the equation (33) in the form

$$\mathscr{L}(\mathbf{x}(0) \mid \mu) = \nu, \quad \mu \in \mathscr{W}_{\mathscr{G},B}, \tag{34}$$

i.e. asking to eastablish a solution μ of a $(\mathscr{G}, B)$-local martingale problem μ with an initial condition $\nu \in \mathscr{P}(\mathbb{R})$, we in fact observe Theorems 3.6 and 3.7 with $T = \mathbb{R}$, and $H : \mathbb{C} \to \mathbb{R}$ to be defined as the projection $H(x) := x(0)$ for $x \in \mathbb{C}(\mathbb{R}^+)$. Corollary 2.13 yields in this case a more neat results compared with 3.6 and 3.7. Having (34) with a point measure $\nu = \varepsilon_y$ agree to say that μ is a solution of a $(\mathscr{G}, B)$-problem with an initial (deterministic) condition $y \in \mathbb{R}$.

3.8. Corollary. *Consider $\mathscr{G} \subset \mathscr{C}$ and $B \in \mathscr{B}(\mathbb{C})$ that are compatible such that $\mathscr{G}$ is a countable set. Then the $(\mathscr{G}, B)$-local martingale problem has a solution μ_ν with an initial condition ν for arbitrary $\nu \in \mathscr{P}(\mathbb{R})$ if and only if it has a solution μ_y with an deterministic initial condition $y \in \mathbb{R}$ for arbitrary $y \in \mathbb{R}$. Moreover, if for any $y \in \mathbb{R}$ there is a* **unique** *solution μ_y of the $(\mathscr{G}, B)$-problem with the initial condition y then $y \to \mu_y$ is a Borel map $\mathbb{R} \to \mathscr{P}(\mathbb{C})$ such that for any $\nu \in \mathscr{P}(\mathbb{R})$ $\mu_\nu := \int_{\mathbb{R}} \mu_y \, \nu(\mathrm{d}y)$ is a* **unique** *solution of the $(\mathscr{G}, B)$-problem with the initial condition ν. The set of solution $\mathscr{W}_{\mathscr{G},B}$ is in this case a simplex, i.e. Choquet simplex, with* ex $\mathscr{W}_{\mathscr{G},B} = \{\mu_y, y \in \mathbb{R}\}$.

Proof. $\mathscr{W}_{\mathscr{G},B}$ is a Choquet set with an Borel extremal boundary by Theorem 3.1 and our assertion would become a special case to Corollary 2.13 if we prove that any $\mu \in \mathscr{W}_{\mathscr{G},B}$ is an H-decomposable measure in the set $\mathscr{W}_{\mathscr{G},B}$ if $H(x) = x(0)$ is defined as the projection of an $x \in \mathbb{C}(\mathbb{R}^+)$ to its initial value $x(0) \in \mathbb{R}$: First observe that it follows by the definition of a local martingale that $\mathscr{W}_{\mathscr{G},B} = \mathscr{W}_{\mathscr{G}',B}$ where $\mathscr{G}' \subset \mathscr{C}$ is a countable set of processes with uniformly bounded trajectories such that $\mathscr{G}'$ and B are compatible (see (2) in [11], for example). Hence, we may assume without loss of generality that $\mathscr{G} = \{G\}$is a singleton such that $|G| \leq c < \infty$ holds on $\mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+)$ in which case

$$\mathscr{W}_{\mathscr{G},B} = \{\mu \in \mathscr{P}(\mathbb{C}) : \mu(B) = 1 \text{ and } G \text{ is an } \mathscr{F}_t^{\mathbf{x},\mu}\text{-martingale on } (\mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)\}.$$

Fix $\mu \in \mathscr{W}_{\mathscr{G},B}$, denote $v := \mathscr{L}(\mathbf{x}(0) \mid \mu)$ and let $\{\mu_y^0, y \in \mathbb{R}\}$ denote a regular conditional distribution of μ given the projection H. It follows that $\mu_y^0(B) = 1$ for $y \notin N_B$ where $N_B \subset \mathbb{R}$ is a Borel set with $v(N_B) = 0$ because $\mu = \int_{\mathbb{R}} \mu_y^0 \, v(\mathrm{d}y)$. Also it follows by the definition of RCD of μ given H that for any fixed $0 \leq s < t < \infty$, $F \in \mathscr{F}_s^{\mathbf{x}}$ and $Z \in \mathscr{B}(\mathbb{R})$

$$\int\limits_Z \int\limits_F G(t) - G(s) \, \mathrm{d}\mu_y^0 \, v(\mathrm{d}y) = \int\limits_{\mathbb{C}(\mathbb{R}^+)} I_{[x(0) \in Z]}(x) \, I_F(x) \, [G(t,x) - G(s,x)] \, \mu(\mathrm{d}x) = 0$$

because $\{x \in \mathbb{C} : x(0) \in Z\} \cap F \in \mathscr{F}_s^{\mathbf{x}}$ and G is an $\mathscr{F}_t^{\mathbf{x},\mu}$-martingale. Thus, for any fixed $0 \leq s < t < \infty$ there is a Borel set $N_{t,s} \subset \mathbb{R}$ with $v(N_{t,s}) = 0$ such that

$$\int\limits_F G(t) - G(s) \, \mathrm{d}\mu_y^0 = 0 \text{ holds for all } F \in \mathscr{F}_s \text{ and } y \notin N_{t,s}$$

because $\mathscr{F}_s^{\mathbf{x}}$ is a countably generated σ-algebra. Putting $N := \bigcup\{N_{t,s}, t, s \in \mathbb{Q}^+\}$, it means that

$$\mathbb{E}_{\mu_y^0}[G(t) \mid \mathscr{F}_s^{\mathbf{x}}] = \mathbb{E}_{\mu_y^0}[G(s) \mid \mathscr{F}_s^{\mathbf{x}}] \quad s < t, \ s, t \in \mathbb{Q}^+, \ y \notin N \tag{35}$$

holds. If $s < t$ are arbitrary in $\mathbb{R}^+$ let $s_n \searrow s, t_n \searrow t$ where $s_n < t_n$ are in $\mathbb{Q}^+$. It follows by (35) that

$$\mathbb{E}_{\mu_y^0}[G(t_n) \mid \mathscr{F}_s^{\mathbf{x}}] = \mathbb{E}_{\mu_y^0}[G(s_n) \mid \mathscr{F}_s^{\mathbf{x}}] \quad \text{if } y \notin N$$

and finally that

$$\mathbb{E}_{\mu_y^0}[G(t) \mid \mathscr{F}_s^{\mathbf{x}}] = \mathbb{E}_{\mu_y^0}[G(s) \mid \mathscr{F}_s^{\mathbf{x}}], \quad 0 \leq s < t < \infty, \ y \notin N,$$

i.e. that G is an $\mathscr{F}_t^{\mathbf{x}}$-premartingale on $(\mathbb{C}, \mathscr{B}(C), \mu_y^0)$ for any $y \notin N$, because G is a continuous bounded process (see [11] for the definition and properties of the premartingale concept). If moreover, $y \notin N_B \cup N$ then G is also an $\mathscr{F}_t^{\mathbf{x},\mu_y^0}$-adapted process as $\mathscr{G}$ and B are assumed to be compatible. Hence, $\mu_y^0 \in \mathscr{W}_{\mathscr{G},B}$ almost surely w.r.t. v because an adapted premartingale is easily seen to be a martingale. We have proved that any $\mu \in \mathscr{W}_{\mathscr{G},B}$ is an H-decomposable measure in $\mathscr{W}_{\mathscr{G},B}$. □

Stroock-Varadhan Theorem on the existence (unique existence) of a weak solution of a (b, σ)-SDE (see 18.10 in [4]) may be made more precise as an application of Corollary 3.8.

3.9. Corollary. *Consider (b, σ)-SDE (7) with $\mathscr{F}_t^{\mathbf{x}}$-progressive coefficients b, σ and the set $\mathscr{W}_{\mathscr{G},B} \subset \mathscr{P}(\mathbb{C})$ of all $\mathscr{L}(X)$ where X goes through all weak solutions of (7). Then (7) has a weak solution with an arbitrary initial condition $v \in \mathscr{P}(\mathbb{R})$ if and only if it has a weak solution with an arbitrary deterministic initial condition*

$y \in \mathbb{R}$. The equation (7) is well posed for any $\nu \in \mathscr{P}(\mathbb{R})$ if and only if it is well posed for any deterministic initial condition $y \in \mathbb{R}$.

If (7) is an SDE that is well posed for any deterministic initial condition $y \in \mathbb{R}$ and $\mu_y \in \mathscr{W}_{\mathscr{G},B}$ is defined by $\mathscr{L}(\mathbf{x}(o) \mid \mu_y) = \varepsilon_y$ then $y \to \mu_y$ is a Borel map $\mathbb{R} \to \mathscr{P}(\mathbb{C})$ such that $\mu_\nu = \int_{\mathbb{R}} \mu_y \, \nu(\mathrm{d}y)$ is the unique $\mu_\nu \in \mathscr{W}_{\mathscr{G},B}$ such that $\mathscr{L}(\mathbf{x}(0) \mid \mu_\nu) = \nu$ holds for any $\nu \in \mathbb{C}(\mathbb{R}^+)$. The set $\mathscr{W}_{\mathscr{G},B}$ is a simplex, i.e. Choquet simplex, with ex $\mathscr{W}_{\mathscr{G},B} = \{\mu_y, y \in \mathbb{R}\}$ *in this case.*

The assertion follows directly by Corollary 3.8 as $\mathscr{W}_{\mathscr{G},B} = \mathscr{W}_{\{G_b, G_{b,\sigma}\}, B_{b,\sigma}}$ where $\mathscr{G} = \{G_b, G_{b,\sigma}\} \subset \mathscr{C}$ and $B_{b,\sigma} \in \mathscr{B}(\mathbb{C})$ defined by (4) are compatible sets.

Remark that the stochastic kernels $\nu \to \mathbb{M}^0$ which existence is stated by Corollaries 3.8 and 3.9 define a Borel affine bijection $\mathscr{P}(\mathbb{R}) \to \mathscr{W}_{\mathscr{G},B}$ and $\mathscr{P}(\mathbb{R}) \to \mathscr{W}_{b,\sigma}$, respectively.

3.10. Example: As an illustration consider $(0, \sigma)$-SDE $\mathrm{d}X(t) = \sigma(X(t))\,\mathrm{d}W(t)$ where $\sigma : \mathbb{R} \to \mathbb{R}$ is a Borel map such that $|\sigma| \geq \varepsilon > 0$. According to Engelbert-Schmidt theorem, see 20.1. in [4], it implies that the equation is well-posed for any initial condition $\nu \in \mathscr{P}(\mathbb{R})$. Checking carefully the proof we verify that there is a $w := \mathscr{L}(W)$-measurable map $H_\sigma : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R})$ such that

$$\mathscr{W}_{0,\sigma} = \mathscr{W}_{\mu_\sigma} := \{\mu_\sigma * \alpha, \alpha \in \mathscr{P}(T)\}, \quad \mu_\sigma = H_\sigma \circ w$$

in the notation introduced in Example 3.4. Hence, the stochastic kernel $y \to \mu_y$ which existence is stated by Corollary 3.9 is simply defined by $\mu_y = \mu_\sigma * \varepsilon_y$ in this case and $\mathscr{W}_{0,\sigma} = \mathscr{W}_{\mu_\sigma}$ is a Choquet simplex with the extremal boundary $\{\mu_y = (H_\sigma \circ w) * \varepsilon_y, \varepsilon \in \mathbb{R}\}$.

Note that the map H_σ becomes the identity on $\mathbb{C}(\mathbb{R}^+)$ if $\sigma = \text{sign}$ and it is generally defined by

$$H(t, x) := x(\tau(t, x)), \quad \tau(t, x) = \inf\left\{s > 0 : \int_0^s \sigma^{-2}(x(u))\,\mathrm{d}u > t\right\}, \; t \geq 0$$

for $x \in B$ where $B \subset \mathbb{C}(\mathbb{R}^+)$ is a Borel set with $P[W \in B] = 1$.

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