Josef Štěpán; P. Ševčík
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Local Martingales Measures

J. ŠTĚPÁN and P. ŠEVČÍK

Praha*

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The space of all pairs of continuous stochastic processes (X, Y) such that Y is a local martingale on the completed canonical filtration of the proces X is proved to be closed w.r.t. the convergence in distribution relatively in the space of all pairs (X, Y) where the Y is an X-adapted stochastic process. As a result, sets of solutions of local martingale problems in Stochastic analysis and Financial mathematics become in many important cases convex and weakly closed sets. We benefit of the property via Krein-Milman Theorem to specify the manner in which the solutions with a deterministic initial condition generate a solution with an arbitrary stochastic initial condition.

1 Introduction

The purpose of the present paper is to investigate the sets of solutions of the following generalization of **Stroock-Varadhan (local) martingale problem:** Given a set $\mathscr{G}$ of continuous processes on the canonical measurable space $(\mathbb{C}(\mathbb{R}^+), \mathscr{B}(\mathbb{C}(\mathbb{R}^+)))$ that start from the origin such a problem is defined by

$$\mathscr{W}_{\mathscr{G}} = \{\mu \in \mathscr{P}(\mathbb{C}(\mathbb{R}^+)) : G \text{ is an } (\mu, \mathscr{F}_t^{x,\mu})\text{-local martingale, } G \in \mathscr{G}\}$$

where $\mathscr{B}(S)$ and $\mathscr{P}(S)$ denote the Borel σ-algebra and the set of all Borel probabilities on S for a metric space S, while $(\mathscr{F}_t^{x,\mu})$ stays for the μ-completion of the canonical filtration $(\mathscr{F}_t^x)$ on $\mathbb{C}(\mathbb{R}^+)$. To cover the sets of weak solution of

* Faculty of Mathematics and Physics, Charles University in Prague, Department of Probability and Statistics, 18600 Praha 8 - Karlín, Sokolovská 83, Czech Republic

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stochastic differential equations (SDE's) we may further specify the $\mathscr{W}_{\mathscr{G}}$-sets by saying that a $B \in \mathscr{B}(\mathbb{C}(\mathbb{R}^+))$ and a set $\mathscr{G}$ is **compatible** if

$$\mu(B) = 1 \Rightarrow G \text{ is } \mathscr{F}_t^{\mathbf{x},\mu}\text{-adapted process for all } G \in \mathscr{G}$$

and writing

$$\mathscr{W}_{\mathscr{G},B} = \mathscr{W}_{\mathscr{G}} \cap \{\mu \in \mathscr{P}(\mathbb{C}(\mathbb{R}^+)) : \mu(B) = 1\}.$$

A distribution $\mu \in \mathscr{W}_{\mathscr{G},B}$ will be called here a **solution to** $(\mathscr{G}, B)$ (local martingale) **problem**, while a continuous stochastic process X defined on complete probability space $(\Omega, \mathscr{F}, P)$ such that $G(X)$ is an $\mathscr{F}_t^{X,P}$-local martingale for any $G \in \mathscr{G}$ and $P[X \in B] = 1$ holds, will be referred to as a **strong solution** to $(\mathscr{G}, B)$ **problem** on $(\Omega, \mathscr{F}, P)$. As usual $(\mathscr{F}_t^{X,P})$ denotes the P-completion of the canonical filtration associated with a process X. It follows by Lemma 2.3 (c) bellow that a continuous process X is a strong solution to $(\mathscr{G}, B)$-problem if and only if $\mathscr{L}(X) \in \mathscr{W}_{\mathscr{G},B}$.

Stochastic analysis provides a variety of examples of $(\mathscr{G}, B)$-problems. Agree to denote by $\mathbf{x}$ the identity map on $\mathbb{C}(\mathbb{R}^+)$ and call it the **canonical process.**

1.1. Example. *If $\mathscr{G}$ is a singleton that consists of the centered canonical process $\mathbf{x} - \mathbf{x}(0)$ on $\mathbb{C}(\mathbb{R}^+)$ then*

$$\mathscr{H} := \{\mathscr{L}(X) : X \text{ is a continuous } \mathscr{F}_t^{\mathbf{x}}\text{-local martingale}\} = \mathscr{W}_{\{\mathbf{x}-\mathbf{x}(0)\}}.$$

Financial mathematics offers good reasons to investigate the geometry of the convex set $\mathscr{H}$ searching namely for a characterization of ex $\mathscr{H}$. See [4] and Chapter V in [7] for a deep discussion of the problem.

1.2. Example. *Consider (b, σ)-SDE (Stochastic differential equation)*

$$\text{(D)} \qquad dX(t) = b(X)\,dt + \sigma(X)\,dW(t)$$

with $\mathscr{F}_t^{\mathbf{x}}$-progressive coefficients b and σ. Denote

$$B_{b,\sigma} = \{x \in \mathbb{C}(\mathbb{R}^+) : \int_0^t |b| + \sigma^2\,\mathrm{d}s < \infty\ \forall t \geq 0\} \in \mathscr{B}(\mathbb{C}(\mathbb{R}^+))$$

$$G_b(x, t) = x(t) - x(0) - \int_0^t b(x)\,\mathrm{d}s, \qquad (t, x) \in \mathbb{R}^+ \times B_{b,\sigma}$$

$$G_{b,\sigma}(x, t) = x^2(t) - x^2(0) - \int_0^t 2xb(x) + \sigma^2(x)\,\mathrm{d}s, \qquad (t, x) \in \mathbb{R}^+ \times B_{b,\sigma}$$

$$G_b = G_{b,\sigma} = 0, \qquad (t, x) \in \mathbb{R}^+ \times (\mathbb{C}(\mathbb{R}^+) \setminus B_{b,\sigma}).$$

Then $\mathscr{G} := \{G_b, G_{b,\sigma}\}$ and $B = B_{b,\sigma}$ are easily seen to be compatible and

$$\mathscr{W}_{b,\sigma} := \{\mathscr{L}(X) : X \text{ is a weak solution of (D)}\} = \mathscr{W}_{\mathscr{G},B}$$

The $(\mathscr{G}, B)$-local martingale problem with the above specification of $\mathscr{G}$ and B is equivalent to Stroock-Varadhan martingale problem (see [5], Chapter 18, for example).

For the sake of completeness we recall briefly the concept of the weak solution and the arguments for the equality $\mathscr{W}_{b,\sigma} = \mathscr{W}_{\mathscr{G},B}$.

A weak solution of (D) is defined as $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W, X)$, where $(\Omega, \mathscr{F}, P)$ is a complete probability space, $(\mathscr{F}_t)$ a complete filtration, W an $\mathscr{F}_t$-Wiener process and X a continuous $\mathscr{F}_t$-semimartingale with $P[X \in B_{b,\sigma}] = 1$ and with the stochastic differential given by (D). To verify $\mathscr{W}_{b,\sigma} = \mathscr{W}_{\mathscr{G},B}$ note first that a continuous process X is a strong solution of $(\mathscr{G}, B)$-problem on a complete probability space $(\Omega, \mathscr{F}, P)$ if and only if $P[X \in B_{b,\sigma}] = 1$ and $M := G_b \circ X$ is an $\mathscr{F}_t^{X,P}$-local martingale with the quadratic variation given by $d\langle M\rangle(t) = \sigma^2(X)\,dt$: Just apply Ito formula to compute

$$dN(t) = 2X\,dM(t) + d\langle M\rangle(t) - \sigma^2(X)\,dt\,, \qquad N := G_{b,\sigma} \circ X\,.$$

Hence, if $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W, X)$ is a weak solution of (b, σ)-SDE then M and N are $\mathscr{F}_t$-local martingales and of course also $\mathscr{F}_t^{X,P}$-local martingales on $(\Omega, \mathscr{F}, P)$ because $\mathscr{F}_t^{X,P} \subset \mathscr{F}_t$, $b(X)$ and $\sigma^2(X)$ are $\mathscr{F}_t^X$-adapted processes and $P[X \in B_{b,\sigma}] = 1$. This proves that X is a strong solution of the $(\mathscr{G}, B)$-problem and also that $\mathscr{W}_{b,\sigma} \subset \mathscr{W}_{\mathscr{G},B}$.

If X' is a strong solution of the $(\mathscr{G}, B)$-problem on $(\Omega', \mathscr{F}', P')$ we construct $(\Omega, \mathscr{F}, P, \mathscr{G}_t, B, X)$ such that $(\Omega, \mathscr{F}, P)$ is a complete probability space, $(\mathscr{G}_t)$ a complete filtration, B an $\mathscr{G}_t$-Wiener process and X a continuous process with $\mathscr{L}(X) = \mathscr{L}(X')$, such that $\mathscr{F}_t^X$ and $\mathscr{G}_t$ are independent filtrations. According to Lemma 2.3 X is a strong solution of the $(\mathscr{G}, B)$-problem which means, that $M := G_b \circ X$ is a $\mathscr{F}_t^{X,P}$-local martingale with $d\langle M\rangle(t) = \sigma^2(X)\,dt$. It follows easily that M is also an $\mathscr{F}_t$-local martingale where $\mathscr{F}_t := \sigma(\mathscr{F}_t^{X,P} \cup \mathscr{G}_t)$ and that (Theorem 16.12 in [5]) there exists an $\mathscr{F}_t$-Wiener proces W such that $dM(t) = \sigma(X)\,dW(t)$. Obviously, $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W, X)$ is a weak solution of (D) and therefore $\mathscr{W}_{\mathscr{G},B} \subset \mathscr{W}_{b,\sigma}$.

Denote by

$$\mathscr{W}_{b,\sigma,0} := \{\mathscr{L}(X(0)), X \text{ is a weak solution of (D)}\} = p_0 \circ \mathscr{W}_{\mathscr{G},B}\,,$$

where $p_t : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}[0, t]$ denotes the projection, the set of initial distributions that may start a weak solution of (D). The "classical weak SDE-theory" offers a variety of results that specify the set of all possible initial distributions $p_0 \circ \mathscr{W}_{\mathscr{G},B} \subset \mathscr{P}(\mathbb{R})$ according to the properties of the coefficients b and σ. For example (Skorochod theorem 18.9 in [5]), if both b and σ are continuous maps $\mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+)$ with locally uniformly bounded trajectories, then $\mathscr{W}_{b,\sigma,0} = p_0 \circ \mathscr{W}_{\mathscr{G},B} = \mathscr{P}(\mathbb{R})$. Recall that a (b, σ)-SDE is well posed for an initial condition $\nu \in \mathscr{P}(\mathbb{R})$ if there is an unique $P_\nu \in \mathscr{W}_{b,\sigma}$ with the initial condition ν. Stroock and Varadhan in [8] (see also Theorem 18.10 in [5]) established an ad hoc Choquet theory which says that a (b, σ)-SDE is well posed for any $\nu \in \mathscr{P}(\mathbb{R})$ if and only if it is well posed for any deterministic initial condition $x \in \mathbb{R}$. In fact they proved that $\mathscr{W}_{\mathscr{G},B}$ and $\mathscr{P}(\mathbb{R})$ are isomorphic simplexes in this case, the Borel measurable affine isomorphism being defined by $\nu \to \int_{\mathbb{R}} P_x \nu(\mathrm{d}x)$, where $P_x := P_{\varepsilon_x}$. A more general treatment may be found in [2].

1.3. Example. *Having continuous processes G and v defined on $\mathbb{C}(\mathbb{R}^+)$ such that the trajectories of v are of finite variation on $\mathbb{R}$ and $G(0) = v(0) = 0$, we put*

$$\mathscr{W}_{G,v} := \{\mu \in \mathscr{P}(\mathbb{C}(\mathbb{R}^+)) : G \text{ is an } \mathscr{F}_t^{\mathrm{x},\mu}\text{-local martingale, } \langle G \rangle \overset{a.s.}{=} v[\mu]\}$$

where $\langle G \rangle$ denotes as usual the quadratic variation of G. The elements of stochastic analysis show, that

$$\mathscr{W}_{G,v} = \mathscr{W}_{\{G,G_v\}} = \mathscr{W}_{\{H_\lambda,\, \lambda \in \mathbb{R}\}}$$

where

$$G_v := G^2 - v \text{ and } H_\lambda = \exp\{\lambda G - \tfrac{\lambda^2}{2} v\} - 1\,.$$

Hence, to construct a probability distribution $\mu \in \mathscr{P}(\mathbb{C}(\mathbb{R}^+))$ under which a given process becomes a local martingale with a given quadratic variation is the same as to construct a solution to a $\mathscr{G}$-local martingale problem with a rich offer for the choice of $\mathscr{G}$.

The sets of solutions $\mathscr{W}_{\mathscr{G},B}$ are Borel and measure convex sets in $\mathscr{P}(\mathbb{C}(\mathbb{R}^+))$ generally – see [10]. Here we shall focus on the $\mathscr{G}$'s for which any $G \in \mathscr{G}$ is a continuous $\mathscr{F}_t^{\mathrm{x}}$-adapted map $\mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+)$. The hypothesis being automatically satisfied by Example 1.1 may be forced to the sets $\mathscr{G}$ that appear in Examples 1.2 and 1.3 causing no serious restrictions there. Corollary 3.4 states that $\mathscr{W}_{\mathscr{G}}$ is a weakly closed convex set under the hypothesis. Especially, for such a set we have an advantage of Krein – Milman and Choquet theorem to be valid. We apply the former one in Theorem 3.7 to deepen and generalize Stroock – Varadhan theorem and further apply the result to an optimization problem in a theoretical economy.

2 Notations and preliminaries

Agree first that until further notice

(1) any **stochastic process** $X = (X(t), t \geq 0)$ we shall treat here, will be **continuous** $\mathbb{R}$ – valued and defined on a **complete** probability space $(\Omega, \mathscr{F}, P)$.

Especially, the X is random variable with values in the Polish space $\mathbb{C} = \mathbb{C}(\mathbb{R}^+)$ of $\mathbb{R}^+ \to \mathbb{R}$ continuous functions with the topology of the uniform convergence on compact intervals in $\mathbb{R}^+$. Our results center around topological and geometrical properties related to the following spaces of $\mathbb{C}(\mathbb{R}^+) \times \mathbb{C}(\mathbb{R}^+)$ random variables[1].

$$\mathbb{A}^0 = \{(X, Y) : Y(0) \overset{a.s.}{=} 0, Y \text{ is an } \mathscr{F}_t^{X,P} \text{ adapted process}\},$$

where

[1] We refer to [5], [6] and [7] for the elements of stochastic analysis we shall employ in our presentation.

$$\mathcal{F}_t^X = \sigma(X(s), s \le t) \text{ is the canonical filtration of } X,$$
$$N_P = \{F \in \mathcal{F} : P(F) = 0\}$$

and

$$\mathcal{F}_t^{X,P} = \sigma(\mathcal{F}_t^X \cup N_P), \qquad \text{is the P-completion of the filtration } \mathcal{F}_t^X.$$

Further denote

$$\mathbb{M}^0 = \{(X, Y) \in \mathbb{A}^0 : Y \text{ is an } \mathcal{F}_t^{X,P} \text{ martingale}\},$$
$$\mathbb{L}^0 = \{(X, Y) \in \mathbb{A}^0 : Y \text{ is an } \mathcal{F}_t^{X,P} \text{ local martingale}\}.$$

Denoting $x \to \tau^c(x)$ and $x \to \alpha^c(x)$ the Borel maps from $\mathbb{C}(\mathbb{R}^+)$ to $[0, \infty]$ and $\mathbb{C}(\mathbb{R}^+)$, respectively, defined by

$$\tau^c(x) = \inf\{s \ge 0 : |x(s)| \ge c\}, \quad \alpha^c(x)(t) = x(t \wedge \tau^c(x)), \ x \in \mathbb{C}(\mathbb{R}^+), \ t \in \mathbb{R}^+, \ c \in \mathbb{R}.$$

We recall the definition of a **local martingale** by

(2) $$\mathbb{L}^0 = \{(X, Y) \in \mathbb{A}^0 : \alpha^c \circ Y \in \mathbb{M}^0 \ \forall c \in \mathbb{N}\}.$$

We shall also agree to call a stochastic process Y, defined on a probability space $(\Omega, \mathcal{F}, P)$ with a possibly uncomplete filtration $(\mathcal{F}_t)$ an **$\mathcal{F}_t$-premartingale**, if

$$Y(t) \in L_1(P), \qquad \mathbb{E}^{\mathcal{F}_s} Y(t) = \mathbb{E}^{\mathcal{F}_s} Y(s) \quad 0 \le s \le t$$

and to call it an **$\mathcal{F}_t$-local premartingale**, if $\alpha^c \circ Y$ is an $\mathcal{F}_t$-premartingale for any $c \in \mathbb{N}$. Denote finally

$$\mathbb{PM} := \{(X, Y) : Y(0) \overset{a.s.}{=} 0 \text{ and } Y \text{ is an } \mathcal{F}_t^X\text{-premartingale}\}$$
$$\mathbb{PL} := \{(X, Y) : Y(0) \overset{a.s.}{=} 0 \text{ and } Y \text{ is an } \mathcal{F}_t^X\text{-local premartingale}\}$$

and observe that

(3) $$\mathbb{M}^0 = \mathbb{A}^0 \cap \mathbb{PM} \quad \text{and} \quad \mathbb{L}^0 = \mathbb{A}^0 \cap \mathbb{PL}$$

because Y is an $\mathcal{F}_t^{X,P}$-premartingale if and only if it is $\mathcal{F}_t^X$-premartingale, while the latter equality follows by (2).

Note that for a pair (X, Y) in $\mathbb{A}^0$ is $\alpha^c(Y) = Y^{\tau^c(Y)}$, where the latter process is the Y stopped by the $\mathcal{F}_t^{X,P}$ Markov time $\tau^c(Y)$ (of the first entry to $[c, \infty)$ by $|Y|$). Obviously

(4) $$(X, Y) \in \mathbb{A}^0 \Rightarrow (X, \alpha^c \circ Y) \in \mathbb{A}^0, \qquad (X, Y) \in \mathbb{L}^0 \Rightarrow (X, \alpha^c \circ Y) \in \mathbb{M}^0.$$

Importantly, we insist that the true identity of a pair $(X, Y) \in \mathbb{A}^0$ is not precise without stating its probability setting given by the underlying complete probability space $(\Omega, \mathcal{F}, P)$. Thus, (X, Y) and (X', Y') are **identical pairs in** $\mathbb{A}^0$ if they are defined on a common complete probability space with a probability P such that $(X, Y) \overset{a.s.}{=} (X', Y')$ $[P]$, i.e. such that (X', Y') is a P-modification of (X, Y).

We abbreviate and mostly even skip the reference to the underlying probability space in the fashion $(X, Y, \Omega, \mathcal{F}, P) = (X, Y, P) = (X, Y)$. The reason for choosing the complete filtration $\mathscr{F}_t^{X,P}$ in our definitions of $\mathbb{A}^0$ and $\mathbb{L}^0$ stems basically from our wish to get them as sets closed w.r.t. the convergence in distribution. See Lemma 2.3 bellow for the first step towards the aim. A communication between complete and uncomplete filtrations $\mathscr{F}_t^{X,P}$ and $\mathscr{F}_t^X$ is suggested by the following simple observation.

2.1. Remark. *Assume that (X, Y) is a pair of processes defined on a possibly uncomplete space $(\Omega, \mathscr{F}, P)$ such that Y is an $\mathscr{F}_t^X$-adapted process ($\mathscr{F}_t^X$-martingale, $\mathscr{F}_t^X$-local martingale) with $Y(0) \overset{a.s.}{=} 0$. Then obviously $(X, Y, \Omega, \mathscr{F}^P, P)$ is in $\mathbb{A}^0$ $(\mathbb{M}^0, \mathbb{L}^0)$ denoting by $(\Omega, \mathscr{F}^P, P)$ the completion of $(\Omega, \mathscr{F}, P)$.*

Thus we agree to consider without further reference the pairs defined on $(\Omega, \mathscr{F}, P)$ with $\mathscr{F}_t^X$-adapted Y ($\mathscr{F}_t^X$-martingale Y, $\mathscr{F}_t^X$-local martingale Y) as members of $\mathbb{A}^0$ $(\mathbb{M}^0, \mathbb{L}^0)$ identified there as $(X, Y, \Omega, \mathscr{F}^P, P)$.

Some more notations may be convenient to introduce: If S is a metric space, $Y: \Omega \to S$ and $\mathscr{F} \subset \exp \Omega$ a σ-algebra, we shall write $Y \in \mathscr{F}$ if $Y^{-1}\mathscr{B}(S) \subset \mathscr{F}$.

2.2. Lemma. *Let $(X, Y) \in \mathbb{A}^0$. Then there exist*

$$\text{a Borel map } g_t : \mathbb{C}[0, t] \to \mathbb{R},\ g_t \in \mathscr{F}_t^X \text{ such that } Y(t) \overset{a.s.}{=} g_t(X(s), s \le t),\ t \ge 0,$$

i.e. the process Y possesses a (possibly uncontinuous) $\mathscr{F}_t^X$-adapted modification.

Proof. It follows that $Y(t) \in \mathscr{F}_t^{X,P} = \sigma\{F \cup N, F \in \mathscr{F}_t^X, N \in N_P\}$ and therefore by the standard extension (Theorem 2.4 in [7]) we exhibit an $Y'(t) \in \mathscr{F}_t^X$ such that $Y'(t) \overset{a.s.}{=} Y(t)$. Finally, apply Lemma 1.13 in [5] to prove the existence of g_t. □

As we have already mentioned any (continuous) process X on an $(\Omega, \mathscr{F}, P)$ is a $\mathbb{C}$ valued random variable and as such it possesses its probability distribution $\mathscr{L}(X \mid P) = X \circ P = \mathscr{L}(X)$ as a Borel probability measure on $\mathbb{C}$. Thus, any pair of processes (X, Y) is a random variable with values in the Polish space $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ with the probability distribution denoted by $\mathscr{L}(X, Y \mid P) = \mathscr{L}(X, Y)$.

The spaces $\mathbb{A}^0$, $\mathbb{M}^0$, $\mathbb{L}^0$ respect the factorization of the space of $\mathbb{C}^2$-valued random variables defined by the equality of probability distributions.

Denote $\mathscr{L}(\mathscr{V}) = \{\mathscr{L}(V), V \in \mathscr{V}\}$ for any set $\mathscr{V}$ of random variables.

2.3. Lemma. *Let (X, Y) be a $\mathbb{C}^2$ random variable. Then*

(a) $\mathscr{L}(X, Y) \in \mathscr{L}(\mathbb{A}^0) \Rightarrow (X, Y) \in \mathbb{A}^0$
(b) $\mathscr{L}(X, Y) \in \mathscr{L}(\mathbb{M}^0) \Rightarrow (X, Y) \in \mathbb{M}^0$
(c) $\mathscr{L}(X, Y) \in \mathscr{L}(\mathbb{L}^0) \Rightarrow (X, Y) \in \mathbb{L}^0$
(d) $\mathscr{L}(X, Y) \in \mathscr{L}(\mathbb{PM}) \Rightarrow (X, Y) \in \mathbb{PM}$
(e) $\mathscr{L}(X, Y) \in \mathscr{L}(\mathbb{PL}) \Rightarrow (X, Y) \in \mathbb{PL}$

Proof. According to (3), (a) $\wedge$ (d) $\Rightarrow$ (b) and (a) $\wedge$ (e) $\Rightarrow$ (c). Moreover, (d) $\Rightarrow$ (e) because $\alpha^c : \mathbb{C}(\mathbb{R}^+) \to \mathbb{C}(\mathbb{R}^+)$ are Borel measurable for any $c \in \mathbb{N}$.

To prove (a) let $\mathscr{L}(X, Y) = \mathscr{L}(\xi, \eta)$ for a $(\xi, \eta) \in \mathbb{A}^0$. Lemma 2.2 yields a Borel map

$$g_t : \mathbb{C}[0, t] \to \mathbb{R}, \qquad g_t \in \mathscr{F}_t^\xi,\ \eta(t) \overset{a.s.}{=} g_t(\xi(s), s \le t), \qquad t \ge 0,$$

which implies immediately that $Y(s) \overset{a.s.}{=} g_t(X(s), s \le t)$ for all $t \ge 0$ and therefore $(X, Y) \in \mathbb{A}^0$.

To prove (d) let $\mathscr{L}(X, Y) = \mathscr{L}(\xi, \eta)$ for a $(\xi, \eta) \in \mathbb{PM}$. Observe that for any $s < t$, $Y(t) \in L_1$,

$$\begin{aligned}\mathbb{E} f_s(X(u), u \le s)\, Y(t) &= \mathbb{E} f_s(\xi(u), u \le s)\, \eta(t) = \mathbb{E} f_s(\xi(u), u \le s)\, \eta(s) \\ &= \mathbb{E} f_s(X(u), u \le s)\, Y(s),\end{aligned}$$

whenever $f_s : \mathbb{C}[0, t] \to \mathbb{R}$ is a bounded Borel map. Hence, $(X, Y) \in \mathbb{PM}$. □

The canonical home for continuous processes is provided by the measurable space $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$, recall that $\mathbb{C} := \mathbb{C}(\mathbb{R}^+)$, filtered by the **canonical filtration** $(\mathscr{F}_t^{\mathbf{x}})$ where $\mathbf{x} = (\mathbf{x}(t), t \ge 0)$ is the **canonical** process defined by $\mathbf{x}(t, x) = x(t)$ for $x \in \mathbb{C}$ and $t \ge 0$. Note that

$$(5) \qquad \mathscr{F}_t^{\mathbf{x}} = p_t^{-1} \mathscr{B}(\mathbb{C}[0, t]), \qquad t \ge 0$$

and

$$(6) \qquad X^{-1} \mathscr{F}_t^{\mathbf{x}} = \mathscr{F}_t^X, \qquad t \ge 0 \quad \text{if } X \text{ is a continuous process.}$$

Denote

$$\begin{aligned}&\mathscr{C} := \{G : \mathbb{C} \to \mathbb{C} \text{ Borel measurable}, G(0) \equiv 0\} \qquad &&\mathscr{C}_a := \{G \in \mathscr{C}, G\ \mathscr{F}_t^{\mathbf{x}} \text{ adapted}\} \\ &\mathscr{C}_c := \{G : \mathbb{C} \to \mathbb{C} \text{ continuous}, G(0) \equiv 0\} &&\mathscr{C}_{c,a} := \mathscr{C}_c \cap \mathscr{C}_a.\end{aligned}$$

Note that putting $G_1(t, x) = x(t^2)$ and $G_2(t, x) = \lambda\{0 \le s \le t, x(s) > 0\}$ we exhibit $G_1 \in \mathscr{C} \setminus \mathscr{C}_a$ and $G_2 \in \mathscr{C}_a \setminus \mathscr{C}_c$. Also, the *stopping maps* α^c's are easily seen to be in $\mathscr{C}_a \setminus \mathscr{C}_c$.

2.4. Remark. *The above spaces are stable with regard to Lebesgue integration: Denote* $I(A)(t) = \int_0^t A(s)\, ds$ *and observe that the following implications hold:*

(i) If A is an $\mathscr{F}_t^{\mathbf{x}}$-progressive process, then $B := \{x \in \mathbb{C} : \int_0^t |A(x)|\, ds < \infty\ \forall t > 0\} \in \mathscr{B}(\mathbb{C})$ and $I_B \cdot I(A) \in \mathbb{C}$ is an $\mathscr{F}_t^{\mathbf{x},\mu}$-adapted process for any $\mu \in \mathscr{P}(\mathbb{C})$, $\mu(B) = 1$.

(ii) $A \in \mathbb{R} \oplus \mathscr{C} \Rightarrow I(A) \in \mathscr{C}$ and $A \in \mathbb{R} \oplus \mathscr{C}_a \Rightarrow I(A) \in \mathscr{C}_a$.[2]

(iii) $A \in \mathbb{R} \oplus \mathscr{C}_c \Rightarrow I(A) \in \mathscr{C}_c$ and $A \in \mathbb{R} \oplus \mathscr{C}_{c,a} \Rightarrow I(A) \in \mathscr{C}_{c,a}$.

(iv) If $x \to A(t, x)$ is a continuous map $\mathbb{C}(\mathbb{R}^+) \to \mathbb{R}$ and $\sup_{(s,x) \in [0,t] \times K} |A(s, x)| < \infty$ for any $t > 0$ and any compact set $K \subset \mathbb{C}(\mathbb{R}^+)$ then $I(A) \in \mathscr{C}_c$.

Denote by $\mathbb{CA}^0$, $\mathbb{CM}^0$ and $\mathbb{CL}^0$ the sets of all $(G, \mu) \in \mathscr{C} \times \mathscr{P}(\mathbb{C})$ such that $(\mathbf{x}, G, \mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)$ is in $\mathbb{A}^0$, $\mathbb{M}^0$ and $\mathbb{L}^0$, respectively. Thus (G, μ) is in $\mathbb{L}^0$ if and only if G is a $(\mu, \mathscr{F}_t^{\mathbf{x},\mu})$-local martingale on $(\mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)$.

2.5. Theorem. *Endow the product $\mathscr{C} \times \mathscr{P}(\mathbb{C})$ with the identity given by $(G, \mu) = (G', \mu')$ if and only if $\mu = \mu'$ and $G \overset{a.s.}{=} G'[\mu]$. Then there exists a unique map $r : \mathbb{A}^0 \to \mathscr{C} \times \mathscr{P}(\mathscr{C})$ such that for any $(G, \mu) \in \mathscr{C} \times \mathscr{P}(\mathscr{C})$ and $(X, Y) \in \mathbb{A}^0$*

[2] $\mathbb{R} \oplus \mathbb{D} := \{a + d, a \in \mathbb{R}, d \in \mathbb{D}\}$, $\mathbb{D} \subset \mathscr{C}$

$$(7) \qquad r(X, Y) = (G, \mu) \quad \Leftrightarrow \quad \mathscr{L}(X, Y) = \mathscr{L}(\mathbf{x}, G \mid \mu)$$

or equivalently

$$(8) \qquad r(X, Y) = (G, \mu) \quad \Leftrightarrow \quad Y \overset{a.s.}{=} G(X), \mathscr{L}(X) = \mu.$$

The pair $r(X, Y) = (G, \mu)$ *will be called* **the canonical representation of** (X, Y). *Moreover,*

$$(9) \quad \mathscr{L}(X, Y) = \mathscr{L}(X', Y') \quad \Leftrightarrow \quad r(X, Y) = r(X', Y'), \quad (X, Y), (X', Y') \in \mathbb{A}^0.$$

Further, $r(\mathbb{A}^0) = \mathbb{C}\mathbb{A}^0$ *and*

$$(10) \qquad (X, Y) \in \mathbb{M}^0 (\in \mathbb{L}^0) \quad \Leftrightarrow \quad r(X, Y) \in \mathbb{C}\mathbb{M}^0 (\in \mathbb{C}\mathbb{L}^0), (X, Y) \in \mathbb{A}^0.$$

Especially,

$$(11) \qquad \mathscr{L}(X) = \mu, \ (G, \mu) \in \mathbb{C}\mathbb{A}^0 \ (\mathbb{C}\mathbb{M}^0, \mathbb{C}\mathbb{L}^0) \Rightarrow (X, G(X)) \in \mathbb{A}^0 \ (\mathbb{M}^0, \mathbb{L}^0).$$

The essence of the above definition is that $r(X, Y, \Omega, \mathscr{F}, P)$ selects the unique member of $\mathbb{A}^0$ stochastically equivalent to (X, Y) that may be identified as $(\mathbf{x}, G, \mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)$ for some $(G, \mu) \in \mathscr{C} \times \mathscr{P}(\mathbb{C})$. Moreover, the canonical representation preserves the information on the presence of the (local) martingale property.

Proof. Assuming that we shall be able to prove that

$$(12) \qquad \forall (X, Y) \in \mathbb{A}^0 \ \exists G \in \mathscr{C} \text{ such that } Y \overset{a.s.}{=} G(X)$$

we observe that (8) defines correctly the unique map $r : \mathbb{A}^0 \to \mathscr{C} \times \mathscr{P}(\mathbb{C})$. Because for any $(X, Y) \in \mathbb{A}^0$ and $(G, \mu) \in \mathbb{C} \times \mathscr{P}(\mathbb{C})$ equality $\mathscr{L}(X, Y) = \mathscr{L}(\mathbf{x}, G \mid \mu)$ holds if and only if $\mathscr{L}(X) = \mu$ and $Y \overset{a.s.}{=} G(X)$ hold simultaneously, we conclude that (7) is the equivalent definition of the map r.

Further, (9) follows by an application of (7) to verify the ($\Rightarrow$) part and by an application of (8) to verify ($\Leftarrow$) part.

To prove (10) apply (7) and then Lemma 2.3 (a), (b), (c), respectively. Finally observe that (11) follows by (8) and (10), because $(G, \mu) \in \mathbb{C}\mathbb{A}^0$ yields $r(\mathbf{x}, G, \mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu) = (G, \mu)$.

Thus, it remains to verify (12): If $(X, Y, P) \in \mathbb{A}^0$, then obviously $Y \in \sigma(X)^P$. Then, Lemma 1.25 in [5] exhibits a $Y' \in \sigma(X)$ such that $Y' \overset{a.s.}{=} Y[P]$ and Lemma 1.13 in [5] a $G \in \mathscr{C}$ such that $G(X) \equiv Y'$. Hence, the G is the process the existence of which is stated by (12). □

Note also, that the canonical representation respects Doob-Mayer decomposition and stochastic integration procedures. Assume that $(X, Y, P), (X, Y_i, P) \in \mathbb{L}^0$, $i = 1, 2$ and recall that $\langle Y_1, Y_2 \rangle$ denotes the **covariation** of local martingales Y_1 and Y_2, the almost surely unique continuous process with finite variation on $\mathbb{R}^+$ such that

$$(13) \qquad (X, \langle Y_1, Y_2 \rangle) \in \mathbb{A}^0 \quad \text{and} \quad (X, Y_1 Y_2 - \langle Y_1, Y_2 \rangle) \in \mathbb{L}^0$$

hold.

As usual we write $\langle Y, Y\rangle = \langle Y\rangle$ and call it a **quadratic variation** of Y observing also that $\langle Y\rangle$ is a nondecreasing process. Also recall that if

$$(14)\qquad A \text{ is } \mathcal{F}_t^{X,P}\text{-progressive process with } \int_0^t A^2\,\mathrm{d}\langle Y\rangle < \infty \ a.s.[P]\ \forall t \geq 0,$$

then $I_t^Y(A) = \int_0^t A\,\mathrm{d}Y$ denotes the almost surely unique continuous process such that

$$(15)\qquad (X, I^Y(A)) \in \mathbb{L}^0 \text{ and } \langle I^Y(A), N\rangle(t) \overset{a.s.}{=} \int_0^t A\,\mathrm{d}\langle Y, N\rangle\ \forall t \geq 0,\ \forall (X, N) \in \mathbb{L}^0.$$

$I^Y(A)$ is called the $\mathcal{F}_t^{X,P}$-stochastic integral of A respect to Y.

2.6. Corollary. *Let* $(X, Y, P), (X, Y_i, P) \in \mathbb{L}^0$ *with* $r(X, Y_i) = (G_i, \mu)$, $r(X, Y) = (G, \mu)$ *and* $r(X, \langle Y_1, Y_2\rangle) = (v, \mu)$, $r(X, \langle Y\rangle) = (q, \mu)$, $i = 1, 2$. *Then*

$$(16)\qquad v \overset{a.s.}{=} \langle G_1, G_2\rangle[\mu], \text{ especially } \langle Y_1, Y_2\rangle \overset{a.s.}{=} \langle G_1, G_2\rangle \circ X[P].$$

Further, if a *is an* $\mathcal{F}_t^{\mathbf{x}}$*-progressive process on* $\mathbb{C}(\mathbb{R}^+)$ *such that* $\int_0^t a^2\,\mathrm{d}q < \infty$ *a.s.* $[\mu]$ *then*

$$(17)\qquad r(X, I^Y(a \circ X)) = (I^G(a), \mu), \text{ i.e. } I^Y(a \circ X) \overset{a.s.}{=} I^G(a) \circ X[P]$$

holds.

Proof. It follows by (7), (13) and (10) that

$$r(X, Y_1Y_2 - \langle Y_1, Y_2\rangle) = (G_1G_2 - v, \mu) \text{ and } (G_1G_2 - v, \mu) \in \mathbb{C}\mathbb{L}^0$$

hold. Because $v \in \mathscr{C}$ is an $\mathcal{F}_t^{\mathbf{x},\mu}$-adapted process of the finite variation, (16) is a consequence of the uniqueness of Doob-Mayer decomposition.

According to Lemma 18.1. in [5], $A := a \circ X$ is an $\mathcal{F}_t^{\mathbf{x}}$-progressive process with

$$\int_0^t A^2\,\mathrm{d}\langle Y\rangle \overset{a.s.}{=} \left(\int_0^t a^2\,\mathrm{d}q\right) \circ X$$

according to (16) and therefore the stochastic integral $I^Y(A)$ exists with $(X, I^Y(A)) \in \mathbb{L}^0$. Choose $(X, N) \in \mathbb{L}^0$ arbitrary, denote $r(X, N) = (H, \mu)$ and apply (16) twice to verify (17):

$$\langle I^G(a) \circ X, N\rangle \overset{a.s.}{=} \langle I^G(a), H\rangle \circ X \overset{a.s.}{=} \int a\,\mathrm{d}\langle G, H\rangle \circ X \overset{a.s.}{=} \int A\,\mathrm{d}\langle Y, N\rangle.$$

□

Theorem 2.5 allows to specify the probability distributions in $\mathscr{L}(\mathbb{L}^0)$ by

$$(18)\qquad \mathscr{L}(\mathbb{L}^0) = \{\mathscr{L}(\mathbf{x}, G \mid \mu) \in \mathscr{P}(\mathbb{C}^2) : (G, \mu) \in \mathbb{C}\mathbb{L}^0\}.$$

That is as to say that $\mathscr{L}(\mathbb{L}^0)$ is the set of distributions $P \in \mathscr{P}(\mathbb{C}^2)$ which are supported by the graph of a process $G \in \mathscr{C}$ that becomes $(\mu, \mathcal{F}_t^{\mathbf{x},\mu})$-local martingale under the first marginal μ of the distribution P. Hence, if we define $\Phi : \mathscr{C} \times \mathscr{P}(\mathbb{C}) \to \mathscr{P}(\mathbb{C}^2)$ by

$$(19) \qquad \Phi(G, \mu) = \mathscr{L}(\mathbf{x}, G \mid \mu), \qquad (G, \mu) \in \mathscr{C} \times \mathscr{P}(\mathbb{C})$$

we get

$$(20) \qquad \mathbb{C}\mathbb{L}^0 = \Phi^{-1}\mathscr{L}(\mathbb{L}^0) \quad \text{(and similarly } \mathbb{C}\mathbb{M}^0 = \Phi^{-1}\mathscr{L}(\mathbb{M}^0)\text{)}.$$

Because any $\mathscr{G}$-local martingale problem may be redefined as $\mathscr{W}_{\mathscr{G}} = \bigcap_{G \in \mathscr{G}} \mathbb{C}\mathbb{L}^0(G)$, where $\mathbb{C}\mathbb{L}^0(G)$ and $\mathbb{C}\mathbb{L}^0(\mu)$ denotes the section of $\mathbb{C}\mathbb{L}^0$ at a $G \in \mathscr{G}$ and at a $\mu \in \mathscr{P}(\mathbb{C})$, respectively, (20) provides an important link between the local martingale problems introduced in Section 1 and the principal set $\mathbb{L}^0$ of Section 2. Moreover, the map Φ may be forced to be "continuous" with respect to the convergence $\overset{cw}{\to}$ introduced by

$$(21) \qquad (G_n, \mu_n) \overset{cw}{\to} (G, \mu) \text{ if}$$
$$G_n \to G \;\; \mu\text{-continuously in } \mathbb{C}(\mathbb{R}^+) \text{ and } \mu_n \to \mu \text{ weakly in } \mathscr{P}(\mathbb{C}).$$

Recall that if $G_n, G : S \to T$, and $\mu \in \mathscr{P}(S)$ where S and T are metric spaces, then the continuous and μ-continuous convergence of the sequence (G_n) to the G, denoted by $G_n \overset{c}{\to} G$ and $G_n \overset{c}{\to} G[\mu]$, is defined by

$$s_n \to s \Rightarrow G_n(s_n) \to G(s) \text{ and } s_n \to s \Rightarrow G_n(s_n) \to G(s), \; s \in F, \; \mu(F) = 1$$

respectively.

2.7. Lemma. *The map $\Phi : \mathscr{C} \times \mathscr{P}(\mathbb{C}) \to \mathscr{P}(\mathbb{C}^2)$ defined by (20) is sequentially continuous with respect to the $\overset{cw}{\to}$ convergence in $\mathscr{C} \times \mathscr{P}(\mathbb{C})$ and the weak convergence in $\mathscr{P}(\mathbb{C}^2)$, i.e.,*

$$(G_n, \mu_n) \overset{cw}{\to} (G, \mu) \;\Rightarrow\; \Phi(G_n, \mu_n) \to \Phi(G, \mu) \textit{ weakly.}$$

Proof. Apply Theorem 3.27 in [5] to $f_n, f : \mathbb{C} \to \mathbb{C}^2$ defined by $f_n(x) = (x, G_n(x))$ and $f(x) = (x, G(x))$ for $x \in \mathbb{C}$. □

The above continuity of Φ together with Proposition in Section 3 deliver the basic tools for proving the existence of solutions of $\mathscr{G}$-local martingale problems. The following assertions suggest possible applications of the $\overset{cw}{\to}$ convergence combined with the information delivered by Proposition.

2.8. Lemma. *Consider $G_n, G \in \mathscr{C}$, (τ_n) a sequence of $\mathscr{F}_t^{\mathbf{x}}$-Markov times and A_n, A processes on $\mathbb{C}(\mathbb{R}^+)$ with locally integrable trajectories. Then the following implications hold:*

(a) If $G \in \mathscr{C}_c$ then $G_n \overset{c}{\to} G \Leftrightarrow \sup_{x \in K} \max_{s \le t} |G_n(s, x) - G(x, x)| \to 0$ for any $t > 0$ and compact set $K \subset \mathbb{C}(\mathbb{R}^+)$.

(b) If $\tau_n \to \infty$ uniformly on any compact set in $\mathbb{C}(\mathbb{R}^+)$ then $\beta_n \overset{c}{\to} \mathbf{x}$, where $\beta_n(t, x) := x(\tau_n(x) \wedge t)$ for $(t, x) \in \mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+)$.

(c) $\alpha^c \overset{c}{\to} \mathbf{x}$ as $c \to \infty$.

(d) If A_n, A are continuous processes such that $A_n \overset{c}{\to} A$ then $I(A_n) \overset{c}{\to} I(A)$.

(e) If $A_n(t, .) \overset{c}{\to} A(t, .)$ and $\sup_{(s,x) \in [0,t] \times K} |A_n(x, x)| \le K < \infty$ for all $t \ge 0$, all compact sets $K \subset \mathbb{C}(\mathbb{R}^+)$ and $n \in \mathbb{N}$ then $I(A_n) \overset{c}{\to} I(A)$.

Note that it follows by (a) that the $\xrightarrow{cw}$ convergence restricted to the subspace $\mathscr{C}_c \times \mathscr{P}(\mathbb{C})$ is exactly the convergence of a sequence with respect to the product of the topology of uniform convergence on compacts in $\mathbb{C}$ and of the weak topology in $\mathscr{P}(\mathbb{C})$.

Proof. (a) is a standard equivalent definition of the continuous convergence and implies (b) directly. To prove (c) observe, that any compact $K \subset \mathbb{C}(\mathbb{R}^+)$ is a locally bounded set and apply (b). As for (d) note that $A_n \xrightarrow{c} A$ implies that $H \circ A_n \xrightarrow{c} H \circ A_n$ for any $H \in \mathscr{C}_c$. Choosing $H := I(\mathbf{x})$, (d) follows then by Remark 2.4 (iii). Verify (e) directly by the definition of the continuous convergence applying the Dominated convergence theorem. □

3 Results

We summarize our main results here and promise to prove them in Section 4.

Proposition. *$\mathscr{L}(\mathbb{L}^0)$ is a relatively weakly closed set in $\mathscr{L}(\mathbb{A}^0)$, i.e. $(X_n, Y_n) \in \mathbb{L}^0$, $n \in \mathbb{N}$, $(X, Y) \in \mathbb{A}^0$, $(X_n, Y_n) \xrightarrow{\mathscr{D}} (X, Y)$ implies $(X, Y) \in \mathbb{L}^0$.*

Unfortunately, a more elegant statement

"$\mathscr{L}(\mathbb{L}^0)$ is a weakly closed set in $\mathscr{P}(\mathbb{C}(\mathbb{R}^+)) \times \mathscr{P}(\mathbb{C}(\mathbb{R}^+)))$"

is not true:

3.1. Example. *Consider a standard Wiener process W and observe that $(n^{-1}W, W) \in \mathbb{M}$ and that $(n^{-1}W, W) \xrightarrow{\mathscr{D}} (0, W)$ because $n^{-1}W \to 0$ in $\mathbb{C}(\mathbb{R}^+)$ everywhere on Ω. Obviously, $(0, W) \notin \mathbb{A}^0$, hence $\mathscr{L}(\mathbb{L}^0)$ is not a weakly closed set in $\mathscr{P}(\mathbb{C}(\mathbb{R}^+) \times \mathbb{C}(\mathbb{R}^+))$.*

Proposition follows directly by a more general

3.2. Theorem. *If $(X_n, Y_n) \in \mathbb{PL}$ for any $n \in \mathbb{N}$ and $(X, Y) \in \mathbb{A}^0$ then $(X_n, Y_n) \xrightarrow{\mathscr{D}} (X, Y)$ in $\mathbb{C}(\mathbb{R}^+) \times \mathbb{C}(\mathbb{R}^+)$ implies $(X, Y) \in \mathbb{L}^0$.*

Observe, that the set of the martingales pairs distributions $\mathscr{L}(\mathbb{M}^0)$ is a dense set in $\mathscr{L}(\mathbb{L}^0)$ (by Lemma 2.8 (c)), hence of course not a closed set. On the other hand Lemma 4.3 proves easily the property for any set of uniformly bounded martingales pairs $\mathbb{M}_c^0 := \{(X, Y) \in \mathbb{M}^0 : |Y| \le c\}$, i.e. the property which would make the proof of Proposition a more simple matter provided that the stopping maps $\alpha_c : \mathbb{C} \to \mathbb{C}$ would be continuous which is not of course the case. We close Section 4 by a discussion of the problem.

3.3. Corollary. *The set $\mathbb{CL}^0$ is a sequentially closed in $\mathbb{CA}^0$ w.r.t. the $\xrightarrow{cw}$ convergence, i.e.*

$(G_n, \mu_n) \xrightarrow{cw} (G, \mu), (G_n, \mu_n) \in \mathbb{CL}^0,\ n \in \mathbb{N},\ (G, \mu) \in \mathbb{CA}^0 \Rightarrow (G, \mu) \in \mathbb{CL}^0,\ G, G_n \in \mathscr{C}$, $\mu, \mu_n \in \mathbb{C}(\mathbb{R}^+)$.

Especially,
$G_n \xrightarrow{c} G,\ X_n \xrightarrow{\mathscr{D}} X,\ (X_n, G_n(X_n)) \in \mathbb{L}^0,\ (X, G(X)) \in \mathbb{A}^0\ \Rightarrow\ (X, G(X)) \in \mathbb{L}^0$ *for any* $G_n,\ G \in \mathscr{C}$ *and any continuous processes* X_n, X.

The assertion is an immediate consequence of Proposition, Lemma 2.7 and (20).

3.4. Corollary. *For any* $\mu \in \mathscr{P}(\mathbb{C}(\mathbb{R}^+))$ *the set* $\mathbb{CL}^0(\mu)$ *of all* $G \in \mathscr{C}$ *such that* G *is an* $\mathscr{F}_t^{\mathbf{x},\mu}$*-local martingale on* $(\mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)$ *is a linear subspace of* $\mathscr{C}$ *closed w.r.t. the convergence in probability on* $(\mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)$, *i.e.*

$$G_n(X) \to G(X) \quad \textit{in probability,} \quad (X, G_n(X)) \in \mathbb{L}^0 \Rightarrow (X, G(X)) \in \mathbb{L}^0$$

for any $G_n,\ G \in \mathscr{C}$ *and arbitrary continuous stochastic process* X *with* $\mathscr{L}(X) = \mu$.

The linearity of $\mathbb{CL}^0(\mu)$ is obvious. The assumption $G_n(X) \to G(X)$ in probability yields that $(X, G_n(X)) \xrightarrow{\mathscr{D}} (X, G(X))$ and also that $(X, G(X)) \in \mathbb{A}^0$. It follows by Proposition that $(X, G(X)) \in \mathbb{L}^0$.

3.5. Corollary. *If* $\mathscr{G}$ *is a subset of* $\mathscr{C}_c$ *compatible with a Borel set* $B \subset \mathbb{C}(\mathbb{R}^+)$ *then the set* $\mathscr{W}_{\mathscr{G},B}$ *of solution of the* $(\mathscr{G}, B)$*-local martingale problem is a weakly closed set in* $\mathscr{P}(B)$. *Particularly,* $\mathscr{W}_{\mathscr{G}}$ *is convex and weakly closed if* $\mathscr{G} \subset \mathscr{C}_{c,a}$.

3.6. Corollary. *If* $\mathscr{G}$ *is at most countable subset of* $\mathscr{C}$ *compatible with a Borel set* $B \subset \mathbb{C}(\mathbb{R}^+)$ *then* $\mathscr{W}_{\mathscr{G},B}$ *is a Borel convex set in* $\mathscr{P}(\mathbb{C})$. *Particularly,* $\mathscr{W}_{\mathscr{G}}$ *is a Borel convex set in* $\mathscr{P}(\mathbb{C}(\mathbb{R}^+))$ *if* $\mathscr{G} \subset \mathscr{C}_a$ *is at most countable set.*

Remark that the sets $\mathscr{W}_{\mathscr{G},B}$ and $\mathscr{W}_{\mathscr{G}}$ in (3.6) are not only convex, their measure convexity will be proved in [10].

Proof. To prove (3.5) and (3.6) we may assume that without loss of generality $\mathscr{G} = \{G\}$ is a singleton set. As for the convexity statements choose $G \in \mathscr{C}$ and check that

$$\mathbb{PM}(G) := \{\mu \in \mathscr{P}(\mathbb{C}) : (\mathbf{x}, G, \mathbb{C}, \mathscr{B}(\mathbb{C}), \mu) \in \mathbb{PM}\}$$

and therefore

$$\mathbb{PL}(G) := \{\mu \in \mathscr{P}(\mathbb{C}) : (\mathbf{x}, G, \mathbb{C}, \mathscr{B}(\mathbb{C}), \mu) \in \mathbb{PL}\}$$

are convex. If the G and a $B \subset \mathscr{B}(\mathbb{C})$ are compatible then $\mathscr{W}_{\mathscr{G},B}$ equals to $\mathbb{PL}(G) \cap \mathscr{P}(B)$, hence it is a convex set.

Fix again $G \in \mathscr{C}$ and define a Borel map $\Phi_G : \mathscr{P}(\mathbb{C}(\mathbb{R}^+)) \to \mathscr{P}(\mathbb{C}^2)$ by $\Phi_G(\mu) = \mathscr{L}(\mathbf{x}, G \mid \mu)$. If the G and a $B \in \mathscr{B}(\mathbb{C})$ are compatible then $\Phi_G(\mathscr{P}(B)) \subset \mathscr{L}(\mathbb{A}^0)$ and therefore the restriction of Φ_G to $\mathscr{P}(B)$, say Φ_G^r, is a Borel map from $\mathscr{P}(B)$ into $\mathscr{L}(\mathbb{A}^0)$. Because $\mathscr{W}_{\mathscr{G},B} = (\Phi_G^r)^{-1}\,\mathbb{L}^0$ and $\mathbb{L}^0$ is a weakly closed set in $\mathscr{L}(\mathbb{A}^0)$ by Proposition, we get the $\mathscr{W}_{\mathscr{G},B}$ as a Borel set in $\mathscr{P}(B)$, hence a Borel set in $\mathscr{P}(\mathbb{C}(\mathbb{R}^+))$.

If $G \in \mathscr{C}_c$ then Φ_G defines a continuous map $\mathscr{P}(\mathbb{C}(\mathbb{R}^+)) \to \mathscr{P}(\mathbb{C}^2)$ and its restriction to $\mathscr{P}(B)$, Φ_G^r, becomes to be a continuous map from $\mathscr{P}(B)$ into $\mathscr{L}(\mathbb{A}^0)$. Because $\mathscr{W}_{\mathscr{G},B} = (\Phi_G^r)^{-1}\,\mathbb{L}^0$ and $\mathbb{L}^0$ is a weakly closed set in $\mathscr{L}(\mathbb{A}^0)$ by Proposition we conclude that $\mathscr{W}_{\mathscr{G},B}$ is a weakly closed set in $\mathscr{P}(B)$. We have proved all statements of 3.5 and 3.6. □

The preceding Corollaries may be applied to the examples we stated in Section 1.

1.1 Example continued

The set $\mathscr{H}$ of local martingale distributions is weakly closed and convex in $\mathscr{P}(\mathbb{C})$, because $\mathscr{H} = \mathscr{W}_{\{\mathbf{x}-\mathbf{x}(0)\}}$ and $\mathbf{x} - \mathbf{x}(0) \in \mathscr{C}_{c,a}$.

1.2 Example continued

The set $\mathscr{W}_{b,\sigma}$ of probability distributions of weak solution of a (b, σ)-SDE is

(22) *a Borel convex set in $\mathscr{P}(\mathbb{C})$ for arbitrary progressive coefficients b and σ*

by Corollary 3.6 and Remark 2.4 (i),

(23) *a weakly closed set in $\mathscr{P}(B_{b,\sigma})$ if $G_b, G_{b,\sigma} \in \mathscr{C}_c$*

by Corollary 3.5. Especially, it follows by Remark 2.4 (iii) and (iv) that

(24)
$b, \sigma \in \mathbb{R}^+ \oplus \mathscr{C}$ (or more generally (25)) $\Rightarrow \mathscr{W}_{b,\sigma}$ is a weakly closed convex set in $\mathscr{P}(B_{b,\sigma})$

where

(25) *$x \to b(t, x), x \to \sigma(t, x)$ are continuous maps $\mathbb{C}(\mathbb{R}^+) \to \mathbb{R}$ and*

$$\sup_{(s,x)\in[0,t]\times K} |b(s, x)| < \infty, \quad \sup_{(s,x)\in[0,t]\times K} |\sigma(s, x)| < \infty, \ \forall t \geq 0, \ K \subset \mathbb{C}(\mathbb{R}^+) \text{ compact.}$$

Note that the requirements (25) are general enough to cover the choices as

$$b(t, x) = \bar{b}(t, x(t)), \qquad \sigma(t, x) = \bar{\sigma}(t, x(t))$$

where $\bar{b}, \bar{\sigma} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

1.3 Example continued

The set $\mathscr{W}_{G,v}$ of probability distributions $\mu \in \mathscr{P}(\mathbb{C})$ such that a $G \in \mathscr{C}$ is an $\mathscr{F}_t^{\mathbf{x},\mu}$-local martingale on $(\mathbb{C}, \mathscr{B}(\mathbb{C})^\mu, \mu)$ with $\langle G \rangle \overset{a.s.}{=} v[\mu]$, where $v \in \mathscr{C}$ is a process of finite variation, is

(26) *a Borel convex set in $\mathscr{P}(\mathbb{C})$ if both G and v are in $\mathscr{C}_a$*

by 3.6 because $\mathscr{W}_{G,v} = \mathscr{W}_{\{G,G_v\}}$ and $G_v := G^2 - v \in \mathscr{C}_a$ and

(27) *a weakly closed convex set in $\mathscr{P}(\mathbb{C})$ if both G and v are in $\mathscr{C}_{c,a}$*

by 3.5 because $G_v \in \mathscr{C}_{c,a}$.

Thus, the local martingale problems in Examples 1.1, 1.2 (with $G_b, G_{b,\sigma} \in \mathscr{C}_{c,a}$) and 1.3 (with $G, v \in \mathscr{C}_{c,a}$) produce sets of solutions $\mathscr{W}$ that are convex and weakly closed in $\mathscr{P}(\mathbb{C})$, which sets are known as a subject to Krein-Milman, or more generally to Choquet property. The former one is specified by Theorems 8 and 9 in [9] by equalities

$$W = \overline{\text{co}}\, \text{ex}\, \mathscr{W} = \overline{\text{co}}_b\, \text{ex}\, \mathscr{W} \tag{28}$$

where $\overline{\text{co}}$ and $\overline{\text{co}}_b$ denote the closed convex hull operators w.r.t. the weak and the $B(\mathbb{C})$-topology in $\mathscr{P}(\mathbb{C})$, respectively. Recall that the $B(\mathbb{C})$-topology is the minimal topology which makes $\mu \to \int_{\mathbb{C}} g \, d\mu$ to be a continuous map $\mathscr{P}(\mathbb{C}) \to \mathbb{R}$ for any Borel bounded function $g : \mathbb{C} \to \mathbb{R}$. Hence, a crucial problem is to find an effective characterization of the extreme boundary ex $\mathscr{W}$. A simple step towards the aim that also suggests possible applications is presented by

3.7. Theorem. *Consider $\mathscr{G} \subset \mathscr{C}_{c,a}$ and denote by $\mathscr{W}_{\mathscr{G},d}$ the set of all solutions of the $\mathscr{G}$-local martingale problem with a deterministic initial condition ε_x. Then*

(a) ex $\mathscr{W}_{\mathscr{G}} \subset \mathscr{W}_{\mathscr{G},d}$ *and* $\mathscr{W}_{\mathscr{G}} = \overline{\text{co}}\, \mathscr{W}_{\mathscr{G},d} = \overline{\text{co}}_b\, \mathscr{W}_{\mathscr{G},d}$.

(b) Assume that for any $x \in \mathbb{R}$ there is a $\mu_x \in \mathscr{W}_{\mathscr{G}}$ with initial condition x ($\mathscr{L}(\mathbf{x}(0) \mid \mu_x) = \varepsilon_x$). Then, given an arbitrary $v \in \mathscr{P}(\mathbb{R})$, there exists a $\mu_v \in \mathscr{W}_{\mathscr{G}}$ with initial condition v.

(c) Let $f : \mathscr{P}(\mathbb{C}) \to \bar{\mathbb{R}}$ be a lower bounded, $B(\mathbb{C})$-lower semicontinuous and convex function. Then

$$\sup \{f(\mu), \mu \in \mathscr{W}_{\mathscr{G}}\} = \sup \{f(\mu), \mu \in \mathscr{W}_{\mathscr{G},d}\}.$$

Note that the set $\mathscr{W}_{\mathscr{G},d} \setminus \text{ex}\, \mathscr{W}_{\mathscr{G}}$ in (a) may be rather complex as in Example 1.1, where $\mu \in \text{ex}\, \mathscr{H}$ is characterized by the predictable representation property as in Proposition (4.6) and Theorem (4.7) in [7]. Further note, that (b) and (c) do not ask the $\mathscr{G}$-local martingale problem to be well posed for the deterministic initial conditions and compare (b) with Theorem 18.10. in [5]. An easy application of (b) is

1.2 Example continued II

If a (b, σ)-SDE with $G_b, G_{b,\sigma} \in \mathscr{C}_{c,a}$ is such that there exists its weak solution for any deterministic initial condition then the equation possesses a weak solution with arbitrary initial condition in $\mathscr{P}(\mathbb{R})$.

3.8. Example. *Consider $\mathscr{G} \subset \mathscr{C}_{c,a}$ and a Borel function $g : \mathbb{R} \to \mathbb{R}^+$. Then for any $t > 0$*

$$\sup \{\mathbb{E} g(X(t)) : \mathscr{L}(X) \in \mathscr{W}_{\mathscr{G}}\} = \sup \{\mathbb{E} g(X(t)) : \mathscr{L}(X) \in \mathscr{W}_{\mathscr{G},d}\}$$

holds.

Fix $t > 0$, put $g_n = g \wedge n$ and define Borel functions $g_t, g_{t,n} : \mathbb{C}(\mathbb{R}^+) \to \bar{\mathbb{R}}$ by

$$g_t(x) := g(x(t)) \quad \text{and} \quad g_{t,n} := g_n(x(t))$$

for $x \in \mathbb{C}(\mathbb{R}^+)$. Also denote

$$f(\mu) = \int_{\mathbb{C}} g_t \, d\mu, \quad f_n(\mu) = \int_{\mathbb{C}} g_{t,n} \, d\mu, \quad \mu \in \mathscr{P}(\mathbb{C}(\mathbb{R}^+))$$

and note that $f : \mathscr{P}(\mathbb{C}(\mathbb{R}^+)) \to \mathbb{R}$ is a lower bounded and $\mathscr{B}(\mathbb{C})$-lower semicontinuous map because $f = \sup_n f_n$ and all f_n's are $\mathscr{B}(\mathbb{C})$-continuous maps $\mathscr{P}(\mathbb{C}(\mathbb{R}^+)) \to \mathbb{R}$ by definition. The equality stated above now follows directly by 3.7 (c).

Note finally that ex $\mathscr{W}_{\mathscr{G}} \subset \mathscr{W}_{\mathscr{G},d}$ holds and will be proved for any $\mathscr{G} \subset \mathscr{C}_a$. Theorem 3.7 will be extended along these lines largely in [10].

Proof of Theorem 3.7. If $\mathscr{G} \subset \mathscr{C}_a$ and $\mu \in \mathscr{W}_{\mathscr{G}}$ is a solution with a non deterministic initial condition $\nu \in \mathscr{P}(\mathbb{R})$, then there is a set $F_0 \in \mathscr{F}_0^x$ with $\mu(F_0) \in (0, 1)$. Denoting $\mu_1 = \mu(.\,|\,F_0)$ and $\mu_2 = \mu(.\,|\,F_0^c)$, $\alpha = \mu(F_0)$, we get $\mu = \alpha\mu_1 + (1 - \alpha)\,\mu_2$ and see that $\mu \notin$ ex $\mathscr{W}_{\mathscr{G}}$ if only μ_1 and μ_2 belong to $\mathscr{W}_{\mathscr{G}}$. To prove this, assume without loss of generality that each $G \in \mathscr{G}$ is an $\mathscr{F}_t^x$-martingale on $(\mathbb{C}, \mathscr{B}(\mathbb{C}), \mu)$. Fix a $G \in \mathscr{G}$, $s < t$ and $F_s \in \mathscr{F}_s^x$. It follows that

$$\int_{F_s} G(t) - G(s) \, d\mu_i = c_i \int_{F_s \cap F_0} G(t) - G(s) \, d\mu = 0, \qquad i = 1, 2,$$

where $c_1 = (\mu(F_0))^{-1}$ and $c_2 = (\mu(F_0^c))^{-1}$ are in $(0, 1)$ and therefore G is an $\mathscr{F}_t^x$-martingale on $(\mathbb{C}, \mathscr{B}(\mathbb{C}), \mu_i)$. Thus, both μ_1 and μ_2 are in $\mathscr{W}_{\mathscr{G}}$ and $\mu \notin$ ex $\mathscr{W}_{\mathscr{G}}$.

The rest of (a) follows by Krein-Milman properties (28) as

$$\mathscr{W}_{\mathscr{G}} \supset \overline{\text{co}}\, \mathscr{W}_{\mathscr{G},d} \supset \overline{\text{co}}_b\, \mathscr{W}_{\mathscr{G},d} \supset \overline{\text{co}}_b \,\text{ex}\, \mathscr{W}_{\mathscr{G}},$$

because $\mathscr{W}_{\mathscr{G}}$ is a weakly closed set and the $B(\mathbb{C})$-topology is finer than the weak one.

To prove (b) note first that $A := \{(\mu, x) \in \mathscr{W}_{\mathscr{G}} \times \mathbb{R} : p_0 \circ \mu = \varepsilon_x\}$ is a closed set in $\mathscr{P}(\mathbb{C}) \times \mathbb{R}$ which $\mathbb{R}$ projection equals to $\mathbb{R}$. According to the Crosssection theorem (see 8.5.3. in [1]) there exists a map $U : \mathbb{R} \to \mathscr{P}(\mathbb{C})$ that is universally measurable ($\mathscr{P}(\mathbb{C})$ being endowed by the weak topology) such that the graph of U is contained in A. Thus, $\mu_0 = \int_{\mathbb{R}} U(x)\, \nu_0(dx)$ is a well-defined measure in $\mathscr{P}(\mathbb{C})$ with $p_0 \circ \mu_0 = \nu_0$ for any $\nu_0 \in \mathscr{P}(\mathbb{R})$. To conclude the proof of (b) we only need to show that $\mu_0 \in \mathscr{W}_{\mathscr{G}}$. Put $\overline{\nu_0} = U \circ \nu_0$ to define a Borel probability measure on $\mathscr{W}_{\mathscr{G}}$ such that $\mu_0 = \int_{\mathscr{W}_{\mathscr{G}}} \mu \, \overline{\nu_0}(d\mu)$. Let $\overline{\nu_n} \to \overline{\nu_0}$ weakly be a sequence of discrete probability measures on $\mathscr{W}_{\mathscr{G}}$. Obviously,

$$\mu_n := \int_{\mathscr{W}_{\mathscr{G}}} \mu \, \overline{\nu_n}(d\mu) \to \int_{\mathscr{W}_{\mathscr{G}}} \mu \, \overline{\nu_0}(d\mu) = \mu_0 \text{ weakly in } \mathscr{P}(\mathbb{C})$$

and therefore $\mu_0 \in \mathscr{W}_{\mathscr{G}}$ because $\mathscr{W}_{\mathscr{G}}$ is a convex weakly closed set.

The assertion (c) follows by (a), because it implies that for any fixed $\mu_0 \in \mathscr{W}_{\mathscr{G}}$ there is a net (μ_α) of convex combinations of measures in $\mathscr{W}_{\mathscr{G},d}$ that converges to μ_0 in $B(\mathbb{C})$-topology and by the properties of f, because

$$f(\mu_0) \le \liminf_{\alpha} f(\mu_\alpha) \le \sup\{f(\mu), \mu \in \mathscr{W}_{\mathscr{G},d}\}.$$

□

3.9. Example (Portfolio process). *Assume, that the value process of our portfolio is described by the SDE (D) from Example 1.2 with initial investment* $X(0) = x$ *for* $x \in \mathbb{R}$. *When there exists a weak solution for any deterministic initial condition (i.e. we know our initial portfolio capital), then there exists a weak solution of SDE with a random initial condition* $\nu \in \mathscr{P}(\mathbb{R})$. *The assumption that we do not know exactly our initial investment is not very common in finance, but we present here two possible situations.*

1. *We want to describe a value process of a portfolio of a selected competing company. Being on the same market, the coefficients* b, σ *can be assumed the same as in our portfolio. As we do not know precisely conditions of the competing company, its initial capital is taken as a random variable* $\nu \in \mathscr{P}(\mathbb{R})$.
2. *We plan to construct a new portfolio with an initial investment resulting from the unknown profit of the contemporary portfolio, which for instance includes one call option. The profit from excercising the option is described by* $\nu \in \mathscr{P}(\mathbb{R})$.

Value of a portfolio is not necessary the only criterion of our success of the financial market. When $X(t)$ *denotes the value of our portfolio at time t, we define a* **utilility function** $g : \mathbb{R} \to \mathbb{R}^+$, *which represents our benefit. When we choose* g and $\mathscr{G}$ *as in Example 3.8, the expected maximal benefit is achieved on the set of processes with deterministic initial conditions. It means, that for a given utility function, we can not increase the maximal expected benefit by assuming random initial condition instead of deterministic one. Notice, that this result does not depend on our more or less precise knowledge of the initial distribution.*

4 Proofs

We shall mainly profit of **Skorochod Theorem** on the representation of the convergence in distribution in terms of the almost sure convergence.

If ξ_n, ξ are S-valued random variables where S is a separable metric space such that $\xi_n \xrightarrow{\mathscr{D}} \xi$, then there are η_n, η on a probability space $(\Omega, \mathscr{F}, P)$ such that $\mathscr{L}(\xi_n) = \mathscr{L}(\eta_n)$, $\mathscr{L}(\xi) = \mathscr{L}(\eta)$ and $\eta_n \to \eta$ a.s. $[P]$ (see Theorem 3.30 in [5]). To prove Theorem 3.2 we need to recognize a martingale among premartingales.

4.1. Lemma. *Let* X *be a continuous bounded* $\mathscr{F}_t$*-premartingale on a complete probability space, where* $(\mathscr{F}_t)$ *is an arbitrary filtration. If* τ *is an* $\mathscr{F}_t$ *Markov time such that* X^τ *is an* $\mathscr{F}_t$*-adapted process, then* X^τ *is an* $\mathscr{F}_t$*-martingale.*

First we shall prove

4.2. Lemma. *Let* X *be as in Lemma 4.1 and* $\tau \leq \nu \leq T < \infty$ *a pair of* $\mathscr{F}_t$ *Markov times. Then* $X(\tau), X(\nu) \in L_1$ *and* $\mathbb{E}^{\mathscr{F}_\tau} X(\nu) = \mathbb{E}^{\mathscr{F}_\tau} X(\tau)$, *where* $\mathscr{F}_\tau$ *is the* σ*-algebra of events up to the Markov time* τ.

Proof. Assume first that the filtration $(\mathscr{F}_t)$ is right-continuous and complete. It follows directly by the definition of a premartingale that the process $(\mathbb{E}^{\mathscr{F}_t} X(t), t \geq 0)$ is an $\mathscr{F}_t$-martingale and such as possesses a right continuous modification $(Y(t), t \geq 0)$ (by Theorem 6.27. (ii) in [5]) that is an $\mathscr{F}_t$-adapted process as the filtration is complete. Thus Y is a right continuous $\mathscr{F}_t$-martingale such that $Y(t) \overset{a.s.}{=} \mathbb{E}^{\mathscr{F}_t} X(t)$ holds for all $t \geq 0$. The properties yield

$$(29) \qquad X(v) \in L_1 \quad \text{and} \quad Y(v) \overset{a.s.}{=} \mathbb{E}^{\mathscr{F}_v} X(v).$$

If v takes its values only in a finite set $0 \leq t_0 < t_1 < \ldots < t_K \leq T$ then

$$\mathbb{E}^{\mathscr{F}_v} X(v) = \sum_{j=1}^{K} I_{[v=t_j]} \mathbb{E}^{\mathscr{F}_v} X(t_j) = \sum_{j=1}^{K} I_{[v=t_j]} \mathbb{E}^{\mathscr{F}_{t_j}} X(t_j) = Y(v)$$

holds almost surely because $[v = t_j] \in \mathscr{F}_v$ and $[v = t_j]$ implies almost surely that $\mathbb{E}^{\mathscr{F}_v} Z = \mathbb{E}^{\mathscr{F}_{t_j}} Z$ for any integrable random variable Z.

To prove (29) for an arbitrary v consider a sequence $v_n \leq T$ and $v_n \searrow v$ of finitely valued $\mathscr{F}_t$ Markov times and perform the L_1-limit in $\mathbb{E}^{\mathscr{F}_v} Y(v_n) \overset{a.s.}{=} \mathbb{E}^{\mathscr{F}_v} X(v_n)$.

It follows by (29) and Stopping theorem 6.29 in [5], which may be applied, because Y is a right continuous $\mathscr{F}_t$-martingale and the filtration is right continuous, that

$$\mathbb{E}^{\mathscr{F}_\tau} X(v) = \mathbb{E}^{\mathscr{F}_\tau} \mathbb{E}^{\mathscr{F}_v} X(v) = \mathbb{E}^{\mathscr{F}_\tau} Y(v) = Y(\tau) = \mathbb{E}^{\mathscr{F}_\tau} X(\tau) \text{ a.s.}$$

To prove the Lemma in general we check first easily that X being a continuous $\mathscr{F}_t$-premartingale it is also $\mathscr{F}_t^a$-premartingale, where $\mathscr{F}_t^a$ is the augmentation of the filtration $\mathscr{F}_t$, i.e. the minimal right continuous and complete filtration with $\mathscr{F}_t^a \supset \mathscr{F}_t$ for all $t \geq 0$. Applying the assertion of our Lemma to X and $(\mathscr{F}_t^a)$ we conclude the proof. □

Proof of 4.1.

First observe that $X(\tau \wedge t) \in \mathscr{F}_{\tau \wedge t} \subset \mathscr{F}_\tau$ for any $t \geq 0$, because the process $X^\tau = (X^\tau)^\tau$ is $\mathscr{F}_{\tau \wedge t}$ adapted. Hence it follows by Lemma 4.2 that for any $s < t$

$$X(\tau \wedge s) = \mathbb{E}^{\mathscr{F}_{\tau \wedge s}} X(\tau \wedge s) = \mathbb{E}^{\mathscr{F}_{\tau \wedge s}} X(\tau \wedge t) = \mathbb{E}^{\mathscr{F}_s} \mathbb{E}^{\mathscr{F}_\tau} X(\tau \wedge t) = \mathbb{E}^{\mathscr{F}_s} X(\tau \wedge t)$$

holds almost surely. Hence, X^τ is an $\mathscr{F}_t$-martingale. □

4.3. Lemma. *If* $(X_n, Y_n) \in \mathbb{PM}$, $|Y_n| \leq c < \infty$ *for all* n *and* $(X_n, Y_n) \xrightarrow{\mathscr{D}} (X, Y)$ *then* $(X, Y) \in \mathbb{PM}$.

Proof.

Just observe that $(\xi, \eta) \in \mathbb{PM} \Leftrightarrow \eta(t) \in L_1$ and $\mathbb{E} g_s(\xi(u), u \leq s)\, \eta(t) = \mathbb{E} g_s(\xi(u), u \leq s)\, \eta(s)$ for any continuous bounded map $g_s : \mathbb{C}[0, s] \to \mathbb{R}$ and any $0 \leq s < t < \infty$. □

Proof of 3.2. Denote

$$\mathbb{Z}_n := (X_n, Y_n, Y_n^1, Y_n^2, \ldots)$$

where

$$Y_n^c := \alpha^c \circ Y_n, \; c, n \in \mathbb{N}$$

and observe that $\mathbb{Z}_1, \mathbb{Z}_2, \ldots$ is a sequence of random variables in $\mathbb{C}^{\mathbb{N}}$ which is a Polish space. Because each α^c maps compact sets in $\mathbb{C}$ to relatively compact sets in $\mathbb{C}$ and $\mathscr{L}(X_n, Y_n)$ converges weakly to $\mathscr{L}(X, Y)$ in $\mathscr{P}(\mathbb{C} \times \mathbb{C})$ we conclude by Prochorov theorem that the sequence $\mathscr{L}(\mathbb{Z}_1), \mathscr{L}(\mathbb{Z}_2), \ldots$ is relatively compact in $\mathscr{P}(\mathbb{C}^{\mathbb{N}})$ endowed by the weak topology. In other words, we may assume without loss of generality, that $\mathbb{Z}_n \overset{\mathscr{D}}{\to} \mathbb{Z}$, where $\mathbb{Z}$ is a $\mathbb{C}^{\mathbb{N}}$ valued random variable represented, again without loss of generality as $\mathbb{Z} = (X, Y, Y^{(1)}, Y^{(2)}, \ldots)$, where $Y^{(c)}$ are continuous processes defined on the same probability space as the pair (X, Y). Finally, we argument by Skorochod theorem that we cause no harm assuming that $\mathbb{Z}, \mathbb{Z}_1, \mathbb{Z}_2, \ldots$ are all defined on a complete probability space $(\Omega, \mathscr{F}, P)$ such that $\mathbb{Z}_n \overset{a.s.}{\to} \mathbb{Z}$ in the metric space $\mathbb{C}^{\mathbb{N}}$ and that $Y_n^c \overset{a.s.}{=} \alpha^c \circ Y_n$ for all c and $n \in \mathbb{N}$. Hence, outside a P-null set N

$$(30) \qquad Y_n \to Y \text{ and } Y_n^c \to Y^{(c)} \text{ in } \mathbb{C}(\mathbb{R}^+) \text{ for all } c \in \mathbb{N}.$$

Because $\tau^c(x) \le \liminf_{n\to\infty} \tau^c(x_n)$ whenever $x_n \to x$ in $\mathbb{C}(\mathbb{R}^+)$, it follows by (30) that outside the P-null set N for any $c \in \mathbb{N}$ and almost all $n \in \mathbb{N}$, Y_n and Y_n^c have trajectories that are identical on the interval $[0, \tau^c(Y)]$. Hence, again by (30), outside N and for any $c \in \mathbb{N}$, $Y = Y^{(c)}$ on $[0, \tau^c(Y)]$ which is as to say that

$$(31) \quad \xi^{(c)}(t) := Y^{(c)}(t \wedge \tau^c(Y)) = Y(t \wedge \tau^c(Y)),\ t \ge 0 \text{ almost surely for any fixed } c \in \mathbb{N}.$$

Because $(X, Y) \in \mathbb{A}^0$ and $\tau^c(Y)$ is an $\mathscr{F}_t^{X,P}$-Markov time it follows by (31) that $(X, \xi^{(c)}) \in \mathbb{A}^0$ and in fact $(X, \xi^{(c)}) \in \mathbb{M}^0$ by Lemma 4.1 because $Y^{(c)}$ is a bounded $\mathscr{F}_t^{X,P}$-premartingale by (30) and Lemma 4.3. Hence, $(X, \alpha^c(Y)) \overset{a.s.}{=} (X, \xi^{(c)})$ are pairs in $\mathbb{M}^0$ for any $c \in \mathbb{N}$, and therefore $(X, Y) \in \mathbb{L}^0$. □

Denoting $S_c := \{x \in \mathbb{C} : x(0) = 0,\ x \text{ is a continuity point of } \alpha^c\}$ a natural question arises in connection with Proposition and its proof: Which are continuous local martingales Y that make the stopping map α^c to be Y-continuous, i.e. for which local martingales $P[Y \in S_c] = 1$ holds? A summary of our knowledge regarding the above problem is as follows:

4.4. Remark. *Denote* $\tau^{c+}(x) = \inf\{t \ge 0 : |x(t)| > c\}$ *for* $x \in \mathbb{C}$ *and* $c > 0$. *Then*

(a) $x_n \to x \Rightarrow \tau^c(x) \le \liminf \tau^c(x_n) \le \limsup \tau^{c+}(x_n) \le \tau^{c+}(x)$,
i.e. τ^c *and* τ^{c+} *are lower and upper semicontinuous maps* $\mathbb{C} \to \mathbb{R}^+ \cup \{+\infty\}$.

(b) $S_c = \{x \in \mathbb{C} : x(0) = 0,\ x \text{ is constant function on } [\tau^c(x), \tau^{c+}(x)]\}$.

(c) $P[Y \in S_c] = 1$ *provided that* Y *is a continuous local martingale either such that* $\langle Y \rangle$ *is a strictly increasing process or* $\langle Y \rangle(+\infty) = +\infty$ *holds almost surely.*

We omit the proof and remark only that (c) follows by an application either of Theorem on Mutual continuity Y and $\langle Y \rangle$ or of Dambis-Dubins-Schwarz theorem.

4.5. Remark. *Example 3.1 and Corollary 3.3 open a problem. Is it true that*

$$G_n \xrightarrow{\mathcal{L}} G, \quad X_n \xrightarrow{\mathcal{D}} X, \quad (X_n, G_n(X)) \in \mathbb{A}^0 \Rightarrow (X, G(X)) \in \mathbb{A}^0,$$

for $G_n, G \in \mathscr{C}$ and X_n, X continuous processes?

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