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# Service Stations Location Problem

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## 1. Description of the model

Let us consider the following system. There are $n$ areas represented by segments in the space, where it is possible to build a service station. Each service station built on such segment is accessible only by edges of this segment.

These edges are determined by access points $(A_j, B_j)$.

Then there are $m$ customers in the space, where each customer is connected by a road with every access point of every service station in the system.

Each road is evaluated by a nonnegative number, which represents the distance of customer's access to a service station $(a_{ij}, b_{ij})$. Each segment has its recommended subsegment $(h_j H_j)$, where is the most suitable place for building of the service station in this segment. (It could be caused for example by some ecological restrictions or prices of land, etc.)

Building a service station somewhere else is penalized by a penalty function. The value of the penalty function is zero in the recommended subsegment and is linearly increasing with distance from the recommended subsegment. The penalty function for each segment is

$$f_j(x_j) \equiv \max\{h_j - x_j; x_j - H_j; 0\} \qquad \forall j = 1, \dots, n$$

where $x_j \in [0; d_j]$, $d_j$ is the length of the segment $A_j B_j$, and $h_j$, $H_j$ are bounds of the recommended subsegment in $[0; d_j]$

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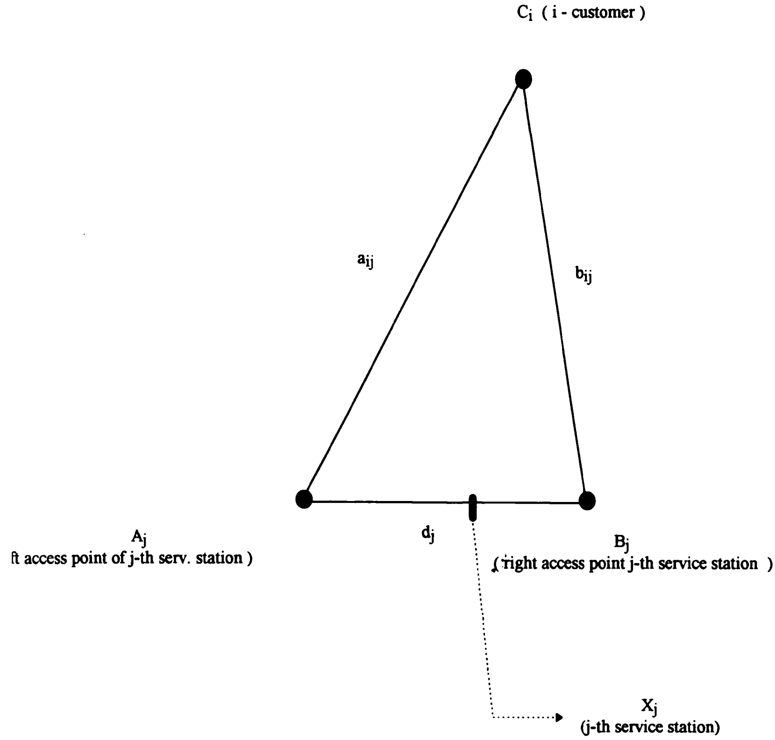
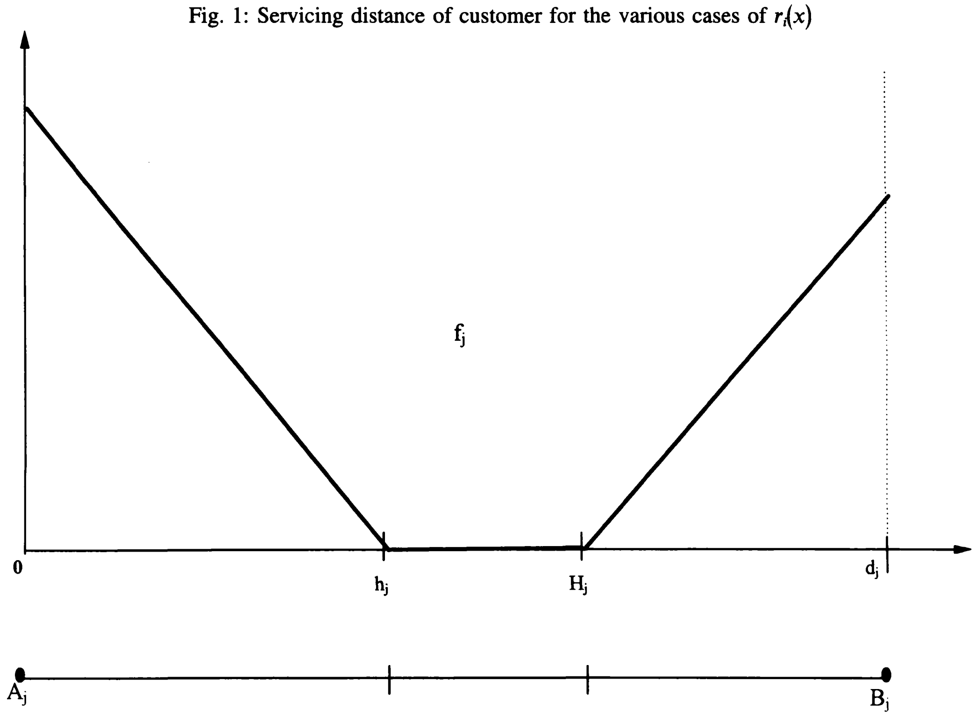



Fig. 1: Servicing distance of customer for the various cases of $r_i(x)$



For each customer, there is a required servicing distance $(b_i)$ given.

Define $r_i(x)$ as a servicing distance of the $i$-th customer, dependent on a position of the service stations (vector $\boldsymbol{x}$). Then the requirement for customer $C_i$ is $r_i(\boldsymbol{x}) \leq b_i$.

Our goal is to find such setting of the service stations which meets the distance requirements of every customer and value of a global penalty function of the system is minimal, where the global penalty function of the system is

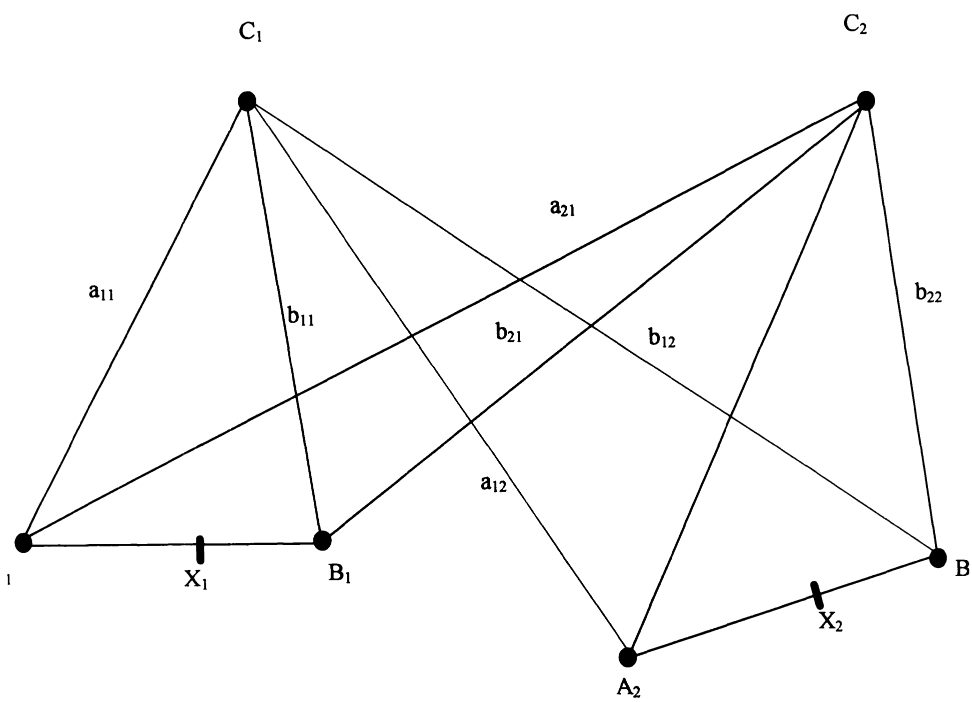
$$f(x) \equiv \max_{1 \leq j \leq n} f_j(x_j) \qquad \text{where} \quad f_j(x_j) \equiv \max\{h_j - x_j; x_j - H_j; 0\}$$

$\approx$ to minimise the worst case.

It means to find a solution of this problem:

(P) $f(x) \to \min$
s.t. $M$
where
$f(x) \equiv \max_{1 \leq j \leq n} f_j(x_j) \quad f_j(x_j) \equiv \max\{h_j - x_j; x_j - H_j; 0\}$
$M = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, r_i(\boldsymbol{x}) \leq b_i \quad \forall i = 1, \ldots, m, \quad 0 \leq x_j \leq d_j \quad \forall j = 1, \ldots, n\}.$

A sample of a system with two service stations and two customers is shown in the following figure:



Setting a servicing distance of customer at the system we can consider several possible strategies:

Define $r_{ij}(x_j)$ as servicing distance of customer $C_i$ by the $j$-th service station. According the measure of quality of the system or the measure of optimism we have, we can choose two marginal possibilities.

1. in optimistic one, we consider that a customer is served by the nearest service station, it means:

$$r_i(\boldsymbol{x}) \equiv \min_{1 \le j \le n} r_{ij}(x_j) \qquad \forall i = 1, \ldots, m$$

2. in pessimistic one, we, on the other way, consider that a customer is served by the service station which is the worst for him, it means:

$$r_i(\boldsymbol{x}) \equiv \max_{1 \le j \le n} r_{ij}(x_j) \qquad \forall i = 1, \ldots, m$$

(for example because of short capacities of service stations, or temporary impossibility of using some roads.)

Similarly way we can think of $r_{ij}(x_j)$ servicing distance of $i$-th customer by $j$-th service station.

In optimistic situation the shorter road will be used, so

$$r_{ij}(x_j) \equiv \min \{x_j + a_{ij}; d_j - x_j + b_{ij}\} \qquad \forall i = 1, \ldots, m \qquad \forall j = 1, \ldots, n,$$

in pessimistic situation the longer road will be used, so

$$r_{ij}(x_j) \equiv \max \{x_j + a_{ij}; d_j - x_j + b_{ij}\} \qquad \forall i = 1, \ldots, m \qquad \forall j = 1, \ldots, n,$$

According to this we can formulate four problems:

Define as before:

$$f(x) \equiv \max_{1 \le j \le n} f_j(x_j) \qquad f_j(x_j) \equiv \max \{h_j - x_j; x_j - H_j; 0\},$$

$$M = \{\boldsymbol{x} \in \mathbb{R}^n \mid r_i(\boldsymbol{x}) \le b_i \ \ \forall i = 1, \ldots, m, \ \ 0 \le x_j \le d_j \ \ \forall j = 1, \ldots, n\}.$$

(P.I)

$f(x) \to \min$
s.t. $M$
where
$r_i(x) \equiv \min_{1 \le j \le n} r_{ij}(x_j)$
$r_{ij}(x_j) \equiv \max \{x_j + a_{ij};\ d_j - x_j + b_{ij}\}$

(P.II)

$f(x) \to \min$
s.t. $M$
where
$r_i(x) \equiv \min_{1 \le j \le n} r_{ij}(x_j)$
$r_{ij}(x_j) \equiv \min \{x_j + a_{ij};\ d_j - x_j + b_{ij}\}$

(P.III) $f(x) \to \min$
s.t. $M$
where
$r_i(x) \equiv \max_{1 \le j \le n} r_{ij}(x_j)$
$r_{ij}(x_j) \equiv \min \{x_j + a_{ij};\ d_j - x_j + b_{ij}\}$

(P.IV) $f(x) \to \min$
s.t. $M$
where
$r_i(x) \equiv \max_{1 \le j \le n} r_{ij}(x_j)$
$r_{ij}(x_j) \equiv \max \{x_j + a_{ij};\ d_j - x_j + b_{ij}\}$

## 2. Solvability of the problems

**(P.III) and (P.IV).** At first, let us consider problems (P.III) and(P.IV).

Both problems are quite easily solvable, which is caused by description of the set of feasible solutions. In both problems, it is possible to express the feasible set $M$ in the following way.

Define: $I \equiv \{1, \ldots, m\}$, $J \equiv \{1, \ldots, n\}$,

$$M = \{x \in \mathbb{R}^n \mid \forall j \in J\, x_j \in W_j\},$$

where for (P.III)

$$W_j = \bigcap_{i \in I} [0, b_i - a_{ij}] \cup [d_j + b_{ij} - b_i, d_j]$$

and for (P.IV):

$$W_j \equiv \bigcap_{i \in I} [\max(0, d_j + b_{ij} - b_i), \min(d_j, b_i - a_{ij})].$$

(It is easy to calculate $W_j$ from conditions of the problems.)

Because function $f_j$ is a partial linear function for each $j$, the optimal solution can be easily computed as

$$x_j : f_j(x_j) = \min_{x_j \in W_j} f_j(x_j) \ \forall j \in J.$$

**(P.I) and (P.II).** Solvability of problems (P.I) and (P.II) is more difficult. In both cases only the problem of finding a feasible solution is NP-hard.

For problem (P.I) define a set

$$V_{ij} \equiv \{x_j \in \mathbb{R} \mid \max\{0, d_j + b_{ij} - b_i\} \le x_j \le \min\{d_j, b_i - a_{ij}\}\},$$

which is the set of $x_j \in [0; d_j]$ which meets distance condition for the $i$-th customer.

Then it is easily seen that we can formulate an equivalence:

$$M \neq \emptyset \Leftrightarrow \left(\forall i \, \exists j(i) \; V_{ij(i)} \neq \emptyset \;\; \& \bigcap_{\{i:j(i)=j_0\}} V_{ij_0} \neq \emptyset \;\; \forall j_0 = 1, \ldots, n\right).$$

Finding a member of a set $M$ means finding a number $j(i)$ for each $i$ where

$$V_{ij(i)} \neq \emptyset \;\; \& \bigcap_{\{i:j(i)=j_0\}} V_{ij_0} \neq \emptyset \;\; \forall j_0 = 1, \ldots, n.$$

Because it need not be generally true that for any $k, l \in J$ and $i \in I$ one can have $V_{ik} \cap V_{il} \neq \emptyset$, so the condition

$$(*) \qquad \bigcap_{\{i:j(i)=j_0\}} V_{ij_0} \neq \emptyset \;\; \forall j_0 = 1, \ldots, n$$

is not met trivially.

Let's call (P.I.A) the problem of finding a feasible solution. We can transfer this problem to an equivalent problem (P.I.B) formulated as follows:

(P.I.B) There is matrix $T\,(mxn)$ and there are given "column conditions", saying which fields of the matrix could not be chosen at the same time. The goal is to choose in each row one field and meet the column conditions (i.e. assign to each $i$ some $j$), where the column conditions for each column are given by the condition (*).

It is easily seen from formulation of the problem (P.I.B), that if we find a solution of this problem we have at the same time a solution of problem (P.I.A), and vice versa.

**Theorem.** *The problem (P.I.B) is NP-hard.*

**Proof.**

* For the proof of NP-hardness of the problem (P.I.B), we use the Coloring problem — colouring a general graph with $K$ colors ($K$-COLOR), which is NP-hard.

* Instantion of $K$-COLOR is given this way: We have a general graph $G(E, V)$ and a number $K$. The goal is to color the points of the graph using $K$ colors where each two points connected by an edge must be colored with different colors.

* From an instance of a problem $K$-COLOR we form an instantion of problem (P.I.B) in the following way:

(1) we form a matrix $T$ of size $|V| \times K$ — rows are given by points and columns are given by colors.

(2) set of column conditions: in each column, it is not possible to choose fields (rows) which represent connected points in the graph $G$.

* This form an instantion of a problem (P.I.B) and finding a solution of the problem (P.I.B) means to choose one column in each row (= to color each point of the graph $G$ with some color), and to meet column conditions (= to color connected points with different colors). So finding a solution of the problem (P.I.B) is finding a solution of the ($K$-COLOR) at the same time. (The solution of the

(P.I.B) gives a solution of ($K$-COLOR). This means that the (P.I.B) is NP-hard too.

Because we know that the problem (P.I.B) is equivalent to the problem (P.I.A), the problem (P.I.A) is NP-hard too.

(It is not possible to use the same reduction for the problem (P.II) NP-hardness proof because its column conditions give only a bipartional graph. But it is possible to use a reduction on SAT — what was in a more general approach done in [4].)

### 3. Algorithms for problem (P.I)

It is possible to find an optimal solution of problem (P.I) by branch-and-bound algorithm, which uses a convexity of functions $f_i$ and recursive processing of a matrix of a partial optimal solutions on intervals $V_{ij}$.

For two intervals $V_{1j}$ and $V_{2j}$ for $1, 2 \in I, j \in J$, where $V_{1j} \cap V_{2j} \neq \emptyset$, the equality

$$\min_{V_{1j} \cap V_{2j}} f_j(x_j) = \max \{\min_{V_{1j}} f_j(x_j);\ \min_{V_{2j}} f_j(x_j)\}$$

follows from the convexity of function $f_j$ for every $j \in J$.

(Because function $f_j$ is a partial linear function for ever $j \in J$ and $V_{ij}$ is an interval, the expression $\min_{V_{ij}} f_j(x_j)$ is easily computable.)

When we make a matrix $T$ where $T[i, j] \equiv \min_{V_{ij}} f_j(x_j)\ \forall i \in I\ \forall j \in J$, then finding an optimal solution of a problem (P.I.) means to choose in the matrix $T$ one field in every row (having $m$ chosen fields), and we want the maximal chosen field (chosen field with maximal value) to be minimal and all chosen fields to meet column conditions given by (*).

The algorithms of finding an optimal solution can be then formulated in a following way.

In every step of the algorithm we cross (not choose) or choose some field of the matrix $T$. Firstly we try to cross the maximal unprocessed (not chosen and not crossed) field of the matrix. In case that we will find solution in such modified matrix, this solution is sure to be better than a solution in which this field is chosen. (At the beginning of the algorithm all fields of the matrix $T$ are unprocessed.) Then the algorithm recursively calls itself on a problem with such modified matrix $T$ with decreased number of unprocessed fields. In the case, when after step of crossing the field there is only one uncrossed field in the row, we must choose this field, and according to column conditions for its column to cross fields which cannot be chosen at the same time with this chosen field.

If there is a situation that in the matrix, there is a row whose all fields are crossed, it means a failure on this level and it is not possible to continue in this way, we go back on a higher level of the recursion (all changes made in the lower level are undone) and we change the crossed field on chosen field (if we choose some field in a row, we have to cross all uncrossed fields in this row). Then we again recursively call the algorithm on such modified matrix $T$.

The algorithm stops in the moment when every field in the matrix $T$ is chosen or crossed (or with the state that there is no feasible solution).

The solution gives $j(i)$ for every $i \in I$, and the maximal chosen field of matrix $T[i,j]$ gives the value of the goal function.

Then we get the solution as follows:

Define sets $S^{(j)} \equiv \{i | j(i) = j\}, H^{(j)} \equiv \bigcap_{i \in S^{(j)}} V_{ij_0} \neq \emptyset$.

Then a vector $\tilde{x}$ defined as

$f_j(\tilde{x}_j) \equiv \min_{x_j \in H^{(j)}} f_j(x_j) \quad$ for $S^{(j)} \neq \emptyset$;

$f_j(\tilde{x}_j) \equiv \min_{0 \leq x_j \leq d_j} f_j(x_j) \quad$ for $S^{(j)} = \emptyset$;

is an optimal solution of the problem (P.I).

**Theorem.** *A solution found by the algorithm is feasible and optimal.*

**Proof.**

1. From a construction of the algorithm we can see that the found solution (S1) is feasible. In every row there is one chosen field, it means for every customer there is a service station able to meet his distance servicing condition.

2. Assume that there is a solution (S2) with a smaller value of the goal function.

The solution (S2) gives a pre-processing of the matrix $T$ and tells us which fields of the matrix are chosen and which are crossed. Because the solution (S2) has a smaller value of the goal function than the solution (S1) found by the algorithm, there is a row in the matrix $T$, where a value of a chosen field of (S2) is smaller than a value of chosen field of (S1), so it means that the algorithm tried at first to choose the field of (S2) but had no success, it means that solution (S2) is not feasible.

It is clear, that time complexity of the algorithm could be in the worst case exponential in size of an input and finding of subclasses of the problems P.I and P.II where the time complexity is "more reasonable" is a subject of author's today investigation. This generally supposes finding of (from practical point of view interpretable) conditions imposed on the system.

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