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Groupoids and the Associative Law V. (Szász–Hájek Groupoids of Type (A, A, B))

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This paper deals with groupoids possessing just one non-associative triple of elements. The triple is of the form (a, a, b).

Článek se zabývá grupoidy, které mají právě jednu neasociativní trojici prvků. Tato trojice je tvaru (a, a, b).

The present paper is a direct continuation of [2] and [3].

IV. 1 Basic arithmetic of SH-groupoids of type (a, a, b).

1.1 In this section, let G be an SH-groupoid of the type (a, a, b). Let $a, b \in G$ be such that $a \,.\, ab \neq a^2b$ and put $c = ab$, $d = ba$, $e = ac$, $f = a^2b$. Then $a \neq b$ and $e \neq f$.

1.2 Proposition. (i) *If $x, y \in G$ are such that $xy = a$ (resp. $xy = b$), then either $x = a$ (resp. $x = b$) or $y = a$ (resp. $y = b$).*
(ii) *If M is a generator set of G, then $\{a, b\} \subseteq M$.*
(iii) *If M is a subgroupoid of G, then either $\{a, b\} \subseteq H$ and H is an* SH-*subgroupoid of type* (a, a, b) *or* $\{a, b\} \nsubseteq H$ *and His a semigroup.*
(iv) *If r is a congruence of G, then either $(e, f) \notin r$ and G/r is an* SH-*groupoid of type* (a, a, b) *or $(e, f) \in r$ and G/r is a semigroup.*

Proof. See III.1.2.

1.3 Lemma. (i) $ad = ca$.
(ii) *If $b \neq d$, then $ea = fa$.*
(iii) *If $b \neq c$ and $a^2 \neq a$, then $ae = af$.*

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Proof. (i) $ad = a \,.\, ba = ab \,.\, a = ca$.
(ii) $ea = (a \,.\, ab)a = a(ab \,.\, a) = a(a \,.\, ba) = a^2 \,.\, ba = a^2b \,.\, a = fa$.
(iii) $ae = a(a \,.\, ab) = a^2 \,.\, ab = a^2a \,.\, b = aa^2 \,.\, b = a \,.\, a^2b = af$.

1.4 Lemma. (i) *If* $x \in G$, *then* $ax = a$ *iff* $xa = a$.
(ii) *If* $y \in G$ *such that* $a \neq y \neq b$, *then* $ay = a$ *iff* $yb = b$.

Proof. (i) It is obvious for $x = a$. For $x \neq a$ suppose that $ax = a$ and $xa \neq a$. Then $e = a \,.\, ab = ax \,.\, ab = a(x \,.\, ab) = a(xa \,.\, b) = (a \,.\, xa)b = (ax \,.\, a)b = a^2b = f$, a contradiction.
Similarly, if $x \neq a \neq ax$ and $xa = a$, then $e = a \,.\, ab = a(xa \,.\, b) = a^2b = f$, a contradiction.
(ii) Suppose that $a \neq y \neq b$, $ay = a$ and $yb \neq b$. Then $e = a \,.\, ab = a(ay \,.\, b) = a(a \,.\, yb) = a^2 \,.\, yb = (a^2y)b = (a \,.\, ay)b = a^2b = f$, a contradiction.
Similarly, if $ay \neq a$, $yb = b$, then $e = a \,.\, ab = a(a \,.\, yb) = a^2b = f$, a contradiction.

1.5 Lemma. *Suppose that* $a^2 = a$. *Then:*
(i) $a \neq c \neq b$, *and* $a \neq d$.
(ii) $c = f$ *and* $da = d$.
(iii) $ae = e = af$.
(iv) $ad = fa$.
(v) *If* $b \neq d$, *then* $ea = fa$.

Proof. (i) If $a = c$, then $e = a \,.\, ab = aa = a = ab = aa \,.\, b = f$, a contradiction. Thus $a \neq c$, and hence $a \neq d$ by 1.4(i). Further, if $b = c$, then $e = a \,.\, ab = ab = aa \,.\, b = f$, again a contradiction.
(ii) $c = ab = aa \,.\, b = f$ and $da = ba \,.\, a = b \,.\, aa = ba = d$.
(iii) $ae = a(a \,.\, ab) = a^2 \,.\, ab = a \,.\, ab = e$, since $b \neq ab$ by (i). Further, $af = ac = a \,.\, ab = e$.
(iv) $fa = ca = ad$ by (ii) and 1.3 (i).
(v) $ea = (a \,.\, ab)a = a(ab \,.\, a) = a(a \,.\, ba) = a^2 \,.\, ba = a \,.\, ba = ad$.

1.6 Lemma. (i) *If* $x \in G$ *such that* $x \neq a \neq xa$, *then* $xe = xf$.
(ii) *If* $x \in G$ *is such that* $x \neq b \neq bx$, *then* $ex = fx$.

Proof. (i) $xe = x(a \,.\, ab) = xa \,.\, ab = (xa \,.\, a)b = xa^2 \,.\, b = x \,.\, a^2b = xf$.
(ii) $ex = (a \,.\, ab)x = a(ab \,.\, x) = a(a \,.\, bx) = a^2 \,.\, bx = a^2b \,.\, x = fx$.

1.7 Lemma. (i) *If* $x \in G$ *is such that* $x \neq a = xa$, *then* $xe = e$ *and* $xf = f$.
(ii) *If* $x \in G$ *is such that* $x \neq b = bx$, $ex = e$ *and* $fx = f$.

Proof. (i) $xe = x(a \,.\, ab) = xa \,.\, ab = a \,.\, ab = e$ and $xf = x \,.\, a^2b = xa^2 \,.\, b = (xa \,.\, a)\, b = a^2b = f$.
(ii) $ex = (a \,.\, ab)x = a(ab \,.\, x) = a(a \,.\, bx) = a \,.\, ab = e$ and $fx = a^2b \,.\, x = a^2 \,.\, bx = a^2b = f$.

1.8 Lemma. *Suppose that either* $a = c$ *or* $a = d$. *Then:*
(i) $a = c = d$.
(ii) $a \neq a^2 = e$ *and* $a^2 \neq f$.
(iii) $ae = a^3 = af$.
(iv) $be = e$ *and* $bf = f$.
(v) $ea = a^3 = fa$.
(vi) $eb = f$, $e^2 = a^4$ *and* $ef = a^4b$.
(vii) $fb = a^2 . b^2$, $f^2 = a^4 . b$ *and* $fe = a^4$.
(viii) *If* $b^2 \neq b$, *then* $fb = e$.
(ix) *If* $b^2 = b$, *then* $fb = f$.

Proof. (i) It follows easily from 1.4(i).
(ii) By 1.5(i), $a \neq a^2$. But $e = a . ab = a^2$ trivially.
(iii) $ae = a . a^2 = a^3 = a^2 . a = a^2b . a = fa$.
(iv) $be = ba^2 = ba . a = a^2 = a$ and $bf = b . a^2b . ba^2 . b = (ba . a)b = a^2b = f$.
(v) $ea = a^2 . a = a^3 = a^2 . a = a^2ba = a^2b . a = fa$.
(vi) $eb = a^2b = f$, $e^2 = a^2a^2 = a^4$ and $ef = a^2 . a^2b = a^4b$.
(vii) $fb = a^2b . b = a^2b^2$, $f^2 = a^2b . a^2b = (a^2b . a^2)b = (a^2 . ba^2)b = (a^2(ba . a))b =$
$= a^2a^2 . b = a^4b$ and $fe = a^2b . a^2 = a^2 . ba^2 = a^4$.
(viii) $a^2b^2 = a . ab^2 = a(ab . b) = a . ab = a^2 = e$.
(ix) $a^2b^2 = a^2b = f$.

1.9 Lemma. *Suppose that* $c = a = d$ *(see 1.8). Then:*
(i) $b \neq e$.
(ii) *If* $b = f$, *then* $b^2 = b = a^2b^2 = a^4b$, $a = a^3$ *and* $e = a^4$.

Proof. (i) If $b = e$, then $e = be = b^2 = eb = f$ (1.8(iv), (vi)), a contradiction.
(ii) See 1.8.

1.10 Lemma. Suppose that $b = c$. Then:
(i) $b = c = e$ *and* $b \neq f$.
(ii) $a^2 \neq a \neq c$ *and* $a \neq d$.
(iii) $ad = d$ *and* $af = f$.
(iv) $bd = b^2a$ *and* $bf = b^2$.
(v) $da = ba^2$, $db = b^2 = df$ *and* $dd = b^2a$.
(vi) *If* $b \neq d$, *then* $fa = d$ *and* $ff = b^2$.
(vii) *If* $b \neq b^2$, *then* $fb = b^2$ *and* $fd = b^2a$.
(viii) *If* $b = d \neq b^2$, *then* $ff = b^2$.

Proof. (i) Obvious.
(ii) Since $b = c$, we have $a \neq c$, and hence $a \neq d$ by 1.8. Finally, $ab =$
$= b = e \neq f = a^2b$ yields $a \neq a^2$.
(iii) $ad = a . ba = ab . a = ba = d$ and $af = a . a^2b = a^3b = a^2a . b = a^2 . ab =$
$= a^2b = f$ (since $a \neq a^2$).

(iv) $bd = b . ba = b^2a$ and $bf = b . a^2b = ba^2 . b = (ba . b)b = ba . ab = ba . b = b . ab = b^2$ (we have $ba \neq a$ by (ii)).
(v) $da = ba . a = ba^2$, $db = ba . b = b . ab = b^2$, $df = ba . a^2b = (ba . a^2)b = ba^3 . b = (ba^2 . a)\, b = ba^2 . ab = (ba . a)b = ba . ab = ba . b = b . ab = b^2$ and $dd = ba . ba = (ba . b)a = b^2a$.
(vi) $fa = a^2b . a = a^2 . ba = a((a . ba) = a(ab . a) = a . ba = ab . a = ba = d$ and further $ff = a^2b . a^2b = (a^2b . a^2)b = ((a^2b . a)a)b = ((a^2 . ba)a)b = ((a(a . ba))a)b = ((a(ab . a))a)b = ((a . ba)a)b) = ((ab . a)a)b = (ba . a)b = ba . ab = ba . b = b . ab = b^2$ (we have $ba \neq a$).
(vii) $fb = a^2b . b = a^2b^2 = a(a . b^2) = a(ab . b) = ab^2 = ab . b = b^2$ and $fd = a^2b . ba = (a^2b . b)a = a^2b^2 . a = b^2a$.
(viii) $ff = a^2b . a^2b = (a^2b . a^2)b = (a^2 . ba^2)b = (a^2(ba . a))b = (a^2 . ba)b = a^2b . b = a^2b^2 = b^2$.

1.11 Lemma *Suppose that $b = c = d$ and $b \neq b^2$. Then:*
(i) $b^2 = b^2a$ *and* $b = ba^2$.
(ii) $fa = f$, $ff = b^2$, $fa^2 = f$ *and* $fb^2 = b^3$.
(iii) $f = a^2f$.

Proof. (i) $b^2a = b . ba = b^2$ and $ba^2 = ba . a = ba = b$.
(ii) $fa = a^2b . a = a^2 . ba = a^2b = f$, $ff = b^2$ by 1.10(ix), $fa^2 = a^2ba^2 = (a^2b . a)a = (a^2 . ba)a = a^2b . a = a^2 . ba = a^2b = f$ and $fb^2 = a^2b . b^2 = a^2b^3 = a^2b^2 . b = (a . ab^2)b = (a(ab . b))b = ab^2 . b = (ab . b)b = b^3$.
(iii) $a^2f = a . af = f$ by 1.10(iii).

1.12 Lemma. *Suppose that $b^2 = b = c$. Then:*
(i) $b = b^2 = c = d = e = ba^2$.
(ii) $bf = b = ba^2$.
(iii) $fa = fb = ff = fa^2 = f = a^2b$ *(and so $a \neq a^2$).*

Proof. (i) First, $a^2 \neq a \neq c, d$ by 1.10(ii). Now, if $b = ba^2$, then $b = bb = b . ab = ba . b = (ba^2 . a)b = ba^3 . b = (ba . a^2)b = ba . a^2b$.
Since $a^2b = f \neq e = b$, we must have $d = ba = b$ by 1.2(i).
Now, let $b \neq ba^2$. Then $e = a . ab = ab = b = bb = b . ab = ba . b = ba . ab = (ba . a)b = ba^2 . b = (ab . a^2)b = (a^2 . ba^2)b = (a^2b . a^2)b = a^2b . a^2b = a^2(b . a^2b) = a^2(ba^2 . b) = a^2((ba . a)b) = a^2(ba . ab) = a^2(ba . b) = a^2((ba . a)b) = a^2(ba . ab) = a^2(ba . b) = a^2(b . ab) = a^2b^2 = a^2b = f$, a contradiction.
(ii) $bf = b . a^2b = ba^2b = bb = b$ by (i).
(iii) $fa = a^2b . a = a^2 . ba = a^2b = f$, $fb = a^2b . b = a^2b^2 = a^2b = f$, $ff = a^2b . a^2b = (a^2b . a^2)b = (a^2 . ba^2)b = (a^2(ba . a))b = a^2b . b = a^2b^2 = a^2b = f$ and $fa^2 = a^2b . a^2 = a^2 . ba^2 = a^2(ba . a) = a^2b = f$.

1.13 Lemma. *Suppose that* $b = d \neq c$. *Then:*

(i) $a \neq c$.

(ii) $bc = be = bf = b^2$ *and* $ba^2 = b$.

(iii) $ca = ca^2 = c$.

(iv) $cb = cc = ce = cf = ab^2$.

(v) $ea = ea^2 = e$.

(vi) $eb = ec = ee = ef = a \cdot ab^2$.

(vii) $fa = fa^2 = f$.

(viii) $fb = fc = fe = ff = a^2b^2$.

Proof. (i) If $a = c$, then $a = d = b$ by a contradiction.

(ii) $bc = b \cdot ab = ba \cdot b = b^2$, $be = b(a \cdot ab) = ba \cdot ab = b \cdot ab = ba \cdot b = b^2$, $bf = b \cdot a^2b = ba^2 \cdot b = bb = b^2$, $ba^2 = ba \cdot a = b$.

(iii) $ca = ab \cdot a = a \cdot ba = ab = c$ and $ca^2 = ca \cdot a = c$.

(iv) $cb = ab \cdot b = ab^2$, $cc = ab \cdot ab = (ab \cdot a)b = ab \cdot b = ab^2$, $ce = (ab)(a \cdot ab) = (ab \cdot a) \cdot ab = (ab)^2 = (ab \cdot a)b = ab^2$, $cf = c \cdot a^2b = cb = ab^2$.

(v) $ea = (a \cdot ab)a = a(ab \cdot a) = a(a \cdot ba) = a \cdot ab = e$ and $ea^2 = ea \cdot a = e$.

(vi) $eb = ac \cdot b = a \cdot cb = a \cdot ab^2$, $ec = ac \cdot c = a \cdot c^2 = a \cdot ab^2$, $ee = e(a \cdot ab) = ea \cdot ab = e \cdot ab = ec = a \cdot ab^2$, $ef = (a \cdot ab)f = a(ab \cdot f)a = (a \cdot bf) = a \cdot ab^2$.

(vii) $fa = a^2b \cdot a = a^2 \cdot ba = a^2b = f$ and $fa^2 = fa \cdot a = f$.

(viii) $fb = a^2b \cdot b = a^2b^2$, $fc = a^2b \cdot c = a^2 \cdot bc = a^2b^2$, $fe = f(a \cdot ab) = fa \cdot ab = f \cdot ab = fc = a^2b^2$, $ff = a^2b \cdot f = a^2 \cdot bf = a^2b^2$.

1.14 Lemma. (i) $a \neq ca = ab \cdot a = a \cdot ba = ad$.

(ii) $a \neq e = a \cdot ab$.

Proof. We have $ab \cdot a = a \cdot ba$. If $a = a \cdot ba$, then $a = ba \cdot a = ba^2$ by 1.4(i). If $a = e$, then $a = ab \cdot a = a \cdot ba$. However, if $a = ba^2$, then $a = a^2$ by 1.2(i), and hence $a = ba = d$, a contradiction with 1.5(i).

1.15 Lemma. $a \neq f$.

Proof. Let $a = f = a^2b$. Then $a = a^2$ by 1.2(i), and hence $a = a^2b = ab = d$, a contradiction with 1.5(i).

1.16 Lemma. $a \notin \{b, b^2, e, f, aba\}$.

Proof. See 1.2(i), 1.14 and 1.15.

1.17 Lemma. *Let* $x \in G$ *and* $n \geq 2$ *and that* $x^n = a$. *Then* $x = a$ *and either* $a^2 = a$ *or* $a^2 \neq a$, $a^3 = a$ *and* $f = a^2b = b$.

Proof. By 1.2(i), $x = a$. Now, assume that n is the smallest integer with $n \geq 2$ and $a^n = a$. Using 1.2(i) again, we see that either $n = 2$ or $n = 3$. If $n = 3$, then $a \neq a^2$ and $b = a^2b$ by 1.4(ii).

1.18 Lemma. *Let $x \in G$ and $n \geq 2$ be such that $x^n = b$. Then $x = b$ and either $b^2 = b$ or $b^2 \neq b$, $b^3 = b$, $a = c = d$.*

Proof. Similar to that of 1.16 (use 1.4).

1.19 Lemma. (i) $b = c$ *iff* $b = e$.
(ii) *If $b = f$, then $a^3 = a$ and $b \neq c$.*

Proof. (i) If $b = e = a \,.\, ab$, then $ab = b$ by 1.2(i).
(ii) If $b = f = a^2b$, then either $a^2 = a$ (and hence $a^3 = a$) or $a^3 = a$ by 1.4(ii). If, moreover, $b = c$, then $e = a \,.\, ab = ab = c = b = f$, a contradiction.

1.20 Lemma. *If $b = b \,.\, ab(=ba \,.\, b)$, then $b = b^2 = d$.*

Proof. If $ba = a$, then $a = ab$ by 1.4(i). But $b = ba \,.\, b = ab$, a contradiction. Thus $ba \neq a$. If ba = b, then $b = ba \,.\, b = b^2$. Now, assume that $ba \neq b$. Then, by 1.4(ii), $a = a \,.\, ba$, a contradiction with 1.16.

1.21 Lemma. *Suppose that $b = b \,.\, a^2b(=ba^2 \,.\, b)$.*
(i) $b = d$ *(then $b = ba = ba^2 = b^2$).*
(ii) $b = f$ *(then $ba^2 = a^2$ and $a = c = d$).*

Proof. First, assume that $ba^2 = a$. Then $a = a^2$, $a = ba$, $a = ab$ and $ba^2 = ba \,.\, a = a^2 = a$, $b = ba^2 \,.\, b = ab = a$, a contradiction.

Now, let $ba^2 = b$. Then $ba \,.\, a = b$, $ba = b$, $ba^2 = ba = b$, $b = ba^2 \,.\, b = b^2$.

Finally, let $a \neq ba^2 \neq b$. Then, by 1.4(ii), $a = ba^2 \,.\, a = ba \,.\, a^2$. If $a = a^2$, then $a = ba = ba^2$ and $b = ba^2 \,.\, b = ab = a$, a contradiction. Thus $a \neq a^2$, and hence $a = ba = ab$ and $ba^2 = ba \,.\, a = a^2$.

1.22 Lemma. *Suppose that $b = f = b^3 \neq b^2$.* Then $b^2a^2 = a^2$.

Proof. We have $b^2a^2 \,.\, b = b^2 \,.\, a^2b = b^2 \,.\, f = b^3 = b$. If $b^2a^2 = b$, then $b^2a \,.\, a = b$, $b^2a = b$, $ba = b$ and $b^2a^2 = b^2a \,.\, a = (b \,.\, ba)a = b^2a = b \,.\, ba = b^2$, a contradiction. If $b^2a^2 = a$, then $a^2 = a$, $b^2a = a$, $b \,.\, ba = a$, $ba = a$. Finally, if $a \neq b^2a^2 \neq b$, then $a = b^2a^2 \,.\, a = b^2 \,.\, a^3$, $a = a^3$, $a = b^2a = b \,.\, ba$, $a = ba$ and $b^2a^2 = (b \,.\, ba)a = a^2$.

V.2 Minimal SH-groupoids of type (a, a, b)

2.1 In this section, let G be a minimal SH-groupoid of type (a, a, b). Let $a, b \in G$ be such that $a \,.\, ab \neq a^2b$ and put $c = ab$, $d = ba$, $e = a \,.\, ab$ and $f = a^2b$.

2.2 Lemma. *Suppose that $a \notin \{c, d, a^2, a^3\}$. Then $a \neq xy$ for all $x, y \in G$.*

Proof. Let, on the contrary $a = xy$ and let W denote an absolutely free groupoid over $\{u, v\}$. Then we have a projective homomorphism $\phi: W \to G$ such that $\phi(u) = a$ and $\phi(v) = b$.

Now, according to 1.2(i) we can consider a term $t \in W$ such that $l(t)$ is minimal with the respect to $a = a\phi(t)$ (or $a = \phi(t) a$ – see 1.4(i)). Clearly, $l(t) \geq 2$, and hence $t = rs$. Then $a = a \,.\, \phi(r)\, \phi(s)$. But $a \neq e = a \,.\, ab$, so that $(\phi(r), \phi(s)) \neq (a, b)$ and $a = a \,.\, \phi(r)\, \phi(s) = a\phi(r) \,.\, \phi(s)$. Due to the minimality of t, we have $a \neq a\phi(r)$, and therefore $a = \phi(s)$ and $a = a\phi(r) \,.\, a = a \,.\, a\phi(r) = a^2\phi(r)$ (again, $\phi(r) \neq b$ and we can use 1.41(i)). Since $a \neq a^2$, we must have $a = \phi(r)$, and hence $a = a\phi(r) \,.\, \phi(s) = aa \,.\, a = a^3$, a contradiction.

2.3 Lemma. *Suppose that* $a = c = d$ (see 1.8). If $x, y = G$ *are such that* $xy = a$, *then* $(x, y) \in \{(a, a^2), (a^2, a), (a, b^n), (b^n, a), n \geq 1\}$.

Proof. We sall proceed similarly as in the proof of 2.2.

Let $t \in W$ be such that $l(t)$ is minimal with respect to $\phi(t) \notin \{a^2, b^n, n \geq 1\}$ and $a = a\phi(t)$. Since $a \neq a^2$ by 1.8(ii), we have $t = rs$ and $a = a \,.\, \phi(r)\, \phi(s) = a\phi(r) \,.\, \phi(s)$.

First, assume that $a = a\phi(r)$ and $\phi(r) = b^n$. Then $a = a\phi(s)$ and either $\phi(s) = b^m$ and $\phi(t) = b^{n+m}$, a contradiction, or $\phi(s) = a^2$ and $\phi(t) = b^n a^2 = b^n a \,.\, a = (b^{n-1} \,.\, ba)\, a = b^{n-1} a \,.\, a = \dots = a^2$, a contradiction.

Next, let $a = a\phi(r)$ and $\phi(r) = a^2$. Then $a = a^3$ and $a = a\phi(r) \,.\, \phi(s) = a\phi(s)$. If $\phi(s) = a^2$, then $\phi(t) = a^4 = a^3 \,.\, a = a^2$, a contradiction. Thus $\phi(s) = b^n$ and $\phi(t) = a^2 b^n = a^2$, again a contradiction.

Finally, let $\phi(s) = a$. Then $a = a\phi(r) \,.\, a = a \,.\, a\phi(r) = a^2\phi(r)$, $\phi(r) = a$ and $\phi(t) = \phi(r)\, \phi(s) = a^2$, a contradiction.

2.4 Lemma. *Suppose that* $a = a^2$. *Then* $xy \neq a$ *for all* $x, y \neq G$, $(x, y) \neq (a, a)$.

Proof. We can proceed similarly as in the proof of 2.2 (take $t \in W$ minimal with respect to $\phi(t) \neq a$ and $a = a\phi(t)$).

2.5 Lemma. *Suppose that* $c \neq a \neq a^2$ *and* $a = a^3$. *Then* $xy \neq a$ *for all* $x, y \in G$, $(x, y) \notin \{(a, a^2), (a^2, a)\}$.

Proof. We can proceed similarly as in the proof of 2.2.

2.6 Proposition. *Let* $x, y \in G$ *be such that* $xy = a$. *Then just one of the following cases takes place:*

(i) $a = c = d$ *and* $(x, y) \in \{(a, a^2), (a^2, a), (a, b^n), (b^n, a), n \geq 1\}$.

(ii) $a = a^2$ *and* $(x, y) = (a, a)$.

(iii) $c \neq a \neq a^2$, $a = a^3$ *and* $(x, y) \in \{(a, a^2), (a^2, a)\}$.

Proof. Combine 2.2, 2.3, 2.4 and 2.5.

2.7 Lemma. *Suppose that* $b \notin \{c, b, b^2, b^3\}$. *Then* $xb \notin b$ *for every* $x \in G$.

Proof. We shall proceed similarly as in the proof of 2.2.

Let $t \in W$ be such that $\mathrm{l}(t)$ is minimal respect to $b = \phi(t)\, b$. Then $t = rs$ and $b = \phi(r)\, \phi(s)\, .\, b$.

Further by 1.4(ii), $a = \phi(r)\, \phi(s)\, .\, a = \phi(r)\, .\, \phi(s)\, a$. If $\phi(r) = a = \phi(s)$, then $b = a^2 b = f$, a contradiction. If $\phi(r) = a \neq \phi(s)$, then $b = a\, .\, \phi(s)\, b$ and $b = \phi(s)\, b$, again a contradiction. If $\phi(r) \neq a$, then $\phi(s)\, a = a$ and $a = \phi(r)\, .\, \phi(s)\, a = \phi(r)\, a$. Since $\phi(r) \neq a$ and $\phi(r)\, b \neq b$, we have $\phi(r) = b$ by 1.4.

Now, $a = c = d$, $a = \phi(s)\, a$, and hence $\phi(s) \in \{a^2, b^n, n \geq 1\}$ by 2.3. If $\phi(s) = a^2$, then $b = ba^2\, .\, b = (ba\, .\, a)\, b = a^2 b = f$, a contradiction. If $\phi(s) = b^n$, then $b = b^{n+2}$, and hence either $b = b^2$ or $b = b^3$ (by 1.18), the final contradiction.

2.8 Lemma. *Suppose that* $b = c \neq b^2$. *If* $x \in G$ *is such that* $xb = b$, *then* $x = a$.

Proof. We have $b = c = e$, and hence $b \neq f$. Furter, $b \neq b^3$ by 1.18.

Now, let $t \in W$ be such that $\mathrm{l}(t)$ is minimal with respect to $\phi(t) \neq a$ and $b = \phi(t)\, b$. Then $t = rs$ and $b = \phi(r)\, \phi(s)\, .\, b = \phi(r)\, .\, \varphi(s)\, b$ (since $b \neq f$). If $\phi(s)\, b = b$, then $\phi(s) = a$, $b = \phi(r)\, .\, ab = \phi(r)\, b$, $\phi(r) = a$ and $\phi(t) = a^2$, $b = a^2 b = f$, a contradiction. Thus $\phi(s)\, b \neq b$, and hence $\phi(s) \neq a$ and $\phi(r) = b$.

Now, $b = b\phi(s)\, .\, b$ and $b\phi(s) \neq a, b$. By 1.4(ii), $a = a\, .\, b\phi(s) = ab\, .\, \phi(s) = b\phi(s)$, a contradiction.

2.9 Lemma. *Suppose that* $b \notin \{b^2, b^3\}$ *and* $b = f$. *If* $x \in G$ *is such that* $xb = b$, *then* $x = a^2$.

Proof. We shall proceed similarly as in the proof of 2.8 (if $b = f$, then $b \neq c$).

Let $t \in W$ be such that $\mathrm{l}(t)$ is minimal with respect to $\phi(t) \neq a^2$ and $b = \phi(t) b$. Then $t = rs$, $b = \phi(r)\, \phi(s)\, .\, b$, $(\phi(r), \phi(s)) \neq (a, b)$ and $b = \phi(r)\, .\, \phi s)\, b$. If $\phi(s)\, b = b$, then $\phi(s) = a^2$, $b = \phi(r)\, .\, \phi(s)\, b = \phi(r)\, b$, $\phi(r) = a^2$ and $\phi(t) = \phi(r)\, \phi(s) = a^4 = a^3\, .\, a = a^2$ (by 1.19(ii)), a contradiction. Thus $\phi(s)\, b \neq b$, and hence $\phi(r) = b$. Now, $b = b\phi(s)\, .\, b$.

If $b\phi(s) = a$, then $b = ba\, .\, b$, a contradiction with 1.20. If $b\phi(s) = b$, then $b = b^2$, again a contradiction. Thus $b\phi(s) \neq a, b$, and hence $a = b\phi(s)\, .\, a = b\, .\, \phi(s)\, a$ and $a = a\, .\, b\phi(s) = ab\, .\, \phi(s)$ by 1.4(ii). Now, by 1.2(i) and 1.4(i), $\phi(s) = a = a\phi(s)$. Clearly, $\phi(s) \neq a, b$ (by 1.20 and 1.18) and consequently $b = \phi(s)\, b$ by 1.4(ii). It follows that $\phi(s) = a^2$ and we have $b = ba^2\, .\, b = b\, .\, a^2 b = bb$, a contradiction.

2.10 Lemma. *Suppose that* $b \notin \{c, f\}$ *and* $b = b^2$. *If* $x \in G$ *is such that* $b = xb$, *then* $x = b$.

Proof. Let $t \in W$ be such that $l(t)$ is minimal with respect to $\phi(t) \neq b$ and $b = \phi(t)\, b$. Then $t = rs$, $b = \phi(r)\,\phi(s)\,.\,b = \phi(r)\,.\,\phi(s)\,b$. Then (since $\phi(t) \neq b = b^2$), $\phi(r) = b$ and $b = b\phi(s)\,b$.

Clearly, $\phi(s) \neq b \neq b\phi(s)$ and $b\phi(s) \neq a$. Then $a = b\phi(s)\,.\,a = b\,.\,\phi(s)\,a = ba$, $\phi(s)\,a = a$. Since $b = b\phi(s)\,.\,b$ and $a = ba$, we must have $\phi(s) \neq a$, and hence $b = \phi(s)\,b$. Thus $\phi(s) = b$, a contradiction.

2.11 Lemma. *Suppose that* $b \notin \{c, f, b^2\}$ *and* $b = b^3$. *If* $x \in G$ *is such that* $b = xb$, *then* $x = b^2$.

Proof. We can proceed similarly as in the proof of 2.10.

2.12 Lemma. *Suppose that* $b = c = b^2$. *If* $x \in G$ *is such that* $b = xb$, *then* $x \in \{a, b\}$.

Proof. We can proceed similarly as in the proof of 2.10.

2.13 Lemma. *Suppose that* $b = f = b^2$. *If* $x \in G$ *is such that* $b = xb$, *then* $x \in \{a^2, b\}$.

Proof. We can proceed similarly as in the proof of 2.10.

2.14 Lemma. *Suppose that* $b = f = b^3 \neq b^2$. *If* $x \in G$ *is such that* $b = xb$, *then* $x \in \{a^2, b^2\}$.

Proof. We can proceed similarly as in the proof of 2.10 (use 1.22).

2.15 Proposition. *Let* $x \in G$ *be such that* $xb = b$. *Then just one of the following cases takes place:*

(i) $b = c \neq b^2$ *and* $x = a$.
(ii) $b = c = b^2$ *and* $x \in \{a, b\}$.
(iii) $b = f \notin \{b^2, b^3\}$ *and* $x = a^2$.
(iv) $b = b^2 \notin \{c, f\}$ *and* $x = b$.
(v) $b = b^3 \notin \{c, f, b^2\}$ *and* $x = b^2$.
(vi) $b = f = b^2$ *and* $x \in \{a^2 b\}$.
(vii) $b = f = b^3 \neq b^2$ *and* $x \in \{a^2, b^2\}$.

Proof. See 2.7, ..., 2.14.

2.16 Lemma. *Suppose that* $b \notin \{c, d, b^2, b^3, f\}$. *Then* $b \neq xy$ *for all* $x, y \in G$.

Proof. Let, on the contrary, $b = xy$. By 2.7, $x = b \neq y$. Now, let $t \in W$ be such that $l(t)$ is minimal with respect to $b = b\phi(t)$. Then $t = rs$, $b = b\,.\,\phi(r)\,\phi(s) = b\phi(r)\,.\,\phi(s)$. Since $b\phi(r) \neq b$, we have $\phi(s) = b$ and $b = b\phi(r)\,.\,b$, a contradiction with 2.7.

2.17 Lemma. *Suppose that* $b = c \notin \{d, b^2\}$.*If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) = (a, b)$.

Proof. Similar to that of 2.16 (use 2.8).

2.18 Lemma. *Suppose that* $b = d \notin \{c, b^2, b^3, f\}$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) \in \{(b, a^n\}; n \geq 1\}$.

Proof. Similar to that of 2.16 (use 2.7).

2.19 Lemma. *Suppose that* $b = b^2 \notin \{c, d, f\}$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) = (b, b)$.

Proof. Similar to that of 2.16 (use 2.10).

2.20 Lemma. *Suppose that* $b = b^3 \notin \{f, b^2\}$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) \in \{(b, b^2), \{b^2, b)\}$.

Proof. We have $b \neq c, d$. Similar to that of 2.16 (use 2.11; if $\phi(s) = b$ and $b\phi(r) = b^2$, then $b = b^3 = b \,.\, b\phi(r) = b^2\phi(r)$, $\phi(r) = b$ and $\phi(t) = \phi(r)\,\phi(s) = b^2$, a contradiction).

2.21 Lemma. *Suppose that* $b = f \notin \{d, b^2, b^3\}$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) = (a^2, b)$.

Proof. Similar to that of 2.16 (use 2.9, 2.6(i) and 2.18; if $\phi(s) = b$ and $b\phi(r) = a^2$, then $a = a^3 = b\phi(r) \,.\, a = b \,.\, \phi(r)\, a$, $a = \phi(r)\, a = c = d$, $\phi(r) = a^2$, $\phi(t) = \phi(r)\,\phi(s) = a^2 b = b$ and $b = b^2$, a contradiction).

2.22 Lemma. *Suppose that* $b = c = d \notin \{b^2, b^3\}$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) \in \{(a, b), (b, a^n); n \geq 1\}$.

Proof. Similar to that of 2.16 (use 2.8).

2.23 Lemma. *Suppose that* $b = c = b^2$. *If* $x, y \in G$ *are such that* $xy = b$, *then* $(x, y) \in \{(a, b), (b, b), (b, f), (b, a^n), n \geq 1\}$.

Proof. By 1.12, $b = c = d = e = b^2 \neq f$. Further, $a \neq a^2$, $af = f = bf = fb = ff$. Now, we can proceed similarly as in the proof of 2.16.

2.24 Lemma. *Suppose that* $b = d = b^2 \notin \{c, f\}$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) \in \{(b, b), (b, e), (b, a^n), (b, a^n b), n \geq 1\}$.

Proof. Similar to that of 2.16 (use 2.10).

2.25 Lemma. *Suppose that* $b = d = f \neq b^2$. *If* $x, y \in G$ *are such that* $b = xy$ *then* $(x, y) \in \{(a^2, b), (b, a^n), n \geq 1\}$.

Proof. Similar to that of 2.16 ($b = f$ implies $a^3 = a$ and if $b = ba \,.\, b$, then $ba = a^2$ by 2.9, and hence $a = bu \,.\, a = b \,.\, ua$, $ua = au = a$, $a = ba = d = b$, a contradiction).

2.26 Lemma. *Suppose that* $b = f = b^2$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) \in \{(b, b), (a^2, b)\}$.

Proof. We have $b \neq c, d$. Now, using 2.13, we can proceed similarly as in the proof of 2.16.

2.27 Lemma. *Suppose that* $b = f = b^3 \neq b^2$. *If* $x, y \in G$ *are such that* $b = xy$, *then* $(x, y) \in \{(b, b^2), (b^2, b), (a^2, b)\}$.

Proof. Similar to that of 2.16 (if $b\phi(r) = b^2$, then $b = b^3 = b \,.\, b\phi(r) = = b^2\phi(r)$, $\phi(r) = b$ and $\phi(t) = \phi(r)\,\phi(s) = b^2$, a contradiction).

2.28 Proposition. *Let* $x, y \in G$ *be such that* $xy = b$. *Then just one of the following cases takes places:*

(i) $b = c \notin \{d, b^2\}$ *and* $(x, y) = (a, b)$.
(ii) $b = c = d \notin \{b^2, b^3\}$ *and* $(x, y) \in \{(a, b), (b, a^n), n \geq 1\}$.
(iii) $b = c = b^2$ *and* $(x, y) \in \{(a, b), (b, b), (b, f), (b, a^n), n \geq 1\}$.
(iv) $b = d \notin \{c, f, b^2, b^3\}$ *and* $(x, y) \in \{(b, a^n), n \geq 1\}$.
(v) $b = d = b^2 \notin \{c, f\}$ *and* $(x, y) \in \{(b, b), (b, e), (b, a^n), n \geq 1\}$.
(vi) $b = d = f \neq b^2$ *and* $(x, y) \in \{(a^2, b), (b, a^n), n \geq 1\}$.
(vii) $b = f \notin \{d, b^2, b^3\}$ *and* $(x, y) = (a^2, b)$.
(viii) $b = f = b^2$ *and* $(x, y) \in \{(b, b), (a^2, b)\}$.
(ix) $b = b^2 \notin \{c, d, f\}$ *and* $(x, y) = (b, b)$.
(x) $b = b^3 \notin \{f, b^2\}$ *and* $(x, y) \in \{(b, b^2), (b^2, b)\}$.

Proof. Combine 2.16, ..., 2.27.

2.29 In the sequel, we shall say that G is of subtype

(α) if $a = c$ and $b = f$;
(β) if $a = c$, $f = a^3$ and $b = b^2$;
(γ) if $a = c$, $a^3 \neq f$ and $f \neq b = b^2$;
(δ) if $a = c$ and $f \neq b \neq b^2$;
(ε) if $a = a^2$ and $d = b = b^2$;
(φ) if $a = a^2$ and $d = b \neq b^2$;
(ψ) if $a = a^2$, $b \neq d$ and $b = b^2$;
(ϱ) if $a = a^2$ and $d \neq b \neq b^2$;
(η) if $c \neq a \neq a^2$, $a = a^3$ and $b = b^3$;
(μ) if $c \neq a \neq a^2$, $a = a^3$ and $b \neq b^2$;
(ν) if $c \neq a \neq a^2$, $a \neq a^3$ and $b = b^2$;
(λ) if $c \neq aba^2$, $a \neq a^3$ and $b \neq b^2$.

Using the preceeding results, one can show easily that G is just one of the preceeding twelve subtypes (α), (β), ..., (λ).

V.3 Minimal SH-groupoids of type (a, a, b) and subtype (α)

3.1 Consider the following three-element groupoid $T_1(\circ)$:

$T_1(\circ)$	a	b	e
a	e	a	a
b	a	b	e
e	a	b	e

It is easy to check that $T_1(\circ)$ is a minimal SH-groupoid of type (a, a, b) and subtype (α). Clearly, $\text{sdist}\,(T_1(\circ)) = 1$ (put $a * a = a$).

3.2 Proposition. $T_1(\circ)$ *is (up to isomorphism) the only minimal* SH-*groupoid of type* (a, a, b) *and subtype* (α).

Proof. Let G a minimal SH-groupoid of type (a, a, b) and subtype (α) (see 1.8). Then $a = c = d$, $a \neq a^2 = e$, $b = f$, $b^2bf = b \,.\, a^2b = ba^2 \,.\, b = (ba \,.\, a)\, b = a^2b = f = b$ and $a = ab = af = a \,.\, a^2b = a^3b = a^2 \,.\, ab = a^3$. The rest is clear.

V.4 Minimal SH-groupoids of type (a, a, b) and subtype (β)

4.1 Consider the following four-element groupoid $T_2(\circ)$:

$T_2(\circ)$	a	b	e	f
a	e	a	f	f
b	a	b	e	f
e	f	f	f	f
f	f	f	f	f

Then $T_2(\circ)$ is (up to isomorphism) the only minimal SH-groupoid of type (a, a, b) and subtype (β).

V.5 Minimal SH-groupoid of type (a, a, b) and subtype (γ)

5.1 Let G be a minimal SH-groupoid of type (a, a, b) and subtype (γ). Then the elements $a, b, f, a^2 = e, a^2 = e, a^2$ are pair-wise different, and hence G contains at least five elements (if $a^2 = a^3$, then $f = a^2b = a^3b = a^3$, a contradiction). Further, $a \neq a^n \neq b$ for every $n \geq 1$ (see 1.2(i) and 1.17).

If $f = a^n$ for some n, then $n \geq 4$, $a^3 = a^{n+1}$, $a^4 = a^{n+2}$, ..., $a^{n+1} = a^{2n-3}$, $a^n = f$ and we see that G is finite.

If $a^2 = a^n$ for some $n \neq 2$, then $n \geq 4$ and $f = a^2b = a^nb = a^{n-1} . ab = = a^{n-1}a = a^n = a^2$, a contradiction.

5.2 Example. Consider the following infinite groupoid $T_3(\circ)$:

$T_3(\circ)$	a	b	f	a^2	a^3	$\dots$	a^n	$\dots$
a	a^2	a	a^3	a^3	a^4	$\dots$	a^{n+1}	$\dots$
b	a	b	f	a^2	a^3	$\dots$	a^n	$\dots$
f	a^3	f	a^4	a^4	a^5	$\dots$	a^{n+2}	$\dots$
a^2	a^3	f	a^4	a^4	a^5	$\dots$	a^{n+2}	$\dots$
a^3	a^4	a^3	a^5	a^5	a^6	$\dots$	a^{n+3}	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^n	a^{n+1}	a^n	a^{n+2}	a^{n+2}	a^{n+3}	$\dots$	a^{2n}	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then $T_3(\circ)$ is (up to isomorphism) the only infinite minimal SH-groupoid of type (a, a, b) and subtype (γ).

V.6 Minimal SH-groupoids of type (a, a, b) and subtypes (ε), (ϕ), (ψ)

6.1 The following groupoid $T_4(\circ)$ is (up to isomorphism) the only minimal S-groupoid of type (a, a, b) and subtype (ε):

$T_4(\circ)$	a	b	c	e
a	a	c	e	e
b	b	b	b	b
c	c	c	c	c
e	e	e	e	e

6.2 Example. The following groupoid $T_5(\circ)$ is a minimal SH-groupoid of type (a, a, b) and subtype (ϕ):

$T_5(\circ)$	a	b	c	e	g
a	a	c	e	e	g
b	b	g	g	g	g
c	c	g	g	g	g
e	e	g	g	g	g
g	g	g	g	g	g

6.3 Exemple. The following groupoid $T_6(\circ)$ is a minimal SH-groupoid of type (a, a, b) and subtype (ψ):

$T_6(\circ)$	a	b	c	d
a	a	c	d	d
b	d	b	d	d
c	d	c	d	d
d	d	d	d	d

V.7 Comnments and open problems

7.1 The methods developed in the preeceding part IV are used here to obtain a description of several minimal SH-groupoids of type (a, a, b). Among others, some results from [1] are reformulated.

7.2 Continue the description of minimal SH-Groupoids of type (a, a, b) and find their semigroup distances.

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