# On Midpoints in Lattices

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This paper introduces the notion of a midpoint in a lattice $L$ having the zero element and the notion of a centred ideal in $L$. It is shown that the study of centred ideals in sectionally complemented lattices is facilitated by a congruence $\theta(I)$. The congruence $\theta(I)$ is compared with some other known congruences. A complete characterization of centred ideals in such lattices is given.

## 1. Introduction

Our aim in this note is to study midpoints in lattices. We try to get some insight as to the nature of centred ideals and we find a full description of these ideals in the class of all sectionally complemented lattices. As a byproduct we establish also a distributivity criterion in sectionally complemented lattices by means of midpoints.

## 2. Preliminaries

Henceforth, $L$ will denote an arbitrary lattice having the least element 0. If $a, b, s \in L$ are such that

$$s = a \vee b \ \& \ a \wedge b = 0\,,$$

we write $s = a \oplus b$.

If $I$ is an ideal of $L$ and if $a, b, c \in L$ are such that

$$c \leq a \oplus b \ \& \ (c \wedge a) \vee (c \wedge b) \in I\,,$$

then we will write $c \in (a \oplus_I b)/2$ and we will say that $c$ is a *midpoint with respect to the ideal* $I$ in $L$. The set of all midpoints with respect to $I$ in $L$ will be denoted by $(L \oplus_I L)/2$.

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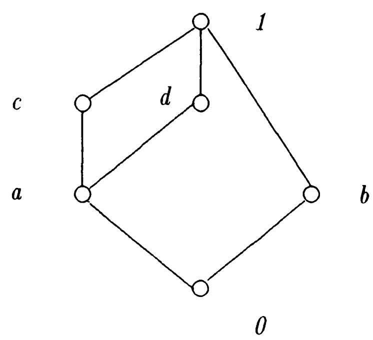
Istead of $(a \oplus_{(0]} b)/2$ or $(L \oplus_{(0]} L)/2$ we will write just $(a \oplus b)/2$ or $(L \oplus L)/2$. Similarly, an element $c$ of $L$ such that $c \in (a \oplus b)/2$ for some $a, b \in L$ is said to be a *midpoint* of $L$.

An ideal $I$ od $L$ is called a *centred ideal* if every midpoint with respect to $I$ in $L$ belongs to $I$, i.e. if

$$(L \oplus_I L)/2 \subset I .$$

Following Rav [8], we define an ideal $I$ of $L$ to be a *semiprime* ideal if for every $a, b, c \in L$, from $a \wedge b \in I$ and $a \wedge c \in I$ it follows necessarily that $a \wedge (b \vee c) \in I$. Note that characterizations of semiprime ideals are already given as well as in general lattices (see e.g. [3] and [4]) as in orthomodular lattices [5]. It is easy to see that any semiprime ideal of $L$ is a centred ideal.

The ideal (0] of the lattice shown in Figure 1 is a centred ideal but it is not a semiprime ideal.




The set of all relative complements of $a \in L$ in the interval $[0, b]$ will be denoted by $C_{[0, b]}(a)$.

If $I$ is an ideal of $L$ and $a \in L$, then we will denote by $a_I^*$ the set of all $b \in L$ such that $a \wedge b \in I$.

For all other notation and terminology, we refer the reader to [6] or [1].

### 3. Ideals in sectionally complemented lattices

Let $I$ be an ideal of $L$ and let $\theta(I)$ be a relation defined on $L$ in such a way that $(a, b) \in \theta(I)$ if and only if $I$ contains every element $c \in C_{[0, a \vee b]}(a \wedge b)$. Clearly, in the even that $a \leq b$, $(a, b) \in \theta(I)$ if and only if any $c \in L$ satisfying

$$c \vee a = b \ \& \ c \wedge a = 0$$

belongs to $I$.

**Theorem 1.** *If $L$ is a sectionally complemented lattice and if $I$ is a centred ideal, then $\theta(I)$ is a congruence relation of $L$.*

**Proof.** It will be useful to prove first the following statement:

(A) *If $a, b, c \in L$ are such that*

$$a \leq c \leq b \ \&\ (a, b) \in \theta(I),$$

*then $(a, c) \in \theta(I)$ and $(c, b) \in \theta(I)$.*

Indeed, let

$$a^+ \in C_{[0,b]}(a), \hat{a} \in C_{[0,c]}(a).$$

Then $a^+ \in I$ and so $\hat{a} \in (a \oplus_I a^+)/2$. Since $I$ is a centred ideal, $\hat{a} \in I$. Hence $(a, c) \in \theta(I)$.

Let $c^+ \in C_{[0,b]}(c)$. From $a^+ \in I$ we have $c^+ \in (a \oplus_I a^+)/2$ which yields $c^+ \in I$. Therefore, $(c, b) \in \theta(I)$.

Now it remains to show that the Grätzer – Schmidt's Theorem [7] can be applied.

First, suppose

$$a \leq b \leq c \ \&\ (a, b) \in \theta(I) \ \&\ (b, c) \in \theta(I).$$

Let

$$a^* \in C_{[0,c]}(a),\ a^+ \in C_{[0,b]}(a),\ b^+ \in C_{[0,c]}(b).$$

By hypothesis, $a^+, b^+ \in I$. From (A) and $(a, b) \in \theta(I)$ we conclude that $(a, a \vee (a^* \wedge b)) \in \theta(I)$. Since $a^* \wedge b \in C_{[0, a \vee (a^* \wedge b)]}(a)$, we have $a^* \wedge b \in I$. From $b^+ \in I$ it follows that $a^* \wedge b^+ \in I$, and, consequently, $a^* \in (b \oplus_I b^+)/2$ and so we obtain $a^* \in I$. Hence $(a, c) \in \theta(I)$.

Next, suppose

$$a \leq b \ \&\ (a, b) \in \theta(I)$$

and let $c$ be an element of $L$.

Let $u^+ \in C_{[0, b \wedge c]}(a \wedge c)$.

It follows from (A) that $(a, a \vee (b \wedge c)) \in \theta(I)$. Now we see that $u^+ \in C_{[0, a \vee (b \wedge c)]}(a)$, so that $u^+ \in I$ which implies $(a \wedge c, b \wedge c) \in \theta(I)$.

Suppose now again that $a \leq b, (a, b) \in \theta(I)$ and $c \in L$. Let

$$w = a \vee c,\ t = b \wedge (a \vee c),\ w^+ \in C_{[0, b \vee c]}(a \vee c),\ t^+ \in C_{[0,b]}(t).$$

It follows from (A) that $(t, b) \in \theta(I)$. Thus $t^+ \in I$ and

$$w^+ \wedge t^+ \in I \ \&\ w^+ \wedge w = 0 \in I.$$

Observe that

$$w \vee t^+ = b \vee c \ \&\ w \wedge t^+ = 0.$$

Therefore $w^+ \in (w \oplus_I t^+)/2$. Consequently, $w^+ \in I$. We thus have $(a \vee c, b \vee c) \in \theta(I)$ which accomplishes the desired results.

Let $I$ be an ideal of $L$ and let $\Phi(I)$ be defined on $L$ by

$$(a, b) \in \Phi(I) \Leftrightarrow (a \wedge b)_I^* = (a \vee b)_I^*.$$

**Corollary 2.** *Let $I$ be a centred ideal of a sectionally complemented lattice. Then*

$$\theta(I) = \Phi(I).$$

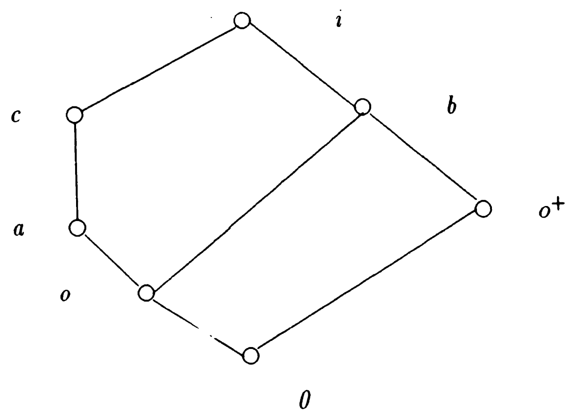
**Proof.** Suppose $(a, b) \in \Phi(I)$ and let $c \in C_{[0, a \vee b]}(a \wedge b)$. Then $c \in (a \wedge b)_I^* = (a \vee b)_I^*$, and so $c \wedge (a \vee b) \in I$. From $c \leq a \vee b$ it follows that $c \in I$. Therefore $(a \wedge b, a \vee b) \in \theta(I)$. Since $\theta(I)$ is a congruence relation, $(a, b) \in \theta(I)$. Thus $\Phi(I) \subset \theta(I)$.

Suppose now that $(a, b) \in \theta(I)$. Observe that $(a \vee b)_I^* \subset (a \wedge b)_I^*$ is always true. If $y \in (a \wedge b)_I^*$, then put $d = y \wedge a \wedge b \in I$ and let $e = (a \vee b) \wedge y$. Since $\theta(I)$ is a congruence relation, $(a \wedge b, (a \wedge b) \vee e) \in \theta(I)$. But the quotients $(a \wedge b) \vee e/a \wedge b$ and $e/d$ are transposed. Therefore $(d, e) \in \theta(I)$.

Let $d^+ \in C_{[0, e]}(d)$. Since $(d, e) \in \theta(I)$, $d^+ \in I$. Now it is evident that $e = y \wedge \wedge (a \vee b) = d \vee d^+ \in I$. Hence $\theta(I) \subset \Phi(I)$ and this, in turn, gives $\theta(I) = \Phi(I)$.

**Lemma 3.** *If $L$ is a sectionally complemented lattice which contains a sublatice isomorphic to the pentagon $N_5$, then there exists a sublattice $N_5'$ of $L$ which is isomorphic to $N_5$ and is such that the zero element $0$ of $L$ belongs to $N_5'$.*

**Proof.** Suppose $N_5 = \{o, a, b, c, i\}$ is a sublattice of $L$. (See Fig. 2.)

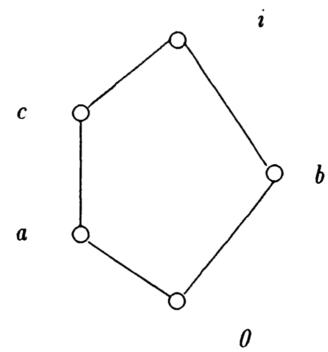



Let $o^+ \in C_{[0, b]}(o)$. The proof is completed by noting that it is sufficient to put $N_5' = \{0, o^+, a, c, i\}$.

**Lemma 4.** *If $L$ is a sectionally complemented lattice which contains the pentagon $N_5$ as a sublattice where the zero of $L$ belongs to $N_5$, then there exists a nonzero midpoint of $L$, i.e.*

$$(L \oplus L)/2 \neq \{0\}.$$

**Proof.** Consider the sublattice $N_5 = \{0, a, b, c, i\}$ as depicted in Figure 3.




Let

$$d \in C_{[0,c]}(a), \;\; e \in C_{[0,i]}(d), \;\; p = a \wedge e, \;\; f \in C_{[0,a]}(p)\,.$$

We will distinguish two cases.

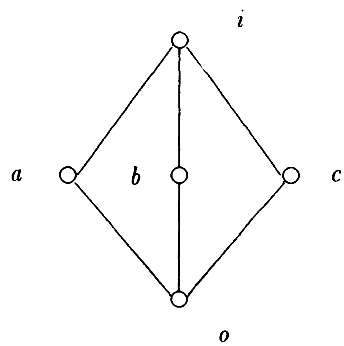
*Case I:* $f \neq 0$. Then $f \in (d \oplus e)/2$.

*Case II:* $f = 0$. Then $a = p = a \wedge e \leq e$. Let $q = b \wedge e$ and let $g \in C_{[0,b]}(q)$. It follows readily that $g \in (d \oplus e)/2$.

Suppose that $g \neq 0$ does not hold. Then $b = b \wedge e \leq e$ and $e \geq a \vee b = i$, i.e. $e = i$. However, this leads to a contradiction, because we could then write $0 = d \wedge e = d \wedge i = d$, obtaining $a = c$.

**Lemma 5.** *If $L$ is a sectionally complemented lattice which contains as a sublattice the diamond $M_5$, then there exists a nonzero midpoint of $L$.*

**Proof.** Consider the sublattice $M_5 = \{o, a, b, c, i\}$ shown in Figure 4.




Let

$$d \in C_{[0,b]}(o), \;\; e \in C_{[0,c]}(o)\,.$$

Then $e \in (a \oplus d)/2$. Since $o \neq c$, $e = 0$ is impossible.

**Theorem 6.** *Let L be a sectionally complemented lattice. Then L is distributive if and only if*

$$(L \oplus L)/2 = \{0\}.$$

**Proof.** If $L$ is not distributive, then $(L \oplus L)/2 \neq \{0\}$ by Lemmas 3, 4 and 5.

If $L$ is distributive and $m \in (a \oplus b)/2$, then $m \leq a \vee b$ and, by distributivity,

$$m = m \wedge (a \vee b) = (m \wedge a) \vee (m \wedge b) = 0\,.$$

**Lemma 7.** *Let L be a sectionally complemented lattice and let I be a centred ideal of L.*

*Let $a/\theta(I)$, $b/\theta(I)$ and $c/\theta(I)$ be elements of the quotient lattice $L/\theta(I)$ satisfying*

(1) $$a/\theta(I) \in (b/\theta(I) \oplus c/\theta(I))/2\,.$$

*Then there exist $p, q \in L$ such that*

$$a \in (p \oplus_I q)/2\,.$$

**Proof.** From $a/\theta(I) \leq b/\theta(I) \vee c/\theta(I)$ we get $(b \vee c)/\theta(I) = (a \vee b \vee c)/\theta(I)$ and so

(2) $$(b \vee c, a \vee b \vee c) \in \theta(I)\,.$$

Let $r \in C_{[0, a \vee b \vee c]}(b \vee c)$. By (2) we have $r \in I$. Let $p = r \vee b$ and let $q \in C_{[0, c]}(p \vee c)$. Then $p \wedge q = 0$.

Now

$$p/\theta(I) = (r \vee b)/\theta(I) = r/\theta(I) \vee b/\theta(I) = 0/\theta(I) \vee b/\theta(I) = b/\theta(I)\,.$$

From (1) we conclude that

$$0/\theta(I) = a/\theta(I) \wedge b/\theta(I) = a/\theta(I) \wedge p/\theta(I) = a \wedge p/\theta(I)\,,$$

so that $a \wedge p \in I$.

Using (1) we see that

$$0/\theta(I) = a/\theta(I) \wedge c/\theta(I) = a \wedge c/\theta(I)\,.$$

Thus we have $a \wedge c \in I$.

But $q \leq c$, and, therefore, $a \wedge q \leq a \wedge c \in I$, that is $a \wedge q \in I$.

Finally,

$$p \vee q = p \vee r \vee b \vee q \geq (p \wedge c) \vee q \vee b \vee r = c \vee b \vee r = a \vee b \vee c \geq a$$

and we are done.

**Theorem 8.** *Let L be a sectionally complemented lattice.*

*Then an ideal I of L is semiprime if and only if it is a centred ideal.*

**Proof.** Let $I$ be a centred ideal. If

$$a/\theta(I) \in (b/\theta(I) \oplus c/\theta(I))/2$$

is true in the quotient lattice $L/\theta(I)$, then $a \in (L \oplus L)/2$ by Lemma 7. From this we conclude that $a \in I$, whence we obtain $a/\theta(I) = 0/\theta(I)$. It follows that

$$(L/\theta(I) \oplus L/\theta(I)) = \{0/\theta(I)\}.$$

Applying Theorem 6 to the quotient lattice $L/\theta(I)$, we conclude that $L/\theta(I)$ is a distributive lattice.

By [1; Theorem 1.10, p. 210] or [2; Theorem 2] we get $\hat{C}(L) \subset \theta(I)$. Therefore $0/\hat{C}(L) \subset 0/\theta(I) = I$. An appeal to [3; Theorem 3.3, p. 226] shows that $I$ is semiprime.

The converse follows easily.

**Corollary 9.** *Let $I$ be a centred ideal of a sectionally complemented lattice $L$. Then $\theta(I) = \Psi(I)$ where $\Psi(I)$ (cf. [8, p. 109]) is defined on $L$ by*

$$(a, b) \in \Psi(I) \Leftrightarrow a_I^* = b_I^*.$$

**Proof.** Apply Theorem 8 and [4; Theorem 3].

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