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Disjointness and Related Properties of Coproducts

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We study the relationship between different properties coproducts may have in a category. Besides universality there are mainly three types of properties: properties o the coproduct injections, properties of the functors $-\ \otimes A$ for all objects, A, and pullback properties of certain diagrams formed by coproducts.

Introduction

This paper is an attempt towards a systematic study of properties of coproducts in a category. The preprint [3] may be regarded as a preliminary version of the present paper; sometimes we go beyond the result there and we correct an error of [3].

The reason for such a study may be various: For the investigation of coproduct preservation by functors in [2] we needed a *pullback property*, which later turned out to be equivalent to the statement that all coproduct injections be *regular-monic* (see 4.3). In particular, it is satisfied in categories with a zero object, as we proved directly in [2, Prop 3.8]. In [4] we introduced *total disjointness*, which is in some sense converse to universality, in order to characterize coprime objects, i.e. objects representing covariant hom-functors that preserve coproducts. Moreover, 7.2 of the present paper can be used to simplify Giraud's characterization of Grothendieck topoi and related results on quasitopoi. On the other hand, disjointness implies the stronger property of total disjointness in the presence of universality (see 7.4).

We always work in a fixed category **A** with *finite coproducts* (including the empty coproduct, i.e. an initial object 0). By a *coproduct injection* we always mean an injection of a binary coproduct. This is justified by the trivial fact that in a category with (finite) coproducts any injection of a (finite) coproduct is also an

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injection of a binary coproduct. In Section 6 and 7 we additionally assume that $\mathbf{A}$ has *pullbacks.*

Section 1 is devoted to the rare case of epic coproduct injections. In Sections 2 through 4 we give criteria when all coproduct injections or all morphisms with domain 0 are (*regular-, extremal-*) *monic. Strong monomorphisms* coincide with extremal ones under mild conditions and have nicer properties in general; but surprisingly extremal monomorphisms seem to be more adequate for our purpose.

Note that a coproduct injection $A \to A \otimes B$ is a unit of the adjunction between the functor $B/\mathbf{A} \to \mathbf{A}$, $(X \to B) \mapsto X$ and its left adjoint $\mathbf{A} \to B/\mathbf{A}$, $A \mapsto \mapsto (A \to A \otimes B)$, where $B/\mathbf{A}$ is the comma category of *objects under B*, i.e. of arrows out of B. In [5] this was used (in dual version) to characterize properties of coproduct injections in terms of the *partial coproducts functors* $- \otimes B : \mathbf{A} \to \mathbf{A}$.

These partial coproduct functors have no nice analogue in the case of arbitrary colimits.

A more formal reason for the exceptional role of coproducts among all colimits may be the fact that coproducts commute with connected limits in **Set** and that connectedness of a category $\mathbf{D}$ means that every constant diagam indexed by $\mathbf{D}$ has the identity natural transformation as a limit.

The conditions that all coproduct injections or all morphisms with domain 0 are regularly epic equivalently be described by the condition that certain *diagrams be pullbacks.* In Section 5 we study a stronger pullback condition, which we call *total disjointness.* Section 6 is devoted to the question of when coproducts commute with pullbacks.

In Section 7 we show that *universality* implies some of the conditions considered before. Moreover, weaker conditions often imply stronger ones under the additional hypothesis of universality. On the other hand, universality is not implied by any of the other properties of coproducts.

We give several counterexamples to show that some implications cannot be *reversed,* but still some questions remain open.

1. Epimorphism

1.1 Throughout this paper, we work in a fixed category $\mathbf{A}$ with (a fixed choice of) *finite coproducts*, including the *empty* coproduct, i.e. and *initial object* 0. For $A, B \in |\mathbf{A}|$ we denote the *coproduct injections* by $\mu(A, B): A \to A \otimes B$, $\nu(A, B): B \to A \otimes B$; note that $\mu(A, B)$ and $\nu(B, A)$ differ only by the canonical isomorphism $A \otimes B \cong B \otimes A$, which we cannot assume to be the identity, since for $A = |\mathbf{A}|$, $o_A: 0 \to A$ denotes the unique morphism.

We call an $\mathbf{A}$-morphism $v: B \to C$ *coconstant,* if $xv = yv$ holds for all pairs (x, y) of parallel morphisms with domain C. The coconstant morphisms form an *ideal*, i.e. uvw is coconstant whenever it exists and v is coconstant.

Moreover, we call a morphism $v\colon B \to C$ *supercoconstant*, if $xv = t$ holds for all morphisms x, t such that xv is defined and parallel to t. The supercoconstant morphisms form a *left ideal*, i.e. uv is supercoconstant, if v is. But in general, they do *not* form a right ideal: In the category of abelian groups, o_A is supercoconstant for every A, but a zero morphism with nonzero domain is not (cf. 2.4 below).

1.2 Proporisition. *For* $A, B \in |\mathbf{A}|$, *the following statements are equivalent:*
(i) $\mu(A, B)$ *is epimorphic.*
(ii) $\nu(A, B)$ *is coconstant.*
(iii) $\nu(A, B)$ *is supercoconstant.*

1.3 Proposition. *For an* $\mathbf{A}$*-morphism* $n\colon B \to C$ *the following statements are equivalent:*
(i) *There exists an* $A \in |\mathbf{A}|$ *such that* $C \cong A \otimes B$ *with an epimorphic first injection* $u\colon A \to C$ *and* $v\colon\ B \to C$ *as second injection.*
(ii) $C \cong B \otimes C$ *with injections* $1\colon C \to C$ *and* $v\colon B \to C$.
(iii) v *is supercoconstant.* ☐

Now we look at the special case $A = \emptyset$ in 1.2, or, equivalently, we consider $o_B\colon = \emptyset \to B$. We call $B \in |\mathbf{A}|$ *pre-initial* (dual notion: *pre-terminal*), if for every $C \in |\mathbf{A}|$ there is at most one morphism $B \to C$.

1.4 Proposition. *For* $B \in |\mathbf{A}|$, *the following statements are equivalent:*
(i) $o_B\colon 0 \to B$ *is epimorphic.*
(ii) $1\colon B \to B$ *is coconstant.*
(iii) $1\colon B \to B$ *is supercoconstant.*
(iv) B *is pre-initial.*
(v) $B \cong B \otimes B$ *with both injections being the identity.*
(vi) *There exist* $A, A' \in |\mathbf{A}|$ *with* $A \otimes A' \cong B$ *such that* $\mu(A, A')$ *and* $\nu(A, A')$ *are both epimorphic.*
(vii) $\mu(B, B) = \nu(B, B)$. ☐

1.5 Proposition. *For* $A, B \in |\mathbf{A}|$, *the following statements are equivalent:*
(i) $\mu(A, B)$ *is an isomorphism.*
(ii) $\mu(A, B)$ *is a split-epimorphism.*
(iii) $\mu(A, B)$ *is epimorphic and* $\mathbf{A}(A, B) \neq \emptyset$. ☐

1.6 Proposition. *The following statements are equivalent:*
(i) *All coproduct injections are epimorphic.*
(ii) $o_B\colon 0 \to B$ *is epimorphic for all* $B \in |\mathbf{A}|$.
(iii) *For all* $B, C, \in |\mathbf{A}|$ *there exists at most one morphism* $B \to C$. ☐

1.7 Proposition. *The following statements are equivalent:*
(i) *All coproduct injections in* $\mathbf{A}$ *are extremal-epimorphic.*
(ii) o_B *is extremal-epimorphic for all* $B \in |\mathbf{A}|$.
(iii) *Every* $\mathbf{A}$*-object is initial.* ☐

2. Monomorphisms

2.1 In the sequel we shall concentrate on properties of *all* coproduct injections or *all* morphisms with domain 0 in **A**. For $B \in |\mathbf{A}|$ we denote by $\uparrow B$ the class of all $C \in |\mathbf{A}|$ with $\mathbf{A}(B, C) \neq \emptyset$. We call a subclass $\mathbf{G} \subset |\mathbf{A}|$ cogenerating, if for every $B, C \in |\mathbf{A}|$, $v, w: B \to C$ it follows that $v = w$ whenever $fv = fw$ for all $G \in \mathbf{G}$, $f \in \mathbf{A}(C, G)$. Rougly speaking, this means that **G** is a "not necessarily small cogenerator".

First we recall – in dual version – from [5, Prop. 7.2] the following:

2.2 Proposition. *For* $B \in |\mathbf{A}|$ *the following statements are equivalent:*

(i) *For all* $A \in |\mathbf{A}|$, $\mu(A, B): A \to A \otimes B$ *is a monomorphism.*

(ii) *The functor* $- \otimes B: \mathbf{A} \to \mathbf{A}$, $A \mapsto A \otimes B$ *is faithful.*

(iii) $\uparrow B$ *is cogenerating.*

2.3 If the above conditions are satisfied for all $B \in |\mathbf{A}|$, we obtain a characterization of supercoconstant morphisms. Obviously, every morphism with pre-initial domain is supercoconstant.

2.4 Proposition. *If the equivalent conditions* (i)–(iii) *of* 2.2 *are satisfied for all* $B \in |\mathbf{A}|$ *then any supercoconstant morphism has pre-initial domain.* □

2.5 Remarks. The hypothesis in 2.4 is necessary. Indeed, for every set X, the inclusion map $\emptyset \hookrightarrow X$ is superconstant in **Set**, i.e. supercoconstant in $\mathbf{Set}^{op}$. But its codomain X is pre-terminal in **Set** only if X has at most one element.

Now we turn to the question of when all o_B are monic. Note that the dual statement fails in **Set**, where $\emptyset \hookrightarrow \{1\}$isnot epic. We use the following mild hypothesis on **A**:

(C) For $A, B \in |\mathbf{A}|$, $x, y \in \mathbf{A}(A, B)$ there exist $C \in |\mathbf{A}|$, $f \in \mathbf{A}(B, C)$ with $fx = fy$.

Obviously, (C) is satisfied, if **A** has coequalizers *or* a terminal object.

2.6 Proposition. *If* 0 *is pre-terminal, then* o_B *is monomorphic for all* $B \in |\mathbf{A}|$. *Conversely, if* **A** *satisfies* (C) *and all* o_B *are monomorphic, then* 0 *is pre-terminal.*

3. Extremal Monomorphisms

3.1 Recall that an **A**-morphism $v: B \to C$ is called an *extremal monomorphism,* if it is monic and additionally satisfies the following condition:

(*) If $D \in |\mathbf{A}|$, $e: B \to D$, $m: D \to C$ are morphisms with $me = v$ and e epic, then e is an isomorphism.

Note that (*) implies that v is monic, provided **A** has coequalizers. Extremal monomorphisms coincide with the *strong monomorphism* (cf. [8]) under mild conditions, e.g. if **A** has pushouts. Since we try to keep our results as general as possible, we shall work with morphisms satisfying (*).

We call an **A**-morphism $f: A \to B$ orthogonal to $D \in |\mathbf{A}|$, if the map $\mathbf{A}(f, D): \mathbf{A}(B, D) \to A\ (A, D)$, $h \mapsto hf$ is bijective; in this case we write $f \perp D$.

Recall that a functor $F: \mathbf{X} \to \mathbf{Y}$ is called *conservative,* if F reflects isomorphisms, i.e. if an **X**-morphism f is invertible whenever Ff is.

3.2 Proposition. *The following statements are equivalent:*

(i) *For all* $A, B \in |\mathbf{A}|$, $\mu(A, B)$ *satisfies* (*).

(ii) *For each* $B \in |\mathbf{A}|$, any **A**-*morphism with* $f \perp D$ *for all* $D \in \uparrow B$ *is an isomorphism.*

(iii) *For every* $B \in |\mathbf{A}|$, *the functor* $- \otimes A: \mathbf{A} \to \mathbf{A}$ *is conservative.*

(iv) *The functor* $- \otimes -: \mathbf{A} \times \mathbf{A} \to \mathbf{A}$, $(A, B) \mapsto A \otimes B$ *is conservative.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is provided like [5, Prop. 7.2] (in dual version).
(iv) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (iv): Let $f: A \to C$, $g: B \to D$ be **A**-morphism with $f \otimes g: A \otimes B \to C \otimes D$ invertible. Since $f \otimes g = (f \otimes 1_D)(1_A \otimes g) = (1_C \otimes g)(f \otimes 1_B)$, we see that $f \otimes 1_D$ is split-epic and $f \otimes 1_B$ is split-monic, hence $(f \otimes 1_D) \otimes 1_B$ is split-epic and $(f \otimes 1_B) \otimes 1_D$ is split-monic. But both morphisms differ only by a canonical isomorphism from $f \otimes 1_{B \otimes D}$, which is therefore invertible. Hence by (iii) f is an isomorphism. Symmetry yields invertibility of g. ☐

3.3 Remarks If **A** has coequalizers, the equivalent conditions of 3.2 imply those of 2.2 for all $B \in |\mathbf{A}|$.

If **A** has even coproducts indexed by some (possibly infinite) set I, then the equivalent conditions of 3.2 imply that the coproduct functor $\mathbf{A}^I \to \mathbf{A}$, $(A_i)_{i \in I} \mapsto \bigotimes_{i \in I} A_i$ is conservative. Indeed if $\bigotimes_{i \in I} f_i: \bigotimes_{i \in I} A_i \to \bigotimes_{i \in I} B_i$ is invertible for a family $(f_i: A_i \to B_i)$ of **A**-morphisms, then for any fixed $i_0 \in I$ we can apply 3.2 (iv) to $f: = f_{i_0}$ and $g: = \bigotimes_{i \in I \setminus \{i_0\}} f_i$ to conclude invertibility of f_{i_0}.

Note that conservativity of other types of colimes is quite rare: if **D** is a small connected category and **A** has all **D**-colimits, then the colimit functor $[\mathbf{D}, \mathbf{A}] \to$ $\to \mathbf{A}$ maps all colimit cones to isomorphisms. Hence the colimit functor is conservative if and only if every diagram $\mathbf{D} \to \mathbf{A}$ is naturally isomorphic to a constant diagram.

Our next results follows immediately from 1.4 and 2.6

3.4 Proposition. *The following statements are equivalent:*

(i) o_B *satisfies* (*) *for all* $B \in |\mathbf{A}|$.

(ii) *Every pre-initial object of* **A** *is initial.*

If these conditions are satisfied and if **A** *has coequalizers, then* 0 *is preterminal.* ☐

Note that for 3.4 we do not even need existence of finite coproducts; an initial object suffices. Using 2.4 we obtain the following:

3.5 Corollary. *Assume that* **A** *has coequalizers. Moreover, let all coproduct injections in* **A** *be monic and let all* o_B, $B \in |\mathbf{A}|$ *even be extremal monomorphisms.*

Then the superconstant morphisms in **A** *are exactly those morphisms whose domain is initial.* □

3.6 Example. The equivalent conditions of 2.2 do not imply those of 3.2. Indeed, if $\mathbf{A} \neq \{0\}$ is a join semilattice with bottom element, considered as a category in the usual way, then all morphisms in **A** are monic, but only the identity morphisms are extremally monic. Note that **A** has equalizers and coequalizers. If **A** is a complete lattice, it is even complete and cocomplete. □

3.7 Example. Let **2** be the two-element lattice, i.e. the category with exactly two objects 0, 1 and one non-identity morphism $0 \to 1$. Then **2** is complete and cocomplete, and so is $\mathbf{Set} \times \mathbf{2}$. The full subcategory $\mathbf{A} \subset \mathbf{Set} \times \mathbf{2}$ of all objects $\neq (\emptyset, 1)$ is coreflective; the coreflection maps $(\emptyset, 1)$ to $(\emptyset, 0)$. Hence **A** is also complete and cocomplete, and colimits (particularly coproducts) in **A** are formed as in $\mathbf{Set} \times \mathbf{2}$, i.e. componentwise. Thus all coproduct injections in **A** are monic. Moreover, we see that the only pre-initial object of **A** is the initial object $(\emptyset, 0)$, hence o_B is extremal-monic for all $B \in |\mathbf{A}|$ by 3.4. On the other hand, the coproduct injection $\mu(A, B)$ is not extremal-monic for $A{:} =(\{0\},0)$, $B{:} =(\{0\},1)$, since $\mu(A, B)$ factors over the unique morphism $A \to B$, which is epic. Therefore, the conditions of 2.2 do not imply those of 3.2, even if all o_B are extremal-monic. □

4. Regular Monomorphisms

4.1 In this section we study the question of when all coproduct injections are regular-monic in the sense of [9]; note that most results remain valid if we define regular monomorphisms in the narrower sense of being an equalizer of a pair of morphisms.

4.2 Lemma. *For all* $A \in |\mathbf{A}|$, *the implications* (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) *hold between the statements below. If all coproduct injections in* **A** *are monomorphisms, then all three statements are equivalent.*

(i) *For all* $B \in |\mathbf{A}|$, $\mu(A, B)$ *is a regular monomorphism.*

(ii) *For all* $B \in |\mathbf{A}|$, $\mu(A, B)$ *is an equalizer of* $1_A \otimes \mu(B, B)$ *and* $1_A \otimes \nu(B, B){:}\ A \otimes B \to A \otimes (B \otimes B)$.

(iii) *For all* B, $C \in |\mathbf{A}|$, *the following diagram is a pullback:*

$$(1)\qquad \begin{tikzcd} A \arrow[r, "\mu(A,B)"] \arrow[d, "\mu(A,C)"'] & A \otimes B \arrow[d, "1_A \otimes \mu(B,C)"] \\ A \otimes C \arrow[r, "1_A \otimes \nu(B,C)"'] & A \otimes (B \otimes C) \end{tikzcd}$$

Proof. (i) ⇔ (ii): The diagram (1) is always a pushout. Apply this to the case $C = B$.
(iii) ⇒ (ii): Apply (iii) to the case $B = C$.
(ii) ⇒ (iii): Consider the following morphisms u, v, w:
$A \otimes (B \otimes C) \to A \otimes ((B \otimes C) \otimes (B \otimes C))$: $u{:} = 1_A \otimes \mu(B \otimes C, B \otimes C)1$, $v{:}\ 1_A \otimes (\mu(B, C) \otimes \nu(B, C))$, $w{:} = 1_A \otimes \nu(B \otimes C, B \otimes C)$. Then we have $u(1_A \otimes \mu(B, C)) = v(1_A \otimes \mu(B, C))$ and $v(1_A \otimes (B, C)) = w(1_A \otimes \nu(B, C))$. By (ii) $\mu(A, B \otimes C){:}\ A \to A \otimes (B \otimes C)$ is an equalizer of u and w.

Now assume $(1_A \otimes \mu(B, C))f = (1_A \otimes \nu(B, C))g =: k$ for some $Z \in |\mathbf{A}|$, $f{:}\ Z \to A \otimes B$, $g{:}\ Z \to A \otimes C$. Since $uk = u(1_A \otimes \mu(B, C))f = v(1_A \otimes \mu(B, C))f = vk = v(1_A \otimes \nu(B, C))g = wk$, the equalizer property renders a unique $h{:}Z \to A$ with $\mu(A, B \otimes C)h = k$. This yields $(1_A \otimes \nu(B, C))\mu(A, B)h = \mu(A, B \otimes C)h = k = (1_A \otimes \mu(B, C))f$ and $(1_A \otimes \nu(B, C))\mu(A, C)h = (1_A \otimes \nu(B, C))g$. But $1_A \otimes \mu(B, C)$ and $I_A \otimes \nu(B, C)$ are coproduct injections; hence they are monic by our additional assumption. This gives $\mu(A, B)h = f$ and $\mu(A, C)h = g$. Obviously, h is uniquely determined even by each of these two equations separately, since $\mu(A, B)$ and $\mu(A, C)$ are both monic. □

4.3 Theorem. *The following statements are equivalent:*

(i) *For all* $A, B \in |\mathbf{A}|$, $\mu(A, B)$ *is a regular monomorphism.*

(ii) *For all* $A, B \in |\mathbf{A}|$, $\mu(A, B)$ *is an equalizer of* $1_A \otimes \mu(B, B)$ *and* $1_A \otimes \mu(B, B)$.

(iii) *For all* $A, B, C \in |\mathbf{A}|$, *the diagram* (1) *is a pullback.*

(iv) *For all* $B \in |\mathbf{A}|$, *the functor* $- \otimes B{:}\ \mathbf{A} \to \mathbf{A}$ *reflects split-monomorphisms into regular monomorphisms, i.e. if* $f \otimes 1_B$ *is split-monic then* f *is regular-monic.*

Proof. (i) ⇔ (ii) ⇔ (iii) follows from 4.2. Note that in 4.2 for (ii) ⇒ (iii) we needed the additional hypothesis that all coproduct injections be monic. Here this hypothesis follows, since (ii) is assumed for all $A \in |\mathbf{A}|$.
(i) ⇔ (iv) follows from [5, Prop. 7.2]. □

4.4 Proposition. *The implications* (i) ⇔ (ii) ⇔ (iv) ⇐ (iii) *hold for the statements below. If all coproduct injections in* $\mathbf{A}$ *are monomorphisms, then all four statements are equivalent:*

(i) *For all* $B \in |\mathbf{A}|$, o_B *is a regular monomorphism.*

(ii) *For all* $B \in |\mathbf{A}|$, $o_B{:}\ 0 \to B$ *is the equalizer of* $\mu(B, B)$ *and* $\nu(B, B){:}\ B \to B \otimes B$.

(iii) *For all* $B, C \in |\mathbf{A}|$, *the diagram*

$$\begin{tikzcd}
0 \arrow[r, "o_B"] \arrow[d, "o_C"'] & B \arrow[d, "\mu(B,C)"] \\
C \arrow[r, "\nu(B,C)"'] & B \otimes C
\end{tikzcd} \tag{2}$$

is a pullback.

(iv) *Every coconstant morphism in* **A** *factors through* 0.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows by specializing 4.2 to the case $A = 0$. (i) $\Leftrightarrow$ (iv) is trivial. ☐

4.5 Remarks. Usually, coproducts are called disjoint, if all coproduct injections are monic and condition (iii) of 4.4 holds. By 4.4, the later condition can be replaced by the simpler condition (i) or (iv).

Next we see that the conditions of 4.3 are stronger than those of 3.2:

4.6 Example *of a complete and cocomplete category, in which all* $\mu(A, B)$ *are extremal-monic, but for some* B, o_B *(and hence* $\mu(0, B)$*) is not regular-monic.* Let $\mathbf{Set}_*$ denote the category of sets with an assigned base point, which we always denote by $*$; the morphisms are base point preserving maps. Now let **A** be the following (non-full!) subcategory of $\mathbf{Set}_* \times \mathbf{Set}$, **A** has the same objects as $\mathbf{Set}_* \times \mathbf{Set}$, and a morphism $(f, g)\colon (X, Y) \to (X', Y')$ belongs to **A** if and only if $Y' \neq \emptyset$ or f reflects the base point, i.e. $f^{-1}\{*\} = \{*\}$.

A is cocomplete, and colimits can be formed as in $\mathbf{Set}_* \times \mathbf{Set}$. Indeed, assume that $(X, Y) \in |\mathbf{A}|$ is a colimit in $\mathbf{Set}_* \times \mathbf{Set}$ of some small diagram in **A**. If $Y \neq \emptyset$, then all colimit injections have codomain $Y \neq \emptyset$ and therefore belong to **A**, and the colimit property follows easily.

Now consider the case $Y = \emptyset$. Then all objects in the diagram have second component $\emptyset$. Therefore all first components of morphisms in the diagram reflect the base points, and we easily see that the colimit injections belong to **A**.

For an arbitrary cocone in **A** of the given diagram with vertex (X', Y'), we get an induced morphism $(X, \emptyset) \to (X', Y')$ in $\mathbf{Set}_* \times \mathbf{Set}$ If $Y' \neq \emptyset$, this morphism belongs to **A** by definition of **A**. If $Y' = \emptyset$, then all injections of the cocone belong to **A**, hence their first components reflect base points; and for the induced $\mathbf{Set} \times \mathbf{Set}_*$-morphism $(X, \emptyset) \to (X', \emptyset)$ it is readily checked that the first component reflects the base point, hence the arrow belongs to **A**. ☐

This proves cocompleteness of **A**. Moreover, we see that $(X, Y) \cong$ $\cong \big(\bigotimes_{x \in X \setminus \{*\}} (\{x, *\}, \emptyset)\big) \otimes \big(\bigotimes_{y \in Y}(\{*\}, \{y\})\big)$ in **A**.

Therefore, every (X, Y) is a colimit (even a coproduct) of copies of the two elements $A\colon = (\{0, *\}, \emptyset)$(where $* \neq 0$) and $B\colon = (\{*\}, \{0\})$. From [1], Thm. 2 or [6], Cor. 5.5 we conclude that **A** is complete.

Now condition (iii) of 3.2 holds for $\mathbf{Set}_*$ and **Set**, hence for $\mathbf{Set}_* \times \mathbf{Set}$. Since **A** contains all isomorphisms of $\mathbf{Set}_* \times \mathbf{Set}$ and since coproducts (even colimits) are formed in the same way as in $\mathbf{Set}_* \times \mathbf{Set}$, it follows that **A** satisfies the equivalent conditions of 3.2.

On the other hand, **A** does not satisfy the equivalent conditions of 4.4, because the unique morphism $A \to B$ is coconstant, but does not factor through the initial object $(\{*\}, \emptyset)$. Indeed, the unique morphism $A \to (\{*\}, \emptyset)$ in $\mathbf{Set}_* \times \mathbf{Set}$ does not belong to **A**, since its first component does not reflect the base point. ☐

Of course, the equivalent conditions of 4.3 imply those of 4.4. On the other hand, even condition (i) of 4.4 does not imply that all coproduct injections are monic, as shown by the following.

4.7 Example *of a complete and cocomplete category with all o_B regularly monic, but which a non-monic coproduct injection:* Let $\mathbf{A}$ be a category whose objects are all pairs (X, i), where X is a set and $i \in \{0, 1\}$; if $i = 1$ we additionally require $X \neq \emptyset$. The morphisms $(X, i) \to (Y, i)$ correspond to the set maps $X \to Y (i \in \{0, 1\})$; the morphisms $(X, 0) \to (Y, 1)$ correspond to the constant maps $X \to Y$; we always consider the inclusion $\emptyset \hookrightarrow Y$ as constant. So $\mathbf{A}((X, 0), (Y, 1))$ has exactly one element in case $X = \emptyset$ and is in bijection with Y if $X \neq \emptyset$. Moreover, we define $\mathbf{A}((X, 1), (Y, 0)) := \emptyset$ for all X, Y.

Let $\mathbf{A}_0 \subset \mathbf{A}$ be the full subcategory of all objects $(X, 0)$; then the projection $\mathbf{A}_0 \to \mathbf{Set}$ to the first component is an isomorphism. Thus all colimits exist in $\mathbf{A}_0$, and they can easily be seen to be colimits even in $\mathbf{A}$.

Let $\mathbf{A}_1 \subset \mathbf{A}$ be the full subcategory, whose objects are $(\emptyset, 0)$ and all $(X, 1)$ $(X \in |\mathbf{Set}|, X \neq \emptyset)$. Then $\mathbf{A}_1$ is also cocomplete. Moreover, $\mathbf{A}_1$ is reflective in $\mathbf{A}$, the reflection maps all objects $(X, 0)$ with $X \neq \emptyset$ to $(\{*\}, 1)$.

Now we show cocompleteness of $\mathbf{A}$. Consider an arbitrary small diagram in $\mathbf{A}$. If it lies in $\mathbf{A}_0$, we have already seen that it admits a colimit in $\mathbf{A}$. If it does not lie in $\mathbf{A}_0$, then all its cocones have vertices with second component 1, which therefore belong to $\mathbf{A}_1$. Now the colimit in $\mathbf{A}$ can be formed by applying the reflector $\mathbf{A} \to \mathbf{A}_1$ to the diagram and then taking the colimit in $\mathbf{A}_1$.

In particular, each $(X, i) \in |\mathbf{A}|$ is the X-th copower of $(\{*\}, i)$, hence $\mathbf{A}$ is also complete by [1, Thm. 2] or [6, Cor. 5.5].

We easily see that any coconstant $\mathbf{A}$-morphism has domain $(\emptyset, 0)$. But for $X \neq \emptyset \neq Y$ we have $(X, 0) \otimes (Y, 1) \cong (Y \cup \{*\}, 1)$ with $* \notin Y$ and injections $\nu((X, 0), (Y, 1))$: $Y \hookrightarrow Y \cup \{*\}$ and $\mu((X, 0), (Y, 1))$ mapping everthing to $*$. If X has at least two points, $\mu((X, 0), (Y, 1))$ is not monic in $\mathbf{A}$. □

4.8 In 3.7 we gave an example of a complete and cocomplete category with all coproduct injections monic, but in general not extremal-monic. One easily sees that the conditions of 4.4 are also satisfied; note that they are equivalent in this situattion. Therefore, even disjointness of coproducts does not imply that all coproduct injections are extremal-monic.

Furthermore, even if all o_B are regular-monic, the equivalent conditions of 4.3 do not follow from those of 3.2, as we see from the following:

4.9 Example *of a complete and cocomplete category with all morphisms with domain* 0 *regularly monic and all coproduct injections extremally monic, but not necessarily regularly monic:* First let $\mathbf{C}$ be the category whose objects are all pairs (X, Y) of sets with $Y \subset X$; a morphism $(X, Y) \to (X', Y')$ is given by a set map $X \to X'$ with $fY \subset Y'$ or, equivalently, $Y \subset f^{-1}Y'$. Then $\mathbf{C}$ is cocomplete. A colimit is constructed by taking the $\mathbf{Set}$-colimit of the first component; the second component is the union of all images of second components of objects in the diagram under colimit injections.

Now let $\mathbf{C}' \subset \mathbf{C}$ be the (non-full!) subcategory consisting of all **C**-objects, but only with those morphism $f\colon (X, Y) \to (X', Y')$ for which $f^{-1}Y' = Y$ holds. Then the functor $\mathbf{C}' \to \mathbf{Set}_* \times \mathbf{Set}$, $(X, Y) \mapsto (Y, X \backslash Y)$ is an equivalence. (Here $X \backslash Y$ denotes the complement of Y in X.) Hence $\mathbf{C}'$ is cocomplete.

Comparing these constructions we see that all colimits in $\mathbf{C}'$ are also colimits in **C**. In particular, copoducts in **C** can be formed componentwise, because discrete diagrams in **C** even belong to $\mathbf{C}'$.

Now consider the (non-full!) subcategory $\mathbf{A} \subset \mathbf{C} \times \mathbf{Set}$, which contains all objects but only those morphisms $(f, g)\colon ((X, Y), Z) \to ((X', Y'), Z')$, for which $Z' \neq \emptyset$ or $f\colon (X, Y) \to (X', Y')$ even belongs to $\mathbf{C}'$.

We claim that **A** is cocomplete and that small colimits in **A** can be formed as in $\mathbf{C} \times \mathbf{Set}$. Indeed, for a diagram in **A**, consider its colimit $((X, Y), Z)$ in $\mathbf{C} \times \mathbf{Set}$. If $Z \neq \emptyset$, then all colimit injections belong to **A**. For an arbitrary cocone from the given diagram to some $((X', Y'), Z') \in |\mathbf{A}|$ we have $Z' \neq \emptyset$; hence the colimit is also a colimit in **A**.

Now we look at the case $Z = \emptyset$. Then all objects in the diagram have last component $\emptyset$, and the first components of the morphisms in the diagram must belong to $\mathbf{C}'$. We can form a colimit in $\mathbf{C}'$, which is even a colimit in **C**, and the colimit injections of our original colimits belong to **A**.

Now consider an arbitrary cocone in **A** from the given diagram to some $((X', Y'), Z') \in |\mathbf{A}|$. If $Z' \neq \emptyset$, the colimit property in $\mathbf{C} \times \mathbf{Set}$ yields a unique morphism $((X, Y), \emptyset) \to ((X', Y'), Z')$, which belongs to **A** because $Z' \neq \emptyset$. If $Z' = \emptyset$, all first components of the cocone must belong to $\mathbf{C}'$, since the cocone lies in **A**. This gives a unique **A**-morphism $((X, Y), \emptyset) \to ((X', Y'), \emptyset)$.

This proves our claim. In particular, coproducts in **A** can be formed componentwise. This implies the equivalent conditions of 3.2. Hence all coproduct injections are extremally monic. Moreover we obtain $((X, Y), Z) \cong (\bigotimes_{y \in Y} A) \otimes (\bigotimes_{z \in Z} B) \otimes (\bigotimes_{x \in X \backslash Z} C)$ for all $(X, Y, Z) \in |\mathbf{A}|$, where $A := ((\{*\}, \{*\}), \emptyset)$, $B := ((\emptyset, \emptyset), \{*\})$, $C := ((\{*\}, \emptyset), \emptyset)$. Form [1, Thm. 2] or [6, Cor. 5.5] we conclude that **A** is also complete.

On the other hand, we have $B \otimes B \cong ((\emptyset, \emptyset), \{0, 1\})$ with the two distinct morphisms $B \to B \otimes B$ as injections; furthermore we get $A \otimes B \cong ((\{*\}, \{*\}), \{*\})$ and $A \otimes (B \otimes B) \cong ((\{*\}, \{*\}), \{0, 1\})$. Since the last component of $A \otimes B$ is $\neq \emptyset$, the unique $\mathbf{C} \times \mathbf{Set}$-morphism $f\colon C \to A \otimes B$ belongs to **A**, and we immediately get $(1_A \otimes \mu(B, B)) f = (1_A \otimes \nu(B, B)) f$. On the other hand, the unique $\mathbf{C} \times \mathbf{Set}$-morphism $g\colon C \to A$ is not in **A**, since the last component of A is $\emptyset$ and since $g^{-1}\{*\} = \{*\} \neq \emptyset$. Therefore $\mu(A, B)$ is not the equalizer of $1_A \otimes \mu(B, B)$ and $1_A \otimes \nu(B, B)$; hence $\mu(A, B)$ is not regular-monic. □

Obviously, everything becomes trivial if all coproduct injections (or at least all o_B) are split-monic:

4.10 Proposition. *The implications* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftarrow$ (v) *hold between the statements below. If* $\mathbf{A}$ *satisfies* (C), *all five statemens are equivalent:*

(i) *For all* $A, B \in |\mathbf{A}|$, $\mu(A, B)$ *is a split-monomorphism.*

(ii) *For all* $B \in |\mathbf{A}|$, o_B *is a split-monomorphism.*

(iii) $\uparrow B = |\mathbf{A}|$ *for all* $B \in |\mathbf{A}|$.

(iv) $\mathbf{A}(B, 0) \neq \emptyset$ *for all* $B \in |\mathbf{A}|$.

(v) $\mathbf{A}$ *has a zero object.* □

5. Total Disjointness

5.1 In this section we shall study a property of coproducts, which is in some sense converse to universality and which was used in [2] in order to characterize coprime objects. We say that coproducts in $\mathbf{A}$ are *totally disjoint* if for all $\mathbf{A}$-morphisms $f: B \to C$, $g: A \to D$ the two squares of the following diagram are pullbacks:

$$\begin{tikzcd}
(3) & A \arrow[r, "\mu(A,B)"] \arrow[d, "g"'] & A \otimes B \arrow[r, "\nu(A,B)"] \arrow[d, "g \otimes f"] & B \arrow[d, "f"] \\
& D \arrow[r, "\mu(D,C)"'] & D \otimes C \arrow[r, "\nu(D,C)"'] & C
\end{tikzcd}$$

By symmetry it suffices to assume the pullback property only for one of the squares in all instances. Moreover, if $\mathbf{A}$ even has arbitrary coproducts, total disjointness even implies that the following diagram with coproduct injections is a pullback for all small families $(f_i: B_i \to C_i)_{i \in I}$ of $\mathbf{A}$-morphisms and all $i_o \in I$:

$$\begin{tikzcd}
(4) & B_{i_o} \arrow[r] \arrow[d, "f_{i_o}"'] & \bigotimes\limits_{i \in I} B_i \arrow[d, "\bigotimes\limits_{i \in I} f_i"] \\
& C_{i_o} \arrow[r] & \bigotimes\limits_{i \in I} C_i
\end{tikzcd}$$

Indeed, just use the decomposition $\bigotimes\limits_{i \in I} B_i \cong B_{i_o} \otimes \left(\bigotimes\limits_{i \in I \setminus \{i\}} B_i \right)$.

We call a full subcategory $\mathbf{C} \subset \mathbf{A}$ *weakly terminal*, if for every $A \in |\mathbf{A}|$ there exists an objects $C \in |\mathbf{C}|$ and a morphism $A \to C$. In particular, **A** is always weakly terminal in itself. If **A** has terminal object T, then $\{T\}$ is weakly terminal in **A**.

5.2 Theorem. *If* $\mathbf{C} \subset \mathbf{A}$ *is a weakly terminal full subcategory, then the following statements are equivalent:*

(i) *Coproducts are totally disjoint in* **A**.

(ii) *The left-hand square of* (3) *is a pullback for all* $A, B \in |\mathbf{A}|$, $C, D \in |\mathbf{C}|$, $f\colon B \to C$, $g\colon A \to D$.

(iii) *For all* $A, B \in |\mathbf{A}|$, $C \in |\mathbf{C}|$, $f\colon B \to C$, *the following diagrams are pullbacks:*

(5)
$$\begin{tikzcd}
A \arrow[r, "\mu(A,B)"] \arrow[d, "1"'] & A \otimes B \arrow[d, "1 \otimes f"] \\
A \arrow[r, "\mu(A,C)"'] & A \otimes C
\end{tikzcd}$$

(6)
$$\begin{tikzcd}
B \arrow[r, "\nu(A,B)"] \arrow[d, "f"'] & A \otimes B \arrow[d, "1 \otimes f"] \\
C \arrow[r, "\nu(A,C)"'] & A \otimes C
\end{tikzcd}$$

Proof. (i) ⇒ (ii) and (i) ⇒ (iii) are trivial; for the latter implication note that (5) and (6) are special cases of the two squares in (3) for $D := A$, $g := 1_A$.
(ii) ⇒(i) Let $A, B, C, D \in |\mathbf{A}|$, $f\colon B \to C$, $g\colon A \to D$. Since **C** is weakly terminal, there exist $C', D' \in |\mathbf{C}|$ and $c\colon C \to C'$, $d\colon D \to D'$. Now the outer rectangle and the right-hand square of the following diagram are pullbacks by (ii):

(7)
$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "\mu(A,B)"'] & D \arrow[r, "d"] \arrow[d, "\mu(D,C)"'] & D' \arrow[d, "\mu(D',C')"] \\
A \otimes B \arrow[r, "g \otimes f"'] & D \otimes C \arrow[r, "d \otimes c"'] & D' \otimes C'
\end{tikzcd}$$

Thus the left-hand square is also a pullback.

(iii) ⇒ (i): For A, B, C, $D \in |\mathbf{A}|$, $f\colon B \to C$, $g\colon A \to D$, we choose C', $D' \in |\mathbf{C}|$ and $c\colon C \to C'$, $d\colon D \to D'$. Then the right-hand square and the outer rectangle of the following diagram are instances of (5) and thus pullbacks by (iii):

(8)
$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, swap, "\mu(A,B)"] & A \arrow[r, "1"] \arrow[d, "\mu(A,C)"] & A \arrow[d, "\mu(A,C')"] \\
A \otimes b \arrow[r, swap, "1 \otimes f"] & A \otimes C) \arrow[r, swap, "1 \otimes c"] & A \otimes C'
\end{tikzcd}$$

Hence the left-hand square is a pullback by cancellation. Moreover, the right-hand side and the outer rectangle of the following diagram are isomorphic to instances of (6) and therefore pullbacks by (ii):

(9)
$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, swap, "\mu(A,C)"] & D \arrow[r, "d"] \arrow[d, swap, "\mu(D,C)"] & D' \arrow[d, "\mu(D',C')"] \\
A \otimes C \arrow[r, swap, "g \otimes 1_C"] & D \otimes C \arrow[r, swap, "d \otimes 1_C"] & D' \otimes C'
\end{tikzcd}$$

Hence the left-hand square is also a pullback by cancellation. Now (iii) follows, because the left-hand square of (3) is a composite of pullbacks:

(10)
$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, swap, "\mu(A,B)"] & A \arrow[r, "g"] \arrow[d, swap, "\mu(A,C)"] & D \arrow[d, "\mu(D,C)"] \\
A \otimes B \arrow[r, swap, "1_A \otimes f"] & A \otimes C \arrow[r, swap, "g \otimes 1_C"] & D \otimes C
\end{tikzcd}$$

5.3 Remarks. Note that (i) ⇔ (ii) is trivial in the case $\mathbf{C} = \mathbf{A}$. On the other hand, (i) ⇔ (iii) gives new information even in this case; it splits total disjointness into two different conditions, which we shall sometimes consider separately in the sequel.

5.4 Proposition. *If diagram* (6) *is a pullback for all* A, B, $C \in |\mathbf{A}|$, $f\colon B \to C$, *then all coproduct injections in* $\mathbf{A}$ *are regular monomorphisms.*

Proof. For A, B, C, the diagram (6) for A replaced by B, B replaced by A, C replaced by $A \otimes C$ and f replaced by $\mu(A, B)$ is isomorphic to (1), the rest follows from 4.3. ☐

5.5 Proposition. *The following statements are equivalent:*

(i) *For all B, $C \in |\mathbf{A}|$, the following diagram is a pullback:*

$$\begin{tikzcd}
0 \arrow[r, "o_B"] \arrow[d] & B \arrow[d, "f"] \\
0 \arrow[r, "o_C"'] & C
\end{tikzcd} \tag{11}$$

(ii) *Every morphism into 0 is invertible.*

(iii) *Every $B \in |\mathbf{A}|$ with $\mathbf{A}(B, 0) \neq \emptyset$ is initial.*

(iv) *Every $B \in |\mathbf{A}|$ with $\uparrow B = |\mathbf{A}|$ is initial.*

Proof. (i) $\Rightarrow$ (ii): Let $f\colon B \to 0$ be an $\mathbf{A}$-morphism. Then by (i) diagram (11) is a pullback for $C := 0$. Since $1\colon 0 \to 0$ is an isomorphism, o_B is also invertible. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) is trivial. (iii) $\Rightarrow$ (i): For $Z \in |\mathbf{A}|$, $u\colon Z \to 0$, $v\colon Z \to B$, assume $o_C u = fv$. Since $u \in \mathbf{A}(Z, 0) \neq \emptyset$, Z is initial by (iii), hence $o_B u = v$. Obviously, we have $1u = u$, and u is uniquely determined by these equations. ☐

5.6 Note that (11) is isomorphic to the special case $A = 0$ of (5). Moreover (5) is obtained from (11) (up to a canonical isomorphism) by an application of the functor $- \otimes A$ to (11).

On the other hand, note that (6) trivially is a pullback for $A = 0$.

5.7 Proposition. *If all $A \in |\mathbf{A}|$ with $\mathbf{A}(A, 0) \neq \emptyset$ are initial, then $o_B\colon 0 \to B$ is a monomorphism for all $B \in |\mathbf{A}|$.*

Proof. The result follows immediately from 2.6. ☐

5.8 Remarks. In contrast to 5.7 the equivalent conditions of 5.5 together with the assumption that all o_B are split-monic imply that $\mathbf{A}$ is trivial by 4.10 and 5.5. ☐

6. Coproducts Commuting with Pullbacks

6.1 From now on, we always assume that $\mathbf{A}$ has pullbacks. We consider the question of when *finite coproducts commute with pullbacks,* i.e. hen each finite pointwise coproduct of pullbacks squares is itself a pullback square. By induction, we can restrict our attention to *nullary* and *binary* coproducts. but nullary coproducts i.e. initial objects trivially even commute always with all connected colimits. Therefore, finite coproducts in **A** commute with pullbacks if and only if the coproduct functor $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$, $(A, B) \mapsto A \otimes B$ preserves pullbacks.

If **A** is a semiadditive category, then finite coproducts coincide with finite products and hence commute with all existing limits (cf. [7, Thm. 40.8] and [7, Thm. 25.4]). In **Set**, arbitrary coproducts commute with all connected limits, in particular with pullbacks.

6.2 Proposition. *In* **A** *finite coproducts commute with pullbacks, if and only if the following two conditions are satisfied:*

(i) *For any* $A \in |\mathbf{A}|$ the functor $-\otimes A$: $\mathbf{A} \to \mathbf{A}$ *preserves pullbacks.*

(ii) *For all* A_i, $B_i \in |\mathbf{A}|$, f_i: $A_i \to B_i$ $(i \in \{0, 1\})$ *the following diagram is a pullback:*

$$(12)\quad\begin{tikzcd}A_0 \otimes A_1 \arrow[r, "1 \otimes f_1"] \arrow[d, "f_0 \otimes 1"'] & A_0 \otimes B_1 \arrow[d, "f_0 \otimes 1"] \\ B_0 \otimes A_1 \arrow[r, "1 \otimes f_1"'] & B_0 \otimes B_1\end{tikzcd}$$

Proof "$\Rightarrow$": (i): Application of the functor $-\otimes A$ to a pullback square yields the pointwise coproduct of the given square with a constant square, which is obviously a pullback.

(ii): (13) is the pointwise coproduct of two pullback squares, one with the vertical arrows being identies, and one square with both horizontal arrows equal to f_1, and identifies as vertical ones.

"$\Leftarrow$": Consider pullback squares

$$(13)\quad\begin{tikzcd}A_i \arrow[r, "f_i"] \arrow[d, "g_i"'] & B_i \arrow[d, "h_i"] \\ C_i \arrow[r, "k_i"'] & D_i\end{tikzcd}$$

for $i \in \{0, 1\}$. Then their pointwise coproduct is the outer square of the following diagram:

(14)

$$\begin{tikzcd}
A_0 \otimes A_1 \arrow[r, "1 \otimes f_1"] \arrow[d, "g_0 \otimes 1"'] \arrow[dr, phantom, "(a)"] & A_0 \otimes B_1 \arrow[r, "f_0 \otimes 1"] \arrow[d, "g_0 \otimes 1"'] \arrow[dr, phantom, "(b)"] & B_0 \otimes B_1 \arrow[d, "h_0 \otimes 1"'] \\
C_0 \otimes A_1 \arrow[r, "1 \otimes f_1"] \arrow[d, "1 \otimes g_1"'] \arrow[dr, phantom, "(c)"] & C_0 \otimes B_1 \arrow[r, "k_0 \otimes 1"] \arrow[d, "1 \otimes h_1"'] \arrow[dr, phantom, "(d)"] & D_0 \otimes B_1 \arrow[d, "1 \otimes h_1"'] \\
C_0 \otimes C_1 \arrow[r, "1 \otimes k_1"] & C_0 \otimes D_1 \arrow[r, "k_0 \otimes 1"] & D_0 \otimes D_1
\end{tikzcd}$$

Now squares (b), (c) are pullbacks by (i), while (a) and (d) are pullbacks by (ii), hence the outer square is also a pullback. □

6.3 Proposition. *Assume that the functor* $-\otimes A$: $\mathbf{A} \to \mathbf{A}$ *preserves pullbacks for every* $A \in |\mathbf{A}|$ *and that* o_B *is a (regular) monomorphism for all* $B \in |\mathbf{A}|$. *Then* $\mu(A, B)$ *is a (regular) monomorphism for all* A, $B \in |\mathbf{A}|$.

Proof. If o_B is a monomorphism, then the pullback of o_B with itself is trivial. (This argument even remains valid, if pullbacks are replaced by arbitrary connected limits.) By hypothesis, this pullback is preserved by $-\otimes A$ for $A \in |\mathbf{A}|$, hence $o_B \otimes 1_A$: $0 \otimes A \to B \otimes A$ and therefore $\mu(A, B)$ is monic.

If all o_B are even regular-monic, then by above argument all coproduct injections are monic. By 4.4, (2) is a pullback for all B, $C \in |\mathbf{A}|$. For any $A \in |\mathbf{A}|$, this diagram is preserved by the functor $-\otimes A$. Hence and all coproduct injections are even regular-monic by 4.3. □

6.4 Proposition. *If* (13) *is a pullback for all* A_i, $B_i \in |\mathbf{A}|$, f_i: $A_i \to B_i$, *then diagram* (6) *is a pullback for all* A, B, $C \in |\mathbf{A}|$, f: $B \to C$.

Proof. (6) is isomorphic to (13) for $A_0 := B, A_1 := 0, B_0 := C, B_1 := A, f_0 := f$, $f_1 := o_A$. □

6.5 Examples. Now it is easy to see that in 6.2 condition (i) does not imply (ii). Let L be a lattice with a bottom element 0, considered as a category in the usual way. Then (i) is satisfied if and only if $L$is distributive. In the other hand, if (ii) holds, then in the special case $A_0 := A_1 := 0$, $B_0 := B_1 := B$ we have $A_0 \otimes B_1 = A_1 \otimes B_0 = A_1 \otimes B_1 = B$, and the pullback property of (13) yields $B = 0$. Therefore only the trivial lattice $C = \{0\}$satisfies (ii).

In a cartesian closed category, the functor $-\pi A$ has a right adjoint and thus preserves colimits, in particular pushouts. Hence, if A has pushouts, $\mathbf{C}^{op}$ satisfies (i). On the other hand, it $\mathbf{C}^{op}$ satisfies (ii), then coproducts commute with pullbacks in $\mathbf{C}^{op}$, thus all product projections in $\mathbf{C}$ are regular-epic by (the dual of) 6.4 and 5.4. But if $\mathbf{C}$ also has an initial object 0, this is preserved by the right-adjoint functor $-\pi A$ for any object A, thus o_A: $0 \to A$ is regular-epic for all A, and $\mathbf{A}$ is trivial by 1.7.

In a semi-additive category $\mathbf{A}$, finite coproducts coincide with finite products; and if they exist, (i) is satisfied by the above argument, thus (6) is always a pullback by 6.4. On the other hand, (12) is a pullback only for $B = 0$, thus (5) is not a pullback for $A = C = 0$, $B \not\cong 0$. This shows that (5) is essential in 5.2 (iii). ☐

6.6 Theorem. *Assume that in* $\mathbf{A}$ *finite coproducts commute with pullbacks. Then coproducts are totally disjoint in* $\mathbf{A}$ *if and only if every* $B \in |\mathbf{A}|$ *with* $\mathbf{A}(B, 0) \neq \emptyset$ *is initial.*

Proof. The "only if" part follows immediately from 5.2 and 5.5. If all $B \in |\mathbf{A}|$ with $\mathbf{A}(B, 0) \neq \emptyset$ are initial, we get from 5.5 that diagram (11) is a pullback for all $B, C \in |\mathbf{A}|$, f: $B \to C$. For $A \in |\mathbf{A}|$, this pullback is preserved by $-\otimes A$ (see 6.2), hence (5) is a pullback. Moreover, (6) is also a pullback by 6.5, since (13) is always a pullback by 6.2. Thus coproducts are totally disjoint by 5.2. ☐

7. Universality

7.1 The notion of *universality* is defined for arbitrary colimits, but we restrict our attention to *finite coproducts*, because we are only interested in the relationship to the previous notions. Note that universality of finite coproducts is a fairly weak condition: in the category of small categories, arbitrary coproducts are universal, but coequalizers are not, and in the category of compact Hausdorff spaces, finite colimits are universal, while countable coproducts are not.

For some type of colimits existing globally in $\mathbf{A}$, these colimits can be formed in $\mathbf{A}/A$ as in $\mathbf{A}$ for all $A \in |\mathbf{A}|$. Here $\mathbf{A}/A$ denotes the comma category of *objects over* A, i.e. of morphisms into A. In this case, universality of these colimits means the same as preservation by all pullbacks functors f^*: $\mathbf{A}/B \to \mathbf{A}/A$ for f: $A \to B$. In particular, *finite coproducts* are universal if and only if *nullary* and *binary* coproducts are universal.

Universality of nullary coproducts, i.e. of the *initial objects* means the same as the equivalent conditions of 5.5 (look at condition (iii)!). Universality of binary coproducts means that $Z \cong X \otimes Y$ with injections $u\colon X \to Z$, $v\colon Y \to Z$ whenever the two squares in the following diagram are pullbacks:

$$(15)\quad \begin{tikzcd} X \arrow[r, "u"] \arrow[d, "g"'] & Z \arrow[d, "f"] & Y \arrow[l, "v"'] \arrow[d, "h"] \\ A \arrow[r, "\mu(A,B)"'] & A \otimes B & B \arrow[l, "\nu(A,B)"] \end{tikzcd}$$

Note that universality of binary coproducts implies universality of the initial object. Indeed, for arbitrary $f\colon Z \to 0 \otimes 0 \cong 0$ consider (19) for $A := B := 0$, $X := Y := Z$, $u := v := 1_Z$; $g := h := \mu(0,\ 0)^{-1}f = \nu(0,\ 0)^{-1}f$. (Note that $\mu(0,\ 0) = \nu(0,\ 0)$ is obviously invertible). Then both squares are pullbacks, universality gives $Z \cong Z \otimes Z$ with both injections being 1_Z. By 1.4, Z is pre-initial and thus even initial, because $f \in \mathbf{A}(Z,\ 0) \neq \emptyset$.

The next result from [3] can be used for a characterization of locally presentable quasi-topoi.

7.2 Proposition. *If finite coproducts are universal in* $\mathbf{A}$, *then all coproduct injections in* $\mathbf{A}$ *are monomorphisms, and for all* A, $B \in |\mathbf{A}|$, *the object* $Y \in |\mathbf{A}|$ *is pre-initial whenever the following square is a pullback:*

$$(16)\quad \begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "h"'] & A \arrow[d, "\mu(A,B)"] \\ B \arrow[r, "\nu(A,B)"'] & A \otimes B \end{tikzcd}$$

Proof. Consider (19) for $Z := A$, $f := \mu(A,\ B)$; then X, Y, g, h, u, v are defined uniquely up to natural isomorphism by the requirement that both squares be pullbacks. By universality we have $A \cong X \otimes Y$ with injections $u\colon X \to A$, $v\colon Y \to A$. The pullback condition on the left-hand square yiedls a unique $l\colon A \to X$ with $ul = gl = 1_A$. Therefore the split-epic coproduct injection u is invertible by 1.5, thus $l = u^{-1}$ and $u = l^{-1} = g$, whence $\mu(A,\ B)$ is monic.

From 1.3 we get that v is supercoconstant, hence Y is pre-initial by 2.4. ☐

7.3 Example. If L is a lattice with bottom element 0, then finite coproducts are universal if and only if L is distributive. Thus universality of coproducts does not imply that all o_B are extremally monic (see also 3.4 above). Thus 7.2 cannot be improved in general. This example is extremal in the sense that all object of L are pre-initial. On the other hand, universal coproducts behave much better under the mild hypothesis that all pre-initial objects are initial. In order to prove this, we use the following.

7.4 Lemma. *Assume that finite coproducts are universal and consider* $A, B, C \in |\mathbf{A}|$, $s\colon B \to C$. *If* $1_A \otimes s\colon A \otimes B \to A \otimes C$ *is an isomorphism, then there exists a pre-initial object* X *such that* $C \cong B \otimes X$ *with* s *as first injection.*

Proof. Consider (19) for $Z := C$, $f := (1_A \otimes s)^{-1}\nu(A, C)$ and X, Y, g, h, u, v defined by the requirement that the two squares be pullbacks. First observe that $\nu(A, C)sh = (1_A \otimes s)\nu(A, B)h = (1_A \otimes s)fv = \nu(A, C)v$. Since $\nu(A, C)$ is monic by 7.2, we get $sh = v$. But $v\colon Y \to C$ is the second injection of the coproduct $C \cong X \otimes Y$; hence v and therefore h is monic.

The pullback property of the right-hand square in (18) yields a morphism $t\colon B \to Y$ with $ht = 1_B$ and $vt = s$. In particular, h is also split-epic and thus invertible. Since we have $C \cong X \otimes Y$ with injections u and $v = sh$, we also get $C \cong B \otimes X$ with injections s and u. Now the functor $A \otimes -$ is faithful by 7.2 and 2.2; hence it reflects epimorphisms. Since $1_A \otimes s$ is epic (even invertible), we can conclude that s is epic. Thus u is supercoconstant by 1.3, hence X is pre-initial by 2.4, because coproduct injections are monic by 7.2. □

7.5 Theorem. *If coproducts are totally disjoint in* $\mathbf{A}$, *then every pre-initial object in* $\mathbf{A}$ *is initial. Conversely, if every pre-initial object is initial and if finite coproducts are universal, then coproducts are totally disjoint.*

Proof. If coproducts are totally disjoint, then all coproduct injections are regular-monic by 5.4, hence every pre-initial object is initial by 3.4. For the converse consider $A, B, C \in |\mathbf{A}|$, $s\colon B \to C$ with $1_A \otimes s$ invertible. By 7.4 we have $C \cong B \otimes X$ with s as first injection for some pre-initial object X.

Now our hypothesis gives $X \cong 0$, hence $C \cong B \otimes 0 \cong B$, and s is invertible. Thus $- \otimes A$ is conservative for any A, and from 3.2 we conclude that $- \otimes -\colon \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ is conservative.

Now let (3) be given and form the pullbacks:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "p"'] & A \otimes B \arrow[d, "g \otimes f"] & Y \arrow[r, "v", "\nu(D,C)"'] \arrow[d, "q"'] & A \otimes B \arrow[d, "g \otimes f"] \\
D \arrow[r, "\mu(D,C)"'] & D \otimes C & C \arrow[r] & D \otimes C
\end{tikzcd}$$

The pullback property yields unique morphisms $h\colon A \to X$, $k\colon B \to Y$ with $uh = \mu(A, B)$, $ph = g$, $vk = \nu(A, B)$, $qk = f$. On the other hand, by universality we get $X \otimes Y \cong A \otimes B$ with injections u, v, i.e. $h \otimes k\colon A \otimes B \to X \otimes Y$ is an isomorphism thus h and k are invertible by conservativity. Thus the two squares of (3) are pullbacks, proving total disjointness. □

7.6 Remarks. Note that [3] contains a faulty counterexample to 7.4. By 3.7 and 4.7 universality is essential.

From 7.5 and 5.4 we see that all coproduct injections are regular-monic in any category with universal finite coproducts and with all pre-initial objects initial. On the other hand, we cannot expect them to be split-monic Indeed, if the initial object is universal and all o_B are split-monic in $\mathbf{A}$, we conclude $B \cong 0$ for all $B \in |\mathbf{B}|$. But note that in **Set** all coproduct injections with non-empty domain are split-monic, arbitrary colimits are universal, and thus totally disjoint by 7.5.

Total disjointness is converse to universality in the following sense: Coproducts are totally disjoint if and only the two squares in (19) are pullbacks whenever $Z \cong X \otimes Y$ with injections $u\colon X \to Z$, $v\colon Y \to Z$.

Universality of finite coproducts does not follow from any of the conditions considered earlier. Indeed, in the category $\mathbf{Rng}_1$ of unital rings products commute with all connected colimits, particularly with pushouts, and products are totally codisjoint in $\mathbf{Rng}_1$ (i.e. coproducts are totally disjoint in $\mathbf{Rng}_1^{op}$). On the other hand the binary product $\mathbb{Z} \times \mathbb{Z}$ is not couniversal along the homomorphism

$$f\colon \mathbb{Z} \times \mathbb{Z} \to A, \quad (x, y) \mapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

into the ring A of (2×2)-matrices over $\mathbb{Q}$.

The above argument heavily rests upon the non-commutativity of A; in the category of commutative rings finite products are couniversal.

But still note that total disjointness enables us to test unviersality on a weakly terminal subcategory:

7.7 Proposition. *Assume that coproducts are totally disjoint in* $\mathbf{A}$ *and let* $\mathbf{C} \subset \mathbf{A}$ *be a weakly terminal full subcategory. Then finite coproducts are universal, if in diagram* (19) *we have* $Z \cong X \otimes Y$ *with injections* $u\colon X \to Z$, $v\colon Y \to Z$, *whenever both squares are pullbacks and* $A, B \in |\mathbf{C}|$.

Proof. The "only if" part is trivial. For the "if" part, let diagram (19) be given with both squares pullbacks, but with arbitrary $A, B \in |\mathbf{A}|$. By weak terminality, we can choose $A', B' \in |\mathbf{A}|$, $a\colon A \to A'$, $b\colon B \to B'$ and consider the diagram:
Then the upper part is diagram (19), hence the two upper squares are pullbacks by hypothesis. But the lower squares are pullbacks by total disjointness; thus the total left hand part and the total right-hand part are pullbacks by composition. Since A', $B' \in |\mathbf{C}|$ we have $Z \cong X \otimes Y$ with injections u, v, proving our claim. □

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "g"'] & Z \arrow[d, "f"'] & Y \arrow[l, "v"'] \arrow[d, "h"'] \\
A \arrow[r, "\mu(A,B)"] \arrow[d, "a"'] & A \otimes B \arrow[d, "a \otimes b"'] & B \arrow[l, "\nu(A,B)"'] \arrow[d, "b"'] \\
A \arrow[r, "\mu(A',B')"] & A' \otimes B & B \arrow[l, "\nu(A',B')"']
\end{tikzcd}$$

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