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A Service Points Location Problem with Min-Max Distance Optimality Criterion

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1. Introduction

The problem considered in this paper generalizes in a certain sense the problem of finding the absolute centre of a graph, which was studied e.g. in [1], [2], [3], [5], [6]. The problem studied in this paper can, similarly to the problem of the absolute centre of a graph, arise in e.g. the selection of sites for emergency service centres and involves the global minimization of certain piecewise-linear non-convex continuous function. We shall bring a motivating example. Let us suppose that m points $Y_1, \ldots, Y_m$ are given, which have to be served (or supplied) from n points $S_1, \ldots, S_n$. Each service point S_j is to be placed on a given segment A_jB_j; the points A_j, B_j are given (for all $j = 1, \ldots, n$). The distances $|Y_iA_j| = a_{ij}$, $|Y_iB_j| = a'_{ij}$, $|A_jB_j| = d_j$ are known non-negative numbers ($i = 1, \ldots, m$, $j = 1, \ldots, n$). In planning emergency transport systems, it is natural to evaluate any given proposal by the worst service it provides. The proposal for which the worst service is as good as possible is then accepted. We shall use this idea in the process of formulating our optimization problem. Let $x_j = |A_jS_j|$ for $j = 1, \ldots, n$, (i.e. x_j is the distance between A_j and S_j, which has to be chosen). It is then obviously $x_j \in [0, d_j]$.

The function

$$r_{ij}(x_j) \equiv \min (a_{ij} + x_j, a'_{ij} + d_j - x_j)$$

expresses thus the minimal distance we can choose if we serve the point Y_i from a chosen point $S_j \in A_jB_j$ with $|A_jS_j| = x_j$ (compare Figure 1).

The value

$$d_i(x_1, \ldots, x_n) = d_i(x) = \max_{1 \leq j \leq n} r_{ij}(x_j)$$

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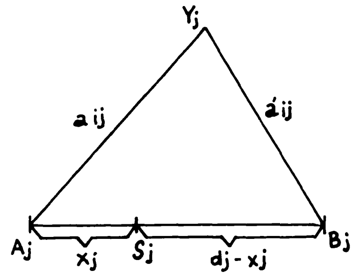

FIG. 1.

expresses thus the worst case, which can occur for the given point Y_i (if S_j's are chosen). Therefore we try to find x_j, $j = 1, \ldots, n$ in such a way that it solves the optimization problem

$$\left.\begin{aligned} &f(x_1, \ldots, x_n) = f(x) \equiv \max_{1 \leqq i \leqq m} d_i(x) \rightarrow \min \\ \text{subject to} \qquad & \\ &d_i(x) = \max_{1 \leqq i \leqq m} \min (a_{ij} + x_j, b_{ij} - x_j) \\ &i = 1, \ldots, m \quad 0 \leqq x_j \leqq d_j \quad j = 1, \ldots, n, \end{aligned}\right\} \qquad \text{(P0)}$$

where $b_{ij} \equiv a'_{ij} + d_j$.

The algorithm we are going to suggest looks for minima of certain piecewise-linear non-convex functions which requires $O(mn \log m)$ time, m being the number of supplied points and n number of located service points.

2. Theoretical background of the algorithm

We shall consider the following optimization problem:

$$\left.\begin{aligned} &f(x_1, \ldots, x_n) \equiv \max_{1 \leqq j \leqq n} \max_{1 \leqq i \leqq m} r_{ij}(x_j) \rightarrow \min \\ \text{subject to} \qquad & \\ &h_j \leqq x_j \leqq H_j \qquad j = 1, \ldots, n, \end{aligned}\right\} \qquad \text{(P1)}$$

where $r_{ij}(x_j) \equiv \min (a_{ij} + x_j, b_{ij} - x_j)$ for all i, j (a_{ij}, b_{ij}, h_j, H_j are given real numbers).

Note that in the objective the order of the operators is interchanged, what does not have effect upon the solution, as in both (P0) and (P1) the maximum distance between the service points and the supplied points is minimized. The problem (P1) is slightly more general than (P0), since a_{ij}, b_{ij}, h_j, H_j are arbitrary real numbers.

Denote $x_j^{(i)} \equiv (b_{ij} - a_{ij})/2$ for all i, j we have

$$r_{ij}(x_j) = a_{ij} + x_j \leqq b_{ij} - x_j \qquad \text{for} \quad x_j \leqq x_j^{(i)}$$

$$r_{ij}(x_j) = b_{ij} - x_j < a_{ij} + x_j \qquad \text{for} \quad x_j > x_j^{(i)}$$

Thus, $r_{ij}(x_j)$ is a piecewise-linear, concave function with slopes ± 1 attaining its maximum in $x_j^{(i)}$. For $x_j \in [h_j, H_j]$, $r_{ij}(x_j)$ is a function in x_j either linear and attaining its maximum at one of the points h_j or H_j, or piecewise-linear with a maximum at the point $x_j^{(i)}$.

Along each segment $[h_j, H_j]$, $j = 1, \ldots, n$,

the function
$$s_j(x_j) \equiv \max_{1 \leq i \leq m} r_{ij}(x_j)$$

is the piecewise-linear upper envelope of the functions $r_{ij}(x_j)$, where $r_{ij}(x_j)$ are the one- or two-piece linear concave functions in x_j. A typical plot of $s_j(x_j)$ along a segment is shown in the Figure 2.

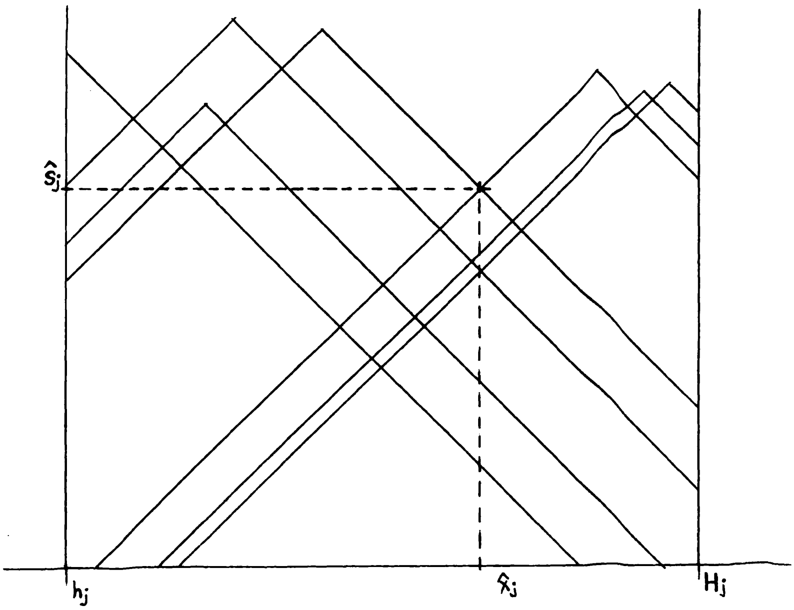

FIG. 2

We will solve the problem (P1) by taking the upper envelope $s_j(x_j)$ of the functions $r_{ij}(x_j)$, $i = 1, \ldots, m$, and inspecting it to find a minimum point $\hat{x}_j$. The optimum value then may be found by choosing the largest of the n values $r_{ij}(\hat{x}_j)$. The latter formulation suggests that the optimum value to the problem (P1) will be found by solving a sequence of the n problems (P2), where (P2) is defined for $j = 1, \ldots, n$ as follows:

$$\left.\begin{aligned} s_j(x_j) &= \max_{1 \leqq i \leqq m} r_{ij}(x_j) \to \min \\ &\text{subject to } h_j \leqq x_j \leqq H_j . \end{aligned}\right\} \tag{P2}$$

It is clear that any point $\hat{x}_j$ where the function $s_j(x_j)$ attains its minimum $\hat{s}_j \equiv s_j(\hat{x}_j)$ is either h_j or H_j, or $\hat{x}_j$ is a break point where two functions $r_{ij}(x_j)$ and $r_{kj}(x_j)$ intersect, $1 \leqq i, k \leqq m$. Furthermore the slopes of such $r_{ij}(x_j)$, $r_{kj}(x_j)$ have the opposite signs. There are at most $m(m-1)/2$ break points and to determine $\hat{x}_j$ is possible to examine at most $m(m-1)/2 + 2$ points on $[h_j, H_j]$ and select the best among them.

3. An algorithm for the problem (P2)

In this section, we present an algorithm for solving the problem (P2) for fixed j, $j \in \{1, \ldots, n\}$. The approach of Hakimi [2] or Cunninghame-Green [1] of finding the absolute centre of a network might be useful to solve the problem (P2). This section presents a simple new technique for determining the minimum value of the upper envelope of the piecewise-linear functions described in the Section 2. A numerical example and the complexity of an algorithm are given — this particular problem is solved in the same time, as in [1].

Let $x_j^{(i)}$, $1 \leqq i \leqq m$, be the point of the maximum of $r_{ij}(x_j)$ and let us define c_j as the minimum of the m values $r_{ij}(x_j^{(i)})$, i.e.,

$$c_j \equiv \min_{1 \leqq i \leqq m} r_{ij}(x_j^{(i)}) .$$

Let $\hat{s}_j$ be the minimum value of the function $s_j(x_j)$. For any function $r_{ij}(x_j)$, $1 \leqq i \leqq m$, let us denote

$$\bar{x}_j^i \equiv c_j - a_{ij}$$
$$\bar{\bar{x}}_j^i \equiv b_{ij} - c_j .$$

Lemma 1. $r_{ij}(\bar{x}_j^i) = r_{ij}(\bar{\bar{x}}_j^i) = c_j$ *for* $i = 1, \ldots, m$.

Proof. It is under our assumptions

$$c_j = \min_{1 \leqq i \leqq m} r_{ij}(x_j^{(i)}) = \min_{1 \leqq i \leqq m} (a_{ij} + x_j^{(i)}) = \min_{1 \leqq i \leqq m} (b_{ij} - x_j^{(i)}) = \min_{1 \leqq i \leqq m} (a_{ij} + b_{ij})/2 .$$

It follows from the definition of the functions $r_{ij}(x_j)$:

$$r_{ij}(\bar{x}_j^i) = \min(a_{ij} + \bar{x}_j^i, b_{ij} - \bar{x}_j^i) = \min(a_{ij} + c_j - a_{ij}, b_{ij} - c_j + a_{ij}) .$$

Since $c_j \leqq (a_{ij} + b_{ij})/2$,

$$b_{ij} - c_j + a_{ij} \geqq b_{ij} + a_{ij} - (a_{ij} + b_{ij})/2 = (a_{ij} + b_{ij})/2 \geqq c_j$$

and we have $r_{ij}(\bar{x}_j^i) = c_j$.

Similarly,

$$r_{ij}(\bar{\bar{x}}_j^i) = \min(a_{ij} + \bar{\bar{x}}_j^i, b_{ij} - \bar{\bar{x}}_j^i) = \min(a_{ij} + b_{ij} - c_j, b_{ij} - b_{ij} + c_j) = c_j .$$

Q.E.D.

Remark 1. If $r_{ij}(\bar{x}_j^{(i)}) = c_j$ for some $i \in \{1, \dots, m\}$ then $\bar{x}_j^i = \bar{\bar{x}}_j^i$.

Let π be an ordering of the set $\{1, \dots, m\}$ such that for $i = 1, \dots, m - 1$ either

$$\bar{x}_j^{\pi(i)} < \bar{x}_j^{\pi(i+1)}$$
$$\text{or} \quad \bar{x}_j^{\pi(i)} = \bar{x}_j^{\pi(i+1)} \quad \text{and} \quad \bar{\bar{x}}_j^{\pi(i)} \leqq \bar{\bar{x}}_j^{\pi(i+1)} .$$

According to the above ordering we can easily find an index which "takes an active part in determining the value $s_j(x_j)$".

We can suppose, without loss of generality, that no pair of indices (i, k), $1 \leqq i$, $k \leqq m$ exists that $\bar{x}_j^i \leqq \bar{x}_j^k$ and $\bar{\bar{x}}_j^i \geqq \bar{\bar{x}}_j^k$. In that case the piecewise-linear function $r_{kj}(x_j)$ corresponding to the index k does not have effect upon the solution and may be omitted.

a) the case $c_j \geqq \hat{s}_j$:

If we suppose $c_j \geqq \hat{s}_j$ then $\hat{s}_j$ may be found as follows:

Remark 2. (compare Fig. 3)

The inequality $c_j \leqq \hat{s}_j$ holds if

either $s_j(h_j) \leqq c_j$

or $s_j(H_j) \leqq c_j$

or $\bar{\bar{x}}_j^{\pi(i)} \leqq \bar{x}_j^{\pi(i+1)}$ for some $i \in \{1, \dots, m - 1\}$.

Along the interval $[\bar{\bar{x}}_j^{\pi(i)}, \bar{x}_j^{\pi(i+1)}]$ the minimum of the function $s_j(x_j)$ is attained at the point $(\bar{x}_j^{\pi(i+1)} + \bar{\bar{x}}_j^{\pi(i)})/2$ because the functions $r_{\pi(i)j}(x_j)$, $r_{\pi(i+1)j}(x_j)$ are there

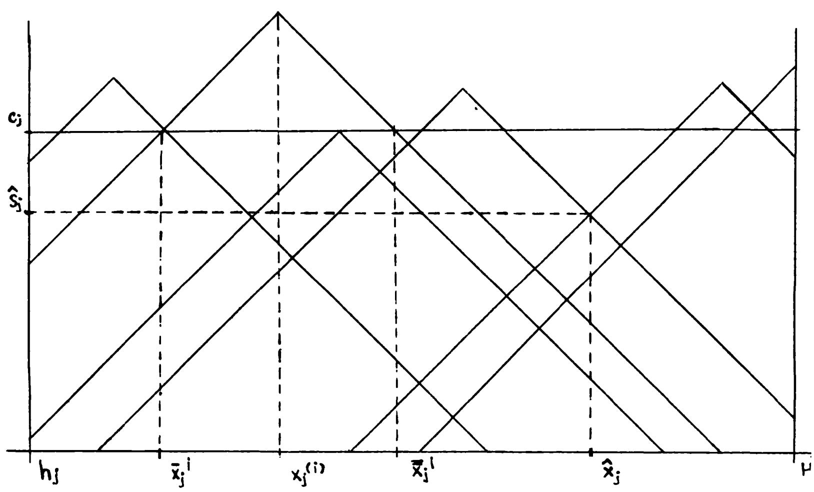

FIG. 3

linear with slopes -1, $+1$. The minimum solution value is then the minimum of the values $s_j((\underline{x}_j^{\pi(i+1)} + \bar{x}_j^{\pi(i)})/2)$ for which $\bar{x}_j^{\pi(i)} \leqq \underline{x}_j^{\pi(i+1)}$, $s_j(h_j)$ and $s_j(H_j)$.

The calculation of $\underline{x}_j^i$, $\bar{x}_j^i$, for $i = 1, \ldots, m$, requires $O(m)$ time. Then the ordering of the set $\{1, \ldots, m\}$ needs $O(m \log m)$ time. Finally, determining $s_j((\underline{x}_j^{\pi(i+1)} + \bar{x}_j^{\pi(i)})/2)$ for $i = 1, \ldots, m$, $s_j(h_j)$, $s_j(H_j)$ and minimum of them can be done in $O(m)$ time. Hence, we can summarize:

Theorem 1. *If $c_j \geqq \hat{s}_j$, the minimum solution value to the problem (P2) can be found in $O(m \log m)$ time.*

b) the case $c_j < \hat{s}_j$:

In the case $c_j < \hat{s}_j$, i.e. $s_j(h_j) > c_j$, $s_j(H_j) > c_j$ and $\bar{x}_j^{\pi(i)} > \underline{x}_j^{\pi(i+1)}$ for $1 \leqq i \leqq m - 1$, we will use a different approach to find the minimum of $s_j(x_j)$ along the interval $[h_j, H_j]$. A point of the minimum is either at the end points h_j or H_j, or at a point $\hat{x}_j$ where two functions $r_{\pi(i)j}(x_j)$, $r_{\pi(i+1)j}(x_j)$ intersect, $1 \leqq i \leqq m - 1$.

Notice, we have supposed that in the case $\underline{x}_j^i \leqq \underline{x}_j^k$ and $\bar{x}_j^i \geqq \bar{x}_j^k$ the index k has been omitted. Then only the following case can occur:

Lemma 2. *$\underline{x}_j^{\pi(i)} < \underline{x}_j^{\pi(i+1)}$ and $\bar{x}_j^{\pi(i)} < \bar{x}_j^{\pi(i+1)}$ for $1 \leqq i \leqq m - 1$.*

Proof. According to the ordering π, either $\underline{x}_j^{\pi(i)} < \underline{x}_j^{\pi(i+1)}$ or $\underline{x}_j^{\pi(i)} = \underline{x}_j^{\pi(i+1)}$ and $\bar{x}_j^{\pi(i)} \leqq \bar{x}_j^{\pi(i+1)}$.

If $\underline{x}_j^{\pi(i)} < \underline{x}_j^{\pi(i+1)}$ and $\bar{x}_j^{\pi(i)} \geqq \bar{x}_j^{\pi(i+1)}$ then the index $\pi(i+1)$ might be omitted.

Similarly, if $\underline{x}_j^{\pi(i)} = \underline{x}_j^{\pi(i+1)}$ and $\bar{x}_j^{\pi(i)} \leqq \bar{x}_j^{\pi(i+1)}$, then the index $\pi(i)$ might be omitted. This completes the proof.

Q.E.D.

After omitting of the needless indices we shall find the indices $\pi(s)$, $\pi(t)$ such that

$$s_j(h_j) = r_{\pi(s)j}(h_j) \quad \text{and} \quad s_j(H_j) = r_{\pi(t)j}(H_j) .$$

Thus, the indices $\pi(s)$, $\pi(t)$ correspond to the functions active in $s_j(x_j)$ at the end points of the interval $[h_j, H_j]$. Then we inspect only the break points of the pairs of the functions

$$[r_{\pi(s)j}(x_j), r_{\pi(s+1)j}(x_j)], [r_{\pi(s+1)j}(x_j), r_{\pi(s+2)j}(x_j)], \ldots, [r_{\pi(t-1)j}(x_j), r_{\pi(t)j}(x_j)]$$

and the points h_j, H_j for finding the minimum s_j of $s_j(x_j)$ along the interval $[h_j, H_j]$. It is obvious that the break point of any pair of the function $[r_{\pi(i)j}(x_j), r_{\pi(i+1)j}(x_j)]$ is

$$(b_{\pi(i)j} - a_{\pi(i+1)j})/2 \qquad \text{for} \quad i = 1, \ldots, m .$$

Thus, in the case $c_j < \hat{s}_j$ the algorithm requires $O(m \log m)$ time for ordering of the set $\{1, \ldots, m\}$, further $O(m)$ time for calculation of $\underline{x}_j^i$, $\bar{x}_j^i$, $i = 1, \ldots, m$. Also $O(m)$ time is required for omitting the needless indices, calculation of the above break points and selecting the minimum. Then the next result holds:

Theorem 2. *The minimum solution value to the problem (P2) can be found in $O(m \log m)$ time.*

We can use now the Theorems 1, 2 to construct an algorithm for solving (P2).

The algorithm AP2. (optimal solution of (P2)).
(1) determine order π of the set $\{1, \dots, m\}$;
(2) omit the "needless" indices;
(3) compute $x_j^{(i)}$, $\vec{x}_j^i$, $\overleftarrow{x}_j^i$ for $i = 1, \dots, m$;
(4) if $c_j < \hat{s}_j$ then go to 6;
(5) determine the minimum value $\hat{s}_j$ as in the Theorem 1, go to 7;
(6) determine the minimum value $\hat{s}_j$ as in the Theorem 2;
(7) end;
It remains to give an algorithm for solving (P1):

Theorem 3. *The problem (P1) can be solved in* $O(nm \log m)$ *time.*

Proof. As we have mentioned in the Section 2, the optimum value $\hat{s}$ to the problem (P1) can be found as follows:

Algorithm AP1. (optimal solution of (P1)).
(1) compute $\hat{x}_j$, $\hat{s}_j$ for $j = 1, \dots, n$ by the algorithm AP2;
(2) find the maximum $\hat{s}$ of $\hat{s}_j$'s, $j = 1, \dots, n$;
(3) end;
Trivially, we use the algorithm AP1 n times what requires $O(nm \log m)$ time.
Q.E.D.

Now we consider the following example. Suppose the case of two intervals and six points which have to be served. The Table 1 gives the entries h_j, H_j, a_{ij}, b_{ij}, for $i = 1, \dots, 6$, $j = 1, 2$.

TABLE 1

	i	1	2	3	4	5	6
$h_1 = 4$	a_{i1}	-3	3	6	-7	-8	-7
$H_1 = 20$	b_{i1}	18	17	16	27	35	22
$h_2 = -3$	a_{i2}	10	14	11	2	0	6
$H_2 = 15$	b_{i2}	10	11	20	24	28	23

The Table 2 gives the computed values $x_j^{(i)}$, $\vec{x}_j^i$, $\overleftarrow{x}_j^i$, c_j for all i, j. Further according to the relation between c_j and $\hat{s}_j$ is the minimum $\hat{s}_1$ computed by the step (5) and $\hat{s}_2$ by the step (6) of the algorithm AP1. Note that $\hat{s}_1$ is realized at the inner point $\hat{x}_1 = 12.5$ of the interval $[4, 20]$ and $\hat{s}_2$ is realized at the inner point $\hat{x}_2 = 0$ as well as at the end point $h_2 = -3$ of the interval $[-3, 15]$. The optimum value to the problem (P1) is then $\hat{s} = \max(7.5, 10) = 10$.

TABLE 2

	i	1	2	3	4	5	6
$c_1 = 7.5$	$x_1^{(i)}$	10.5	7	5	17	21.5	14.5
$\hat{x}_1 = 12.5$	$\underline{x}_1^i$	10.5	4.5	1.5	14.5	15.5	7.5
$\hat{s}_1 = 5.5$	$\bar{x}_1^i$	10.5	9.5	8.5	19.5	27.5	7.5
$c_2 = 10$	$x_2^{(i)}$	0	-1.5	4.5	11	14	8.5
$\hat{x}_2 = -3, 0$	$\underline{x}_2^i$	0	-4	-1	8	10	4
$\hat{s}_2 = 11$	$\bar{x}_2^i$	0	1	10	14	18	13

The optimum value: $\hat{s} = \max(\hat{s}_1, \hat{s}_2) = 11$.
Location: $(\hat{x}_1, \hat{x}_2) = (12.5, -3)$ or $(12.5, 0)$.

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