Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 1, 3--11

Persistent URL: `http://dml.cz/dmlcz/142599`

1989 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 30. NO. 1

Quasigroups Having at most Three Inner Mappings

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Received 2 February 1988

In the paper, the quasigroups with at most three inner mappings are described.

V článku se popisují kvazigrupy, které mají nejvýše tři vnitřní permutace.

В статье изучаются квазигруппы имеющие не более 3 внутренных подстановок.

1. Introduction

Let Q be a quasigroup. We denote by $\mathcal{M}_l(Q)$ ($\mathcal{M}_r(Q)$) the left (right) multiplication group of Q, i.e. the permutation group generated by all $L_a(R_a)$, $a \in Q$; here, $L_a(x) = ax$ and $R_a(x) = xa$ for every $x \in Q$. Futher, let $\mathcal{M}(Q)$ be the multiplication group of Q. For $a \in Q$, we put $\mathcal{I}(Q, a) = \{f \in \mathcal{M}(Q); f(a) = a\}$. Clearly, the inner mapping groups $\mathcal{I}(Q, a)$ are isomorphic and we can define $\mathrm{i}(Q) = \mathrm{card}\,(\mathcal{I}(Q, a))$.

1.1. Proposition. Let Q be a loop with $\mathrm{i}(Q) \leqq 3$. Then Q is an abelian group and $\mathrm{i}(Q) = 1$.

Proof. For $\mathrm{i}(Q) \leqq 2$, the result is proved in [1]. Hence, assume that $\mathrm{i}(Q) = 3$. Then $\mathcal{I}(Q, 1) = \{1, g, g^2\}$, where $g^3 = 1$. Further, assume for a moment that Q is not commutative. Then $ab \neq ba$ for some $a, b \in Q$. We have $f(b) \neq b$, where $f = R_a^{-1}L_a$. But $f(1) = 1$, $f(a) = a$, and hence either $f = g$ or $f = g^2$. Similarly, $h = R_b^{-1}L_b$ is equal either to g or to g^2. In particular, either $f = h$ or $f = h^2$ and, anyway, $f(b) = b$, a contradiction. We have proved that Q in commutative. Put $f_{a,b} = L_b^{-1}L_a^{-1}L_{ab}$ for all $a, b \in Q$. Then $f_{a,b}(a) = a$. We have $f_{a,b} \in \{1, g, g^2\}$, and so either $f_{a,b} = 1$ or $f_{a,b} \in \{g, g^2\}$ and $g(a) = a$. Now, let $a, b, c \in Q$ be such that $a \,.\, bc \neq ab \,.\, c$. Then $f_{b,c}(a) \neq a$, so that $f_{b,c} \in \{g, g^2\}$, $f_{a,b} = 1$ and $a \,.\, bc = ab \,.\, c$, a contradiction. We have proved that Q is associative.

1.2. Corollary. Let Q be a quasigroup with $\mathrm{i}(Q) \leqq 3$. Then Q is isotopic to an abelian group.

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A quasigroup Q is said to be
-- medial if it satisfies $xy \,.\, uv = xu \,.\, yv$,
-- left modular if it satisfies $x \,.\, yz = z \,.\, yx$,
-- right modular if it satisfies $xy \,.\, z = zy \,.\, x$,
-- left permutable if it satisfies $x \,.\, yz = y \,.\, xz$,
-- right permutable if it satisfies $xy \,.\, z = xz \,.\, y$,
-- an LIP – quasigroup if there is a mapping f of Q into Q such that $f(x) \,.\, xy = y$ for all $x, y \in Q$,
-- an RIP – quasigroup if there is a mapping f of Q into Q such that $yx \,.\, f(x) = y$ for all $x, y \in Q$,
-- an IP – quasigroup if it is both left and right IP – quasigroup.

2. Auxiliary results

In this section, let $G(+)$ be an abelian group containing at least three elements and let g be a permutation of G such that $g \neq \mathrm{id}_G = g^3$ and $g(0) = 0$.

Put $A = \{a \in G;\ g(a + x) = g(a) + x \text{ for every } x \in G\}$, $B = \{b \in G;\ g(b + x) = g(b) + g(x) \text{ for every } x \in G\}$ and $C = \{c \in G;\ g(c + x) = g(c) + g^2(x) \text{ for every } x \in G\}$.

2.1. Lemma. $A = \emptyset$ and $C \neq G$.

Proof. Let, on the contrary, $a \in A$. Then $0 = g(0) = g(a - a) = g(a) - a$, so that $g(a) = a$. Consequently, $g(a + x) = a + x$ for every $x \in G$, and therefore $g = \mathrm{id}_G$, a contradiction. Proceeding similarly, we can show that $C \neq G$.

2.2. Lemma. $0 \in B$ and $B \cap C = \emptyset$.

Proof. Obvious.

2.3. Lemma. $B = G$ iff g is an automorphism of $G(+)$.

Proof. Obvious.

In the rest of this section, we shall assume that $G = B \cup C$ and $C \neq \emptyset$. Then $\emptyset \neq B \neq G$.

2.4. Lemma. $g \mid B = \mathrm{id}_G$.

Proof. Let $b \in B$ and $c \in C$. Then $g^2(b) + g(c) = g(b + c) = g(b) + g(c)$, so that $g^2(b) = g(b)$ and $b = g(b)$.

2.5. Lemma. $g(C) \subseteq C$.

Proof. Let, on the contrary, $c \in C$ be such that $g(c) \in B$. By 2.4, $g^2(c) = g(c)$, hence $c = g(c) \in B \cap C$, a contradiction with 2.2.

2.6. Lemma. $g(c) \neq c$ for every $c \in C$.

Proof. Let, on the contrary, $g(c) = c$ for some $c \in C$. Then, for every $a \in C$, $c + g^2(a) = g(c + a) = g(a) + g^2(c) = g(a) + c$, and so $g^2(a) = g(a)$ and $g(a) = a$. We have proved $g \mid c = \mathrm{id}_C$ and hence $g = \mathrm{id}_G$ by 2.4, a contradiction.

2.7. Lemma. $B = \{b \in G; g(b) = b\}$.

Proof. The result follows from 2.4 and 2.6.

2.8. Lemma. Let $a, b \in B$ (resp. $a, b \in C$). Then $a + b \in B$.

Proof. If $a, b \in B$, then $g(a + b) = g(a) + g(b) = a + b$ by 2.4 and $a + b \in B$ by 2.7. Now, let $a, b \in C$. Then $g(a) \in C$ by 2.5 and we have $g(a + b) = g^2(a) + g(b) = g(g(a) + g^2(b)) = g^2(a + b)$, so that $a + b = g(a + b)$ and $a + b \in B$ by 2.7.

2.9. Lemma. B is a subgroup of index 2 in $G(+)$.

Proof. If $b \in B$, then $0 = g(b - b) = g(b) + g(-b)$, $g(-b) = -g(b) = -b$ and $-b \in B$ by 2.7. Now, from 2.8 it follows that B is a subgroup of $G(+)$. Again by 2.8, B is of index 2.

2.10. Lemma. Let $u \in C$ and $v = g(u)$. Then $v \in C$, $u \neq v$, $3u = 3v$, $C = B + u$ and $g(b + u) = b + v$ for each $b \in B$.

Proof. By 2.9 and 2.4, $C = B + u$, $g(b + u) = b + v$ for each $b \in B$. Since $g \neq \mathrm{id}_G$, $u \neq v$. Finally, $u = g^3(u) = g^2(v) = g(g(v - u + u)) = g(2v - u) = g(2v - 2u + u) = 3v - 2u$, and hence $3u = 3v$.

2.11. Lemma. There is an element $0 \neq w \in B$ such that $3w = 0$ and $g(c) = c + w$ for every $c \in C$.

Proof. The result follows from 2.10.

3. Auxiliary results

In this section let Q be a quasigroup with $\mathrm{i}(Q) = 3$. Suppose that Q is not a right loop. Take an element $0 \in Q$, put $g = R_z$ (where $z \in Q$ is such that $0 \,.\, z = 0$), $h = L_0$ and $x + y = g^{-1}(x)\, h^{-1}(y)$ for all $x, y \in Q$. Then $Q(+)$ is an abelian group. Since Q is not a right loop, $g \neq \mathrm{id}_Q$. Clearly, $g(0) = 0$, and hence $\mathscr{I}(Q, 0) = \{1, g, g^2\}$. In particular, $g^3 = \mathrm{id}_Q$.

We have $xy = g(x) + h(y)$ for all $x, y \in Q$. Put $h(0) = v$.

3.1. Lemma. Just one of the following three cases takes place:

(i) $h(x) = x + v$ for every $x \in Q$.
(ii) $h(x) = g(x) + v$ for every $x \in Q$.
(iii) $h(x) = g^2(x) + v$ for every $x \in Q$.

Proof. Let $u \in Q$ be such that $u\,0 = 0$. Then $L_u \in \{1, g, g^2\}$. However, $u = g^2(-h(0)) = g^2(-v)$ and $L_u = L^+_{-v}h$. The rest is clear.

Further, define the sets A, B, C similarly as in the preceding section.

3.2. Lemma. $Q = A \cup B \cup C$.

Proof. Let $a \in Q$ and $k = L^+_{-g(a)}\, g L^+_a$. Then $k(0) = -g(a) + g(a) = 0$, so that $k \in \{1, g, g^2\}$. If $k = 1$, then $a \in A$; if $k = g$, then $a \in B$; if $k = g^2$, then $a \in C$.

Now, suppose that $B \neq Q$, i.e. $C \neq Q$. Then, by 2.9 and 2.11, B is a subgroup of $Q(+)$, B is of index 2, $g \mid B = \mathrm{id}_B$ and there is an element $0 \neq w \in B$ such that $3w = 0$ and $g(c) = c + w$ for every $c \in C$ (then $g^2 \mid B = \mathrm{id}_B$ and $g^2(c) = c + 2w = c - w$).

4. Auxiliary results

In this section, let $G(+)$ be an abelian group having at least six elements, let B be a subgroup of index two, $C = G - B$, and let $0 \neq w \in B$ be such that $3w = 0$. Further, let $v \in G$ be arbitrary.

Define a multiplication on G by $bx = b + x + v$ and $cx = c + x + w + v$ for all $b \in B$, $c \in C$, $x \in G$. Then we get a groupoid which is clearly a quasigroup. We denote this quasigroup by $G = G[+, B, w, v, 1]$.

4.1. Lemma. (i) For $x, y \in G$, $xy = yx$ iff either $x, y \in B$ or $x, y \in C$.

(ii) The quasigroup G is not commutative.

(iii) G is a left loop and G is not a right loop.

(iv) If $v \in B$, then $-v$ is the left unit element of G.

(v) If $v \in C$, then $-w - v$ is the left unit element of G.

(vi) $r = (B \times B) \cup (C \times C)$ is a congruence of G and G/r is a two-element group.

(vii) Q is an LIP $-$ quasigroup and Q is not an RIP $-$ quasigroup.

(viii) Q is left permutable and is not right permutable.

Proof. Easy.

4.2. Lemma. If $v \in B$, then the mapping $x \to x + v$ is an isomorphism of $G[+, B, w, v, 1]$ onto $G[+, B, w, 0, 1]$.

Proof. The assertion may be checked easily.

4.3. Lemma. If $v \in C$, then the mapping $x \to x + v + w$ is an isomorphism of $G[+, B, w, v, 1]$ onto $G[+, B, w, 0, 1]$.

Proof. The assertion may be checked easily.

We have proved that the quasigroups $G[+, B, w, v, 1]$ and $G[+, B, w, 0, 1]$ are isomorphic. In the rest of this section, we shall assume that $v = 0$ and we put $G = G[+, w, 0, 1]$.

4.4. Lemma. G is not medial.

Proof. Let $c \in C$. Then $c0 \,.\, c0 = (c + w)(c + w) = 2c + 3w \neq 2c + w = (2c + w) \,.\, 0 = cc \,.\, 00$.

Put $g(b) = b$ and $g(c) = c + w$ for all $b \in B$, $c \in C$. Then g is a permutation of G, $g \neq \mathrm{id}_G$ and $g^3 = \mathrm{id}_G$. Further, let $h(b) = b + w$ and $h(c) = c$ for all $b \in B$, $c \in C$. Again, $h \neq \mathrm{id}_G$ and $h^3 = \mathrm{id}_G$. Clearly, $gh = hg = L_w^+$, $L_b^+ g = gL_b^+$ and $L_c^+ h = gL_c^+$ for all $b \in B$, $c \in C$.

4.5. Lemma. $\mathcal{M}_l(G) = \mathcal{M}(G +)) = \{L_a^+ ; a \in G\}$ and $\mathcal{M}_r(G) = \mathcal{M}(G) = \{L_a^+, L_a g, L_a^+ g^2; a \in G\}$.

Proof. Easy.

4.6. Lemma. $\mathcal{I}(G, 0) = \{1, g, g^2\}$, and so $\mathrm{i}(G) = 3$.

Proof. This follows from 4.5.

Finally, let B' be a subgroup of index 2 of an abelian group $G'(+)$, let $0 \neq w' \in B'$, $3w' = 0$, and let $f\colon G \to G'$ be a mapping.

4.7. Lemma. The following conditions are equivalent:

(i) f is an isomorphism of $G = G[+, B, w, 1]$ onto $G' = G'[+, B', w', 1]$.

(ii) f is an isomorphism of $G(+)$ onto $G'(+)$, $f(B) = B'$ and $f(w) = w'$.

Proof. (i) implies (ii). Clearly, $f(0) = 0$ (f preserves left units). Further, let $b \in B$. If $f(b) \notin B'$, then $g(b) = f(b\,0) = f(b)\,0 = f(b) + w'$, a contradiction. Consequently, $f(B) \subseteq B'$ and, conversely, $f^{-1}(B') \subseteq B$, so that $f(B) = B'$. Now, for any $x \in G$, $f(b + x) = f(bx) = f(b)f(x) = f(b) + f(x)$. On the other hand, if $c \in C$, then $f(c + x + w) = f(cx) = f(c)f(x) = f(c) + f(x) + w'$ for every $x \in G$. In particular, $f(c + w) = f(c) + w'$ and $f(c + w + x) = f(c + w + x) = f(c + w) + f(x)$. We have proved that f is an isomorphism of $G(+)$ onto $G'(+)$. Finally, $f(w) = f(c\,.\,(-c)) = f(c) - f(c) + w' = w'$. (ii) implies (i). This implication is evident.

Finally, we put $G[+, B, w, 2] = G[+, B, w, 1]^{\mathrm{op}}$.

5. Auxiliary results

In this section, let $G(+)$ be an abelian group with at least six elements, let B be a subgroup of index two, $C = G - B$, and let $O \neq w \in B$ be such that $3w = 0$. Further, let $v \in G$ be arbitrary.

Define a multiplication on G by $bb' = b + b' + v$, $cc' = c + c' + v - w$ and $bc = cb = b + c + w + v$ for all $b, b' \in B$, $c, c' \in C$. We get a groupoid which is a quasigroup and we denote it by $G = G[+, B. w, v, 3]$.

5.1. Lemma. (i) The quasigroup G is commutative.

(ii) If $v \in B$, then $(-v)\,.\,(-v) = -v$ and $-v$ is the only idempotent of G.

(iii) If $v \in C$, then $(w - v)(w - v) = w - v$ and $w - v$ is the only idempotent of G.

(iv) $r = (B \times B) \cup (C \times C)$ is a congruence of G and G/r is a two – element group.
(v) G is not an IP-quasigroup.

Proof. Easy.

5.2. Lemma. If $v \in B$, then the mapping $x \to x + v$ is an isomorphism of $G[+, B, w, v, 3]$ onto $G[+, B, w, 0, 3]$.

Proof. The assertion may be checked easily.

5.3. Lemma. If $v \in C$, then the mapping $x \to x + v - w$ is an isomorphism of $G[+, B, w, v, 3]$ onto $G[+, B, w, 0, 3]$.

Proof. The assertion may be checked easily.

We have proved that the quasigroups $G[+, B, w, v, 3]$ and $G[+, B, w, 3] = G[+, B, w, 0, 3]$ are isomorphic. In the rest of this section, we shall assume that $v = 0$.

5.4. Lemma. G is not medial.

Proof. For $c \in C, 0c \,.\, 0c = (c + w)(c + w) = 2c + w \neq 2c - w = 0 \,.\, (2c - w) = 00 \,.\, cc$.

Put $g(b) = b$ and $g(c) = c + w$ for all $b \in B$, $c \in C$. Further, let $h(b) = b + w$ and $h(c) = c$. Then $L_b^+ g = g L_b^+$ and $L_c^+ h = g L_c^+$.

5.5. Lemma. $\mathscr{M}(G) = \{L_a^+, L_a^+ g, L_a^+ g^2; a \in G\}$.

Proof. Easy.

5.6. Lemma. $\mathscr{I}(G, 0) = \{1, g, g^2\}$ and $\mathrm{i}(G) = 3$.

Proof. This follows from 4.5.

Finally, let B' be a subgroup of index 2 of an abelian group $G'(+)$, let $0 \neq w' \in B'$, $3w' = 0$, and let $f: G \to G'$ be a mapping.

5.7. Lemma. The following conditions are equivalent:
(i) f is an isomorphism of $G = G[+, B, w, 3]$ onto $G' = G'[+, B', w', 3]$.
(ii) f is an isomorphism of $G(+)$ onto $G'(+)$, $f(B) = B'$ and $f(w) = w'$.

Proof. Similar to that of 4.7.

6. Auxiliary results

In this section, let $G(+)$ be an abelian group with at least six elements, let B be a subgroup of index two, $C = G - B$, and let $0 \neq w \in B$ be such that $3w = 0$. Furher, let $v \in G$ be arbitrary.

Define a multiplication on G by $bb' = b + b' + v$, $cc' = c + c' + v$, $bc = b + c + v - w$ and $cb = b + c + v + w$ for all $b, b' \in B$, $c, c' \in C$. We get a groupoid $G = G[+, B, w, v, 4]$ which is a quasigroup.

6.1. Lemma. (i) For $x, y \in G$, $xy = yx$ iff either $x, y \in B$ or $x, y \in C$.
(ii) $-v$ is the only idempotent of G.
(iii) G is neither a left nor a right loop.
(iv) $r = (B \times B) \cup (C \times C)$ is a congruence of G and G/r is a two – element group.
(v) G is neither an LIP-quasigroup nor an RIP-quasigroup.

Proof. Easy.

6.2. Lemma. Let $v \in B$. The mapping $x \to x + v$ is an isomorphism of $G[+, B, w, v, 4]$ onto $G[+, B, w, 0, 4]$.

Proof. Easy.

6.3. Lemma. Let $v \in C$. The mapping $x \to -x - v$ is an isomorphism of $G[+, B, w, v, 4]$ onto $G[+, B, w, 0, 4]$.

Proof. Easy.
In the rest of this section, we shall assume that $v = 0$.

6.4. Lemma. G is not medial.

Proof. Let $c \in C$. Then $00 \, . \, cc = 0.2c = 2c - w \neq 2c - 2w = (c - w)(c - w) = 0c \, . \, 0c$.
Put $g(b) = b$ and $g(c) = c + w$ for all $b \in B$, $c \in C$.

6.5. Lemma. $\mathscr{M}_l(G) = \mathscr{M}_r(G) = \mathscr{M}(G) = \{L_a^+, L_a^+ g, L_a^+ g^2; a \in G\}$.

6.6. Lemma. $\mathscr{I}(G, 0) = \{1, g, g^2\}$ and $\mathrm{i}(G) = 3$.

Proof. See 6.5.

6.7. Lemma. The opposite quasigroup G^{op} is equal to $G[+, B, -w, 4]$. In particular G and G^{op} are isomorphic.

Proof. Obvious.
Finally, let B' be a subgroup of index 2 of an abelian group $G'(+)$, let $0 \neq w' \in B'$, $3w' = 0$, and let $f: G \to G'$ be a mapping.

6.8. Lemma. The following conditions are equivalent:
(i) f is an isomorphism of $G = G[+, B, w, 4]$ onto $G' = G'[+, B', w', 4]$.
(ii) f is an isomorphism of $G(+)$ onto $G'(+)$, $f(B) = B'$ and $f(w) = w'$.

Proof. Similar to that of 4.7.

7. Quasigroups with $\mathrm{i}(Q) \leqq 3$

The following statements are well known.

7.1. Proposition. A quasigroup Q is medial iff there exist an abelian group $Q(+)$, commuting automorphisms g, h of $Q(+)$ and an element $e \in Q$ such that $xy =$

$= g(x) + h(y) + e$ for all $x, y \in Q$. In this case:

(i) Q is commutative iff $g = h$;
(ii) Q is a left (right) loop iff $h = \mathrm{id}_Q$ ($g = \mathrm{id}_Q$).
(iii) Q is left (right) modular iff $g = h^2$ ($h = g^2$).
(iv) Q is modular iff $g^3 = \mathrm{id}_Q$ and $h = g^2$.
(v) Q is left (right) permutable iff $h = \mathrm{id}_Q$ ($g = \mathrm{id}_Q$).
(vi) If $g = h$, then Q satisfies the identity $x(y \,.\, uv) = v(y \,.\, ux)$ iff $g^2 = \mathrm{id}_Q$.
(vii) If $g = h$, then $g^3 = \mathrm{id}_Q$ iff satisfies the identity $x(y(u \,.\, vw)) = w(y(u \,.\, vx))$.
(viii) If $h = \mathrm{id}_Q$, then $g^3 = \mathrm{id}_Q$ iff Q satisfies the identity $(xy \,.\, u)\, v = (vy \,.\, u)\, x$.
(ix) Q is an LIP – quasigroup (RIP – quasigroup) iff $h^2 = \mathrm{id}_Q$ ($g^2 = \mathrm{id}_Q$).
(x) $\mathscr{I}(Q, 0)$ is just the permutation group generated by g and h.

7.2. Theorem. Let Q be a quasigroup.

(i) $\mathrm{i}(Q) = 1$ iff Q is an abelian group.
(ii) $\mathrm{i}(Q) = 2$ iff Q is not an abelian group and at least one (and then just) one of the following cases takes place:
(a) Q is a commutative medial quasigroup satisfying the identity $x(y \,.\, uv) = v(y \,.\, ux)$.
(b) Q is left modular and right permutable (then Q is a right loop).
(c) Q is right modular and left permutable (then Q is a left loop).

In all these cases, Q is a medial IP – quasigroup.

Proof. See 4.1 and [1, Theorem 4.6].

7.3. Theorem. The following conditions are equivalent for a quasigroup Q:

(i) Q is medial and $\mathrm{i}(Q) = 3$.
(ii) Q is not an abelian group and at least one (then just one) of the following cases takes place:
(a) Q is commutative and satisfies the identity $x(y(u \,.\, vw)) = w(y(u \,.\, vx))$.
(b) Q is left permutable and satisfies the identity $(xy \,.\, u)\, v = (vy \,.\, u)\, x$ (then Q is a left loop).
(c) Q is right permutable and satisfies the identity $x(y \,.\, uv) = v(y \,.\, ux)$ (then Q is a right loop).
(d) Q is modular.

Proof. Apply 4.1.

7.4. Theorem. Let Q be a quasigroup such that Q is not medial and $\mathrm{i}(Q) = 3$. Then there exist an abelian group $Q(+)$, its subgroup B of index 2 and an element $0 \neq w \in B$, $3w = 0$, such that Q is equal to at least one (and then to exactly one) from the quasigroups $Q[+, B, w, 1]$, $Q[+, B, w, 2]$, $Q[+, B, w, 3]$, $Q[+, B, w, 4]$.

Proof. Apply the results of the preceding sections.

7.5. Proposition. Let $G(+)$ be an abelian group with at least six elements, B its subgroup of index 2 and $0 \neq w \in B$, $3w = 0$. Then:

(i) None of the quasigroups $G_j = G[+, B, w, j]$, $1 \leqq j \leqq 4$, is medial and $\mathrm{i}(G_j) = 3$.
(ii) G_1 is a left loop, G_2 is a right loop, G_3 is commutative, $G_4 \neq G_4^{\mathrm{op}}$ and G_4 is isomorphic to G_4^{op}.
(iii) None of the quasigroups G_1, G_2, G_3, G_4 is simple.
(iv) $G[+, B, w, j]$ is isomorphic to $G'[+, B', w', j']$ iff $j = j'$ there and is an isomorphism $f\colon G(+) \to G'(+)$ such that $f(B) = B'$ and $f(w) = w'$.

Proof. Apply the results of the preceding sections.

Reference

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