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Luděk Zajíček

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Alternative Definitions of J-Density Topology

L. ZAJÍČEK

Department of Mathematical Analysis, Charles University*)

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Some new equivalent definitions of the Wilczyński's J-density topology on the real line are given.

V článku je ukázáno několik nových ekvivalentních definic pro Wilczyńského kategoriální hustotní topologii na přímce.

Показываются некоторые новые определения топологии Вилчиньского.

1. Introduction

The present article contains proofs of some results presented in my lecture on Scuola di Analisi Reale, Ravello 1985.

W. Wilczyński [5] defined the J-density topology on $\mathbb{R}$ which is in a sense a category analogue of the density topology on $\mathbb{R}$. The properties of the J-density topology and its generalization to $\mathbb{R}^n$ were investigated in several articles (for a survey of results see [6]). The J-density topology is an interesting example of an "abstract category density topology" (cf. [3] and [7]).

In [7] was given a definition of p^*-topology ($*$-porosity topology) which provides a natural generalization of the J-density topology to an arbitrary metric space. The fact that the p^*-topology coincides on the real line with the J-density topology is not proved in [7]; there it is observed only that this statement can be easily proved on the base of a still unpublished result of E. Lazarow ([2], cf. [5], Theorem 44). Our main Theorem gives a proof of the mentioned fact, which is independent on the Lazarow's result; it shows also a new alternative definition of the J-density topology (condition (4) in Theorem).

*) Sokolovská 83, 186 00 Praha 8, Czechoslovakia.

2. Preliminaries

In the sequel we use the usual notations $A + x = \{a + x;\ a \in A\}$ and $\lambda A = \{\lambda x;\ x \in A\}$. The characteristic function of a set A will be denoted by X_A. The symbol $\mathbb{N}$ stands for the set of all positive integers. The symbol $A \subset^* B$ means that $A \setminus B$ is a first category set.

Let $A \subset \mathbb{R}$ have the Baire property. Then we say [5] that O is an J-density point of A if and only if for any increasing sequence of positive integers (n_k) there exists a subsequence (n_{k_p}) such that $\lim_{p \to \infty} X_{n_{k_p} A} = 1$ on a residual subset of $(-1, 1)$. We say that $x \in \mathbb{R}$ is an J-density point of A if and only if O is an J-density point of $A - x$.

It is proved in [4] that the system τ of all sets $A \subset \mathbb{R}$ such that A has the Baire property and all points of A are J-density points of A forms a topology on $\mathbb{R}$. The topology τ is called J-density topology.

A set $V \subset \mathbb{R}$ is a τ-neighbourhood of a point $x \in V$ if and only if there exists a set $U \subset V$ which has the Baire property and x is an J-density point of U (cf. [6]).Thus we see that any alternative definition of J-density points yields an alternative definition of the J-density topology.

Let (P, ϱ) be a metric space. The open ball with the centre $x \in P$ and the radius $r > 0$ will be denoted by $U(x, r)$.

Let $M \subset P$, $x \in P$, $R > 0$. Then we denote the supremum of the set of all $r > 0$ for which there exists $y \in P$ such that $U(y, r) \subset U(x, R) \setminus M$ by $\gamma(x, R, M)$. We say that M is porous at x if $\limsup_{R \to 0+} \gamma(x, R, M)/R > 0$. We say that $E \subset P$ is superporous at x if $E \cup F$ is porous at x whenever F is porous at x.

A set $G \subset P$ is said to be p-open (porosity open) if $P \setminus G$ is superporous at any point of G.

A set $H \subset P$ is said to be p^*-open ($*$-porosity open) if it is of the form $H = G \setminus N$, where G is a p-open set and N is a first category set.

The systems of p-open sets and p^*-open sets form topologies [7] which are labeled as the porosity topology (p-topology) and $*$-porosity topology (p^*-topology), respectively. The following characterization [7] of p-interior points is useful for applications.

Proposition A. A set $V \subset P$ is a p-neighbourhood of a point $x \in V$ if and only if the following condition (C) is satisfied. (C) For any $u > 0$ there exist $d > 0$ and $v > 0$ such that whenever $U(y, r) \subset U(x, R)$ are balls for which $x \notin U(y, r)$, $R < d$ and $r/R > u$, then there exists a ball $U(z, a) \subset U(y, r) \cap V$ such that $a/r > v$.

This proposition easily implies the following characterization of p^*-interior points.

Proposition A*. A set $V \subset P$ is a p^*neigbourhood of a point $x \in V$ if and only if the following condition (C*) is satisfied. (C*) For any $u > 0$ there exist $d > 0$ and $v > 0$ such that whenever $U(y, r) \subset U(x, R)$ are balls for which $x \notin U(y, r)$, $R < d$

and $r/R > u$, then there exists a ball $U(z, a) \subset U(y, r)$ such that $a/r > v$ and $U(z, a) \subset^* V$.

Proof. Let V be a p^*-neighbourhood of x. Then there clearly exists a p-neighbourhood U of x such that $U \subset^* V$. Applying Proposition A to U we immediately obtain that the condition (C*) is satisfied. Now, conversely, suppose that the condition (C*) is satisfied. Denote by N the set of all points $x \in P \setminus V$ such that $P \setminus V$ is of the first category at x (a.e. $(P \setminus V) \cap U(x, r)$ is a first category set for an $r > 0$). Since N is of the first category at all its points we obtain (cf. [1]) that N is a first category set. Now put $\tilde{V} = V \cup N$. It is easy to see that (C*) implies that (C) is satisfied for the set $\tilde{V}$. Thus by Proposition A $\tilde{V}$ is a p-neighbourhood of x and consequently V is a p^*-neighbourhood of x.

3. Main theorem

Theorem. Let $A \subset \mathbb{R}$ have the Baire property. Then the following conditions are equivalent. (1) O is an J-density point of A.

(2) For any increasing sequence of positive integers (n_k) there exists a subsequence (n_{k_p}) such that $\lim X_{n_{k_p} A} = 1$ on a residual subset of $\mathbb{R}$.

(3) For any open interval $\emptyset \neq I \subset (-1, 1)$ and for any infinite set $K \subset \mathbb{N}$ there exist an open interval $\emptyset \neq J \subset I$ and an infinite set $L \subset K$ such that $(1/n)\, J \subset^* A$ for all $n \in L$.

(4) For any open interval $\emptyset \neq I \subset \mathbb{R}$ and any infinite set $K \subset \mathbb{N}$ there exists an open interval $\emptyset \neq J \subset I$ and an infinite set $L \subset K$ such that $(1/n)\, J \subset^* A$ for all $n \in L$.

(5) For any $0 < c < 1$ there exist $\varepsilon > 0$ and $\delta > 0$ such that for any $0 < x < \delta$ there exist open intervals I_1, I_2 of the length at least εx such that

$$I_1 \subset^* A \cap (x - cx, x) \quad \text{and} \quad I_2 \subset^* A \cap (-x, -x + cx)\,.$$

(6) O is a p^*-interior point of $A \cup \{0\}$.

Proof. We shall proceed by the following scheme:

$$(2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (4) \Rightarrow (2)\,, \quad 5 \Leftrightarrow 6\,.$$

The implication $(2) \Rightarrow (1)$ is trivial.

$(1) \Rightarrow (3)$: Let an open interval $\emptyset \neq I \subset (-1, 1)$ and an infinite set $K \subset \mathbb{N}$ be given. Suppose that $K = \{n_1, n_2, \ldots\}$, where (n_k) is an increasing sequence and find a subsequence (n_{k_p}) by (1). For this subsequence we have that the set

$$\{x \in (-1, 1);\ \lim_{p \to \infty} X_{n_{k_p} A}(x) = 1\} = \Big(\bigcup_{s=1}^{\infty} \bigcap_{p=s}^{\infty} n_{k_p} A\Big) \cap (-1, 1)$$

is residual in $(-1, 1)$. Consequently there exists $s_0 \in \mathbb{N}$ such that the set $T =$

$= \left(\bigcap_{p=s_0}^{\infty} n_{k_p} A\right) \cap (-1, 1)$ is of the second category. Since T has clearly the Baire property, we conclude that there is an open interval $\emptyset \neq J \subset I$ such that $J \cap T$ is residual in J. This implies that for $p \geqq s_0$ $J \subset^* n_{k_p} A$ and consequently $(1/n_{k_p}) J \subset^* A$. Therefore it is sufficient to put $L = \{n_{k_{s_0}}, n_{k_{s_0+1}}, \ldots\}$.

(3) $\Rightarrow$ (5): Suppose on the contrary that (3) holds and (5) does not hold. Since the desired properties of the intervals I_1, I_2 are symmetrical, we can suppose without a loss of generality that we have $0 < c < 1$ such that for all $\varepsilon > 0$ and $\delta > 0$ there exists $0 < x < \delta$ such that any interval $I \subset^* A \cap (x - cx, x)$ has the length less than εx. Consequently we can find a sequence $0 < x_n < 1$, $x_n \to 0$ such that any interval $I \subset^* A \cap (x_n - cx_n, x_n)$ has the length less than x_n/n. Let (m_n) be the sequence of positive integers determined by the inequalities $(m_n + 1)^{-1} < x_n \leqq \leqq (m_n)^{-1}$. An elementary computation gives

$$\lim_{n\to\infty} (x_n - (m_n)^{-1})(x_n)^{-1} = 0\,. \tag{7}$$

Using the condition (3) to $I = (1 - c, 1)$ and $K = \{m_1, m_2, \ldots\}$ we find corresponding $J \subset I$ and $L \subset K$. We can suppose that $J = (a, b)$, where $1 - c < a < < b < 1$. Now we shall prove that for sufficiently large n we have $(m_n)^{-1}(a, b) \subset \subset (x_n - cx_n, x_n)$. In fact,

$$a(m_n)^{-1} - (x_n - cx_n) = x_n(a - 1 + c) + a((m_n)^{-1} - x_n) > 0$$

and (7) implies that for sufficiently large n

$$\begin{aligned} x_n - (m_n)^{-1} b &= (1 - b) x_n + b(x_n - (m_n)^{-1}) = \\ &= x_n(1 - b + b(x_n - (m_n)^{-1})(x_n)^{-1}) > 0\,. \end{aligned}$$

Thus we can choose $k \in L$ such that $(b - a) > k^{-1}$ and $(m_k)^{-1}(a, b) \subset (x_k - cx_k, x_k)$. By the definition of L and (a, b) we have $(m_k)^{-1}(a, b) \subset^* (x_k - cx_k, x_k) \cap A$ and since the length of the interval $(m_k)^{-1}(a, b)$ equals $(m_k)^{-1}(b - a) > x_k k^{-1}$ we obtain a contradiction with the definition of the sequence (x_n).

(5) $\Rightarrow$ (4): To prove (4) suppose that an open interval $I \neq \emptyset$ and an infinite set $K \subset \mathbb{N}$ are given. We can clearly suppose without a loss of generality that I is of the form $I = (y - cy, y)$, $0 < y$, $0 < c < 1$. Find numbers $\varepsilon > 0$, $\delta > 0$ which corresponds to c by the condition (5). Let $y - cy = z_0 < z_1 < z_2 < \ldots < z_m = y$ be an equidistant division of I for which $\Delta = z_{i+1} - z_i < (\varepsilon/2)\,\mu(I)$. The condition (5) implies that for any $n \in \mathbb{N}$ such that $n^{-1}y < \delta$ we can find an interval $T_n \subset^* \subset^* n^{-1}I \cap A$ for which $\mu(T_n) \geqq \varepsilon n^{-1}\,\mu(I)$. Since clearly $nT_n \subset I$ and $\mu(nT_n) \geqq \varepsilon\,\mu(I)$, we can choose an index $i_n \in \{1, \ldots, m\}$ such that $J_{i_n} := (z_{i_n - 1}, z_{i_n}) \subset nT_n$. Since the set K is infinite we can find an index $i \in \{1, \ldots, m\}$ for which the set $L := \{n \in K;\ n^{-1}y < \delta,\ i_n = i\}$ is infinite. If we put $J = J_i$, we have in fact for $n \in L$ $n^{-1}J = = n^{-1}J_{i_n} \subset T_n \subset^* A$.

(4) $\Rightarrow$ (2): Let an increasing sequence of positive integers (n_k) be given. Let $(I_k)_{k=1}^{\infty}$

be a sequence of all open intervals with rational endpoints. Put $L_0 = \{n_1, n_2, \ldots\}$. Applying (4) to $K = L_0$ and $I = I_1$ we obtain an open interval $\emptyset \neq J_1 \subset I_1$ and an infinite set $L_1 \subset K$ such that $n^{-1}J_1 \subset^* A$ for all $n \in L_1$. In the second step we apply (4) to $K = L_1$ and $I = I_2$; we obtain correspondding L_2 and $J_2 \subset I_2$. In this way we obtain a sequence of infinite sets $L_0 \supset L_1 \supset L_2 \supset \ldots$ and a sequence of open intervals $(J_n)_{n=1}^{\infty}$ such that $J_n \subset I_n$ and there exists a sequence of first category sets $(Z_{k,n})_{k,n=1}^{\infty}$ such that $k^{-1}J_n \setminus Z_{k,n} \subset A$ for any $k \in L_n$. Now choose an increasing sequence (m_j) for which $m_j \in L_j$. Clearly (m_j) is a subsequence of (n_k). Let now a point $x \in V := \bigcup_{n=1}^{\infty} J_n \setminus \bigcup\{kZ_{k,n};\ n \in \mathbb{N},\ k \in L_n\}$ be given. Choose an index n such that $x \in J_n$. Since $x \notin Z_{k,n}$ for any $k \in L_n$ we have that $k^{-1}x \in A$ for any $k \in L_n$. Consequently we have that $x \in m_j A$ for any $j \geqq n$. Therefore $\lim_{j \to \infty} X_{m_j A} = 1$ on V. Since $\bigcup_{n=1}^{\infty} J_n$ is clearly an open dense set we have that V is residual which proves (2).

(5) $\Rightarrow$ (6): Suppose that (5) holds. We shall prove that the condition (C*) from Proposition A^* holds. Let $u > 0$ be given. Put $c = \min(1/2, u)$ and find by (5) corresponding $\varepsilon > 0$ and $\delta > 0$. Put $d = \delta$ and $v = \varepsilon/2$ and suppose that balls $U(y, r) \subset U(0, R)$ such that $0 \notin U(y, r)$, $R < d$ and $r/R > u$ are given. Since the condition (5) is symmetrical, we can suppose without a loss of generality that $0 \leqq y - r < y + r \leqq R$. If we apply (5) to $x = y + r$ we obtain that there exists an open interval $I \subset^* A \cap (x - cx, x)$ of the length at least εx. Since $r(y + r)^{-1} \geqq$ $\geqq rR^{-1} > u \geqq C$, we have $x - cx = (y + r) - c(y + r) \geqq y$ and consequently $I \subset^* A \cap U(y, r)$. Obviously $I \subset U(y, r)$. Since $x \geqq 2r$, we see that for $U(z, a) := I$ we have $a \geqq \varepsilon r$ and consequently $a/r \geqq \varepsilon > v$.

(6) $\Rightarrow$ (5): Suppose that (6) holds and choose a $0 < c < 1$. Put $u = c/4$ and find by the condition (C*) corresponding $d > 0$, $v > 0$. Put $\delta = d$, $\varepsilon = vc$ and suppose that a number $0 < x < \delta$ si given. Put $R = x$, $y = x - cx/2$ and $r = cx/2$. Then $\emptyset \notin (x - cx, x) = U(y, r) \subset U(x, R) = (-x, x)$, $R = x < \delta = d$ and $r/R = c/2 >$ $> c/4 = u$. Consequently there exists a ball $U(z, a) \subset U(y, r) = (x - cx, x)$ such that $U(z, a) \subset^* V$ and $a > rv$. Consequently we have for the length of the open interval $I_1 := U(z, a)$ the inequality $2a > 2rv = cxv = \varepsilon x$. In the symmetrical way we can find the interval I_2.

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