On Generalized Fuzzy Relation Equations: Necessary and Sufficient Conditions for the Existence of Solutions

E. TURUNEN

University of Tampere, Department of Mathematics*)

Received 29 May 1986

The characterization of solvability of the relational equation $R \circ X = T$, where R, X, T are fuzzy relations, X the unknown one, and $\circ$ the minimum-induced composition given by Sanchez ([6]), is extended to compositions induced by more general products in general value lattices. Moreover, the procedure also applies to systems of equations.

V práci je rozšířena Sanchezova charakteristika ([6]) řešitelnosti rovnice $R \circ X = T$, kde R, X, T jsou fuzzy-relace (X neznámá) a $\circ$ je skládání indukované minimem, na rovnice se skládáním indukovaným obecnějšími produkty na obecném svazu hodnot. Výsledky jsou dále rozšířeny na systémy rovnic.

В работе расширяется характеристика Санчеза ([6]) решимости уравнения $R \circ X = T$, где R, X, T фаззи-отношения (X неизвестная) и $\circ$ — композиция индуцированная минимумом, для уравнений с композицией индуцированной более общими продуктами в общих структурах значений. Результаты далее обобщаются на системы уравнений.

These results were obtained in 1985–86 at the Charles University when the author received a post-graduate exchange scholarship of The Academy of Finland and Czechoslovak Academy of Sciences.

In [6], Sanchez studied the solvability of the equations $R \circ X = T$ and $X \circ S = T$, where R, S, T and X are fuzzy relations, X the unknown one, and the composition is the sup-min composition (see below). In this paper we show that these results can be easily generalized for all compositions generated by residuation structures on the value lattice. The procedure can be applied also to the systems of relation equations, to generalize the results of Gottwald [3].

Introduction

Let U, V be sets. Recall that a binary *fuzzy relation* R on $U \times V$ is a fuzzy set on $U \times V$, i.e., R is a function

$$R: U \times V \to L,$$

where L is a lattice. In such a case we will write $R \subseteq U \times V$.

*) Kalevantie 4, 331 00 Tampere, Finland.

The *inverse* of R, denoted by R^{-1}, is the fuzzy relation on $V \times U$ defined by $R^{-1}\langle v, u\rangle = R\langle u, v\rangle$. The *identical* fuzzy relation I is defined by

$$I\langle u, v\rangle = \begin{cases} 1 & \text{for} \quad u = v \\ 0 & \text{otherwise}. \end{cases}$$

The *sup-min composition* of two fuzzy relations $R \subsetneqq U \times V$, $S \subsetneqq V \times W$ is defined by

$$R \circ S\langle u, w\rangle = \bigvee_{v \in V} (R\langle u, v\rangle \wedge S\langle v, w\rangle) \quad \text{for each} \quad u \in U, \quad w \in W,$$

where $\bigvee$ stands for sup and $\wedge$ stands for min.

E. Sanchez [6] has studied the fuzzy relation equations

(1) $$X \circ S = T$$

and

(2) $$R \circ Y = T,$$

where $R \subsetneqq U \times V$, $S \subsetneqq V \times W$ and $T \subsetneqq U \times W$ are fixed and $X \subsetneqq U \times V$, $Y \subsetneqq V \times W$ are unknown fuzzy relations with values in a Brouwerian lattice (i.e., in a complete lattice satisfying the complete distributivity rule $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$). He introduced the operation α to compose fuzzy relations by

$$R\alpha S\langle u, w\rangle = \bigwedge_{v \in V} (R\langle u, v\rangle \, \alpha S\langle v, w\rangle) \quad \text{for each} \quad u \in U, \quad w \in W,$$

where $\bigwedge$ stands for inf and α is an operator on L defined by

$$a\alpha b = \bigvee\{x \in L / a \wedge x \leqq b\} \quad \text{for each} \quad a, b \in L.$$

Sanchez proved

Theorem A. The fuzzy relation equation (1) has a solution iff $(S\alpha T^{-1})^{-1}$ is a solution. If a solution exists, then $(S\alpha T^{-1})^{-1}$ is the greatest one.
and

Theorem B. The fuzzy relation equation (2) has a solution iff $R^{-1}\alpha T$ is a solution. If a solution exists, then $R^{-1}\alpha T$ is the greatest one.

On Generalized Fuzzy Relation Equations

Let L be a generalized residuated lattice (cf. [1], [2], [5]) i.e., a lattice endowed by binary operations μ, h_1, h_2 such that
(3) μ is isotone and associative,
(4) $\mu(x, y) \leqq z$ iff $x \leqq h_1(y, z)$, $\mu(x, y) \leqq z$ iff $y \leqq h_2(x, z)$.

The operation μ will be usually denoted simply as

$$\mu(x, y) = xy$$

and will be called the product in L.

Since we have the Galois correspondences (4) between $\mu(-, y)$ and $h_1(y, -)$, and between $\mu(x, -)$ and $h_2(x, -)$, the product μ preserves suprema in both variables.

For examples of residuated lattices see e.g. [1], [2], [4].

Remark. Any Brouwerian lattice can be viewed as a residuated lattice with $\mu = \bigwedge$. For more examples of the structure see e.g. [4].

Definition. Let L be a residuated lattice and let $R \subsetneq U \times V$, $S \subsetneq V \times W$ and $T \subsetneq U \times W$ be binary fuzzy relations with values in L. The *μ-composition* of R and S is a fuzzy relation $R\mu S \subsetneq U \times W$ defined by

$$R\mu S\langle u, w\rangle = \bigwedge_{v \in V} (R\langle u, v\rangle\, S\langle v, w\rangle) \quad \text{for each} \quad u \in U\,, \quad w \in W\,.$$

Fuzzy relation $H_1(S, T) \subsetneq U \times V$ and $H_2(R, T) \subsetneq V \times W$ are defined by

$$H_1(S, T)\,\langle u, v\rangle = \bigwedge_{w \in W} h_1(S\langle v, w\rangle, T\langle u, w\rangle) \quad \text{for each} \quad u \in U\,, \quad v \in V\,,$$

$$H_2(R, T)\,\langle v, w\rangle = \bigwedge_{u \in U} h_2(R\langle u, v\rangle, T\langle u, w\rangle) \quad \text{for each} \quad v \in V\,, \quad w \in W\,.$$

Observation. One sees easily that the operation μ on relations is associative.

Remark. In the case of μ commutative we have of course, $h_1 = h_2$. It is easy to see, that in this case

$$(R\mu S)^{-1} = S^{-1}\mu R^{-1}$$

and

$$H_2(R, T) = (H_1(R^{-1}, T^{-1}))^{-1}\,.$$

Theorem 1. We have $R\mu S \leqq T$ iff $R \leqq H_1(S, T)$ and $R\mu S \leqq T$ iff $S \leqq H_2(R, T)$.

Proof. $R\mu S \leqq T$

$$\Leftrightarrow$$

(5) $$\bigvee_{v} (R\langle u, v\rangle\, S\langle v, w\rangle) \leqq T\langle u, w\rangle \quad \text{for each} \quad u \in U\,, \quad w \in W$$

$$\Leftrightarrow$$

$$R\langle u, v\rangle\, S\langle v, w\rangle \leqq T\langle u, w\rangle \quad \text{for each} \quad u \in U\,, \quad v \in V\,, \quad w \in W$$

$$\Leftrightarrow$$

$$R\langle u, v\rangle \leqq h_1(S\langle v, w\rangle, T\langle u, w\rangle) \quad \text{for each} \quad u \in U\,, \quad v \in V\,, \quad w \in W$$

$$\Leftrightarrow$$

$$R\langle u, v\rangle \leqq \bigwedge_{w} h_1(S\langle v, w\rangle, T\langle u, w\rangle) \quad \text{for each} \quad u \in U\,, \quad v \in V$$

$$\Leftrightarrow$$

$$R \leqq H_1(S, T)\,.$$

The (in-) equality (5) is also equivalent to

$$S\langle v, w\rangle \leqq h_2(R\langle u, v\rangle, T\langle u, w\rangle) \quad \text{for each} \quad u \in U\,, \quad v \in V\,, \quad w \in W$$

$$\Leftrightarrow$$

$$S\langle v, w\rangle \leqq \bigwedge_u h_2(R\langle u, v\rangle, T\langle u, w\rangle) \quad \text{for each} \quad v \in V\,, \quad w \in W$$

$$\Leftrightarrow$$

$$S \leqq H_2(R, T)\,. \quad \square$$

Since $H_1(S, T) \leqq H_1(S, T)$ we obtain

Corollary 1. $H_1(S, T)\,\mu S \leqq T.$
and similarly, since $H_2(R, T) \leqq H_2(R, T)$.

Corollary 2. $R\mu H_2(R, T) \leqq T.$

Theorem 2. The fuzzy relation equation

$$X\mu S = T \tag{6}$$

has a solution X iff $H_1(S, T)$ is a solution. If a solution exists, then $H_1(S, T)$ is the largest one.

Proof. If $H_1(S, T)$ is a solution, then (6) has a solution. Let R be a solution of (6). Then

$$R \leqq H_1(S, T) \tag{7}$$

and, by the isotonicity of μ, we have

$$T = R\mu S \leqq H_1(S, T)\,\mu S \leqq T \quad \text{(by Corollary 1)}\,,$$

i.e., $H_1(S, T)$ is a solution and, by (7), also the largest one. $\square$
Similarly, using Corollary 2, we easily obtain

Theorem 3. The fuzzy relation equation

$$R\mu Y = T \tag{8}$$

has a solution Y iff $H_2(R, T)$ is a solution. It is also the largest one, if a solution exists.
In the similar way one immediately obtains

Theorem 4. A system of fuzzy relation equations

$$X\mu S_i = T_i\,, \quad i = 1, \ldots, n \tag{9}$$

has a solution X iff $C = \bigcap_{i=1}^{n} H_1(S_i, T_i)$ is a solution. If a solution exists, then C is the largest one.
and

Theorem 5. A system of fuzzy relation equations

$$R_i \mu Y = T_i\,, \quad i = 1, \ldots, n \tag{10}$$

has a solution Y iff $D = \bigcap_{i=1}^{n} H_2(R_i, T_i)$ is a solution. If a solution exists, then D is the largest one.

Theorem 6. The fuzzy relation equation (6) has a solution X for each $T \subsetapprox U \times W$ iff $H_1(S, I)\,\mu S = I$.

Proof. If (6) has a solution for each T, it has it in particular for $T = I$ and hence

$$H_1(S, I)\,\mu S = I\,. \tag{11}$$

On the other hand if (11) holds, we have, for $X = T\mu H_1(S, I)$

$$X\mu S = T\mu(H_1(S, I)\,\mu S) = T\,. \quad \square$$

Similarly, we have

Theorem 7. The fuzzy relation equation (8) has a solution Y for each $T \subsetapprox U \times W$ iff $R\mu H_2(R, I) = I$.

References

[1] Dilworth, R. P.: Abstract residuation over lattices. Bull. Amer. Math. Soc. 44 (1938) 262—268.

[2] Dilworth, R. P. and Ward.: Residuated lattices. Trans. Amer. Math. Soc. 45 (1939) 335—354.

[3] Gottwald, S.: On the existence of solutions of systems of fuzzy relations. Fuzzy Sets and Systems 12 (1984) 301—302.

[4] Menu, J. and Pavelka, J.: A note on tensor products on the unit interval. Comment. Math. Univ. Carolinae. 17,1 (1976) 71—83.

[5] Pavelka, J.: On fuzzy logic II, Z. f Math. Logik 25 (1979) 119—134.

[6] Sanchez, E.: Resolution of composite fuzzy relation equation. Inf. Control 30 (1976) 38—48.